

STABILITY ANALYSIS FOR NONLOCAL EVOLUTION EQUATIONS INVOLVING INFINITE DELAYS

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ABSTRACT. We deal with the inquiry about stability for nonlocal differential equations involving infinite delays. The dissipativity, stability and weak stability of solutions are addressed by using local estimates, fixed point arguments and a new Halanay type inequality. Our analysis is based on suitable assumptions on the phase space and nonlinearity function. An application to nonlocal partial differential equations will be shown to demonstrate our abstract results.

1. INTRODUCTION

We consider the following problem

$$\frac{d}{dt}[k * (u - u_0)](t) + Au(t) = f(t, u_t), \quad t > 0, \quad (1.1)$$

$$u_0 = \varphi \in \mathcal{B}, \quad (1.2)$$

where the unknown function u takes values in a separable Hilbert space H , the kernel $k \in L^1_{loc}(\mathbb{R}^+)$, the notation ‘ $*$ ’ denotes the Laplace convolution, A is an unbounded linear selfadjoint operator and $f : \mathbb{R}^+ \times \mathcal{B} \rightarrow H$ is a given nonlinear function. The admissible phase space \mathcal{B} satisfies certain conditions that will be defined later. In our model, u_t stands for the history of the state function up to the time t , i.e. $u_t(s) = u(t + s)$, $s \leq 0$.

It is worth pointing out that the system under consideration includes some important classical cases with respect to the kernel function k being of special ones (see, e.g. [12, 14]). Namely, if $k(t) = g_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha)$, $\alpha \in (0, 1)$ then equation (1.1) is in the form of fractional differential equations since the convolution represents D_0^α , the Caputo fractional derivative of order α . Regarding the fractional differential systems involving finite delays in Banach spaces, some results on (weak) stability and decay solutions were established in [1, 10, 11]. Based on the special features (e.g., the analyticity, subordinate principle) associated with the kernel $g_{1-\alpha}(t)$, the fractional differential equations can be considered in a more general framework:

$$\begin{cases} D_0^\alpha[x(t) - h(t, x_t)] = Bx(t) + f(t, x(t), x_t), t > 0 \\ x(\theta) = \varphi(\theta), \theta \leq 0. \end{cases} \quad (1.3)$$

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where B is the infinitesimal generator of an analytic semigroup on a Banach space X . Considering abstract neutral functional differential equations like (1.3) involving infinite delays, we refer the reader to [16, 18] for the existence of integral solutions, and [2] for existence of integral solutions with decay rate. Noting that the approach in the mentioned works heavily relies on the point-wise decaying of the Mittag-Leffler functions $E_{\alpha,\beta}(z)$, which is no longer available for the general nonlocal derivatives. Nevertheless, system (1.1) without delay has received considerably attention over decades. It has been employed to model different problems related to processes in materials with memory (see, e.g. [4, 6, 19]). Particularly, Vergara and Zacher [23] mentioned that equation (1.1) with an appropriate class of kernels can be used to describe the anomalous diffusion phenomena including slow/ultraslow diffusions when $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$, and $A = -\Delta$ is the Laplace operator associated with the homogeneous boundary condition of Dirichlet/Neumann type. We also refer to [15, 12] and the references therein for recent development on this trend.

It should be mentioned that, in modeling of physical/biological processes, the formulated system is usually subject to the history information, that is, a delay term comes into the model. The authors in the recent work [14] studied (1.1) in the case of finite delay, i.e. $\varphi \in C([-h, 0]; H)$, where some stability results were obtained. As far as we know, this is the first attempt dealing with stability analysis for nonlocal differential equations involving delays. In this work, we consider the case that φ belongs to the fading memory spaces, which were axiomatically introduced by Hale and Kato in [7]. This situation is entirely different from that in [14] due to the complicated structure of phase spaces. Our aim is to find a class of admissible phase spaces and conditions on the nonlinearity function f under which our problem is solvable, and its solution is stable/weakly stable. To this end, we first make the following fundamental hypotheses.

- (A0) *The operator $A : D(A) \rightarrow H$ is self-adjoint on H and its spectral $\sigma(A)$ is bounded from below, that is, there exists $\lambda_1 := \lambda_1(A) \in \mathbb{R}$ such that $\sigma(A) \subset [\lambda_1, +\infty)$.*
- (K) *The kernel function $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0, \infty)$.*

Hypothesis (K) enables us to get a representation of solutions for (1.1)-(1.2). This hypothesis has been used in a wide range of works (see [15, 12, 14, 20, 23, 24]).

Let us give a brief on our approach. The well-posedness of linear equation is followed by Prüss' theory of resolvent families and Lemma 2 below, which extends the recent result [12, Lemma 2.3]. Namely, the existence and qualitative properties of the solution operators are established for a general semibounded self-adjoint operator. The assumption (A0) also covers the case A has a negative spectrum, see Lemma 2.2(1). The solvability of (1.1)-(1.2) is obtained by fixed point argument. This will be done by proving the compactness of the Cauchy operator in Proposition 2.3 without regularity assumption on the kernel, see Remark 2.1. The stability of solution to (1.1)-(1.2) is proved by applying a new Halanay type inequality, which is more flexible, in comparison with the one in [14]. In addition, we make use of the technique developed in [3, 14] to get the weakly asymptotic stability result, which is based on the fixed point principle for condensing maps on a special constructed subset. We find that the sufficient conditions for weak stability result, Theorem 4.4, only depend on the asymptotic behavior of the coefficients of the system for large time. This phenomenon provides a compatible observation of existence result

in finite time, where one has no restriction on the magnitude of Lipschitz constant of the nonlinear function. Our setting is more practical and relaxes some conditions proposed by previous works in the literature.

The paper is organized as follows. In Section 2, we collect some necessary results on the theory of resolvent, establish a compactness of the Cauchy operator and propose a new Halanay type inequality. Section 3 is devoted to studying the existence of mild solutions and the dissipativity via the existence of absorbing sets. In section 4, the stability results and the weakly asymptotic stability of the zero solution are formulated under suitable assumptions on the nonlinearity as well as on the phase space. The last section presents an application to a class of nonlocal partial differential equations with infinite delays.

2. PRELIMINARIES

2.1. Phase spaces. We will utilize the axiomatic definition of the phase space \mathcal{B} introduced by Hale and Kato in [7]. The space \mathcal{B} is a linear subspace of functions mapping $(-\infty, 0]$ into H endowed with a suitable seminorm $|\cdot|_{\mathcal{B}}$ and satisfying the following fundamental axioms. If a function $v : (-\infty, T + \sigma] \rightarrow H$ is such that $v|_{[\sigma, T + \sigma]} \in C([\sigma, T + \sigma]; H)$ and $v_{\sigma} \in \mathcal{B}$, then

- (B1) $v_t \in \mathcal{B}$ for $t \in [\sigma, T + \sigma]$;
- (B2) the function $t \mapsto v_t$ is continuous on $[\sigma, T + \sigma]$;
- (B3) $|v_t|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} \|v(s)\| + M(t - \sigma)|v_{\sigma}|_{\mathcal{B}}$, where $K, M : [0, \infty) \rightarrow [0, \infty)$ are independent of v , and K is continuous, M is locally bounded.

In this work, we use a further assumption on \mathcal{B} :

- (B4) there exists $\varrho > 0$ such that $\|\varphi(0)\| \leq \varrho|\varphi|_{\mathcal{B}}$, for all $\varphi \in \mathcal{B}$.

We give here some examples of phase spaces. We refer to the book by Hino, Mukarami and Naito [8] for more details. The first one is given by

$$C_{\gamma} = \{\varphi \in C((-\infty, 0]; H) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } H\},$$

for a given positive number γ . This phase space satisfies (B1)-(B3) with

$$K(t) = 1, \quad M(t) = e^{-\gamma t},$$

and it is a Banach space with the following norm

$$|\varphi|_{\mathcal{B}} = \sup_{\theta \leq 0} e^{\gamma\theta} \|\varphi(\theta)\|.$$

Considering another typical example, assume that $1 \leq p < +\infty, 0 \leq r < +\infty$ and $g : (-\infty, -r] \rightarrow \mathbb{R}$ is nonnegative, Borel measurable function on $(-\infty, -r)$. Let CL_g^p be a class of functions $\varphi : (-\infty, 0] \rightarrow H$ such that φ is continuous on $[-r, 0]$ and $g(\theta)\|\varphi(\theta)\|^p \in L^1(-\infty, -r)$. The associated seminorm in CL_g^p is given by

$$|\varphi|_{CL_g^p} = \sup_{-r \leq \theta \leq 0} \|\varphi(\theta)\| + \left[\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta \right]^{\frac{1}{p}}.$$

Assume further that

$$\int_s^{-r} g(\theta)d\theta < +\infty, \text{ for every } s \in (-\infty, -r) \text{ and} \quad (2.1)$$

$$g(s + \theta) \leq G(s)g(\theta) \text{ for } s \leq 0 \text{ and } \theta \in (-\infty, -r), \quad (2.2)$$

where $G : (-\infty, 0] \rightarrow \mathbb{R}^+$ is locally bounded function. We know from [8], CL_g^p satisfies (B1)-(B3) provided that (2.1)-(2.2) hold true. Moreover, one can take

$$K(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq r, \\ 1 + \left[\int_{-t}^{-r} g(\theta) d\theta \right]^{\frac{1}{p}} & \text{for } t > r; \end{cases} \quad (2.3)$$

$$M(t) = \begin{cases} \max \left\{ 1 + \left[\int_{-t}^{-r} g(\theta) d\theta \right]^{\frac{1}{p}}, G(-t)^{\frac{1}{p}} \right\} & \text{for } 0 \leq t \leq r, \\ \max \left\{ \left[\int_{-t}^{-r} g(\theta) d\theta \right]^{\frac{1}{p}}, G(-t)^{\frac{1}{p}} \right\} & \text{for } t > r. \end{cases} \quad (2.4)$$

2.2. The resolvent families. Consider the following scalar Volterra equations which describe the relaxation functions

$$s(t) + \lambda(l * s)(t) = 1, \quad t \geq 0, \quad (2.5)$$

$$r(t) + \lambda(l * r)(t) = l(t), \quad t > 0. \quad (2.6)$$

The existence and uniqueness of s and r were mentioned in [17]. Denote by $s(\cdot, \lambda)$ and $r(\cdot, \lambda)$ the solutions of (2.5) and (2.6), respectively. Recall that the function l is called a completely positive kernel iff $s(\cdot)$ and $r(\cdot)$ take nonnegative values for every $\lambda > 0$. The complete positivity of l is equivalent to that (see [4, Theorem 2.2]), there exist $\alpha \geq 0$ and $k \in L_{loc}^1(\mathbb{R}^+)$ nonnegative and nonincreasing which satisfy $\alpha l(t) + l * k(t) = 1$ for all $t > 0$. So the hypothesis (K) implies that l is completely positive and particularly, l takes nonnegative values by [4, Proposition 2.1 (1)]. Consequently, the functions $s(\cdot, \lambda)$ and $r(\cdot, \lambda)$ take nonnegative values (for even $\lambda \leq 0$, see also explanation in [24]). We remind some additional properties of these functions.

Proposition 2.1. [12, 24] *Let the hypothesis (K) hold. Then for every $\lambda \in \mathbb{R}$, $s(\cdot, \lambda), r(\cdot, \lambda) \in L_{loc}^1(\mathbb{R}^+)$. In addition, we have:*

- (1) *The function $s(\cdot, \lambda)$ is nonnegative and nonincreasing. Moreover, for $\lambda > 0$*

$$s(t, \lambda) \left[1 + \lambda \int_0^t l(\tau) d\tau \right] \leq 1, \quad \forall t \geq 0. \quad (2.7)$$

Hence if $l \notin L^1(\mathbb{R}^+)$ then $\lim_{t \rightarrow \infty} s(t, \lambda) = 0$ for every $\lambda > 0$.

- (2) *The function $r(\cdot, \lambda)$ is nonnegative and one has*

$$s(t, \lambda) = 1 - \lambda \int_0^t r(\tau, \lambda) d\tau = k * r(\cdot, \lambda)(t), \quad t \geq 0,$$

so $\int_0^t r(\tau, \lambda) d\tau \leq \lambda^{-1}$, $\forall t > 0$. If $l \notin L^1(\mathbb{R}^+)$ then $\int_0^\infty r(\tau, \lambda) d\tau = \lambda^{-1}$ for every $\lambda > 0$.

