DISCRETE DATA ASSIMILATION FOR THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS USING LOCAL OBSERVABLES

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Abstract. We study the data assimilation for the two-dimensional Navier-Stokes equations when the local measurements are obtained discretely in time and may be contaminated by systematic errors. Under suitable conditions on the relaxation (nudging) parameter, the spatial mesh resolution, and the time step between successive measurements, with the spatial resolution N is sufficiently large (in the case using a spectral inequality), the complement of the full domain Ω_0 and the sub-domain Ω is small enough (in the case using no spectral operator) we obtain an asymptotic in time estimate of the difference between the approximating solution and the unknown reference solution corresponding to the measurements, in an appropriate norm, which shows exponential convergence up to a term which depends on the size of the errors.

1. Introduction

Suppose that the evolution of u is governed by the two-dimensional Navier-Stokes equations, subject to periodic boundary conditions on $\Omega_0 = [0, L]^2$ or no-slip boundary conditions ($u = 0$ on $\partial\Omega_0$) if Ω_0 is a C^2 bounded domain in $\mathbb{R}^2$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

on the interval $[t_0, \infty)$, where the initial data $u(t_0) = u_0$ is unknown. Here $u = u(x, t)$ represents the velocity of the fluid, called the filtered velocity, p is the pressure, and f is a body force which is assumed, for simplicity, to be time-independent.

Data assimilation is a methodology to estimate weather or ocean variables combining (synchronizing) information from observational data with a numerical dynamical (forecast) model. In the pioneering work [2] Titi et al. introduced a new continuous data assimilation algorithm (this algorithm is often called as the AOT algorithm) for the two-dimensional Navier-Stokes equations (1.1) based on the ideas that have been developed for designing finite-dimensional feedback controls for disipative dynamical systems. With generalized interpolant operators I_h, the AOT algorithm is to construct $v(t)$ from the observational measurements $I_h(u(t))$ for $t\in[t_0, T]$, $t_0\geq 0$, is given by

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla q = f - \mu I_h(v) + \mu I_h(u), \\ \nabla\cdot v = 0, \\ v(t_0) = v_0, \end{cases} \tag{1.2}$$

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where μ is a positive relaxation parameter, which relaxes the coarse spatial scales of v toward those of the observed data, and v_0 is taken to be arbitrary. Here ν and f are the same kinematic viscosity parameter and forcing term from (1.1), q is a modified pressure. The result is that when μ is large enough depending on parameters of systems and forcing function f, and h is small enough depending on μ then the solutions $v(t)$ of (1.2) goes to $u(t)$ in suitable phase spaces as $t \to \infty$ at exponential rate. After this poineering paper, continuous data assimilation problems for many important equations in fluid mechanics have been extensively studied, see e.g., [1, 2, 4, 9, 16, 17, 18, 19, 22, 23, 24].

On the other hand, in [21] the authors introduced the following discrete data assimilation algorithm for finding v of the following system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla q = f - \mu \sum\limits_{n=0}^{\infty} \left(I_h(v(t_n)) - \tilde{u}(t_n) \right) \chi_n, \\ \nabla \cdot v = 0, \\ v(t_0) = v_0, \end{cases}$$

where the increasing sequence of instants of time $\{t_n\}_{n\in\mathbb{N}}$ in $[t_0, \infty)$ at which measurements are taken which is assumed that

$$\begin{aligned} &t_n < t_{n+1}, \ \forall n \in \mathbb{N}, \text{ and } t_n \to \infty \text{ as } n \to \infty, \\ &|t_{n+1} - t_n| \le \kappa, \ \forall n \in \mathbb{N}, \end{aligned}$$

and the observational measurements $\tilde{u}(t_n)$ at the time t_n given by

$$\tilde{u}(t_n) = I_h(u(t_n)) + \eta_n.$$

Here η_n is the error associated to the measurements at time t_n, χ_n is the characteristic function of the interval $[t_n, t_{n+1})$, and κ is called the step size between successive measurements. Under suitable conditions on μ, κ and h, then $v(t) \to u(t)$ in H or in V as $t \to \infty$ at exponential rate. Then the similar discrete data assimilation schemes for some other models were studied later in [3, 5]. The discrete-in-time downscaling data assimilation algorithm was also studied in a recent work [13].

Recently, Biswas et al. [8] introduced the continuous data assimilation using local observables for the two-dimensional Navier-Stokes equations. To overcome the difficulty due to the local observations, the authors used spectral inequality and Gevrey regularity of solutions in the periodic case, and the assumption of the complement of the full domain Ω_0 and the sub-domain Ω is small enough in the no-slip boundary conditions case. In a very recent work [6], combining ideas in [8] and in [16, 17], we study continuous data assimilation for the three-dimensional Leray-α model using local observables on any two components of the three-dimensional velocity field, and without any information of the rest component.

To the best of our knowledge, there is no result on the dicrete data assimilation using local observables. In the present paper, using some ideas in [8] and [21], we set up the discrete data assimilation problem for the two-dimensional Navier-Stokes equations (1.1) using local observables. Our aim here is to prove similar results as in the continuous data assimilation with local observables in [8].

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some results on function spaces and results on the two-dimensional Navier-Stokes equations which will be used in the proof. Section 3 is devoted to the discrete data assimilation using local observables in the no-slip boundary conditions case when the complement of the full domain Ω_0 and the sub-domain Ω is small enough. In both two cases of periodic and no-slip boundary conditions, the finite-dimensional discrete data assimilation using local observables is studied in the last section by exploiting spectral inequalities.

2. Preliminaries

We begin by defining a suitable domain Ω_0 and space $\mathcal{V}$ of smooth functions which satisfy each type of boundary conditions.

