Event-triggered state estimation for recurrent neural networks with unknown time-varying delays

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***Abstract*—We consider the problem of event-triggered state estimation for recurrent neural networks subject to unknown time-varying delays by proposing a robust dynamic event-triggered state observer. A method based on a novel state observer and a dynamic event-triggered mechanism (ETM) is proposed to provide robust state estimation of the delayed recurrent neural networks. The significance of the new dynamic ETM is that it helps to reduce unnecessary transmissions from the sensors to the observer. A sufficient condition for the existence of the dynamic event-triggered state observer in terms of a convex optimization problem is proposed based on Lyapunov theory combined with free-weighting matrix technique and some useful inequalities such as Wirtinger-based integral inequality, Cauchy matrix inequality and reciprocally convex combination inequality. The effectiveness of the proposed estimation method is demonstrated by two numerical examples and simulation results.**

***Index Terms*—Neural networks, event-triggered state estimation, dynamic event-triggering mechanism, unknown time-varying delays.**

I. Introduction

Neural network systems have received considerable attention in recent years due to their advantages in learning ability, parallel computation, industrial automation, and function approximation [1], [2], [3]. A wide range of applications of neural network systems in fault diagnosis, pattern recognition, signal processing, communication, image processing has been reported [2], [4]. It has been well realized that the information of state vectors is necessary for many neural network analysis problems. However, information of the state of neural networks is often unavailable for many reasons such as technical, economic. Therefore, the problem of estimating state vectors for neural networks has been widely considered (see, for example, [5], [6], [7]). For example, in [5], state estimation problem was investigated for a class of delay neural networks. By using Lyapunov theory combined with the Jensen inequality, the authors in [6] designed an exponential state estimator for switched Hopfield neural networks. Estimating the state vector of static delay neural networks has been studied in [7] by using an improved reciprocally convex inequality.

Note that in all the above works [5], [6], [7], state observers for neural networks are typically implemented based on a time-triggered mechanism. On the other hand, event-triggered mechanism (ETM) is a new approach in state estimation which can improve the efficiency in resource utilization of the network components [8]. So far, this new control and estimation approach based on ETMs has attracted a great deal of research attention. For instance, in [9], the author proposed an event-triggered scheduler based on a state feedback controller in order to solve stabilization problem on embedded processors while event-triggered adaptive control technique was investigated in [10], [11] and event-triggered output regulation problem was solved in [12]. The state estimation problem has also attracted significant research attention (see, for examples, [13], [14], [15], [16], [17], [8], [18], [19], [20]). In particular, for stochastic dynamical systems, in [13], an event-triggered particle filter was proposed for smart grids with limited communication bandwidth infrastructure, while an event-triggered cubature Kalman filter was designed in [14] for estimating the set vectors of power systems. Especially, in [15], the authors provided some interesting event-triggered nonlinear filters in order to estimate state of wide-area measurement systems in power grid. For deterministic dynamical systems, methods based on a state observer and an ETM for estimating state vectors of dynamical systems were proposed in [16], [17], [8], [18]. Reference [19] provided a method to estimate state vectors of interconnected systems with disturbances based on event-triggered functional observers, while a robust state observer on the basis of the continuous dynamic ETM was discussed in [20]. It is worth noting that the event-driven conditions in [16], [17], [8], [18], [19] only depend on output information, while the one in [20] depends on continuous supervision, which could cause wastage of communication resources. Up to now, the continuous dynamic ETM in [20] has not been developed to the case discrete supervision combined with observers for estimation the state vectors of neural networks.

We study the event-triggered state estimation problem for neural networks subject to unknown time-varying delays in this paper. The contributions of this paper are: (1) For the first time, a dynamic ETM is designed and utilized for recurrent neural networks; (2) the structure of the observer is new and can reduce the utilization of communication resources while maintaining the desired robust estimation performance; and (3) based on Wirtinger-based integral inequality combined Lyapunov theory, a sufficient condition for the existence of the dynamic event-triggered state observer is established and the unknown observer matrices are obtained via solving a convex optimization problem.