- (3) *For each $t > 0$, the functions $\lambda \mapsto s(t, \lambda)$ and $\lambda \mapsto r(t, \lambda)$ are nonincreasing in \mathbb{R} .*
(4) *Equation (2.5) is equivalent to the problem*

$$\frac{d}{dt} [k * (s - 1)] + \lambda s = 0, \quad s(0) = 1.$$

- (5) *Let $v(t) = s(t, \lambda)v_0 + (r(\cdot, \lambda) * g)(t)$, here $g \in L_{loc}^1(\mathbb{R}^+)$. Then v solves the problem*

$$\frac{d}{dt} [k * (v - v_0)](t) + \lambda v(t) = g(t), \quad v(0) = v_0.$$

Using spectral theorem for self-adjoint operator [22, Theorem 1.7], the hypothesis (A0) implies that there exist a measure space $(\Xi, d\mu)$, a unitary map $U : L^2(\Xi, d\mu) \rightarrow H$ and a real-valued function a on Ξ such that

$$U^{-1}AUf(\xi) = a(\xi)f(\xi), \quad Uf \in D(A). \quad (2.8)$$

Note that for $f \in L^2(\Xi, d\mu)$, $Uf \in D(A)$ iff $M_a f(\cdot) = a(\cdot)f(\cdot)$ belongs to $L^2(\Xi, d\mu)$.

Based on this spectral representation of A , the Borelian functional calculus of A is given by

$$(U^{-1}g(A)Uv)(\xi) = g(a(\xi))v(\xi), \quad \text{for almost every } \xi \in \Xi, \quad (2.9)$$

for arbitrary Borel function $g : \mathbb{R} \rightarrow \mathbb{C}$. In general, $g(A)$ is unbounded linear self-adjoint in H for an arbitrary Borel real-valued function g . If g is bounded in \mathbb{R} then so is $g(A)$ and $\|g(A)\|_{L(H)} \leq \|g\|_{L^\infty(\mathbb{R})}$.

The spectral boundedness from below of A implies that

$$a(\xi) \in [\lambda_1, +\infty), \quad \text{for almost every } \xi \in \Xi.$$

Therefore, the functional $g(A)$ only depends on the essential value of g in $[\lambda_1, +\infty)$. In particular, if $g \in L^\infty([\lambda_1, \infty), d\mu)$ then $g(A)$ is a bounded linear map in H and furthermore

$$\|g(A)\|_{L(H)} \leq \text{ess sup}_{\xi \in \Xi} |g(a(\xi))|. \quad (2.10)$$

Note that if $\lambda_1 \geq 0$ then for $\gamma \geq 0$, one can define the fractional power of A as follows

$$D(A^\gamma) = \{Uw \in H : w \in L^2(\Xi, d\mu), (a(\xi))^\gamma w(\xi) \in L^2(\Xi, d\mu)\},$$

$$U^{-1}A^\gamma U w(\xi) = a(\xi)^\gamma w(\xi), \quad Uw \in D(A^\gamma).$$

Let $V_\gamma = D(A^\gamma)$. Then V_γ is a Banach space endowed with the norm

$$\|v\|_{D(A^\gamma)} = \left(\int_{\Xi} (1 + |a(\xi)|^{2\gamma}) |U^{-1}v(\xi)|^2 d\mu \right)^{\frac{1}{2}}.$$

For $\lambda_1 > 0$, this is equivalent to the following norm

$$\|v\|_\gamma = \|A^\gamma v\|_H = \left(\int_{\Xi} |a(\xi)|^{2\gamma} |U^{-1}v(\xi)|^2 d\mu \right)^{\frac{1}{2}}. \quad (2.11)$$

Furthermore, for $\gamma > 0$, we can identify $V_{-\gamma}$ with the dual space V_γ^* of V_γ .

By formula (2.9) and properties of the functions $s(t, \mu), r(t, \mu)$, we now define the following operators

$$U^{-1}S(t)Uv(\xi) = s(t, a(\xi))v(\xi), \quad \xi \in \Xi, t \geq 0, Uv \in H, \quad (2.12)$$

$$U^{-1}R(t)Uv(\xi) = r(t, a(\xi))v(\xi), \quad \xi \in \Xi, t > 0, Uv \in H. \quad (2.13)$$

Obviously, $S(t)$ and $R(t)$ are linear self-adjoint operators in H and satisfy the following relation

$$S'(t, A) = -AR(t, A), \quad t > 0, \quad (2.14)$$

due to Proposition 2.1(2) and the relation (2.12)-(2.13). In the following lemma, we prove some basic properties of these operators which extend the recent result [12, Lemma 3.2]. By relation (2.14), the statement in Lemma 2.2 (2) below implies that $S(t, A)$ is differentiable in the sense of Prüss [19, Definition 1.4]. A consequence of (2.17) is a smoothing effect of the solution operator. This estimate plays an important role in analyzing semilinear nonlocal evolution equations since the assumption on the nonlinearity can be relaxed considerably, as mentioned in [19, Section 13.5].

Lemma 2.2. *Let $\{S(t)\}_{t \geq 0}$ and $\{R(t)\}_{t > 0}$, be the families of linear operators defined by (2.12) and (2.13), respectively and $T > 0$ be given.*

(1) *For each $v \in H$, $S(\cdot)v \in C([0, T]; H)$ and $AS(\cdot)v \in C((0, T]; H)$. Moreover,*

$$\|S(t)v\| \leq s(t, \lambda_1)\|v\|, \quad t \in [0, T];$$

$$\|AS(t)v\| \leq \begin{cases} \frac{\|v\|}{(1 * l)(t)}, & \text{if } \lambda_1 \geq 0, \\ |\lambda_1|s(t, -|\lambda_1|)\|v\|, & \text{if } \lambda_1 < 0, \end{cases}$$

for $t \in (0, T]$.

(2) *Let $v \in H$. Then $R(\cdot)v \in C((0, T]; H)$. Furthermore,*

$$\|R(t)v\| \leq r(t, \lambda_1)\|v\|, \quad t \in (0, T], \quad (2.15)$$

$$\|AR(t)v\| \leq r(t, \lambda_1)\|Av\|, \quad v \in D(A), t > 0. \quad (2.16)$$

In particular, for $t > 0$ one has $\|S'(t)v\| \leq r(t, \lambda_1)\|v\|_{D(A)}$ for all $v \in D(A)$.

(3) *Assume further that $\lambda_1 > 0$. Then the convolution with R possesses a smoothing effect in the sense that if $g \in C([0, T]; V_\gamma)$, $\gamma \geq 0$ then $A(R * g) \in C([0, T]; V_{\gamma - \frac{1}{2}})$*

$$\|A(R * g)(t)\|_{V_{\gamma - \frac{1}{2}}} \leq \left(\int_0^t r(t - \tau, \lambda_1) \|g(\tau)\|_{V_\gamma}^2 d\tau \right)^{\frac{1}{2}}, \quad t \in [0, T]. \quad (2.17)$$

Proof. The first part in (1) and (2) follow by the same argument as in [12, Lemma 2.3], so we verify only the remain statement in (1). By (2.10), one has

$$\|s(t, A)\|_{L(H)} \leq \sup_{\lambda \geq \lambda_1} s(t, \lambda) = s(t, \lambda_1), \quad (2.18)$$

where the last relation follows from the monotonicity of $s(t, \cdot)$ with respect to λ . Analogously, one also gets

$$\|As(t, A)\|_{L(H)} \leq \sup_{\xi} |a(\xi)|s(t, a(\xi)) \quad (2.19)$$

$$= \sup_{\lambda \geq \lambda_1} |\lambda|s(t, \lambda) \leq \sup_{\lambda \geq -|\lambda_1|} |\lambda|s(t, \lambda) \quad (2.20)$$

$$= \max \left\{ \sup_{-|\lambda_1| \leq \lambda \leq 0} |\lambda|s(t, \lambda), \sup_{\lambda > 0} \lambda s(t, \lambda) \right\} \quad (2.21)$$

$$\leq \max \left\{ |\lambda_1|s(t, -|\lambda_1|), \frac{1}{1 * l(t)} \right\}, \quad (2.22)$$

here the last inequality follows from Proposition 2.1 (1).

The proof of the first part in (3) goes similarly the one above, hence we show only the last estimate (2.17). Using the representation

$$U^{-1}A(R * U\hat{g})(t, \xi) = \int_0^t a(\xi)r(t - \tau, a(\xi))\hat{g}(\tau, \xi)d\tau,$$

where $\hat{g}(t, \cdot) = U^{-1}g(t)$, we have

$$\|AR * g(t)\|_{V_{\gamma - 1/2}}^2 = \int_{\Xi} a(\xi)^{2\gamma - 1} \left(\int_0^t a(\xi)r(t - \tau, a(\xi))\hat{g}(\tau, \xi)d\tau \right)^2 d\mu,$$

thanks to (2.11). Then utilizing the Hölder inequality, we get

$$\begin{aligned}
& \|AR * g(t)\|_{V_{\gamma-1/2}}^2 \\
& \leq \int_{\Xi} a(\xi)^{2\gamma-1} \left(\int_0^t a(\xi)r(t-\tau, a(\xi))d\tau \right) \left(\int_0^t a(\xi)r(t-\tau, a(\xi))|\hat{g}(\tau, \xi)|^2 d\tau \right) d\mu \\
& \leq \int_{\Xi} (1-s(t, a(\xi))) \left(\int_0^t a(\xi)^{2\gamma}r(t-\tau, a(\xi))|\hat{g}(\tau, \xi)|^2 d\tau \right) d\mu \\
& \leq \int_0^t \left(\int_{\Xi} r(t-\tau, a(\xi))a(\xi)^{2\gamma}|\hat{g}(\tau, \xi)|^2 d\mu \right) d\tau \\
& \leq \int_0^t \left(r(t-\tau, \lambda_1) \int_{\Xi} a(\xi)^{2\gamma}|\hat{g}(\tau, \xi)|^2 d\mu \right) d\tau = \int_0^t r(t-\tau, \lambda_1)\|g(\tau)\|_{V_{\gamma}}^2 d\tau,
\end{aligned}$$

which ensures (2.17). In particular, $\|R * g\|_{C([0, T]; V_{\gamma+1/2})} \leq \frac{1}{\sqrt{\lambda_1}}\|g\|_{C([0, T], V_{\gamma})}$. This is the half smoothing effect of the resolvent operator R . \square

Let E and F be Banach spaces. The notations $\mathcal{L}(E, F), \mathcal{K}(E, F)$ stand for spaces of bounded linear operators, linear compact operators from E to F , respectively. Note that $\mathcal{K}(E, F)$ is closed subset in $\mathcal{L}(E, F)$ with respect to the operator norm.

To gain the compactness of the solution operators, we need further assumption on the operator A as follows:

(A) *The operator $A : D(A) \rightarrow H$ is nonnegative, self-adjoint on H with compact resolvent.*

The assumption (A) yields the existence of an orthonormal basis of H consisting of eigenfunctions $\{e_n\}_{n=1}^{\infty}$ of operator A and we have

$$Av = \sum_{n=1}^{\infty} \lambda_n v_n e_n, \quad v_n = (v, e_n),$$

where $\lambda_n > 0$ is the eigenvalue corresponding to the eigenfunction e_n of A ,

$$D(A) = \left\{ v = \sum_{n=1}^{\infty} v_n e_n : \sum_{n=1}^{\infty} \lambda_n^2 v_n^2 < \infty \right\}.$$

We can assume that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. In this case, Ξ is the set \mathbb{N} of natural numbers and the measure $d\mu$ is thus the counting measure, the function $a(n) = \lambda_n, \forall n \in \mathbb{N}$ and the unitary map $U : L^2(\mathbb{N}, d\mu) \rightarrow H$ is given by

$$f = (f_1, f_2, \dots, f_n, \dots) \in L^2(\mathbb{N}) \mapsto Uf = \sum_{k=1}^{\infty} f_k e_k \in H.$$

For a real number s we denote $X_s = C([0, T]; D(A^s))$. We need the following result to investigate the existence results.

Proposition 2.3. *Let assumptions (A) and (K) hold. Then the operator*

$$Q : C([0, T]; D(A^s)) \rightarrow C([0, T]; D(A^{s+1/2})), f \mapsto Qf(t) := R * f(t)$$

is compact for any $s \in \mathbb{R}$.

Proof. Based on the approximation argument, the proof is divided into several steps.

Step 1. We first remind that for a given $g \in C[0, T]$, the convolution map $\mathcal{C}_g : C[0, T] \rightarrow C[0, T], v \mapsto g * v$ is compact. This is classical result, but for convenience

of the readers, we provide a proof here. Fix any bounded subset $D \subset C[0, T]$, that is, there exists a positive constant $R > 0$ such that

$$\|v\| \leq R, \forall v \in D.$$

Clearly, $|g * v(t)| \leq \|g\|_{L^1(0, T)} \max_{t \in [0, T]} |v(t)|$, $v \in D$, which implies the point-wise bounded of $\mathcal{C}_g(D)$.