- In the periodic boundary conditions case: $\Omega_0 = [0, L]^2$, we denote by $\mathcal{V}$ the set of all vector valued trigonometric polynomials defined in Ω_0, which are divergence-free and have average zero.
- In the no-slip boundary conditions case: Let Ω_0 be an open, bounded and connected domain with C^2 boundary. We denote by $\mathcal{V}$ the set of all C^∞ vector fields from Ω_0 to $\mathbb{R}^2$ that are divergence free and compactly supported.

Then we denote by H and V the closures of $\mathcal{V}$ in the $L^2(\Omega_0)^2$ and $H^1(\Omega_0)^2$, respectively. Then H and V are Hilbert spaces with inner products given by

$$(u, v) = \sum_{i=1}^{2} \int_{\Omega_0} u_i v_i dx \text{ and } ((u, v)) = \sum_{i,j=1}^{2} \int_{\Omega_0} \partial_j u_i \partial_j v_i dx,$$

respectively, and the associated norms

$$|u| = (u, u)^{1/2} \text{ and } \|u\| = ((u, u))^{1/2}.$$

With the Leray projector $\mathcal{P}$, we denote the Stokes operator $A = -\mathcal{P}\Delta$ with domain $D(A) = H^2(\Omega_0)^2 \cap V$. In the case of periodic boundary conditions, $A = -\Delta|_{D(A)}$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{\phi_j\}_{j=1}^{\infty} \subset H$, such that $A\phi_j = \lambda_j \phi_j$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

We have the following versions of the Poincaré inequalities:

$$|v|^2 \leq \lambda_1^{-1} \|v\|^2, \ \forall v \in V, \tag{2.1}$$

$$\|v\|_{V'}^2 \leq \lambda_1^{-1} |v|^2, \quad \forall v \in H. \tag{2.2}$$

For every $u, v \in \mathcal{V}$, we write $B(u, v) = \mathcal{P}[(u \cdot \nabla)v]$. The bilinear operator B can be extended continuously from $V \times V$ with values in V'.

Let us now recall some algebraic properties of the nonlinear term $B(u, v)$ that play an important role in our analysis. For $u, v, w \in V$ we have that

$$\langle B(u, v), w \rangle_{V',V} = - \langle B(u, w), v \rangle_{V',V},$$

and consequently

$$\langle B(u, v), v \rangle_{V',V} = 0. \tag{2.3}$$

Furthermore,

$$| \langle B(u, v), w \rangle_{V',V} | \leq c_0 |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\|, \ \forall u, v, w \in V. \tag{2.4}$$

From (2.4), we have

$$\|B(u, v)\|_{V'} \leq c_0 |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2}, \ \forall u, v \in V. \tag{2.5}$$

With the above notations, we can rewrite the two-dimensional Navier-Stokes equations in the following functional form

$$\frac{du}{dt} + \nu A u + B(u, u) = \mathcal{P} f, \tag{2.6}$$

with initial condition $u(t_0) = u_0 \in H$.

We first have the following result about the global existence and long-time behavior of solutions to two-dimensional Navier-Stokes equations (1.1) (see for instance [14, 20]). The results concerning some uniform bounds of the attractor with respect to the H and V norms. In the statement below and in the remainder of this work, we denote by c a generic absolute constant, whose value may change from line to line and $G = \nu^{-2}\lambda_1^{-1}|f|$ is the Grashoff number.

Theorem 2.1. *Let $f \in H$ and $u_0 \in H$. Then system (2.6) has a unique global solution u that satisfies*

$$u \in C([t_0,\infty);H) \cap L^2_{loc}(t_0,\infty;V), \frac{du}{dt} \in L^2_{loc}(t_0,\infty;V'). \tag{2.7}$$

Furthermore, the associated continuous semigroup $S(t) : H \to H$ has a global attractor $\mathcal{A}$ in H, which is bounded in V. Additionally, for any $u \in \mathcal{A}$, we have

$$|u| \le M_0 := \nu G, \quad \|u\| \le M_1. \tag{2.8}$$

where

$$M_1 := \begin{cases} \nu\lambda_1^{1/2}G & \textit{in the case of periodic boundary conditions,} \\ c\nu\lambda_1^{1/2}Ge^{\frac{G}{4}} & \textit{in the case of no-slip boundary conditions.} \end{cases}$$

We denote by $\lambda_1(\Omega_0\setminus\Omega)$ the first eigenvalue of the Stokes operator on the domain $\Omega_0 \setminus \overline{\Omega}$ with no-slip boundary conditions, i.e.,

$$\lambda_1(\Omega_0\setminus\Omega) := \inf\left\{\int_{\Omega_0\setminus\Omega} |\nabla\varphi|^2 dx \mid \forall\varphi \in H_0^1(\Omega_0\setminus\Omega) \text{ with } \int_{\Omega_0\setminus\Omega} |\varphi|^2 dx = 1\right\}.$$

Then we have the following lemma.

Lemma 2.2. [7, Lemma 1] *Let Ω and Ω_0 be bounded domains with smooth boundary so that $\Omega \subset \Omega_0$. For any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon) > 0$ so that for $\ell > \ell_0$, the following inequality holds*

$$\int_{\Omega_0} \left(|\nabla\varphi|^2 + \ell 1_\Omega|\varphi|^2\right) dx \ge (\lambda_1(\Omega_0\setminus\Omega) - \varepsilon)\int_{\Omega_0} |\varphi|^2 dx, \tag{2.9}$$

for $\varphi \in V$.

We note that here

$$\lambda_1(\Omega_0\setminus\Omega) \ge C\left(\sup_{x\in\Omega_0\setminus\Omega} \operatorname{dist}(x,\partial\Omega_0)\right)^{-2}. \tag{2.10}$$

We have the following spectral inequality (one can see [8] for the periodic boundary conditions case concerning with spectral inequality to thick sets in [15], or in [11, Theorem 3.1] for the no-slip boundary conditions case): If $\varphi \in \operatorname{span}(\phi_1,\dots,\phi_N)$, then

$$\|\varphi\|^2_{L^2(\Omega_0)^2} \le C_\Omega e^{C_\Omega\sqrt{N}}\|\varphi\|^2_{L^2(\Omega)^2}. \tag{2.11}$$

where C_Ω presents a positive constant which is independent of N.