In the next Section, we present some preliminaries. In Section III, the event-triggered state estimation problem for neural networks is solved based on the design of a dynamic

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ETM and a state observer. The effectiveness of the proposed method is demonstrated in Section IV by two examples and simulations. Finally, the conclusion of the paper is given in Section V.

The symbols used in this paper are listed in the following table:

M^T	matrix transpose
$\|\| \cdot \|\|$	Euclidean norm of a matrix or a vector
$\mathbb{R}^n$	$n-$ dimensional linear vector space over $\mathbb{R}$
$P > 0$	$x^T P x > 0, \forall x \neq 0$
sym$\{M\}$	$M + M^T$
$*$	the entries of a matrix implied by symmetry
$\|\|f\|\|_{\mathcal{L}_2^n}$	$(\int_0^\infty \|\|f(t)\|\|^2 dt)^{\frac{1}{2}}$
$\mathcal{L}_2^n$	$\{f : [0, \infty) \to \mathbb{R}^n : \|\|f\|\|_{\mathcal{L}_2^n} < \infty\}$

II. Preliminaries and problem statement

Let us consider the following recurrent neural networks subject to unknown time-varying delay $\tau(t)$

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-\tau(t)) \\ &\quad +Bu(t) \;\; t \geq 0, \quad (1) \\ x(\theta) &= \phi(\theta), \;\; \theta \in [-\tau_2, 0], \quad (2) \\ y(t) &= Cx(t), \quad (3) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state vector, control input vector, output vector, respectively. $f(x(t)) = \begin{bmatrix} f_1(x_1(t)) \\ \vdots \\ f_n(x_n(t)) \end{bmatrix}$, $g(x(t-\tau(t))) = \begin{bmatrix} g_1(x_1(t-\tau(t))) \\ \vdots \\ g_n(x_n(t-\tau(t))) \end{bmatrix}$ denote the neuron activation functions with $f_i(0) = 0, g_i(0) = 0 (i = 1, \ldots, n)$, $A = \text{diag}\{a_1, a_2, \ldots, a_n\} \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, $W_0 \in \mathbb{R}^{n \times n}$ is the weight matrix, $W_1 \in \mathbb{R}^{n \times n}$ is the delayed weight matrix. $B \in \mathbb{R}^{n \times m}$ is a constant matrix, $\phi(\theta)$ is the initial function. The activation functions f, g and time-varying delay $\tau(t)$ satisfy the following assumptions:

(H_1) There are positive scalars $\ell_i > 0, k_i > 0$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} |f_i(v_1) - f_i(v_2)| &\leq \ell_i |v_1 - v_2|, \; \forall v_1, v_2 \in \mathbb{R}, \\ |g_i(v_1) - g_i(v_2)| &\leq k_i |v_1 - v_2|, \; \forall v_1, v_2 \in \mathbb{R}. \end{aligned} \quad (4)$$

(H_2) $\tau(t)$ is unknown but bounded within the interval $[\tau_1, \tau_2]$, where $\tau_1 > 0$ and $\tau_2 > 0$ are known.

Lemma 1. ([21] Wirtinger-based integral inequality) Let matrix $R > 0$ and $v : [a, b] \to \mathbb{R}^n$ be a differentiable function. Then,

$$\int_a^b \dot{v}^T(s) R \dot{v}(s) ds \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2, \quad (5)$$

where

$$\Omega_1 = v(b) - v(a), \quad (6)$$

$$\Omega_2 = v(b) + v(a) - \frac{2}{b-a} \int_a^b v(s) ds. \quad (7)$$

Lemma 2. ([22] Reciprocal convexity lemma) Given $\delta \in [0, 1]$, matrices $R > 0$, S and vectors x_1, x_2. The following inequality holds

$$\begin{aligned} &-\frac{1}{\delta} x_1^T R x_1 - \frac{1}{1-\delta} x_2^T R x_2 \\ \leq \quad & -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} R & S \\ * & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (8)$$

subject to

$$\begin{bmatrix} R & S \\ * & R \end{bmatrix} > 0. \quad (9)$$

Lemma 3. ([23]) For $r, s \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{n \times n}$, $M > 0$, the following inequality holds

$$\pm 2 r^T s \leq r^T M r + s^T M^{-1} s. \quad (10)$$

III. Main Results

In the following, we assume that the measurement output $y(t)$ of delayed recurrent neural networks (1)-(3) is time-triggered with data sampling technique and zero-order hold (ZOH),

$$y^\star(t) = y(kh) = Cx(kh), \; t \in [kh + \nu_k, kh + h + \nu_{k+1}), \quad (11)$$

where $h > 0$ is a sampling period, ν_k is the network-induced delay, $\nu_k \in [0, \nu]$, $k = 1, 2, 3, \ldots$.