On the other hand, for any $\epsilon > 0$, by the uniform continuity of g on $[0, T]$, one chooses a positive number $\delta < \epsilon / (2RT\|g\| + 1)$ such that

$$|g(s_1) - g(s_2)| \leq \frac{\epsilon}{2RT} \text{ for any } s_1, s_2 \in [0, T], |s_1 - s_2| \leq \delta.$$

For any $t_1, t_2 \in [0, T]$, $0 < t_2 - t_1 < \delta$ and $v \in D$, one has

$$\begin{aligned} |g * v(t_2) - g * v(t_1)| &\leq \int_0^{t_1} |g(t_1 - \tau) - g(t_2 - \tau)| |v(\tau)| ds + \int_{t_1}^{t_2} |g(t_2 - \tau) v(\tau)| ds \\ &\leq \frac{\epsilon}{2RT} \int_0^T |v(\tau)| ds + \max_{\tau \in [0, T]} |v(\tau)| |t_1 - t_2| \max_{\tau \in [0, T]} |g(\tau)| \\ &\leq \left(\frac{\epsilon}{2RT} T + \|g\| \delta \right) \sup_{v \in D} \|v\| \leq \epsilon, \end{aligned}$$

for any $v \in D$. Therefore, the equi-continuity of $\mathcal{C}_g(D)$ is testified. Thus, $\mathcal{C}_g(D)$ is relatively compact in $C[0, T]$ by Azelà-Ascoli Theorem.

Step 2. Extend the statement above to the singular kernel. For a given $g \in L^1(0, T)$ Then $g * : C[0, T] \rightarrow C[0, T]$ is compact. Indeed, by density of smooth function in $L^1(0, T)$ one can choose a sequence of continuous function g_n such that g_n converges to g in $L^1(0, T)$. Then we have

$$\begin{aligned} \|(\mathcal{C}_{g_n} - \mathcal{C}_g)v\| &= \sup_{t \in [0, T]} \left| \int_0^t (g_n(t - \tau) - g(t - \tau))v(\tau) d\tau \right| \\ &\leq \sup_{t \in [0, T]} \int_0^t |g_n(t - \tau) - g(t - \tau)| \cdot |v(\tau)| d\tau \\ &\leq \|g_n - g\|_{L^1(0, T)} \|v\|. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \|\mathcal{C}_{g_n} - \mathcal{C}_g\|_{\mathcal{L}(C[0, T])} = 0$, which implies the compactness of \mathcal{C}_g .

Step 3. Let denote

$$Q_n f(t) = \sum_{k=1}^n \left(\int_0^t r(t - \tau, \lambda_k) f_k(\tau) d\tau \right) e_k$$

for any $f = \sum_{k=1}^{\infty} f_k(t) e_k \in C([0, T]; D(A^s))$.

By Step 2, Q_n is a compact operator from $C([0, T]; D(A^s))$ to $C([0, T]; D(A^{s+1/2}))$.

Step 4. By Step 3, it reduces to show that Q_n converges to Q with respect to the operator norm in $\mathcal{L}(X_s, X_{s+1/2})$. Indeed, we have

$$\begin{aligned}
\|(Q - Q_n)v(t)\|_{D(A^{s+1/2})}^2 &= \sum_{k>n} |\lambda_k^{1+2s} \int_0^t r(t-\tau, \lambda_k) v_k(\tau) d\tau|^2 \\
&\leq \sum_{k>n} \int_0^t \lambda_k r(t-\tau, \lambda_k) d\tau \cdot \int_0^t r(t-\tau, \lambda_k) |\lambda_k^s v_k(\tau)|^2 d\tau \quad (\text{by Hölder's inequality}) \\
&\leq \int_0^t r(t-\tau, \lambda_n) \sum_{k>n} |\lambda_k^s v_k(\tau)|^2 d\tau \\
&\quad (\text{since } \int_0^t \lambda_k r(\tau, \lambda_k) d\tau = 1 - s(t, \lambda_k) \leq 1 \text{ and } r(\cdot, \lambda_k) \leq r(\cdot, \lambda_n) \text{ for } k > n) \\
&\leq \int_0^t r(t-\tau, \lambda_n) \|v(\tau)\|_{D(A^s)}^2 d\tau \\
&\leq \left(\int_0^t r(t-\tau, \lambda_n) d\tau \right) \sup_{s \in [0, T]} \|v(\tau)\|_{D(A^s)}^2 \\
&= \frac{1 - s(t, \lambda_n)}{\lambda_n} \|v\|_{X_s}^2.
\end{aligned}$$

Hence, we obtain

$$\|Q_n - Q\|_{X_{s+1/2}} = \sup_{t \in [0, T]} \|(Q - Q_n)v(t)\|_{D(A^{s+1/2})}^2 \leq \frac{1}{\lambda_n} \|v\|_{X_s}^2.$$

In other word, we get

$$\|Q - Q_n\|_{\mathcal{L}(X_s, X_{s+1/2})} \leq \lambda_n^{-1/2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This completes the proof. \square

Remark 2.1. *The standard argument for checking the compactness of a subset in $C([0, T], D(A^{s+1/2}))$ is applying Azelà-Ascoli Theorem directly. However, due to the singularity of the kernel l (so $r(\cdot, \lambda)$), it is difficult to testify the equicontinuity of $Q(D)$ directly without further regularity assumption on the function l . So the proof of compactness for Q in this work requires less conditions than those in [13, Lemma 3.5].*

2.3. Gronwall and Halanay type inequalities. The following proposition shows a Gronwall type inequality.

Proposition 2.4. [12] *Let v be a nonnegative function satisfying*

$$v(t) \leq s(t, \mu)v_0 + \int_0^t r(t-\tau, \mu)[\alpha(\tau) + \beta v(\tau)]d\tau, \quad t \geq 0,$$

for $\mu > 0, v_0 \geq 0, \beta > 0$ and $\alpha \in L_{loc}^1(\mathbb{R}^+)$. Then

$$v(t) \leq s(t, \mu - \beta)v_0 + \int_0^t r(t-\tau, \mu - \beta)\alpha(\tau)d\tau.$$

Let X be a Banach space. We denote by $BC(\mathbb{R}^+; X)$ the space of continuous and bounded functions defined on \mathbb{R}^+ taking values in X . It is a Banach space with the norm given by $\|y\|_{BC} = \sup_{t \geq 0} \|y(t)\|$. Let $BC(\mathbb{R}^+) = BC(\mathbb{R}^+; \mathbb{R})$ and

$BC_0 := \{v \in BC(\mathbb{R}^+) \mid \lim_{t \rightarrow \infty} v(t) = 0\}$. We prove the following Halanay type inequality, which plays an important role in our analysis.

Proposition 2.5 (Halanay type inequality). *Let v be a continuous and nonnegative function on \mathbb{R}^+ . Assume that for any $t > 0$,*

$$v(t) \leq p(t) + \int_0^t r(t-\tau, a)q(\tau)d\tau + b \int_0^t r(t-\tau, a) \sup_{\xi \in [\tau-\rho(\tau), \tau]} v(\xi)d\tau, \quad (2.23)$$

where $p, q \in BC(\mathbb{R}^+)$, $a > b \geq 0$ and ρ is a given function such that $t \geq \rho(t)$ for $t \geq 0$. Then $v \in BC(\mathbb{R}^+)$ and

$$v(t) \leq (\|p\|_{BC} + \|r(\cdot, a) * q\|_{BC}) \frac{a}{a-b}, \quad \forall t \geq 0. \quad (2.24)$$

Moreover, if $\lim_{t \rightarrow \infty} (t - \rho(t)) = \infty$ then

$$\limsup_{t \rightarrow \infty} v(t) \leq \limsup_{t \rightarrow \infty} \left(p(t) + (r(\cdot, a) * q)(t) \right) \frac{a}{a-b}. \quad (2.25)$$

In particular, for any $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that

$$v(t) \leq \frac{ar^*}{a-b} + \epsilon, \quad \forall t \geq T(\epsilon),$$

here $r^* = \limsup_{t \rightarrow \infty} p(t) + \limsup_{t \rightarrow \infty} r(\cdot, a) * q(t)$.

Proof. First, we prove (2.24). By (2.23), for all $t \in [0, T]$, one has

$$\begin{aligned} v(t) &\leq \|p + r(\cdot, a) * q\|_{BC} + b \sup_{\xi \in [0, T]} v(\xi) \int_0^t r(t-\tau, a) d\tau \\ &\leq \|p\|_{BC} + \|r(\cdot, a) * q\|_{BC} + b \sup_{\xi \in [0, T]} v(\xi) \frac{1-s(t, a)}{a} \\ &\leq \|p\|_{BC} + \|r(\cdot, a) * q\|_{BC} + \frac{b}{a} \sup_{\xi \in [0, T]} v(\xi). \end{aligned}$$

It implies

$$\sup_{\xi \in [0, T]} v(\xi) \leq \left(\|p\|_{BC} + \|r(\cdot, a) * q\|_{BC} \right) \frac{a}{a-b}.$$

Let $T \rightarrow \infty$, we get

$$\sup_{\xi \in [0, \infty)} v(\xi) \leq \left(\|p\|_{BC} + \|r(\cdot, a) * q\|_{BC} \right) \frac{a}{a-b}.$$

Thus (2.24) is verified.

We next show that (2.25) holds. Since $t - \rho(t) \rightarrow \infty$ as $t \rightarrow \infty$, it follows that for any $T > 0$ there exists $T_1 = T_1(T) > 0$ such that

$$t - \rho(t) \geq T, \quad \forall t \geq T_1,$$

and $T_1 \rightarrow \infty$ as $T \rightarrow \infty$. Using (2.23) with $t \geq T_1$ yields

$$\begin{aligned}
v(t) &\leq p(t) + (r(\cdot, a) * q)(t) + b \int_0^{T_1} r(t - \tau, a) \sup_{\xi \in [\tau - \rho(\tau), \tau]} v(\xi) d\tau \\
&\quad + b \int_{T_1}^t r(t - \tau, a) \sup_{\xi \in [\tau - \rho(\tau), \tau]} v(\xi) d\tau \\
&\leq p(t) + (r(\cdot, a) * q)(t) + b \|v\|_{BC} \int_0^{T_1} r(t - \tau, a) d\tau \\
&\quad + b \sup_{\xi \geq T} v(\xi) \int_{T_1}^t r(t - \tau, a) d\tau \\
&\leq p(t) + (r(\cdot, a) * q)(t) + C \int_{t-T_1}^t r(\xi, a) d\xi + \frac{b}{a} \sup_{\xi \geq T} v(\xi),
\end{aligned}$$

here $C = \left(\|p\|_{BC} + \|r(\cdot, a) * q\|_{BC} \right) \frac{ab}{a-b}$. Taking the supremum over $[2T_1, \infty)$, we have

$$\sup_{t \geq 2T_1} v(t) \leq \sup_{t \geq 2T_1} \left(p(t) + (r(\cdot, a) * q)(t) \right) + \frac{C}{a} s(T_1, a) + \frac{b}{a} \sup_{\xi \geq T} v(\xi).$$

Let $T \rightarrow \infty$, then $T_1 \rightarrow \infty$. So we obtain

$$\begin{aligned}
\lim_{T_1 \rightarrow \infty} \sup_{t \geq 2T_1} v(t) &\leq \lim_{T_1 \rightarrow \infty} \sup_{t \geq 2T_1} \left(p(t) + (r(\cdot, a) * q)(t) \right) \\
&\quad + \lim_{T_1 \rightarrow \infty} \frac{C}{a} s(T_1, a) + \lim_{T \rightarrow \infty} \frac{b}{a} \sup_{\xi \geq T} v(\xi).
\end{aligned}$$

Hence, it implies

$$\limsup_{t \rightarrow \infty} v(t) \leq \left(\limsup_{t \rightarrow \infty} p(t) + \limsup_{t \rightarrow \infty} (r(\cdot, a) * q)(t) \right) \frac{a}{a-b}.$$

Consequently, the last statement in Proposition 2.5 holds. The proof is complete. \square

Corollary 2.6. *If $p \in BC_0$ and $q(t) = q_0(t) + q_\infty(t)$, $t \geq 0$, $q_0 \in BC_0, q_\infty \in BC(\mathbb{R}^+)$ then there exists $T(\epsilon) > 0$, for each given $\epsilon > 0$, such that*

$$v(t) \leq \frac{\|q_\infty\|_{BC}}{a-b} + \epsilon, \quad t \geq T(\epsilon).$$

2.4. Definition of mild solutions. For $\varphi \in \mathcal{B}$, we define

$$\mathbf{C}_\varphi = \{u \in C([0, T]; H) : u(0) = \varphi(0)\}$$

as a closed subset of $C([0, T]; H)$ with respect to the supremum norm denoted by $\|\cdot\|_\infty$.