3. Discrete data assimilation using local observables in the case of no-slip boundary conditions

In this section, the observational measurements at each time t_n are represented by

$$\tilde{u}(t_n) = P_m 1_\Omega(u(t_n)) + \eta_n, \tag{3.1}$$

where u is the unknown reference solution of of the two-dimensional Navier-Stokes equations (1.1), 1_Ω is the characteristic function of the sub-domain Ω of Ω_0, $P_m : H \to \operatorname{span}\{\phi_1,\dots,\phi_m\}$ is the low Fourier modes projector, which is defined as the orthogonal projector of H onto the subspace $H_m = \operatorname{span}\{\phi_1,\dots,\phi_m\}$ generated by m first eigenfunctions of the Stokes operator A, and η_n is the error associated to the measurements at time t_n. We assume that $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega_0)^2$.

We now follow the approach in [21] to introduce the following discrete data assimilation algorithm for finding an approximate solution v of the unknown reference solution u: Given an arbitrary initial data $v_0 \in H$, we look for a function v satisfying the same boundary conditions for u, and the following system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla q = f - \mu \sum\limits_{n=0}^{\infty} \left(P_m 1_\Omega(v(t_n)) - \tilde{u}(t_n)\right)\chi_n, \\ \nabla\cdot v = 0, \\ v(t_0) = v_0, \end{cases} \tag{3.2}$$

where ν and f are the same kinematic viscosity parameter and forcing term from (1.1), q is a modified pressure, $\tilde{u}(t_n)$ represents the observational measurements at time t_n given in (3.1), and $\mu > 0$ is a relaxation (nudging) parameter. The purpose of μ is to force the coarse spatial scales of v toward those of the reference solution v. As mentioned in [21], one of the advantages of this algorithm is that the initial data v_0 of the approximate solution can be chosen to be arbitrary.

Using the definition of $\tilde{u}(t_n)$ given in (3.1) and the functional setting from Section 2, we can rewrite system (3.2) in the following equivalent form

$$\begin{cases} \dfrac{dv}{dt} + \nu Av + B(v,v) = \mathcal{P}f - \mu \sum\limits_{n=0}^{\infty} \left(P_m 1_\Omega(v(t_n) - u(t_n))\right)\chi_n + \mu \sum\limits_{n=0}^{\infty} \mathcal{P}\eta_n\chi_n, \\ v(t_0) = v_0. \end{cases} \tag{3.3}$$

We will show that for any initial data $v_0 \in H$, the data assimilation equation (3.3) has a unique solution v defined on the whole interval $[t_0, \infty)$, and under suitable conditions of μ, κ, E_0, m and $\lambda_1(\Omega_0 \setminus \Omega)$, this approximate solution will converge to the reference solution u of the two-dimensional Navier-Stokes equations as time goes to ∞.

The existence and uniqueness of a global weak solution for the initial value problem associated to data assimilation equation (3.3) is given in the following theorem.

Theorem 3.1. *Let $v_0 \in H, f \in H$ and let u be a trajectory in the global attractor $\mathcal{A}$ of the two-dimensional Navier-Stokes equations. Then, there exists a unique solution v of equation* (3.3) *on* $[t_0, \infty)$ *satisfying $v(t_0) = v_0$ and*

$$v \in C([t_0, \infty); H) \cap L^2_{loc}(t_0, \infty; V), \frac{dv}{dt} \in L^2_{loc}(t_0, \infty; V').$$

Proof. The proof is very similar to that of Theorem 3.1 in [21], so we can omit it here.

□

Let us set

$$B_H(M_0) := \{u \in H : |u| \leq M_0\}.$$

The next theorem is the main result of this section.

Theorem 3.2. *Let Ω be a sub-domain of a C^2 bounded domain Ω_0. Let u be a trajectory in the global attractor $\mathcal{A}$ of the two-dimensional Navier-Stokes equations and let M_0 be positive constants related to estimates of the solution u given in* (2.8)*. Consider $v_0 \in B_H(M_0)$, and let v be the unique solution of* (3.3) *on the interval $[t_0, \infty)$ satisfying $v(t_0) = v_0$. Assume that $\{\eta_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\Omega_0)^2$, namely, there exists a constant $E_0 \geq 0$ such that*

$$\|\eta_n\|_{L^2(\Omega_0)^2} \leq E_0, \ \forall n \in \mathbb{N}. \tag{3.4}$$

If $\lambda_1(\Omega_0\setminus\Omega)$ and μ are large enough such that

$$\lambda_1(\Omega_0\setminus\Omega)\geq\frac{4\mu}{\nu}\ \textit{with}\ \mu\geq\max\left\{\frac{12c_0^2M_1^2}{\nu},\frac{\left(\nu^2\ell^2+24c_0^2M_1^2\ell\right)^{1/2}+\nu\ell}{8}\right\},\tag{3.5}$$

κ is small enough such that

$$\kappa\leq\frac{c}{\mu}\min\left\{1,\frac{\nu^{1/2}}{\left(\nu+c_0\left(M_0+\frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0\right)^2\right)^{1/2}},\left(\frac{\nu^2\lambda_1\mu}{\nu^2\mu^2+\lambda_1(c_0M_0M_1)^2}\right)^{1/2},\right.$$

$$\left.\frac{1}{\nu^2+c_0\left(M_0+\frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0\right)^2+\frac{(c_0M_0M_1)^2}{\nu^2\lambda_1}+\frac{\mu^2}{\lambda_1^2}+\frac{\mu}{\nu\lambda_1}}\right\},\tag{3.6}$$

and m is large enough such that

$$\lambda_{m+1}\geq\frac{6\mu}{\nu},\tag{3.7}$$

then

$$\limsup_{t\to\infty}|v(t)-u(t)|\leq cE_0.$$

Moreover, if $E_0=0$, then $v(t)\to u(t)$ in H, exponentially, as $t\to\infty$.