In the following, we propose a new event-triggered condition and design an event-triggered state observer for system (1)-(3) with requirement that information $y(kh)$ is only transmitted when the event-triggered condition holds.

Consider the following event-triggered state observer for delayed recurrent neural networks (1)-(3):

$$\begin{aligned} \dot{z}(t) &= -Az(t) + W_0 f(z(t)) + W_1 g(z(t - \tau_z)) + Bu(t) \\ &\quad +L(y(t_k) - Cz(t_k), \\ &\quad \tau_z = \frac{\tau_1 + \tau_2}{2}, \; t \in [t_k + \nu_k, t_{k+1} + \nu_{k+1}), \quad (12) \\ z(s) &= \psi(s), \; s \in [-\tau_2, 0], \quad (13) \end{aligned}$$

where L is the gain matrix to be designed and $z(t) \in \mathbb{R}^n$ is the state vector of observer, $\{t_k\}_{k \in \mathbb{N}}$ is the triggering sequence which is defined as below:

$$\begin{aligned} &t_0 = 0, \; t_{k+1} = t_k + \ell_k h, \\ &\ell_k = \min\Big\{ \ell \in \mathbb{N}^+ \; |\mathcal{F}(\epsilon_{xz}(t_{k_\ell}), e_{xz}(t_k)) > \gamma(t_k) \Big\}, \end{aligned} \quad (14)$$

where $\mathcal{F}(\epsilon_{xz}(t_{k_\ell}), e_{xz}(t_k)) = \alpha[\epsilon_{xz}^T(t_{k_\ell}) \Delta \epsilon_{xz}(t_{k_\ell}) - \beta e_{xz}^T(t_k) \Delta e_{xz}(t_k)]$, $e_{xz}(t) = x(t) - z(t)$, $\epsilon_{xz}(t_{k_\ell}) = e_{xz}(t_k) - e_{xz}(t_k + \ell h)$, $\alpha > 0$, $\beta > 0$ are positive scalars, Δ is a positive definite matrix, $\gamma(t)$ is the internal dynamic variable satisfying the following condition

$$\dot{\gamma}(t) = -\rho \gamma(t) + \beta e_{xz}^T(t_k) \Delta e_{xz}(t_k) - \epsilon_{xz}^T(t_{k_\ell}) \Delta \epsilon_{xz}(t_{k_\ell}), \quad (15)$$

$$\gamma(0) = 0, \; \rho > 0. \quad (16)$$

Lemma 4: Assume that β is a positive scalar, ρ, α are positive scalars satisfying condition

$$e^{\rho \eta h} \leq \alpha \rho + 1, \quad (17)$$

where h is the sampling period and η is the smallest integer number such that $h \leq t_{k+1} - t_k \leq \eta h$. Then, the internal dynamic variable $\gamma(t)$ satisfies $\gamma(t) \geq 0$ for all $t > 0$.

Proof. From condition (14), we see that

$$\gamma(t_k) + \alpha[\beta e_{xz}^T(t_k)\Delta e_{xz}(t_k) - \epsilon_{xz}^T(t_{k_\ell})\Delta \epsilon_{xz}(t_{k_\ell})] \geq 0, \quad \forall t > 0. \tag{18}$$

Since $\alpha > 0$, we obtain

$$\beta e_{xz}^T(t_k)\Delta e_{xz}(t_k) - \epsilon_{xz}^T(t_{k_\ell})\Delta \epsilon_{xz}(t_{k_\ell}) \geq -\frac{1}{\alpha}\gamma(t_k). \tag{19}$$