For any $v \in \mathbf{C}_\varphi$, we define the function $v[\varphi] : \mathbb{R} \rightarrow H$ as follows

$$v[\varphi](t) = \begin{cases} \varphi(t), & -\infty < t \leq 0, \\ v(t), & t > 0. \end{cases}$$

Then, clearly

$$v[\varphi]_t(\theta) = \begin{cases} \varphi(t + \theta), & -\infty < \theta < -t, \\ v(t + \theta), & \theta \in [-t, 0]. \end{cases}$$

Motivated by arguments in [12], we have the following definition of mild solution to the system (1.1)-(1.2).

Definition 2.1. A function $u \in C((-\infty, T]; H)$ is said to be a mild solution to (1.1)-(1.2) on $(-\infty, T]$ iff $u(t) = \varphi(t)$ for $t \in (-\infty, 0]$ and

$$u(t) = S(t)\varphi(0) + \int_0^t R(t-\tau)f(\tau, u[\varphi]_\tau)d\tau,$$

for $t \in [0, T]$.

3. EXISTENCE RESULTS AND DISSIPATIVITY OF SOLUTIONS

We use the fixed point method to get our results by considering the operator defined by

$$\begin{aligned} \mathcal{F} : \mathbf{C}_\varphi &\rightarrow \mathbf{C}_\varphi \\ \mathcal{F}(v)(t) &= S(t)\varphi(0) + \int_0^t R(t-\tau)f(\tau, v[\varphi]_\tau)d\tau, \quad t \in [0; T]. \end{aligned}$$

It is clear that if v is a fixed point of \mathcal{F} then $v[\varphi]$ is a mild solution to (1.1)-(1.2). So \mathcal{F} is referred to as the solution operator.

The first result is obtained in the case that f has a superlinear growth and the initial datum is sufficiently small.

Theorem 3.1. Let (A) and (K) hold. Assume that f is a continuous function and satisfies the following condition

$$\|f(t, w)\| \leq \beta|w|_{\mathcal{B}} + \Psi(|w|_{\mathcal{B}}), \forall t \geq 0, w \in \mathcal{B}, \quad (3.1)$$

where $\beta > 0$, $\Psi \in C(\mathbb{R}^+; \mathbb{R})$ such that $\lim_{r \rightarrow 0} \frac{\Psi(r)}{r} = 0$. If $\beta < \lambda_1 \left(\sup_{s \in [0; T]} K(s) \right)^{-1}$, then there is a positive number δ such that a mild solution to (1.1)-(1.2) exists globally provided $|\varphi|_{\mathcal{B}} < \delta$. Moreover, if f is locally Lipschitzian, i.e., for each $\bar{r} > 0$, there is $L(\bar{r}) > 0$ such that

$$\|f(t, w_1) - f(t, w_2)\| \leq L(\bar{r})|w_1 - w_2|_{\mathcal{B}}, \quad (3.2)$$

for all $t \geq 0$, $|w_i|_{\mathcal{B}} \leq \bar{r}$, $i \in \{1, 2\}$, then the mild solution is unique.

Proof. By definition of \mathcal{F} , we see that it is a continuous map from \mathbf{C}_φ into itself. We employ the Schauder theorem to prove that \mathcal{F} has a fixed point in \mathbf{C}_φ . Firstly, we find a number $\eta > 0$ such that $\mathcal{F}(B_\eta) \subset B_\eta$, provided that $|\varphi|_{\mathcal{B}}$ is small enough. Here B_η is the closed ball in \mathbf{C}_φ with radius η and center at origin.

Due to the assumption on f , for $\theta \in (0; \frac{\lambda_1}{K_T} - \beta)$, where $K_T = \sup_{[0; T]} K(s)$, there exists $\bar{\eta} > 0$ such that

$$\|f(t, w)\| \leq (\beta + \theta)|w|_{\mathcal{B}}, \text{ for all } w \in \mathcal{B}, |w|_{\mathcal{B}} \leq \bar{\eta}.$$

Now we choose $\eta = \frac{\bar{\eta}}{2K_T}$ and let $\|u\|_\infty \leq \eta$. If $|\varphi|_{\mathcal{B}} \leq \delta_1 := \frac{\bar{\eta}}{2M_T}$, here $M_T = \sup_{s \in [0; T]} M(s)$, then $|u[\varphi]_\tau|_{\mathcal{B}} \leq \bar{\eta}$ for $\tau > 0$.

One gets

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1)(\beta + \theta)|u[\varphi]_\tau|_{\mathcal{B}}d\tau \\
&\leq s(t, \lambda_1)\|\varphi(0)\| \\
&\quad + (\beta + \theta) \int_0^t r(t-\tau, \lambda_1)[K(\tau) \sup_{s \in [0; \tau]} \|u(s)\| + M(\tau)|\varphi|_{\mathcal{B}}]d\tau \\
&\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} \\
&\quad + (\beta + \theta) \int_0^t r(t-\tau, \lambda_1)[K_T\eta + M_T|\varphi|_{\mathcal{B}}]d\tau.
\end{aligned}$$

Using Proposition 2.1, we have

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} + (\beta + \theta)\lambda_1^{-1}(1 - s(t, \lambda_1))[K_T\eta + M_T|\varphi|_{\mathcal{B}}] \\
&\leq \varrho|\varphi|_{\mathcal{B}} + \frac{\beta + \theta}{\lambda_1}(K_T\eta + M_T|\varphi|_{\mathcal{B}}) \\
&\leq \varrho|\varphi|_{\mathcal{B}} + \frac{\beta + \theta}{\lambda_1}(K_T\eta + M_T|\varphi|_{\mathcal{B}}) \\
&= \left(\varrho + \frac{M_T(\beta + \theta)}{\lambda_1}\right)|\varphi|_{\mathcal{B}} + \frac{K_T(\beta + \theta)}{\lambda_1}\eta.
\end{aligned}$$

Putting $\delta_2 := \eta \frac{(\lambda_1 - (\beta + \theta)K_T)}{\varrho\lambda_1 + (\beta + \theta)M_T}$, we obtain $\|\mathcal{F}(u)(t)\| \leq \eta$ if $|\varphi|_{\mathcal{B}} \leq \delta_2$.

Thus $\mathcal{F}(B_\eta) \subset B_\eta$ if $|\varphi|_{\mathcal{B}} < \delta := \min\{\delta_1, \delta_2\}$. Employing Proposition 2.3, we see that \mathcal{F} is a compact operator. Applying the Schauder theorem for $\mathcal{F} : B_\eta \rightarrow B_\eta$, we have the desired conclusion on solvability.

Finally, assume that f satisfies the Lipschitz condition (3.2). If $u_i, i \in \{1, 2\}$, are solutions of (1.1)-(1.2), then

$$u_i(t) = S(t)\varphi(0) + \int_0^t R(t-\tau)f(\tau, u_i[\varphi]_\tau)d\tau.$$

Let $\bar{r} = \max\{|u_i[\varphi]|_{\mathcal{B}} : i = 1, 2\}$, then

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leq \int_0^t r(t-\tau, \lambda_1)L(\bar{r})|(u_1 - u_2)[\varphi]_\tau|_{\mathcal{B}}d\tau \\
&\leq L(\bar{r}) \int_0^t r(t-\tau, \lambda_1)K_T \sup_{[0, \tau]} \|u_1(\xi) - u_2(\xi)\|d\tau.
\end{aligned}$$

Since the last term is nondecreasing in t , we get

$$\sup_{[0, t]} \|u_1(\xi) - u_2(\xi)\| \leq L(\bar{r}) \int_0^t r(t-\tau, \lambda_1)K_T \sup_{[0, \tau]} \|u_1(\xi) - u_2(\xi)\|d\tau.$$

Employing the Gronwall type inequality stated in Proposition 2.4, we conclude that $u_1 = u_2$. The proof is complete. \square

In the next result, we get a global existence to problem (1.1)-(1.2) by relaxing the smallness condition on both the initial datum and coefficients. However, the nonlinearity part must satisfy the sublinear condition.

Theorem 3.2. *Assume the hypotheses (A) and (K). Let f be continuous and obey the condition given by*

$$\|f(t, w)\| \leq \alpha(t) + \beta|w|_{\mathcal{B}}, \forall t \geq 0, w \in \mathcal{B},$$

where $\alpha \in L^1_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ and β is a nonnegative number. Then the problem (1.1)-(1.2) admits at least one global mild solution.

Proof. Since \mathcal{F} is a compact operator, we just find a closed convex set which is invariant under the solution operator. On the other words, we look for a closed convex set $D \subset \mathbf{C}_\varphi$ such that $\mathcal{F}(D) \subset D$.

Indeed, from the formulation of \mathcal{F} , we obtain

$$\begin{aligned} \|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| \\ &\quad + \int_0^t r(t-\tau, \lambda_1) \left(\alpha(\tau) + \beta|u|_{\mathcal{B}} \right) d\tau \\ &\leq s(t, \lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1) \alpha(\tau) d\tau \\ &\quad + \int_0^t r(t-\tau, \lambda_1) \beta \left(K_T \sup_{\xi \in [0; \tau]} \|u(\xi)\| + M_T |\varphi|_{\mathcal{B}} \right) d\tau, \quad t \in [0, T]. \end{aligned}$$

Then, in view of Proposition 2.1, we have

$$\begin{aligned} \|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| + (r(\cdot, \lambda_1) * \alpha)(t) + \beta M_T \lambda_1^{-1} (1 - s(t, \lambda_1)) |\varphi|_{\mathcal{B}} \\ &\quad + \beta K_T \int_0^t r(t-\tau, \lambda_1) \sup_{[0, \tau]} \|u(\xi)\| d\tau \\ &\leq (\varrho + \beta M_T \lambda_1^{-1}) |\varphi|_{\mathcal{B}} + \sup_{[0, T]} (r(\cdot, \lambda_1) * \alpha)(t) \\ &\quad + \beta K_T \int_0^t r(t-\tau, \lambda_1) \sup_{[0, \tau]} \|u(\xi)\| d\tau, \quad t \in [0, T]. \end{aligned}$$

By the fact that the function $\tau \mapsto \sup_{[0, \tau]} \|u(\xi)\|$ is nondecreasing, the last integral is nondecreasing in t . Therefore, one gets

$$\sup_{[0, t]} \|\mathcal{F}(u)(\xi)\| \leq M_0 + \beta K_T \int_0^t r(t-\tau, \lambda_1) \sup_{[0, \tau]} \|u(\xi)\| d\tau, \quad t \in [0, T]. \quad (3.3)$$

where $M_0 = (\varrho + \beta M_T \lambda_1^{-1}) |\varphi|_{\mathcal{B}} + \sup_{[0, T]} (r(\cdot, \lambda_1) * \alpha)(t)$.

Let $v \in C([0, T]; \mathbb{R}^+)$ be the unique solution of the integral equation

$$v(t) = M_0 + \beta K_T \int_0^t r(t-\tau, \lambda_1) v(\tau) d\tau, \quad t \in [0, T].$$

We define the set

$$D = \{w \in \mathbf{C}_\varphi : \sup_{[0, t]} \|w(\xi)\| \leq v(t), \forall t \in [0, T]\}.$$

Obviously, D is a bounded closed convex set. Then inequality (3.3) implies that $\mathcal{F}(D) \subset D$. The proof is complete. \square

The rest of this section is devoted to showing the dissipativity of the system. Denote by $\mathbb{S}(\varphi)$ the solution set with respect to the given initial datum φ .

The problem (1.1)-(1.2) is said to be uniformly dissipative with an absorbing set B_σ if we can find a constant $\sigma > 0$ such that: For each bounded set $D \subset \mathcal{B}$ there exists $T(D) > 0$ such that $\forall u \in \mathbb{S}(\varphi)$, $\varphi \in D$, we have

$$|u_t|_{\mathcal{B}} \leq \sigma, \quad \forall t > T(D).$$

We now in a position to states a dissipativity result for (1.1)-(1.2).

Theorem 3.3. *Let (A) and (K) hold. Assume that f is continuous and satisfies the condition*

$$\|f(t, w)\| \leq \alpha(t) + \beta|w|_{\mathcal{B}}, \quad \forall t \geq 0, w \in \mathcal{B},$$

where β is nonnegative numbers such that $\beta K_\infty < \lambda_1$, $K_\infty = \sup_{s \geq 0} K(s)$, $\alpha \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative nondecreasing function such that $r(\cdot, \lambda_1) * \alpha \in BC(\mathbb{R}^+)$. If $l \notin L^1(\mathbb{R}^+)$ and $M \in BC_0$, then the system (1.1)-(1.2) is dissipative with the absorbing set B_σ for any σ satisfying

$$\sigma > \frac{\lambda_1 \alpha^* K_\infty}{\lambda_1 - \beta K_\infty},$$

where $\alpha^* = \sup_{\mathbb{R}^+} (r(\cdot, \lambda_1) * \alpha)(t)$.