Here, c_0 is the best constant in estimate (2.4) *and c is a suitable positive constant independent of parameters of system.*

Remark 3.1. Comparing with the corresponding conditions in the discrete data assimilation using global observables [21], one can see that condition (3.7) on m is exactly the same, condition (3.5) on μ is slightly larger, condition on κ is also slightly changed. This comes from the fact that the inequality (2.9) is used to overcome the essential difficulty caused by local observables.

By inequality (2.10), condition $\lambda_1(\Omega_0\setminus\Omega)\geq\frac{4\mu}{\nu}$ will hold if the sub-domain Ω is large enough.

Proof of Theorem 3.2. Denote $w=u-v$. Subtracting (3.3) from (2.6) to obtain

$$\frac{dw}{dt}+\nu Aw+B(u,w)+B(w,u)+B(w,w)=-\mu\sum_{n=0}^{\infty}P_m1_\Omega(w(t_n))\chi_n+\mu\sum_{n=0}^{\infty}\mathcal{P}\eta_n\chi_n.\tag{3.8}$$

Here we have used

$$B(v,v)-B(u,u)=B(u,w)+B(w,u)+B(w,w).$$

Multiplying (3.8) by w, then integrating over Ω_0 and using property (2.3) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|w|^2+\nu\|w\|^2+\langle B(w,u),w\rangle_{V',V}\\&=-\mu\sum_{n=0}^{\infty}(P_m1_\Omega(w(t_n)),w)\chi_n+\mu\sum_{n=0}^{\infty}(\eta_n,w)\chi_n\\&=-\mu|P_m1_\Omega w|^2-\mu\sum_{n=0}^{\infty}(P_m1_\Omega(w(t_n)-w),w)\chi_n+\mu\sum_{n=0}^{\infty}(\eta_n,w)\chi_n.\end{aligned}\tag{3.9}$$

Since u is a trajectory in the global attractor $\mathcal{A}$, we can use the bound from (2.8). Using (2.4), and the Cauchy inequality, we obtain

$$|\langle B(w,u),w\rangle_{V',V}|\leq c_0|w|\|w\|\|u\|$$

$$\leq \frac{3c_0^2M_1^2}{2\nu}|w|^2 + \frac{\nu}{6}\|w\|^2. \tag{3.10}$$

Using hypothesis (3.7) we get

$$\begin{aligned}
-\mu|P_m 1_\Omega w|^2 &= -\mu\|w\|^2_{L^2(\Omega)^2} + \mu\|Q_m w\|^2_{L^2(\Omega)^2}\\
&\leq -\mu\|w\|^2_{L^2(\Omega)^2} + \mu|Q_m w|^2\\
&\leq -\mu\|w\|^2_{L^2(\Omega)^2} + \frac{\mu}{\lambda_{m+1}}\|w\|^2\\
&\leq -\mu\|w\|^2_{L^2(\Omega)^2} + \frac{\nu}{6}\|w\|^2.
\end{aligned}\tag{3.11}$$

Also, using the Cauchy inequality and the bound from hypothesis (3.4), we have

$$\begin{aligned}
\mu|(\eta_n, w)| &\leq \mu\|\eta_n\|_{L^2(\Omega_0)^2}|w|\\
&\leq \mu E_0|w|\\
&\leq \frac{\mu}{2}E_0^2 + \frac{\mu}{2}|w|^2.
\end{aligned}\tag{3.12}$$

Moreover,

$$\begin{aligned}
\mu|(P_m 1_\Omega(w(t_n) - w(t)), w(t))| &= \mu|(w(t_n) - w(t), P_m 1_\Omega w(t))|\\
&= \mu\left|\left(\int_{t_n}^t \frac{dw}{ds}(s)ds, P_m 1_\Omega w(t)\right)\right|\\
&\leq \mu\left\|\int_{t_n}^t \frac{dw}{ds}(s)ds\right\|_{V'}\|P_m 1_\Omega w(t)\|\\
&\leq \mu\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\ \|w(t)\|\\
&\leq \frac{3\mu^2}{2\nu}\left(\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\right)^2 + \frac{\nu}{6}\|w(t)\|^2.
\end{aligned}\tag{3.13}$$

From (3.8), we obtain

$$\begin{aligned}
\left\|\frac{dw}{ds}(s)\right\|_{V'} &\leq \nu\|w(s)\| + \|B(u,w)\|_{V'} + \|B(w,u)\|_{V'} + \|B(w,w)\|_{V'}\\
&\quad + \mu\|P_m(w(t_n) - w(s))\|_{V'} + \mu\|P_m(w(s))\|_{V'} + \mu\|\eta_n\|_{V'}.
\end{aligned}$$

Then, by same as in the proof of (4.16) in [21], we deduce that

$$\left(\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\right)^2 \leq c\kappa\int_{t_n}^t \varphi(s)ds + \frac{c\mu^2\kappa^2}{\lambda_1}E_0^2, \tag{3.14}$$

where

$$\varphi(s) = \nu^2\|w(s)\|^2 + 2c_0M_0M_1|w(s)|\|w(s)\| + c_0|w(s)|^2\|w(s)\|^2 + \frac{\mu^2}{\lambda_1}|w(s)|^2. \tag{3.15}$$

Substituting estimates (3.10), (3.11), (3.12), (3.13) and (3.14) into (3.9) we have

$$\begin{aligned}
\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\leq -2\mu\|w\|^2_{L^2(\Omega)^2} + \frac{3c_0^2M_1^2}{\nu}|w|^2 + \mu|w|^2\\
&\quad + \frac{c\mu^2\kappa}{\nu}\sum_{n=0}^{\infty}\chi_n\int_{t_n}^t \varphi(s)ds + \mu\left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1}\right)E_0^2.
\end{aligned}\tag{3.16}$$