On the other hand, for all $t \in [t_k, t_{k+1})$, from (16) we obtain

$$\begin{aligned} &\gamma(t) = e^{-\rho(t-t_k)}\gamma(t_k) \\ &+ \int_{t_k}^{t} e^{-\rho(t-s)}(\beta e_{xz}^T(t_k)\Delta e_{xz}(t_k) - \epsilon_{xz}^T(t_{k_\ell})\Delta \epsilon_{xz}(t_{k_\ell}))ds. \end{aligned} \tag{20}$$

By using (19), we obtain

$$\gamma(t) \geq e^{-\rho(t-t_k)}\gamma(t_k) - \frac{1}{\alpha}\gamma(t_k)\int_{t_k}^{t} e^{-\rho(t-s)})ds \tag{21}$$

It follows from (21) that

$$\begin{aligned} \gamma(t) &\geq \Big(e^{-\rho(t-t_k)}(1 + \frac{1}{\alpha\rho}) - \frac{1}{\alpha\rho}\Big)\gamma(t_k) \\ &\geq \Big(e^{-\rho\eta h}(1 + \frac{1}{\alpha\rho}) - \frac{1}{\alpha\rho}\Big)\gamma(t_k). \end{aligned} \tag{22}$$

It follows from (17) that $e^{-\rho\eta h}(1 + \frac{1}{\alpha\rho}) - \frac{1}{\alpha\rho} > 0$. Therefore, from (18), (19) and (22), we obtain

$$\begin{aligned} \dot{\gamma}(t) &\geq -\rho\gamma(t) - \frac{1}{\alpha}\gamma(t_k) \\ &\geq -\Big(\rho + \frac{1}{\alpha(e^{-\rho\eta h}(1 + \frac{1}{\alpha\rho}) - \frac{1}{\alpha\rho})}\Big)\gamma(t), \end{aligned} \tag{23}$$

for all $t > 0$ and thus $\gamma(t) \geq 0$. The proof is completed.

Remark 1: Under assumption (H_2), instead of using $z(t - \tau(t))$ in (12), which is not possible due to the unknown nature of the time-varying delay, we use its estimate $z(t - \tau_z)$. For this, there is an error, where

$$\omega(t) = z(t - \tau(t)) - z(t - \tau_z) = -\int_{-\tau(t)}^{-\tau_z} \dot{z}(t + \theta)d\theta. \tag{24}$$

The error $\omega(t)$ satisfies inequality [24]:

$$||\omega||_{\mathcal{L}_2^n} \leqslant \frac{\tau_2 - \tau_1}{2}||v||_{\mathcal{L}_2^n}, \tag{25}$$

for all $v(t) \in \mathcal{L}_2^n$. Moreover, we obtain the following relation:

$$\begin{aligned} &||x(t - \tau(t)) - z(t - \tau_z))||^2 \\ &\leq \frac{3}{2}\Big(||e_{xz}(t - \tau(t))||^2 + ||\omega(t)||^2\Big). \end{aligned} \tag{26}$$

Remark 2: Unlike the event-triggered conditions in [17], [16], [19], which only depend on output information of the systems, the one in (14) in this paper depends on not only error vectors $e_{xz}(t)$, $\epsilon_{xz}(t_{k_\ell})$ but also dynamic variable $\gamma(t)$. This indicates that, ETM (14) can enlarge the time intervals between two consecutive triggering events. Moreover, in comparison with the continuous-time event-triggered condition in [20], the event-triggered mechanism (14) offers more advantages since the event-triggered condition is dependent on discrete supervision and therefore can reduce the utilization of communication resources while maintaining the desired control performance.

Noting that for

$$s = \min_{\ell \in \mathbb{N}}\Big\{\ell \mid t_k + (\ell + 1)h + \nu_k \geq t_{k+1} + \nu_{k+1}\Big\},$$

we can divide interval $[t_k + \nu_k, t_{k+1} + \nu_{k+1})$ into the following subintervals

$$[t_k + \nu_k, t_{k+1} + \nu_{k+1}) = \cup_{n=0}^{s} I_n, \tag{27}$$

where $I_n = [t_k + nh + \nu_k, t_k + (n + 1)h + \nu_k)$ for $n = 1, 2, \ldots, s - 1$ and $I_s = [t_k + sh + \nu_k, t_{k+1} + \nu_{k+1})$.