Proof. Let $D \subset \mathcal{B}$ be a bounded set, $\varphi \in D$ and $u \in \mathbb{S}(\varphi)$. Then

$$u(t) = S(t)\varphi(0) + \int_0^t R(t-\tau)f(\tau, u[\varphi]_\tau) d\tau, \quad t \geq 0.$$

Thus

$$\begin{aligned} \|u(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1)[\alpha(\tau) + \beta|u[\varphi]_\tau|] d\tau \\ &\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau, \lambda_1)\left[\alpha(\tau) + \beta M(\tau)|\varphi|_{\mathcal{B}} + \beta K_\infty \sup_{\xi \in [0; \tau]} \|u(\xi)\|\right] d\tau \\ &\leq \left(s(t, \lambda_1)\varrho + \beta \int_0^t r(t-\tau, \lambda_1)M(\tau) d\tau\right)|\varphi|_{\mathcal{B}} \\ &\quad + \int_0^t r(t-\tau, \lambda_1)\alpha(\tau) d\tau + \int_0^t r(t-\tau, \lambda_1)\beta K_\infty \sup_{\xi \in [0; \tau]} \|u(\xi)\| d\tau \\ &\leq \left(\varrho + \frac{\beta M_\infty}{\lambda_1}\right)|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau, \lambda_1)\alpha(\tau) d\tau + \frac{\beta K_\infty}{\lambda_1} \sup_{\xi \in [0; t]} \|u(\xi)\| \end{aligned}$$

It implies

$$\begin{aligned} \sup_{\xi \in [0; t]} \|u(\xi)\| &\leq \frac{\lambda_1}{\lambda_1 - \beta K_\infty} \left(\varrho + \frac{\beta M_\infty}{\lambda_1}\right)|\varphi|_{\mathcal{B}} + \frac{\lambda_1}{\lambda_1 - \beta K_\infty} \sup_{\mathbb{R}^+} (r(\cdot, \lambda_1) * \alpha)(t) \\ &\leq \frac{\lambda_1}{\lambda_1 - \beta K_\infty} \left(\varrho + \frac{\beta M_\infty}{\lambda_1}\right)|D|_{\mathcal{B}} + \frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_\infty}, \end{aligned}$$

here $|D|_{\mathcal{B}} := \sup\{|w|_{\mathcal{B}} : w \in \mathcal{B}\}$ and $\alpha^* = \sup_{\mathbb{R}^+} (r(\cdot, \lambda_1) * \alpha)(t)$.

Combing the last estimate and the formulation of solution, we have

$$\begin{aligned}
\|u(t)\| &\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} \\
&+ \int_0^t r(t-\tau, \lambda_1) \left[\alpha(\tau) + \beta M(\tau/2)|u_{\frac{\tau}{2}}|_{\mathcal{B}} + \beta K(\tau/2) \sup_{\xi \in [\tau/2; \tau]} \|u(\xi)\| \right] d\tau \\
&\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau, \lambda_1) \left[\alpha(\tau) + \beta M(\tau/2)|u_{\frac{\tau}{2}}|_{\mathcal{B}} \right] d\tau \\
&\quad + \beta K_{\infty} \int_0^t r(t-\tau, \lambda_1) \sup_{\xi \in [\tau/2; \tau]} \|u(\xi)\| d\tau \\
&\leq s(t, \lambda_1)\varrho|D|_{\mathcal{B}} \\
&+ \int_0^t r(t-\tau, \lambda_1) \left[\alpha(\tau) + \beta M(\tau/2)(K_{\infty}K_D + M_{\infty}|D|_{\mathcal{B}}) \right] d\tau \\
&\quad + \beta K_{\infty} \int_0^t r(t-\tau, \lambda_1) \sup_{\xi \in [\tau/2; \tau]} \|u(\xi)\| d\tau,
\end{aligned}$$

where $K_D = \frac{\lambda_1}{\lambda_1 - \beta K_{\infty}} \left(\varrho + \frac{\beta M_{\infty}}{\lambda_1} \right) |D|_{\mathcal{B}} + \frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_{\infty}}$.

In view of Proposition 2.5 with $v(t) = \|u(t)\|$, $p(t) = s(t, \lambda_1)\varrho|D|_{\mathcal{B}}$, and $q(t) = \alpha(t) + \beta(K_{\infty}K_D + M_{\infty}|D|_{\mathcal{B}})M(t/2)$, we have $\|u(\cdot)\| \in BC(\mathbb{R}^+)$ and

$$\begin{aligned}
\limsup_{t \rightarrow \infty} \|u(t)\| &\leq \left[\limsup_{t \rightarrow \infty} s(t, \lambda_1)\varrho|D|_{\mathcal{B}} + \limsup_{t \rightarrow \infty} (r(\cdot, \lambda_1) * \alpha)(t) \right] \frac{\lambda_1}{\lambda_1 - \beta K_{\infty}} \\
&+ \limsup_{t \rightarrow \infty} \left[r(\cdot, \lambda_1) * M\left(\frac{\cdot}{2}\right)(t) \right] \beta(K_{\infty}K_D + M_{\infty}|D|_{\mathcal{B}}) \frac{\lambda_1}{\lambda_1 - \beta K_{\infty}} \\
&\leq \frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_{\infty}},
\end{aligned}$$

thanks to the fact that $M(\frac{\cdot}{2}) \in BC_0$. So for $\epsilon > 0$ there exists $T_1(\epsilon) > 0$ such that

$$\sup_{\xi \geq t} \|u(\xi)\| \leq \frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_{\infty}} + \frac{\epsilon}{2K_{\infty}}, \quad \forall t \geq T_1(\epsilon).$$

Thus

$$|u_t|_{\mathcal{B}} \leq K_{\infty} \left(\frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_{\infty}} + \frac{\epsilon}{2K_{\infty}} \right) + M\left(\frac{t}{2}\right)|u_{\frac{t}{2}}|_{\mathcal{B}}, \quad \forall t \geq 2T_1(\epsilon). \quad (3.4)$$

On the other hand, we have

$$M\left(\frac{t}{2}\right)|u_{\frac{t}{2}}|_{\mathcal{B}} \leq M\left(\frac{t}{2}\right)(K_{\infty}K_D + M_{\infty}|D|_{\mathcal{B}}).$$

By virtue of $M \in BC_0$, there exists $T_2(D, \epsilon) > 0$ such that

$$M\left(\frac{t}{2}\right)|u_{\frac{t}{2}}|_{\mathcal{B}} \leq \frac{\epsilon}{2}, \quad \forall t \geq T_2(D, \epsilon). \quad (3.5)$$

From (3.4) and (3.5), we arrive at

$$|u_t|_{\mathcal{B}} \leq \frac{\lambda_1 \alpha^* K_{\infty}}{\lambda_1 - \beta K_{\infty}} + \epsilon, \quad \forall t \geq T(D, \epsilon),$$

where $T(D, \epsilon) = \max\{T_1(\epsilon), T_2(D, \epsilon)\}$. We choose a fixed number ϵ small enough to get the uniform dissipativity with any $\sigma > \frac{\lambda_1 \alpha^* K_{\infty}}{\lambda_1 - \beta K_{\infty}}$. The proof is complete. \square

4. ASYMPTOTIC STABILITY AND WEAKLY ASYMPTOTIC STABILITY

In this section, we will establish the asymptotic stability for our system when it has unique solution and the weak stability when the uniqueness of solution is not guaranteed.

4.1. Asymptotic stability. In the following theorem, we prove the asymptotic stability of solution to (1.1)-(1.2) when the nonlinearity is globally Lipschitzian.

Theorem 4.1. *Let (A) and (K) hold. Assume that f satisfies the Lipschitz condition*

$$\|f(t, w_1) - f(t, w_2)\| \leq \beta|w_1 - w_2|_{\mathcal{B}},$$

for all $t \geq 0, w_i \in \mathcal{B}, i \in \{1, 2\}$, where $\beta > 0$ such that $\beta K_\infty < \lambda_1$. If $l \notin L^1(\mathbb{R}^+)$ and $M \in BC_0$, then the solution of (1.1)-(1.2) is asymptotically stable.

Proof. Let $u(\cdot, \varphi)$ and $v(\cdot, \psi)$ be solutions of (1.1)-(1.2) with initial data φ and ψ , respectively. Then

$$\begin{aligned} \|u(t) - v(t)\| &\leq s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\ &\quad + \int_0^t r(t - \tau, \lambda_1)\|f(\tau, u[\varphi]_\tau) - f(\tau, v[\psi]_\tau)\|d\tau \\ &\leq s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\ &\quad + \int_0^t r(t - \tau, \lambda_1)\beta|u[\varphi]_\tau - v[\psi]_\tau|_{\mathcal{B}}d\tau \\ &\leq s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\ &\quad + \beta \int_0^t r(t - \tau, \lambda_1)[K_\infty \sup_{\xi \in [0; \tau]} \|u(\xi) - v(\xi)\| + M_\infty|\varphi - \psi|_{\mathcal{B}}]d\tau \\ &\leq \frac{\beta K_\infty}{\lambda_1} \sup_{\xi \in [0; t]} \|u(\xi) - v(\xi)\| + \left(\varrho + \frac{\beta M_\infty}{\lambda_1}\right)|\varphi - \psi|_{\mathcal{B}}. \end{aligned}$$

Therefore

$$\sup_{\xi \in [0; t]} \|u(\xi) - v(\xi)\| \leq \frac{\varrho\lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty}|\varphi - \psi|_{\mathcal{B}}. \quad (4.1)$$

Using (B3) and the estimate above, one gets

$$\begin{aligned} |u_t - v_t|_{\mathcal{B}} &\leq K(t) \sup_{\xi \in [0; t]} \|u(\xi) - v(\xi)\| + M(t)|\varphi - \psi|_{\mathcal{B}} \\ &\leq (K_\infty \frac{\varrho\lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} + M_\infty)|\varphi - \psi|_{\mathcal{B}}, \quad \forall t \geq 0. \end{aligned} \quad (4.2)$$

Thus

$$\begin{aligned}
\|u(t) - v(t)\| &\leq s(t, \lambda_1) \|\varphi(0) - \psi(0)\| + \beta \int_0^t r(t - \tau, \lambda_1) M(\tau/2) |u_{\frac{\tau}{2}} - v_{\frac{\tau}{2}}|_{\mathcal{B}} d\tau \\
&\quad + \beta \int_0^t r(t - \tau, \lambda_1) K(\tau/2) \sup_{\xi \in [\tau/2; \tau]} \|u(\xi) - v(\xi)\| d\tau \\
&\leq s(t, \lambda_1) \|\varphi(0) - \psi(0)\| \\
&\quad + \beta (K_\infty \frac{\varrho \lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} + M_\infty) |\varphi - \psi|_{\mathcal{B}} \int_0^t r(t - \tau, \lambda_1) M(\tau/2) d\tau \\
&\quad + \beta K_\infty \int_0^t r(t - \tau, \lambda_1) \sup_{\xi \in [\tau/2; \tau]} \|u(\xi) - v(\xi)\| d\tau
\end{aligned}$$

By Proposition 2.5, we have

$$\begin{aligned}
\limsup_{t \rightarrow \infty} \|u(t) - v(t)\| &\leq \frac{\lambda_1}{\lambda_1 - \beta K_\infty} \limsup_{t \rightarrow \infty} s(t, \lambda_1) \|\varphi(0) - \psi(0)\| \\
&\quad + C \limsup_{t \rightarrow \infty} (r(\cdot, \lambda_1) * M(\cdot/2))(t),
\end{aligned}$$

here $C = \beta (K_\infty \frac{\varrho \lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} + M_\infty) |\varphi - \psi|_{\mathcal{B}}$. Then

$$\limsup_{t \rightarrow \infty} \|u(t) - v(t)\| = 0,$$

thanks to $M \in BC_0$.

It follows that

$$\forall \epsilon > 0, \exists T_1(\epsilon) > 0 : \sup_{\xi \geq t} \|u(\xi) - v(\xi)\| \leq \frac{\epsilon}{2K_\infty}, \forall t \geq T_1(\epsilon).$$

Consequently,

$$|u_t - v_t|_{\mathcal{B}} \leq K_\infty \frac{\epsilon}{2K_\infty} + \frac{\varrho \lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} |\varphi - \psi|_{\mathcal{B}} M(t/2), \forall t \geq 2T_1(\epsilon),$$

due to (4.1). Since $M \in BC_0$, there exists $T_2 > 0$ such that

$$\frac{\varrho \lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} |\varphi - \psi|_{\mathcal{B}} M(t/2) \leq \frac{\epsilon}{2}, \forall t \geq T_2.$$

This gives

$$|u_t - v_t|_{\mathcal{B}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \forall t \geq T, \tag{4.3}$$

here $T = \max\{2T_1(\epsilon), T_2\}$.