We denote $R = 2M_0 + \frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0$ then since $w \in C([t_0,\infty);H)$, and

$$|w(t_0)| \leq |v(t_0)| + |u(t_0)| \leq 2M_0 \leq R,$$

there exists $\tau \in (t_0, \infty)$ such that

$$|w(t)| \leq 2R, \;\; \forall t \in [t_0, \tau].$$

Define

$$\tilde{t} = \sup \left\{ \tau \in [t_0, \infty) \; : \; \sup_{t \in [t_0, \tau]} |w(t)| \leq 2R \right\}. \tag{3.17}$$

Suppose that $\tilde{t} < t_1$. Then, integrating (3.16) from t_0 to $t \leq \tilde{t}$, we obtain that

$$\begin{aligned} &|w(t)|^2 - |w(t_0)|^2 + \nu \int_{t_0}^{t} \|w(s)\|^2 ds \\ \leq & -2\mu \int_{t_0}^{t} \|w(s)\|^2_{L^2(\Omega)^2} ds + \left(\frac{3c_0^2 M_1^2}{\nu} + \mu \right) \int_{t_0}^{t} |w(s)|^2 ds \\ &+ \frac{c\mu^2\kappa^2}{\nu} \int_{t_0}^{t} \varphi(s) ds + \mu\kappa \left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1} \right) E_0^2. \end{aligned} \tag{3.18}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty} \chi_n \int_{t_n}^{t} \varphi(s) ds \leq \int_{t_0}^{t} \varphi(s) ds, \;\; \forall t \in [t_0, \tilde{t}].$$

Since $|w(t)| \leq 2R$ for all $t \in [t_0, \tilde{t}]$, and using the Cauchy inequality we have

$$\varphi(s) \leq \left(2\nu^2 + 4c_0R^2\right) \|w(s)\|^2 + \left(\frac{(c_0 M_0 M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1} \right) |w(s)|^2. \tag{3.19}$$

Hence, (3.18) becomes

$$\begin{aligned} &|w(t)|^2 - |w(t_0)|^2 + \nu \left(1 - c\mu^2\kappa^2 \left(1 + \frac{c_0 R^2}{\nu} \right) \right) \int_{t_0}^{t} \|w(s)\|^2 ds \\ \leq & -2\mu \int_{t_0}^{t} \|w(s)\|^2_{L^2(\Omega)^2} ds + \mu\kappa \left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1} \right) E_0^2 \\ &+ \left(\frac{3c_0^2 M_1^2}{\nu} + \mu + \left(\frac{c_0^2 (M_0 M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1} \right) \frac{c\mu^2\kappa^2}{\nu} \right) \int_{t_0}^{t} |w(s)|^2 ds. \end{aligned} \tag{3.20}$$

Using condition (3.3) and applying (2.9) then we deduce from (3.20) that

$$\begin{aligned} &|w(t)|^2 - |w(t_0)|^2 \\ &+ \left\{ \frac{\nu}{2} - \left(\frac{3c_0^2 M_1^2}{\nu} + \mu + \left(\frac{c_0^2 (M_0 M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1} \right) \frac{c\mu^2\kappa^2}{\nu} \right) \frac{1}{\lambda_1(\Omega_0 \setminus \Omega)} \right\} \int_{t_0}^{t} \|w(s)\|^2 ds \\ &\leq \left\{ -2\mu + \left(\frac{3c_0^2 M_1^2}{\nu} + \mu \right.\right. \\ &\qquad \left.\left. + \left(\frac{c_0^2 (M_0 M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1} \right) \frac{c\mu^2\kappa^2}{\nu} \right) \frac{\ell}{\lambda_1(\Omega_0 \setminus \Omega)} \right\} \int_{t_0}^{t} \|w(s)\|^2_{L^2(\Omega)^2} ds \\ &\quad + \mu\kappa \left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1} \right) E_0^2. \end{aligned} \tag{3.21}$$

From condition (3.6) on κ and condition (3.5) on μ and $\lambda_1(\Omega_0 \setminus \Omega)$, we deduce from (3.21) that

$$|w(t)|^2 - |w(t_0)|^2 + \frac{\nu}{2} \int_{t_0}^{t} \|w(s)\|^2 ds \leq cE_0^2,$$

which implies in particular that

$$\int_{t_0}^{t} \|w(s)\|^2 ds \leq \frac{2}{\nu} |w(t_0)|^2 + \frac{c}{\nu} E_0^2, \;\; \forall t \in \left[t_0, \tilde{t}\right]. \tag{3.22}$$

Furthermore, by using the Poincaré inequality (2.1), we get from (3.19) that

$$\varphi(s) \leq \left(2\nu^2 + 4c_0R^2 + \lambda_1^{-1}\left(\frac{c_0^2(M_0M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1}\right)\right)\|w(s)\|^2. \tag{3.23}$$

Substituting (3.23) into (3.16) and using (2.9) then we have that for all $t \in \left[t_0, \tilde{t}\right]$:

$$\begin{aligned}
&\frac{d}{dt}|w|^2 + \left(\nu - \left(\frac{3c_0^2M_1^2}{\nu} + \mu\right)\frac{1}{\lambda_1(\Omega_0 \setminus \Omega)}\right)\|w\|^2 \\
\leq &- \left(2\mu - \left(\frac{3c_0^2M_1^2}{\nu} + \mu\right)\frac{\ell}{\lambda_1(\Omega_0 \setminus \Omega)}\right)\|w\|^2_{L^2(\Omega)^2} \\
&+ \frac{c\mu^2\kappa}{\nu}\left(2\nu^2 + 4c_0R^2 + \frac{c_0^2(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\int_{t_0}^t \|w(s)\|^2 ds \\
&+ \mu\left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1}\right)E_0^2.
\end{aligned} \tag{3.24}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty}\int_{t_n}^t \varphi(s)ds \leq \int_{t_0}^t \varphi(s)ds, \ \forall t \in [t_0, \tilde{t}].$$