By denoting $f_{xz}^{\Delta}(t) = f(x(t)) - f(z(t))$ and $g_{xz}^{\Delta}(t) = g(x(t - \tau(t))) - g(z(t - \tau_z))$, from (1) and (12), we obtain

$$\begin{aligned} \dot{e_{xz}}(t) &= -Ae_{xz}(t) + W_0 f_{xz}^{\Delta}(t) + W_1 g_{xz}^{\Delta}(t) \\ &\quad -LCe_{xz}(t_k),\ t \in [t_k + \nu_k, t_{k+1} + \nu_{k+1}), \end{aligned} \tag{28}$$

$$e_{xz}(s) = \phi(s) - \psi(s),\ s \in [-\tau_2, 0]. \tag{29}$$

Next, we define time-varying delay $\zeta(t)$ in the interval $[t_k + \nu_k, t_{k+1} + \nu_{k+1})$ as follows:

$$\zeta(t) = t - t_k - \ell h,\ t \in I_n. \tag{30}$$

It follows from (30) that $\epsilon_{xz}(t_{k_\ell}) = e_{xz}(t_k) - e_{xz}(t - \zeta(t))$ and $0 \leq \zeta(t) \leq h + \tau_2 = \zeta$ for all $t \in [t_k + \nu_k, t_{k+1} + \nu_{k+1})$. Therefore, the error system (28)-(29) can be expressed as follows:

$$\begin{aligned} \dot{e_{xz}}(t) &= -Ae_{xz}(t) + W_0 f_{xz}^{\Delta}(t) + W_1 g_{xz}^{\Delta}(t) \\ &\quad -LCe_{xz}(t - \zeta(t)) - LC\epsilon_{xz}(t_{k_\ell}), \\ &\quad t \in [t_k + \nu_k, t_{k+1} + \nu_{k+1}), \end{aligned} \tag{31}$$

$$e_{xz}(s) = \phi(s) - \psi(s),\ s \in [-\zeta, 0]. \tag{32}$$

Remark 3: Our newly proposed event-triggered state observer (12)-(13) is different from the ones in [17], [16], [19], [20], due to the term $Cz(t)$ is replaced by $Cz(t_k + nh) = Cz(t - \zeta(t))$. This change is significant due to the fact that the measurement received by the observers after a network delay $\zeta(t)$.

Provided that ETM (14) is designed, our objective is to design gain observer L such that:

(i) The error dynamical system (31)-(32) with two time-varying delays $\tau(t)$ and $\zeta(t)$ is asymptotically stable for the case where $\omega(t) \equiv 0$;

(ii) For the case where $\omega(t) \neq 0$ and initial condition of $e_{xz}(t)$ is zero, the following inequality is guaranteed:

$$||e_{xz}||_{\mathcal{L}_2^n} \leq \lambda||\omega||_{\mathcal{L}_2^n}, \tag{33}$$

where λ is a positive scalar being optimised.

Remark 4: Unlike the method reported in [20], where error vector $e_{xz}(t)$ holds condition

$$||e_{xz}||_{\mathcal{L}_2^n} \leq \delta||x||_{\mathcal{L}_2^n} + \lambda||\omega||_{\mathcal{L}_2^{\ell}}, \tag{34}$$

where $0 < \delta < 1$ and $\lambda > 0$, the state estimation error $e_{xz}(t)$ in this paper satisfies (33), which is more advantage since the error between the state of neural networks and its estimation does not depend on the state $x(t)$.

In the following, we employ Wirtinger-based integral, the reciprocally convex approach, the free-weighting matrix technique inequality and Cauchy matrix inequality combined Lyapunov theory to establish a sufficient condition in term linear matrix inequalities (LMIs) to guarantee (33) with a minimized level $\lambda > 0$ and gain matrix L.