Combining (4.2) and (4.3) gives us the desired conclusion. The proof is complete. \square

The next theorem states a result on the asymptotic stability of the zero solution when f satisfies the hypotheses in Theorem 3.1 with K_∞ in place of K_T .

Theorem 4.2. *Let the hypotheses of Theorem 3.1 hold where K_T is replaced by K_∞ . If $l \notin L^1(\mathbb{R}^+)$ and $M \in BC_0$, then the zero solution of (1.1) is asymptotically stable.*

Proof. Fix the numbers $\theta \in (0, \frac{\lambda_1}{K_\infty} - \beta)$, η and δ as in the proof of Theorem 3.1. Then under the assumptions (3.1)-(3.2), for $\|\varphi\|_\infty < \delta$, there exists a unique solution $u \in B_\eta$ of (1.1)-(1.2), which satisfies the following estimate

$$\begin{aligned} \|u(t)\| &\leq (t, \lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1)(\beta + \theta)|u[\varphi]_\tau|_{\mathcal{B}}d\tau \\ &\leq s(t, \lambda_1)\|\varphi(0)\| \\ &\quad + (\beta + \theta) \int_0^t r(t-\tau, \lambda_1)[K_\infty \sup_{s \in [0; \tau]} \|u(s)\| + M(\tau)|\varphi|_{\mathcal{B}}]d\tau \\ &\leq \frac{(\beta + \theta)K_\infty}{\lambda_1} \sup_{s \in [0; t]} \|u(s)\| + \left(\varrho + (\beta + \theta) \int_0^t r(t-\tau, \lambda_1)M(\tau)d\tau\right)|\varphi|_{\mathcal{B}}, \end{aligned}$$

for all $t \geq 0$. Then

$$\sup_{s \in [0; t]} \|u(s)\| \leq \frac{\lambda_1}{\lambda_1 - (\beta + \theta)K_\infty} \left(\varrho + (\beta + \theta)M^*\right)|\varphi|_{\mathcal{B}},$$

for all $t \geq 0$, where $M^* = \sup_{t \geq 0} \int_0^t r(t-\tau, \lambda_1)M(\tau)d\tau$.

From the last estimate and (B3), we see that

$$|u_t|_{\mathcal{B}} \leq \left(\frac{\lambda_1 K_\infty (\varrho + (\beta + \theta)M^*)}{\lambda_1 - (\beta + \theta)K_\infty} + M_\infty\right)|\varphi|_{\mathcal{B}}. \quad (4.4)$$

By the same arguments as in Theorem 4.1, one gets

$$\limsup_{t \rightarrow \infty} \|u(t)\| = 0.$$

Then

$$\lim_{t \rightarrow \infty} |u_t|_{\mathcal{B}} = 0. \quad (4.5)$$

From (4.4) and (4.5), we obtain the asymptotic stability of the zero solution. \square

4.2. Weakly asymptotic stability. In this subsection, we relax the Lipschitz condition on f and study the weak stability for the zero solution due to the lack of uniqueness. The concept of weakly asymptotic stability is given below.

Definition 4.1. [3] *Let $\mathbb{S}(\varphi)$ be the solution set of (1.1)-(1.2) with respect to the initial datum φ . Assume that $0 \in \mathbb{S}(0)$, that is (1.1) admits zero solution. The zero solution of (1.1) is said to be weakly asymptotically stable iff*

- (1) *It is stable, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi|_{\mathcal{B}} < \delta$ then $|u_t|_{\mathcal{B}} < \varepsilon$ for all $u \in \mathbb{S}(\varphi)$;*
- (2) *It is weakly attractive, i.e. for each $\varphi \in \mathcal{B}$, there exists $u \in \mathbb{S}(\varphi)$ such that $|u_t|_{\mathcal{B}} \rightarrow 0$ as $t \rightarrow \infty$.*

In order to analyze the weak stability for our system, we make use of the fixed point theory for condensing maps. So we now collect some notations and results related to measure of noncompactness (MNC) and fixed point principles.

Definition 4.2. [9] *Let E be a Banach space and $\mathcal{B}(E)$ the collection of all nonempty and bounded subsets of E . A function $\omega : \mathcal{B}(E) \rightarrow \mathbb{R}^+$ is said to be a measure of noncompactness (MNC) if $\omega(\overline{\text{co}} D) = \omega(D)$ for all $D \in \mathcal{B}(E)$. An MNC is called*

- *nonsingular if $\omega(D \cup \{x\}) = \omega(D)$ for all $D \in \mathcal{B}(E)$, $x \in E$.*

- *monotone if $\omega(D_1) \leq \omega(D_2)$ provided that $D_1 \subset D_2$.*

The Hausdorff measure of noncompactness is a MNC which is defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ admits a finite } \varepsilon\text{-net}\}.$$

Definition 4.3. [9] *Let E be a Banach space and $D \in \mathcal{B}(E)$. A continuous map $\mathcal{F} : D \rightarrow E$ is said to be condensing with respect to MNC ω (ω -condensing) iff the relation $\omega(B) \leq \omega(\mathcal{F}(B)), B \subset D$, implies that B is relatively compact.*

The following theorem states a fixed point principle for condensing maps.

Theorem 4.3. [9] *Let ω be a monotone and nonsingular MNC on E . Assume that $D \subset E$ is a closed convex set and $\mathcal{F} : D \rightarrow D$ is ω -condensing. Then \mathcal{F} admits a fixed point.*

In this subsection, we consider the solution operator on $BC_0(\mathbb{R}^+; H)$ where

$$BC_0(\mathbb{R}^+; H) = \{y \in BC(\mathbb{R}^+; H) : \lim_{t \rightarrow \infty} \|y(t)\| = 0\}.$$

It is known that $BC_0(\mathbb{R}^+; H)$ is a closed subspace of $BC(\mathbb{R}^+; H)$. Given $\varphi \in \mathcal{B}$, put $\mathcal{BC}_0^\varphi = \{u \in BC_0(\mathbb{R}^+; H) : u(0) = \varphi(0)\}$. Then \mathcal{BC}_0^φ is a closed convex set of $BC_0(\mathbb{R}^+; H)$.

Let D be a bounded set in \mathcal{BC}_0^φ and $\pi_T : \mathcal{BC}_0^\varphi \rightarrow C([0, T]; H)$ the restriction operator on \mathcal{BC}_0^φ , i.e. $\pi_T(u)$ is the restriction of $u \in \mathcal{BC}_0^\varphi$ to the interval $[0, T]$. Define

$$\begin{aligned} d_\infty(D) &= \lim_{T \rightarrow \infty} \sup_{u \in D} \sup_{t \geq T} \|u(t)\|, \\ \chi_\infty(D) &= \sup_{T > 0} \chi_T(D), \end{aligned}$$

where $\chi_T(\cdot)$ is the Hausdorff MNC in $C([0, T]; H)$. Put

$$\chi^*(D) = d_\infty(D) + \chi_\infty(D).$$

It is shown in [2] that χ^* possesses all properties stated in Definition 4.2. In addition, if $\chi^*(D) = 0$ then D is relatively compact in $BC_0(\mathbb{R}^+; H)$. Especially, if $u \in C(\mathbb{R}^+; H)$, then $d_\infty(\{u\}) = 0$ iff $u \in BC_0(\mathbb{R}^+; H)$.

The main result in this subsection is the following theorem.

Theorem 4.4. *Assume (A) and (K) hold. Let f be continuous and satisfy the condition*

$$\|f(t, w)\| \leq \beta(t)|w|_{\mathcal{B}}, \quad \forall t \geq 0, w \in \mathcal{B}, \quad (4.6)$$

where $\beta \in BC(\mathbb{R}^+)$ is a nonnegative function. Suppose that $l \notin L^1(\mathbb{R}^+)$, $K \in BC(\mathbb{R}^+)$ and $M \in BC_0$. Then the zero solution of (1.1) is weakly asymptotically stable provided that

$$\ell = \limsup_{t \rightarrow \infty} \int_{t/2}^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) d\tau < 1. \quad (4.7)$$

Let us outline the proof. The idea is employing the fixed point theorem for condensing maps. We first establish the well-defined and condensing property of the solution operator in Lemma 4.5. Then Lemma 4.6 reduces the condition on functions K and M to the auxiliary functions K_1 and M_1 . Using this preparation, Lemma 4.7 constructs a closed bounded invariant subset of the solution operator. Combining all these results, the proof is finished by following the standard argument. We are now in a position to show the proof in details.

Lemma 4.5. *Assume the hypotheses of Theorem 4.4 hold. Then*

$$d_\infty(\mathcal{F}(D)) \leq \ell \cdot d_\infty(D)$$

for all bounded set $D \subset \mathcal{BC}_0^\varphi$. In particular, $\mathcal{F}(\mathcal{BC}_0^\varphi) \subset \mathcal{BC}_0^\varphi$.

Proof. We first show that

$$\limsup_{t \rightarrow +\infty} \int_0^t r(t-\tau, \lambda_1) M(\tau/2) \beta(\tau) d\tau = 0. \quad (4.8)$$

Indeed, we see that

$$\begin{aligned} \int_0^t r(t-\tau, \lambda_1) M(\tau/2) \beta(\tau) d\tau &\leq M_\infty \beta_\infty \int_0^{t/2} r(t-\tau, \lambda_1) d\tau \\ &\quad + \beta_\infty \int_{t/2}^t r(t-\tau, \lambda_1) M(\tau/2) d\tau, \end{aligned}$$

where $M_\infty = \sup_{t \geq 0} M(t)$ and $\beta_\infty = \sup_{t \geq 0} \beta(t)$. Moreover,

$$\int_0^{t/2} r(t-\tau, \lambda_1) d\tau = \int_{t/2}^t r(\tau, \lambda_1) d\tau \rightarrow 0 \text{ as } t \rightarrow \infty,$$

according to $r(\cdot, \lambda_1) \in L^1(\mathbb{R}^+)$. In addition,

$$\begin{aligned} \int_{t/2}^t r(t-\tau, \lambda_1) M(\tau/2) d\tau &\leq \sup_{\tau \geq t/4} M(\tau) \int_0^{t/2} r(\tau, \lambda_1) d\tau \\ &\leq \lambda_1^{-1} \sup_{\tau \geq t/4} M(\tau) \rightarrow 0 \text{ as } t \rightarrow \infty, \end{aligned}$$

thanks to the assumption $M \in BC_0$. Thus (4.8) takes place.

Now let $D \subset \mathcal{BC}_0^\varphi$ be a bounded set and $u \in D$. Put $R_D = \sup_{u \in D} \|u\|_{BC} + |\varphi|_{\mathcal{B}}$.

Then

$$\begin{aligned} \|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1) \|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1) \beta(\tau) |u[\varphi]_\tau|_{\mathcal{B}} d\tau \\ &\leq s(t, \lambda_1) \|\varphi(0)\| \\ &\quad + \int_0^t r(t-\tau, \lambda_1) \beta(\tau) \left(M(\tau/2) |u[\varphi]_{\tau/2}|_{\mathcal{B}} + K(\tau/2) \sup_{\xi \in [\tau/2, \tau]} \|u(\xi)\| \right) d\tau \\ &\leq s(t, \lambda_1) \|\varphi(0)\| \\ &\quad + r * \widetilde{M}(t) [M_\infty |\varphi|_{\mathcal{B}} + K_\infty R_D] + \int_0^t r(t-\tau, \lambda_1) \widetilde{K}(\tau) \sup_{\xi \in [\tau/2, \tau]} \|u(\xi)\| d\tau, \quad (4.9) \end{aligned}$$

for all $t \geq 0$, where $\widetilde{M}(\tau) = \beta(\tau) M(\tau/2)$, $\widetilde{K}(\tau) = \beta(\tau) K(\tau/2)$.