Substituting (3.22) into (3.24) and using condition (3.5) to deduce

$$\begin{aligned}
&\frac{d}{dt}|w|^2 + \frac{\nu}{2}\|w\|^2 + \ell\|w\|^2_{L^2(\Omega)^2} \\
\leq &\left(2\nu^2 + 4c_0R^2 + \frac{c_0^2(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\frac{c\mu^2\kappa}{\nu}\left(|w(t_0)|^2 + E_0^2\right) + \mu\left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1}\right)E_0^2.
\end{aligned} \tag{3.25}$$

Using (2.9) and note that $R = 2M_0 + \frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0$ we deduce from (3.25) that

$$\begin{aligned}
&\frac{d}{dt}|w|^2 + \lambda_1(\Omega_0 \setminus \Omega)\min\left\{\frac{\nu}{2}, 1\right\}|w|^2 \\
\leq &\left(\nu^2 + c_0\left(M_0 + \frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0\right)^2 + \frac{c_0^2(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\frac{c\mu^2\kappa}{\nu}\left(|w(t_0)|^2 + E_0^2\right) \\
&+ \mu\left(1 + \frac{c\mu^3\kappa^2}{\nu\lambda_1}\right)E_0^2.
\end{aligned} \tag{3.26}$$

Using the Gronwall inequality to (3.26) in $[t_0, t], t < \tilde{t}$, with noting that $\lambda_1(\Omega_0 \setminus \Omega) \geq \frac{4\mu}{\nu}$, we have

$$\begin{aligned}
|w(t)|^2 \leq &\left(1 - e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)}\right)\left(\gamma_1|w(t_0)|^2 + \gamma_2E_0^2 + \frac{\nu}{4\min\left\{\frac{\nu}{2},1\right\}}E_0^2\right) \\
&+ |w(t_0)|^2e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)},
\end{aligned} \tag{3.27}$$

where

$$\gamma_1 = \frac{c\mu\kappa\nu}{4\min\left\{\frac{\nu}{2},1\right\}}\left(\nu^2 + c_0\left(M_0 + \frac{\nu^{1/2}}{\left(\min\left\{\frac{\nu}{2},1\right\}\right)^{1/2}}E_0\right)^2 + \frac{c_0^2(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right),$$

and

$$\gamma_2 = \gamma_1 + \frac{c\mu^3\kappa^2}{4\lambda_1\min\left\{\frac{\nu}{2},1\right\}}.$$

Since $|w(t_0)| \leq R$, then (3.27) becomes

$$|w(t)|^2 \leq \left(1 - e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)}\right)\left(\gamma_1 R^2 + \gamma_2 E_0^2 + \frac{\nu}{4\min\left\{\frac{\nu}{2},1\right\}}E_0^2\right) + R^2 e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)}.$$

Using the choice of κ in (3.6) with suitable constant c, we get $\gamma_1 \leq 1/2, \gamma_2 \leq \frac{\nu}{4\min\left\{\frac{\nu}{2},1\right\}}$ so that

$$\gamma_1 R^2 + \gamma_2 E_0^2 + \frac{\nu}{4\min\left\{\frac{\nu}{2},1\right\}}E_0^2 \leq R^2.$$

Thus

$$|w(t)| \leq R,\ \forall t \in [t_0, \tilde{t}].$$

In particular, $|w(\tilde{t})| \leq R$, and from the definition of $\tilde{t}$ in (3.17) we conclude that $\tilde{t} \geq t_1$. Therefore, we also have $|w(t_1)| \leq R$ and we can apply the same previous arguments to obtain that $\tilde{t} \geq t_2$ and $|w(t_2)| \leq R$. Continuing inductively, we obtain that $\tilde{t} \geq t_n$, for all $n \geq 0$. Furthermore, we get the same as (3.27) that

$$|w(t)|^2 \leq \left(1 - e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)}\right)\left(\gamma_1 |w(t_n)|^2 + \gamma_2 E_0^2 + \frac{\nu}{4\min\left\{\frac{\nu}{2},1\right\}}E_0^2\right) + |w(t_n)|^2 e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}(t-t_0)}, \tag{3.28}$$

for all $t \in [t_n, t_{n+1}]$ and for all $n \in \mathbb{N}$. Therefore,

$$|w(t_{n+1})|^2 \leq \theta |w(t_n)|^2 + cE_0^2,\ \forall n \geq 0,$$

where

$$\theta = e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}\kappa} + \gamma_1\left(1 - e^{-\frac{4\mu}{\nu}\min\left\{\frac{\nu}{2},1\right\}\kappa}\right) < 1.$$

Thus,

$$|w(t_n)|^2 \leq \theta^n |w(t_0)|^2 + cE_0^2 \sum_{j=0}^{n-1} \theta^j,\ \forall n \geq 1. \tag{3.29}$$

Combining (3.28) and (3.29) we deduce that

$$|w(t)|^2 \leq \theta^n |w(t_0)|^2 + cE_0^2\left(1 + \sum_{j=0}^{n-1}\theta^j\right), \forall t \in [t_n, t_{n+1}],\ \forall n \geq 1.$$

Therefore

$$\limsup_{t\to\infty} |w(t)|^2 \leq cE_0^2.$$

Moreover, if $E_0 = 0$, we have

$$|w(t)|^2 \leq \theta^n |w(t_0)|^2,\ \forall t \in [t_n, t_{n+1}],\ \forall n \geq 1,$$

and thus $w(t)$ converges exponentially to 0 in H as $t \to \infty$. □

4. Finite-dimensional discrete data assimilation using local observables

We now consider the two-dimensional Navier-Stokes equations (1.1) in the both two cases, periodic boundary conditions and no-slip boundary conditions. For any positive integer N, we denote P_N the projection onto the finite-dimensional subspace generated by N first eigenvectors of the Stokes operator. For any $N \in \mathbb{N}$, we will consider the measurements as follows

$$\tilde{u}(t_n) = P_N 1_\Omega(u(t_n)) + P_N \eta_n, \tag{4.1}$$

where u is the unknown solution of (1.1).