Theorem 1. Under Assumptions (H_1)-(H_2) and the assumption of Lemma 4, for a given positive scalar μ, system (31)-(32) is asymptotically stable with a minimized level $\lambda > 0$, such that condition (33) holds if there exist matrices $P > 0$, $Q > 0$, $R > 0$, $\Delta > 0$, X, non-singular H and positive scalars λ_ω, δ_1, δ_2 which solve problem:

$$\min(\lambda_\omega) \tag{35}$$

subject to

$$\Omega^\star(\sigma) = \begin{bmatrix} \Omega(\sigma) & \Theta \\ * & \Xi \end{bmatrix} < 0, \ \forall \sigma \in \{0, 1\}, \tag{36}$$

$$\Phi = \begin{bmatrix} \mathrm{diag}(R, R) & Z \\ * & \mathrm{diag}(R, R) \end{bmatrix} > 0, \tag{37}$$

where

$$\begin{aligned}
\Omega(\sigma) &= \Omega_1(\sigma) + \Omega_2(\sigma) \\
&\quad + \delta_1 \ell_{\max}^2 g_1^T g_1 + \delta_2 \frac{3}{2} k_{\max}^2 g_7^T g_7, \\
&\quad \ell_{\max} = \max\{\ell_1, \ldots, \ell_n\}, \\
&\quad k_{\max} = \max\{k_1, \ldots, k_n\}, \\
\Omega_1(\sigma) &= \mathrm{sym}\{\Psi_\sigma^T P \Gamma\} + g_1^T Q g_1 - g_3^T Q g_3 \\
&\quad + \zeta^2 g_4^T R g_4 - \Lambda^T \Phi \Lambda + g_1^T g_1 \\
&\quad - \lambda_\omega g_9^T g_9 + \delta_2 \frac{3}{2} k_{\max}^2 g_9^T g_9 + \beta g_2^T \Delta g_2 \\
&\quad + \beta g_7^T \Delta g_7 - \mathrm{sym}\Big\{ \begin{bmatrix} g_1^T & g_4^T \end{bmatrix} \begin{bmatrix} \mu H \\ H \end{bmatrix} g_4 \Big\},
\end{aligned}$$

$$\begin{aligned}
\Omega_2(\sigma) &= \mathrm{sym}\Big\{ \begin{bmatrix} g_1^T & g_4^T \end{bmatrix} \begin{bmatrix} \mu H \\ H \end{bmatrix} A g_1 \\
&\quad - \begin{bmatrix} g_1^T & g_4^T \end{bmatrix} \begin{bmatrix} \mu X \\ X \end{bmatrix} C g_2 \\
&\quad - \begin{bmatrix} g_1^T & g_4^T \end{bmatrix} \begin{bmatrix} \mu X \\ X \end{bmatrix} C g_8, \\
&\quad \Psi_\sigma = \begin{bmatrix} g_1^T & \sigma \zeta g_5^T + (1-\sigma) \zeta g_6^T \end{bmatrix}^T, \\
&\quad \Gamma = \begin{bmatrix} g_4^T & (g_1^T - g_3^T) \end{bmatrix}^T, \ \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}^T, \\
&\quad \Lambda_1 = \begin{bmatrix} (g_1 - g_2)^T & \sqrt{3}(g_1 + g_2 - 2g_5)^T \end{bmatrix}, \\
&\quad \Lambda_2 = \begin{bmatrix} (g_2 - g_3)^T & \sqrt{3}(g_2 + g_3 - 2g_6)^T \end{bmatrix}, \\
&\quad g_i = \begin{bmatrix} 0_{n \times (i-1)n} & I_n & 0_{n \times (9-i)n} \end{bmatrix} \in \mathbb{R}^{n \times 9n}, \\
&\qquad\qquad i = 1, \ldots, 9, \\
&\quad \Theta = \begin{bmatrix} g_1^T & g_4^T \end{bmatrix} \begin{bmatrix} \mu H \\ H \end{bmatrix} (W_0 h_1 + W_1 h_2), \\
&\quad h_1 = \begin{bmatrix} I_n & 0_{n \times n} \end{bmatrix}, \ h_2 = \begin{bmatrix} 0_{n \times n} & I_n \end{bmatrix}, \\
&\quad \Xi = -\mathrm{diag}(\delta_1 I_n, \delta_2 I_n).
\end{aligned}$$

The optimal disturbance attenuation level λ and observer gain matrix L are obtained as

$$\lambda = \sqrt{\lambda_\omega}, \ L = H^{-1} X. \tag{38}$$

Algorithm 1

Step 1: Check condition (4).
Step 2: Given a scalar $\mu > 0$, solve the convex optimization problem (35)-(37), $P > 0$, $Q > 0$, $R > 0$, $\Delta > 0$, X, non-singular H and positive scalars λ_ω, δ_1, δ_2, minimized level λ and observer gain L.
Step 3: Given scalars $h > 0$, α, $\beta > 0$, $\rho > 0$, obtain ETM (14) and observer (12)-(13).