In order to estimate the last term, fixed a $T > 0$, for $t > 4T$, we get

$$\begin{aligned} \int_0^t r(t-\tau, \lambda_1) \widetilde{K}(\tau) \sup_{\xi \in [\tau/2, \tau]} \|u(\xi)\| d\tau &\leq R_D \widetilde{K}_\infty \int_0^{t/2} r(t-\tau, \lambda_1) d\tau \\ &\quad + \sup_{\xi \geq T} \|u(\xi)\| \int_{t/2}^t r(t-\tau, \lambda_1) \widetilde{K}(\tau) d\tau. \quad (4.10) \end{aligned}$$

Combining (4.9) and (4.10) yields

$$\begin{aligned} \sup_{u \in D} \sup_{t \geq 4T} \|\mathcal{F}(u)(t)\| &\leq \sup_{t \geq 4T} s(t, \lambda_1) \|\varphi(0)\| \\ &\quad + \sup_{t \geq 4T} \left(r * \widetilde{M}(t) (M_\infty |\varphi|_{\mathcal{B}} + K_\infty R_D) + R_D \widetilde{K}_\infty \frac{s(t/2, \lambda_1) - s(t, \lambda_1)}{\lambda_1} \right) \\ &\quad + \left(\sup_{u \in D} \sup_{\xi \geq T} \|u(\xi)\| \right) \sup_{t \geq 4T} \int_{t/2}^t r(t - \tau, \lambda_1) \widetilde{K}(\tau) d\tau. \end{aligned}$$

Letting $T \rightarrow \infty$ we conclude

$$d_\infty(\Phi(D)) \leq \ell \cdot d_\infty(D),$$

thanks to (4.7)-(4.8) and

$$\sup_{t \geq 4T} s(t, \lambda_1) = s(4T, \lambda_1) \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Consequently, if $D = \{u\}$ then $d_\infty(\{\mathcal{F}(u)\}) \leq \ell \cdot d_\infty(\{u\}) = 0$. This ensures $\mathcal{F}(u) \in \mathcal{BC}_0^\varphi$ for all $u \in \mathcal{BC}_0^\varphi$. It follows that $\mathcal{F}(\mathcal{BC}_0^\varphi) \subset \mathcal{BC}_0^\varphi$. \square

Lemma 4.6. *Assume that (4.7) is satisfied. Let $K_1(t) = K(t/2) + K(t/2)M(t/2)$. Then there exist two positive numbers $T_1 > 0$ and $l_1 < 1$ such that $r(\cdot, \lambda_1) * (\beta K_1)(t) \leq l_1$ for all $t \geq T_1$.*

Proof. We get

$$\begin{aligned} &r(\cdot, \lambda_1) * (\beta K_1)(t) \\ &= \int_0^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) d\tau + \int_0^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) M(\tau/2) d\tau \\ &\leq \int_0^{t/2} r(t - \tau, \lambda_1) \beta(\tau/2) K(\tau/2) d\tau + \int_{t/2}^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) d\tau \\ &\quad + \int_0^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) M(\tau/2) d\tau \\ &\leq K_\infty \beta_\infty \int_{t/2}^t r(\tau, \lambda_1) d\tau + \int_{t/2}^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) d\tau \\ &\quad + K_\infty \int_0^t r(t - \tau, \lambda_1) \beta(\tau) M(\tau/2) d\tau. \end{aligned}$$

Obviously, the first term tends to zero when $t \rightarrow \infty$ thanks to $r \in L^1(\mathbb{R}^+)$. The last term goes to zero by (4.8). Thus it follows from (4.7) that

$$\limsup_{t \rightarrow \infty} r(\cdot, \lambda_1) * (\beta K_1)(t) = \limsup_{t \rightarrow \infty} \int_{t/2}^t r(t - \tau, \lambda_1) \beta(\tau) K(\tau/2) = l < 1.$$

The last inequality implies the desired result. \square

Lemma 4.7. *Assume the hypotheses of Theorem 4.4 hold. Then there exists a bounded closed convex set, which is invariant under the solution operator.*

Proof. Take T_1 and l_1 from Lemma 4.6, that is,

$$r(\cdot, \lambda_1) * (\beta K_1)(t) \leq l_1, \quad \forall t \geq T_1.$$

The construction of the invariant set consists of two steps. We first use a suitable weight function to obtain the invariant set for finite time. Then combining the estimate on this finite time interval with Lemma 4.6 to get the estimate for the large time, which gives us the invariant set for all time.

Step 1 (Estimate on the interval $[0, T_1]$). We have

$$\begin{aligned} \int_0^t r(t-\tau, \lambda_1) \beta(\tau) K_1(\tau) e^{-\mu(t-\tau)} d\tau &\leq \|\beta K_1\|_{L^\infty(0, T_1)} \int_0^t r(t-\tau, \lambda_1) e^{-\mu(t-\tau)} d\tau \\ &\leq \|\beta K_1\|_{L^\infty(0, T_1)} \int_0^{T_1} r(\tau, \lambda_1) e^{-\mu\tau} d\tau, \forall t \in (0, T_1]. \end{aligned}$$

Observing that

$$\lim_{\mu \rightarrow +\infty} \int_0^{T_1} r(\tau, \lambda_1) e^{-\mu\tau} d\tau = 0,$$

one can take a positive number μ such that

$$r(\cdot, \lambda_1) * (\beta K_1 m)(t) < \frac{m(t)}{2}, \text{ for all } t \in [0, T_1], \quad (4.11)$$

where $m(t) = e^{-\mu t}$. Let $M_1(t) = M(t/2)^2$, then one sees that

$$\begin{aligned} |u_t|_{\mathcal{B}} &\leq K(t/2) \sup_{\xi \in [t/2, t]} \|u(\xi)\| + M(t/2) |u_{t/2}|_{\mathcal{B}} \\ &\leq K(t/2) \sup_{\xi \in [t/2, t]} \|u(\xi)\| + M(t/2) \left(K(t/2) \sup_{\xi \in [0, t/2]} \|u(\xi)\| + M(t/2) |\varphi|_{\mathcal{B}} \right) \\ &\leq K_1(t) \sup_{\xi \in [0, t]} \|u(\xi)\| + M_1(t) |\varphi|_{\mathcal{B}}, \end{aligned} \quad (4.12)$$

according to the axiom (B3) of phase spaces.

Choose

$$R_1 \geq 2 \sup_{t \in [0, T_1]} \frac{(\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}}}{m(t)}. \quad (4.13)$$

Then for all $u \in BC_0(\mathbb{R}^+; H)$ such that $\sup_{t \in [0, T_1]} \frac{\|u(t)\|}{m(t)} \leq R_1$, one has

$$\begin{aligned} \|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1) \|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1) \beta(\tau) |u[\varphi]_\tau|_{\mathcal{B}} d\tau \\ &\leq \|\varphi(0)\| + R_1 \int_0^t r(t-\tau, \lambda_1) \beta(\tau) K_1(\tau) m(\tau) d\tau \\ &\quad + |\varphi|_{\mathcal{B}} \int_0^t r(t-\tau, \lambda_1) \beta(\tau) M_1(\tau) d\tau \\ &\leq (\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}} + R_1 \frac{m(t)}{2}, \end{aligned}$$

here we employed (4.11) and (4.12). Hence, combining with (4.13) one gets

$$\sup_{t \in [0, T_1]} \frac{\|\mathcal{F}(u)(t)\|}{m(t)} \leq R_1.$$

Step 2 (Estimate on the infinite interval $[T_1, \infty)$) Fix a number R_2 such that

$$(\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}} + R_1 m(T_1) \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC} \leq (1 - l_1) R_2. \quad (4.14)$$

Then for all $u \in BC_0(\mathbb{R}^+; H)$ such that $\sup_{t \geq T_1} \|u(t)\| \leq R_2$ and for $t \geq T_1$, we obtain

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1)\beta(\tau)|u[\varphi]_\tau|_{\mathcal{B}}d\tau \\
&\leq \|\varphi(0)\| + R_1m(T_1) \int_0^{T_1} r(t-\tau, \lambda_1)\beta(\tau)K_1(\tau)d\tau \\
&\quad + \int_{T_1}^t r(t-\tau, \lambda_1)\beta(\tau)K_1(\tau) \left[\max_{\xi \in [0, T_1]} \|u(\xi)\| + \max_{T_1 \leq \xi \leq \tau} \|u(\xi)\| \right] d\tau \\
&\quad + |\varphi|_{\mathcal{B}} \int_0^t r(t-\tau, \lambda_1)\beta(\tau)M_1(\tau)d\tau \\
&\leq (\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}} + R_1m(T_1)\|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC} + l_1R_2 \\
&\leq R_2,
\end{aligned}$$

thanks to (4.14).

Finally, consider the set

$$\mathbf{D} = \left\{ u \in \mathcal{BC}_0^\varphi : \sup_{[0, T_1]} \frac{\|u(t)\|}{m(t)} \leq R_1; \sup_{t \geq T_1} \|u(t)\| \leq R_2 \right\}. \quad (4.15)$$

Then \mathbf{D} is a closed bounded convex subset of \mathcal{BC}_0^φ satisfying $\mathcal{F}(\mathbf{D}) \subset \mathbf{D}$. This completes the proof. \square

Proof of Theorem 4.4. By Lemma 4.7, we have

$$\mathcal{F}(\mathbf{D}) \subset \mathbf{D},$$

where \mathbf{D} is given by (4.15). Let $D^* = \overline{\text{co}}\mathcal{F}(\mathbf{D})$, then $\pi_T(D^*)$ is compact for all $T > 0$ thanks to Proposition 2.3, and it is also a convex set. In addition, we get $\mathcal{F}(D^*) \subset D^*$. Considering $\mathcal{F} : D^* \rightarrow D^*$, we show that \mathcal{F} is χ^* -condensing. Obviously, if $D \subset D^*$ then $\chi_T(D) = 0$, which implies $\chi_\infty(D) = 0$. Using Lemma 4.5, we have

$$\chi^*(\mathcal{F}(D)) = \chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D)) = d_\infty(\mathcal{F}(D)) \leq \ell \cdot d_\infty(D) \leq \ell \cdot \chi^*(D).$$

If $\chi^*(D) \leq \chi^*(\mathcal{F}(D))$ then $\chi^*(D) \leq \ell \cdot \chi^*(D)$, which yields $\chi^*(D) = 0$, since $\ell < 1$. This implies the relative compactness of D . Therefore, \mathcal{F} is χ^* -condensing and it admits a fixed point due to Theorem 4.3.

We now show that for all $u \in \mathbb{S}(\varphi)$, $|u_t|_{\mathcal{B}} \leq C|\varphi|_{\mathcal{B}}$ for some $C > 0$.

Let $t \in [0, T_1]$. The following estimate holds

$$\begin{aligned}
\|u(t)\| &\leq s(t, \lambda_1)\|\varphi(0)\| \\
&\quad + \int_0^t r(t-\tau, \lambda_1)\beta(\tau) \left(K_1(\tau) \sup_{\xi \in [0, \tau]} \|u(\xi)\| + M_1(\tau)|\varphi|_{\mathcal{B}} \right) d\tau \\
&\leq s(t, \lambda_1)\varrho|\varphi|_{\mathcal{B}} + |\varphi|_{\mathcal{B}} \int_0^t r(t-\tau, \lambda_1)\beta(\tau)M_1(\tau)d\tau \\
&\quad + \beta_\infty K_{1\infty} \int_0^t r(t-\tau, \lambda_1) \left(\sup_{[0, \tau]} \|u(\xi)\| \right) d\tau \\
&\leq (\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}} + \beta_\infty K_{1\infty} \int_0^t r(t-\tau, \lambda_1) \sup_{[0, \tau]} \|u(\xi)\| d\tau.
\end{aligned}$$

Since the last integral is nondecreasing in t , one can take supremum over $[0, t]$ to get

$$\sup_{[0, t]} \|u(\xi)\| \leq (\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}} + \beta_\infty K_{1\infty} \int_0^t r(t - \tau, \lambda_1) \sup_{[0, \tau]} \|u(\xi)\| d\tau.$$

The Gronwall type inequality gives

$$\sup_{[0, t]} \|u(\xi)\| \leq Y(t) C_1(\varphi), \forall t \in [0, T_1],$$

where $C_1(\varphi) = (\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty) |\varphi|_{\mathcal{B}}$ and $Y(t)$ is the unique solution of Volterra equation

$$Y(t) = 1 + \beta_\infty K_{1\infty} \int_0^t r(t - \tau, \lambda_1) Y(\tau) d\tau.$$

Particularly,

$$\|u(t)\| \leq Y(T_1) C_1(\varphi), \forall t \in [0, T_1]. \quad (4.16)$$

Now estimating for $t \geq T_1$, we have

$$\begin{aligned} \|u(t)\| &\leq s(t, \lambda_1) \|\varphi(0)\| + \int_0^t r(t - \tau, \lambda_1) \beta(\tau) \left(K_1(\tau) \sup_{\xi \in [0, \tau]} \|u(\xi)\| + M_1(\tau) |\varphi|_{\mathcal{B}} \right) d\tau \\ &\leq s(t, \lambda_1) \varrho |\varphi|_{\mathcal{B}} + |\varphi|_{\mathcal{B}} \int_0^t r(t - \tau, \lambda_1) \beta(\tau) M_1(\tau) d\tau \\ &\quad + \int_0^t r(t - \tau, \lambda_1) \beta(\tau) K_1(\tau) \left(\sup_{[0, T_1]} \|u(\xi)\| + \sup_{[T_1, \tau]} \|u(\xi)\| \right) d\tau \\ &\leq C_1(\varphi) + \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC} Y(T_1) C_1(\varphi) \\ &\quad + \sup_{[T_1, t]} \|u(\xi)\| \int_0^t r(t - \tau, \lambda_1) \beta(\tau) K_1(\tau) d\tau \\ &\leq C_1(\varphi) [1 + Y(T_1) \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC}] + l_1 \sup_{[T_1, t]} \|u(\xi)\|, \end{aligned}$$

thanks to Lemma 4.7. Let t vary on $[T_1, T]$ for an arbitrary $T > T_1$, one concludes that

$$\sup_{[T_1, T]} \|u(t)\| \leq C_1(\varphi) (1 + Y(T_1) \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC}) + l_1 \sup_{[T_1, T]} \|u(\xi)\|.$$

Consequently,

$$\sup_{t \geq T_1} \|u(t)\| \leq \frac{1}{1 - l_1} C_1(\varphi) (1 + Y(T_1) \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC}). \quad (4.17)$$

Combing (4.16) with (4.17), we finally obtain

$$\|u(t)\| \leq C_2 |\varphi|_{\mathcal{B}}, \forall t > 0,$$

where

$$C_2 = \frac{\varrho + \|r(\cdot, \lambda_1) * (\beta M_1)\|_\infty}{1 - l_1} (1 + Y(T_1) \|r(\cdot, \lambda_1) * (\beta K_1)\|_{BC}).$$

This implies

$$|u_t|_{\mathcal{B}} \leq M(t) |\varphi|_{\mathcal{B}} + K(t) \sup_{[0, t]} \|u(\xi)\| \leq [M(t) + K(t) C_2] |\varphi|_{\mathcal{B}} \leq C |\varphi|_{\mathcal{B}}, \quad (4.18)$$

where $C = M_\infty + K_\infty C_2$.