We now follow the approach in [8] to introduce the following data assimilation algorithm for finding an approximate solution v_N of the unknown reference solution v: Given information about a reference solution u by using the interpolant operator $P_N 1_\Omega$, we look for a function v_N satisfies the following system

$$\begin{cases} \dfrac{dv_N}{dt} + \nu A v_N + P_N B(v_N, v_N) = P_N f - \mu \sum\limits_{n=0}^{\infty} \left(1_\Omega v_N(t_n) - \tilde{u}(t_n)\right) \chi_n, \\ v_N(t_0) = v_0. \end{cases} \tag{4.2}$$

Using (4.1) then (4.2) can be rewritten as following system

$$\begin{cases} \dfrac{dv_N}{dt} + \nu A v_N + P_N B(v_N, v_N) = P_N f - \mu \sum\limits_{n=0}^{\infty} 1_\Omega (v_N(t_n) - P_N u(t_n))\chi_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mu \sum\limits_{n=0}^{\infty} P_N \eta_n \chi_n, \\ v_N(t_0) = v_0. \end{cases} \tag{4.3}$$

By similar arguments as in the proof of Theorem 3.1 we also have that (4.3) has a unique global weak solution $v_N(t)$ on $[t_0, \infty)$. We can now state the main result of this section.

Theorem 4.1. *Let Ω be any open subset of the domain Ω_0. Let u be a trajectory in the global attractor $\mathcal{A}$ of the two-dimensional Navier-Stokes equations and let M_0 be the positive constant related to estimates of the solution u given in* (2.8). *Consider $v_0 \in B_H(M_0)$, and let v_N be the unique solution of* (4.3) *on the interval $[t_0, \infty)$ satisfying $v_N(t_0) = v_0$. Let $\varepsilon > 0$ be given. Assume that $\{\eta_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\Omega_0)^2$, namely, there exists a constant $E_1 \geq 0$ such that*

$$\|\eta_n\|_{L^2(\Omega_0)^2} \leq E_1, \ \forall n \in \mathbb{N}. \tag{4.4}$$

If N is large enough such that

$$N \geq c \max\left\{ \frac{c_0^2 M_0^2 M_1^6 (c_0 + M_1)^2}{(\varepsilon \nu \lambda_1)^2}, \frac{M_1^2}{\varepsilon} \right\}, \tag{4.5}$$

μ is defined by

$$\mu = \frac{3 c_0^2 M_1^2 C_\Omega e^{C_\Omega \sqrt{N}}}{\nu}, \tag{4.6}$$

and κ is small enough such that

$$\begin{aligned} \kappa \leq \frac{c}{\mu} \min\Bigg\{ & 1, \nu \left(\nu^2 + M_0^2 + \frac{\mu^2}{\nu^2 \lambda_1^2} E_1^2 \right)^{-1/2}, c_0 M_1 \left(\frac{(M_0 M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1} \right)^{-1}, \\ & \frac{\nu\lambda_1}{\mu\overline{\mu}}, \frac{\mu}{\mu + \lambda_1 \overline{\mu}}, \frac{(\nu\lambda_1)^2}{\mu\lambda_1\overline{\mu} + \mu^2} \Bigg\}, \end{aligned} \tag{4.7}$$

where

$$\overline{\mu} := \nu^2 + M_0^2 + \frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + \frac{(M_0 M_1)^2}{\nu^2 \lambda_1} + \frac{\mu^2}{\lambda_1^2},$$

then

$$|v_N(t) - u(t)|^2 \leq c \frac{\mu^2}{\nu \lambda_1} E_1^2 + \varepsilon, \tag{4.8}$$

for t sufficiently large.

Remark 4.1. We see that if we choose

$$N = N_0(\varepsilon) = \left[c \max\left\{ \frac{c_0^2 M_0^2 M_1^6 (c_0 + M_1)^2}{(\varepsilon \nu \lambda_1)^2}, \frac{M_1^2}{\varepsilon} \right\} \right] + 1,$$

then $\mu = \mu_0(\varepsilon)$ has the form (4.6) which depends on ε. So if the size of errors E_1 satisfies $E_1 \leq \frac{c\varepsilon\nu\lambda_1}{\mu_0(\varepsilon)}$, then we obtain that $|v_N(t) - u(t)| \leq 2\varepsilon$ for t sufficiently large.

Proof of Theorem 4.1. Let ε be given and we set $\overline{\varepsilon} = \frac{\varepsilon\nu\lambda_1}{8}$. Let u and v_N be as in the statement of Theorem 4.1. Note that for any $N \in \mathbb{N}$, $P_N u$ satisfies

$$\frac{d}{dt}P_N u + \nu A P_N u + P_N B(P_N u, u) = P_N f - P_N B(Q_N u, u), \tag{4.9}$$

where $Q_N = I - P_N$. Let $w = v_N - P_N u$. Subtracting (4.2) from (4.9) to obtain

$$\begin{aligned}&\frac{dw}{dt} + \nu A w + P_N B(P_N u, w) + P_N B(w, w) + P_N B(w, P_N u)\\ =& -\mu \sum_{n=0}^{\infty} 1_\Omega w(t_n)\chi_n + \mu \sum_{n=0}^{\infty} P_N \eta_n \chi_n + P_N B(Q_N u, u) + P_N B(P_N u, Q_N u).\end{aligned} \tag{4.10}$$