IV. Numerical Examples

Example 1. Consider recurrent neural networks (1)-(3), where $\tau(t) = |\sin t|$ and

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \ B = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.1 & -0.02 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0.01 & 0 & -0.1 & 0 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0.02 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0.1 & 0.1 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ f(x(t)) = \begin{bmatrix} \tanh x_1(t) \\ \tanh x_2(t) \\ \tanh x_3(t) \\ \tanh x_4(t) \end{bmatrix}$$

and

$$\begin{aligned}
&g(x(t - \tau(t))) \\
&= 0.5 \begin{bmatrix} |x_1(t-\tau(t)) + 1| - |x_1(t-\tau(t)) - 1| \\ |x_2(t-\tau(t)) + 1| - |x_2(t-\tau(t)) - 1| \\ |x_3(t-\tau(t)) + 1| - |x_3(t-\tau(t)) - 1| \\ |x_4(t-\tau(t)) + 1| - |x_4(t-\tau(t)) - 1| \end{bmatrix}.
\end{aligned}$$

It is easy to check that condition (4) is satisfied with $\ell_i = 1$, $k_i = 1$ for all $i = 1, 2, 3, 4$, and $\ell_{\max} = k_{\max} = 1$.

According to Step 2 of Algorithm 1, for given $\mu = 2$, we obtain

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0.3282 & -0.0073 & 0.0040 & -0.0053 \\ -0.0073 & 1.6807 & -0.6807 & 0.9012 \\ 0.0040 & -0.6807 & 21.9232 & 20.2009 \\ -0.0053 & 0.9012 & 20.2009 & 22.7257 \\ -0.8710 & 0.0357 & -0.0213 & 0.0233 \\ -0.0056 & -4.2873 & 0.4247 & -1.7198 \\ -0.0039 & 1.4821 & -39.7292 & -27.3105 \\ -0.0081 & -2.2636 & -39.0349 & -34.8083 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -0.8710 & -0.0056 & -0.0039 & -0.0081 \\ 0.0357 & -4.2873 & 1.4821 & -2.2636 \\ -0.0213 & 0.4247 & -39.7292 & -39.0349 \\ 0.0233 & -1.7198 & -27.3105 & -34.8083 \\ 2.5108 & 0.0082 & 0.0251 & 0.0281 \\ 0.0082 & 27.1551 & -6.3393 & 17.9094 \\ 0.0251 & -6.3393 & 330.2087 & 297.1027 \\ 0.0281 & 17.9094 & 297.1027 & 323.2255 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.0335 & 0.0024 & -0.0023 & 0.0002 \\ 0.0024 & 2.4157 & -1.9085 & 0.8602 \\ -0.0023 & -1.9085 & 35.3096 & 32.6135 \\ 0.0002 & 0.8602 & 32.6135 & 36.4605 \end{bmatrix},$$

$$R = \begin{bmatrix} 3.6579 & -0.0273 & 0.0063 & -0.0227 \\ -0.0273 & 8.5368 & -3.3849 & 2.9111 \\ 0.0063 & -3.3849 & 79.9630 & 67.5294 \\ -0.0227 & 2.9111 & 67.5294 & 71.8374 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 0.0078 & 0 & 0 & 0 \\ 0 & 0.0164 & 0 & 0 \\ 0 & 0 & 0.0164 & 0 \\ 0 & 0 & 0 & 0.0164 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.0558 & 0 \\ -0.0078 & 0.006 \\ 0.0036 & 0 \\ -0.0037 & -0.004 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.0779 & -0.0062 & 0.0029 & -0.0033 \\ 0.0123 & 0.3205 & -0.2121 & 0.1272 \\ -0.0031 & -0.1953 & 3.7800 & 3.4332 \\ 0.0020 & 0.0847 & 3.0339 & 3.3685 \end{bmatrix},$$