We now show that $\lim_{t \rightarrow \infty} |u_t|_{\mathcal{B}} = 0$. By properties of phase space, we have

$$|u_t|_{\mathcal{B}} \leq K(t/2) \sup_{[t/2, t]} \|u(\xi)\| + M(t/2)|u[\varphi]_{t/2}|_{\mathcal{B}} \quad (4.19)$$

$$\leq K_{\infty} \sup_{[t/2, t]} \|u(\xi)\| + M(t/2)C|\varphi|_{\mathcal{B}}, \quad (4.20)$$

thanks to (4.18).

Because $\lim_{t \rightarrow \infty} \|u(t)\| = 0$ and $M \in BC_0$, for any $\epsilon > 0$, there exists a positive $T(\epsilon) > 0$ such that

$$\|u(t)\| < \epsilon, \quad \|M(t)\| < \epsilon, \quad \text{for all } t > T(\epsilon).$$

Combining with the inequality (4.20) gives

$$|u_t|_{\mathcal{B}} \leq (K_{\infty} + C|\varphi|_{\mathcal{B}})\epsilon, \quad \forall t > 2T(\epsilon).$$

The proof is complete. \square

Remark 4.1. *The statement in Theorem 4.4 presents a new observation, even for bounded delays. In [14, Theorem 7], the weakly asymptotic stability was proved under the condition on magnitude of coefficients on the half-line. In contrast to the latter, the conditions in Theorem 4.4 involves only the asymptotic information of $\beta(t)$ near infinity. For example, if $\beta \in BC(\mathbb{R}^+)$ such that*

$$\limsup_{t \rightarrow \infty} \beta(t) < \lambda_1/K_{\infty},$$

then the assumptions in Theorem 4.4 are testified even for β possessing large values in a finite interval.

5. EXAMPLE

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We apply the obtained results to the following multi-term fractional-in-time PDE:

$$\sum_{i=1}^m \mu_i \partial_t^{\alpha_i} u(t, x) - \Lambda u(t, x) = b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) \kappa(y, u(t + \theta, y)) dy d\theta, \quad (5.1)$$

$$\text{for } t > 0, x \in \Omega,$$

$$u(t, x) = 0, \quad \text{for } t \geq 0, x \in \partial\Omega, \quad (5.2)$$

$$u(s, x) = \varphi(s, x), \quad x \in \Omega, s \in (-\infty, 0], \quad (5.3)$$

where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_m < 1$, $\mu_i > 0$, $\partial_t^{\alpha_i}$ stand for the Caputo fractional derivatives of order α_i in t , for $i = \overline{1, m}$. The operator Λ is defined by

$$D(\Lambda) = \{u \in H_0^1(\Omega) : \Lambda u \in L^2(\Omega)\}, \quad \Lambda u = \sum_{i,j=1}^N \partial_{x_i} (a_{ij}(x) \partial_{x_j} u),$$

where $a_{ij} \in L^{\infty}(\Omega)$, $a_{ij} = a_{ji}$, $1 \leq i, j \leq N$, subject to the condition $\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$, for some $\theta > 0$. Apply the Friedrichs theory [21, Prop. 8.5], $-\Lambda$ is a positive self-adjoint operator with the compact resolvent.

Let $H = L^2(\Omega)$ with the inner product $(u, v) = \int_{\Omega} u(x)v(x)dx$. Put

$$k(t) = \sum_{i=1}^m \mu_i g_{1-\alpha_i}(t),$$

$$A = -\Lambda.$$

Clearly k is completely monotone, so the associated kernel l exists. Moreover, the Laplace transform of l is given by

$$\hat{l}(\lambda) = \lambda^{-1} \hat{k}(\lambda)^{-1} = \frac{1}{\mu_i \sum_{i=1}^m \lambda^{\alpha_i}}.$$

Thus

$$(\widehat{1 * l})(\lambda) = \frac{1}{\mu_i \sum_{i=1}^m \lambda^{\alpha_i+1}} \sim \frac{1}{\mu_1 \lambda^{\alpha_1+1}} \text{ as } \lambda \rightarrow 0.$$

Hence $l \notin L^1(\mathbb{R}^+)$ which follows from the asymptotic expansion

$$(1 * l)(t) \sim \frac{t^{\alpha_1}}{\mu_1 \Gamma(\alpha_1 + 1)} \rightarrow \infty \text{ as } t \rightarrow \infty,$$

thanks to the Karamata-Feller Tauberian theorem (see [5]).

We are now in a position to give the description for the nonlinearity:

(A1) $b \in BC(\mathbb{R}^+; L^2(\Omega))$.

(A2) $\nu : (-\infty, 0] \times \Omega \rightarrow \mathbb{R}$ is a continuous function and there exist a nonnegative function $\omega \in L^2(\Omega)$ and $\nu_0 \in (0, 1)$ such that

$$|\nu(t, x)| \leq \omega(x)e^{\nu_0 t}, \text{ for all } t \in (-\infty, 0], x \in \Omega.$$

(A3) $\kappa : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist a nonnegative function $p \in L^2(\Omega)$ and $q \in \mathbb{R}^+$ satisfying

$$|\kappa(y, z)| \leq p(y) + q|z|.$$

In this example, we choose the phase space $\mathcal{B} = CL_g^1$ with $r = 0$ and $g(s) = e^{\nu_0 s}$. The seminorm in \mathcal{B} is given by

$$|w|_{\mathcal{B}} = \|w(0)\| + \int_{-\infty}^0 e^{\nu_0 \theta} \|w(\theta)\| d\theta.$$

Then one can see that (2.1)-(2.2) are satisfied with $G(s) = g(s)$. Then \mathcal{B} satisfies (B1)-(B3) with

$$K(t) = 1 + \nu_0^{-1}(1 - e^{-\nu_0 t}), \quad M(t) = e^{-\nu_0 t},$$

thanks to the expressions of K and M in (2.3) and (2.4), respectively. Obviously, $M \in BC_0$ and $K_{\infty} = 1 + \nu_0^{-1}$.

Let $f : \mathbb{R}^+ \times \mathcal{B} \rightarrow L^2(\Omega)$ be defined as

$$f(t, \phi)(x) = b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) \kappa(y, \phi(\theta, y)) dy d\theta.$$

Then the problem (5.1)-(5.3) is in the form of (1.1)-(1.2).

We now testify the conditions related to f in Theorem 3.3 and 4.4.

For $\phi \in \mathcal{B}$, we have

$$\begin{aligned}
\|f(t, \phi)\|^2 &= \int_{\Omega} \left| b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) \kappa(y, \phi(\theta, y)) dy d\theta \right|^2 dx \\
&\leq \|b(t, \cdot)\|^2 \left[\int_{-\infty}^0 \int_{\Omega} |\nu(\theta, y)| (p(y) + q|\phi(\theta, y)|) dy d\theta \right]^2 \\
&\leq \|b(t, \cdot)\|^2 \left[\int_{-\infty}^0 e^{\nu_0 \theta} \int_{\Omega} (\omega(y)p(y) + q\omega(y)|\phi(\theta, y)|) dy d\theta \right]^2, \\
&\leq \|b(t, \cdot)\|^2 \left[\int_{-\infty}^0 e^{\nu_0 \theta} \|\omega\| (\|p\| + q\|\phi(\theta, \cdot)\|) d\theta \right]^2 \\
&\leq \|b(t, \cdot)\|^2 \|\omega\|^2 \left[\nu_0^{-1} \|p\| + q \int_{-\infty}^0 e^{\nu_0 \theta} \|\phi(\theta, \cdot)\| d\theta \right]^2,
\end{aligned}$$

thanks to (A2) and (A3) and the Hölder inequality.

Hence

$$\|f(t, \phi)\| \leq \|b(t, \cdot)\| \|\omega\| \left[\nu_0^{-1} \|p\| + q\|\phi\|_{\mathcal{B}} \right].$$

By taking

$$\alpha(t) = \|b(t, \cdot)\| \|\omega\| \nu_0^{-1} \|p\|, \quad \beta = q\|\omega\| \sup_{t \geq 0} \|b(t, \cdot)\|,$$

we see that $r(\cdot, \lambda_1) * \alpha \in BC(\mathbb{R}^+)$ due to (A1).

Applying Theorem 3.3, if $\beta(1 + \nu_0^{-1}) < \lambda_1$ then our system is dissipative.

On the other hand, let $p = 0$ then (4.6) takes place with

$$\beta(t) = q\|\omega\| \|b(t, \cdot)\|.$$

Let $\tilde{\beta} = \limsup_{t \rightarrow \infty} \beta(t)$. Then the condition

$$\tilde{\beta}(1 + \nu_0^{-1}) < \lambda_1 \tag{5.4}$$

implies (4.7). By Theorem 4.4, we conclude that the zero solution of (1.1) is weakly asymptotically stable if the last inequality holds. Note that condition (5.4) holds even for $\sup_{t \geq 0} \beta(t)$ being large.

We now replace (A3) with the following one

(A3b) $\kappa : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and there exists $q > 0$ such that

$$|\kappa(y, z_1) - \kappa(y, z_2)| \leq q|z_1 - z_2|.$$

Then we have the following estimates

$$\begin{aligned}
& \|f(t, \phi_1) - f(t, \phi_2)\|^2 \\
&= \int_{\Omega} \left| b(t, x) \int_{-\infty}^0 \int_{\Omega} \nu(\theta, y) (\kappa(y, \phi_1(\theta, y)) - \kappa(y, \phi_2(\theta, y))) dy d\theta \right|^2 dx \\
&\leq \|b(t, \cdot)\|^2 q^2 \left[\int_{-\infty}^0 \int_{\Omega} |\nu(\theta, y)| |\phi_1(\theta, y) - \phi_2(\theta, y)| dy d\theta \right]^2 \\
&\leq \|b(t, \cdot)\|^2 q^2 \left[\int_{-\infty}^0 e^{\nu_0 \theta} \int_{\Omega} \omega(y) |\phi_1(\theta, y) - \phi_2(\theta, y)| dy d\theta \right]^2 \\
&\leq \|b(t, \cdot)\|^2 q^2 \left[\int_{-\infty}^0 e^{\nu_0 \theta} \|\omega\| \|\phi_1(\theta, \cdot) - \phi_2(\theta, \cdot)\| d\theta \right]^2 \\
&\leq q^2 \|b(t, \cdot)\|^2 \|\omega\|^2 \|\phi_1 - \phi_2\|_{\mathcal{B}}^2.
\end{aligned}$$

It leads to

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq q \|\omega\| \|b(t, \cdot)\| \|\phi_1 - \phi_2\|_{\mathcal{B}},$$

that is, f satisfies the Lipschitz condition with Lipschitz constant

$$\beta = q \|\omega\| \sup_{t \geq 0} \|b(t, \cdot)\|.$$

Employing Theorem 4.1, one concludes that the solution to (5.1)-(5.3) is asymptotically stable provided that $\beta(1 + \nu_0^{-1}) < \lambda_1$.

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