$$\lambda_\omega = 0.0.1169,\ \delta_1 = 0.0032,\ \delta_2 = 0.0777,$$

and therefore minimized level $\lambda = 0.3419$ and gain matrix

$$L = \begin{bmatrix} 0.7127 & 0.0019 \\ -0.0607 & 0.0437 \\ -0.0089 & 0.0237 \\ 0.0081 & -0.0236 \end{bmatrix}.$$

According to Step 3 of Algorithm 1, given $h = 0.02$, $\alpha = 0.3$, $\beta = 0.01$, $\rho = 0.4$, $\gamma(0) = 0$, ETM (14) and observer (12)-(13) are obtained.

Example 2. Let us consider the following recurrent neural networks of the form (1)-(3), where $\tau(t) = 0.98 \sin t$ and

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 0.2 & -0.4 & 0.4 \\ -0.4 & 0.2 & 0.2 \\ 0.2 & 0.4 & -0.4 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$f(x(t)) = \begin{bmatrix} \sin x_1(t) \\ \sin x_2(t) \\ \sin x_3(t) \end{bmatrix}.$$

We can check that condition(4) is satisfied with $\ell_i = 1$ for all $i = 1, 2, 3$, and $\ell_{\max} = 1$.

Given $\mu = 10$, solving (35)-(37), we obtain

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1.0377 & -0.2778 & -0.1745 \\ -0.2778 & 1.3360 & -0.2412 \\ -0.1745 & -0.2412 & 1.0205 \\ -2.0813 & -0.2664 & 0.1463 \\ 0.6373 & -1.9977 & 0.6040 \\ 0.2110 & -0.4154 & -2.0580 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -2.0813 & 0.6373 & 0.2110 \\ -0.2664 & -1.9977 & -0.4154 \\ 0.1463 & 0.6040 & -2.0580 \\ 9.4023 & -1.1802 & 0.0294 \\ -1.1802 & 6.6105 & -1.0521 \\ 0.0294 & -1.0521 & 9.5240 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.5611 & -0.2117 & -0.3085 \\ -0.2117 & 0.4786 & -0.2190 \\ -0.3085 & -0.2190 & 0.5125 \end{bmatrix},$$

$$R = \begin{bmatrix} 2.9906 & -0.9781 & -1.3164 \\ -0.9781 & 4.3396 & -0.9409 \\ -1.3164 & -0.9409 & 2.7195 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 0.4082 & -0.0119 & -0.0247 \\ -0.0119 & 0.4210 & -0.0128 \\ -0.0247 & -0.0128 & 0.4046 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.0047 & -0.0008 \\ 0.0024 & 0.0078 \\ -0.0002 & -0.0007 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.0645 & -0.0360 & -0.0226 \\ -0.0041 & 0.1048 & 0.0019 \\ -0.0206 & -0.0356 & 0.0619 \end{bmatrix},$$

$$\lambda_\omega = 0.063,\ \delta_1 = 0.2002,\ \delta_2 = 0.0419,$$

and therefore minimized level $\lambda = 0.2509$ and gain matrix

$$L = \begin{bmatrix} 0.1037 & 0.0466 \\ 0.0261 & 0.0753 \\ 0.0457 & 0.0482 \end{bmatrix}.$$

According to Step 3 of Algorithm 1, given $h = 0.02$, $\alpha = 0.4$, $\beta = 0.2$, $\rho = 0.5$, and $\gamma(0) = 0$, we obtain ETM (14) and observer (12)-(13).

V. Conclusion

In this paper, we have considered the problem of designing event-triggered state estimation for delayed recurrent neural networks. A new dynamic ETM has been first designed and then based on it a state observer has been proposed to estimate state vectors of the delayed recurrent neural networks. Next, a sufficient condition in term LMIs for the existence of the dynamic event-triggered state observer has been established. Finally, two numerical examples and simulation results have been provided to illustrate the effectiveness of the results.

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