Here we have used the fact that

$$\begin{aligned}P_N B(v_N, v_N) - P_N B(P_N u, u) = & P_N B(P_N u, w) + P_N B(w, w)\\ &+ P_N B(w, P_N u) - P_N B(P_N u, Q_N u).\end{aligned}$$

Multiplying (4.10) by w, then integrating over Ω_0 and using (2.3) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + \langle B(w, P_N u), w\rangle_{V',V}\\ =& -\mu \sum_{n=0}^{\infty} (1_\Omega w(t_n), w)\,\chi_n + \mu \sum_{n=0}^{\infty} (P_N\eta_n, w)\chi_n\\ &+ \langle B(Q_N u, u), w\rangle_{V',V} + \langle B(P_N u, Q_N u), w\rangle_{V',V}\\ =& -\mu\|w\|^2_{L^2(\Omega)^2} - \mu \sum_{n=0}^{\infty} (1_\Omega (w(t_n) - w), w)\,\chi_n + \mu \sum_{n=0}^{\infty} (\eta_n, w)\chi_n\\ &+ \langle B(Q_N u, u), w\rangle_{V',V} + \langle B(P_N u, Q_N u), w\rangle_{V',V}.\end{aligned} \tag{4.11}$$

We now estimate for the right hand side of (4.11). We first apply (2.4), the Cauchy inequality, (2.8) and the spectral inequality (2.11) to obtain

$$\begin{aligned}\langle B(w, P_N u), w\rangle_{V',V} &\le c_0|w|\|w\|\|P_N u\|\\ &\le \frac{2c_0^2}{\nu}\|u\|^2|w|^2 + \frac{\nu}{8}\|w\|^2\\ &\le \frac{2c_0^2 M_1^2}{\nu}|w|^2 + \frac{\nu}{8}\|w\|^2\\ &\le \frac{2c_0^2 M_1^2 C_\Omega e^{C_\Omega\sqrt{N}}}{\nu}\|w\|^2_{L^2(\Omega)^2} + \frac{\nu}{8}\|w\|^2.\end{aligned} \tag{4.12}$$

Applying (2.4), (2.8) and the Cauchy inequality, we obtain

$$\begin{aligned}&\langle B(Q_N u, u), w\rangle_{V',V} + \langle B(P_N u, Q_N u), w\rangle_{V',V}\\ &\le c_0\left(|Q_N u|^{1/2}\|Q_N u\|^{1/2}|u|^{1/2}\|u\|^{1/2} + |P_N u|^{1/2}\|P_N u\|^{1/2}|Q_N u|^{1/2}\|Q_N u\|^{1/2}\right)\|w\|\\ &\le 2c_0|Q_N u|^{1/2}\|Q_N u\|^{1/2}|u|^{1/2}\|u\|\|w\|\\ &\le c_0^2|Q_N u|\|Q_N u\||u|\|u\| + \frac{\nu}{8}\|w\|^2\\ &\le c_0^2\lambda_N^{-1/2}M_0 M_1^3 + \frac{\nu}{8}\|w\|^2.\end{aligned} \tag{4.13}$$

By the Cauchy inequality and using condition (4.4), we have

$$\mu(\eta_n, w) \le \frac{2\mu^2}{\nu\lambda_1}E_1^2 + \frac{\nu\lambda_1}{8}|w|^2 \le \frac{2\mu^2}{\nu\lambda_1}E_1^2 + \frac{\nu}{8}\|w\|^2. \tag{4.14}$$

Moreover, we have

$$
\begin{aligned}
\mu|(1_\Omega(w(t_n)-w(t)),w(t))| &= \mu\left|\left(\int_{t_n}^t \frac{dw}{ds}(s)ds, 1_\Omega w(t)\right)\right| \\
&\le \mu\left\|\int_{t_n}^t \frac{dw}{ds}(s)ds\right\|_{V'}\|1_\Omega w(t)\| \\
&\le \mu\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\ \|w(t)\| \\
&\le \frac{2\mu^2}{\nu}\left(\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\right)^2 + \frac{\nu}{8}\|w(t)\|^2. \qquad (4.15)
\end{aligned}
$$

From (4.10), using (2.5) and (2.2) we obtain

$$
\begin{aligned}
\left\|\frac{dw}{ds}(s)\right\|_{V'} &\le \nu\|w(s)\| + \|B(P_Nu,w)\|_{V'} + \|B(w,P_Nu)\|_{V'} + \|B(w,w)\|_{V'} \\
&\quad + \mu\|1_\Omega(w(t_n)-w(s))\|_{V'} + \mu\|1_\Omega w(s)\|_{V'} \\
&\quad + \mu\|\eta_n\|_{V'} + \|P_NB(Q_Nu,u)\|_{V'} + \|P_NB(P_Nu,Q_Nu)\|_{V'} \\
&\le \nu\|w(s)\| + 2c_0|P_Nu|^{1/2}\|P_Nu\|^{1/2}|w|^{1/2}\|w\|^{1/2} + c_0|w|\|w\| \\
&\quad + \mu\int_{t_n}^s \left\|\frac{dw}{d\tau}(\tau)\right\|_{V'} d\tau + \frac{\mu}{\lambda_1^{1/2}}|w(s)| + \frac{\mu}{\lambda_1^{1/2}}\|\eta_n\|_{L^2(\Omega_0)^2} \\
&\quad + 2c_0|Q_Nu|^{1/2}\|Q_Nu\|^{1/2}|u|^{1/2}\|u\| \\
&\le \nu\|w(s)\| + 2c_0(M_0M_1)^{1/2}|w|^{1/2}\|w\|^{1/2} + c_0|w|\|w\| \\
&\quad + \mu\int_{t_n}^s \left\|\frac{dw}{d\tau}(\tau)\right\|_{V'} d\tau + \frac{\mu}{\lambda_1^{1/2}}(|w(s)|+E_1) + 2c_0\lambda_N^{-1/4}M_0^{1/2}M_1^2.
\end{aligned}
$$

Integrating with respect to s from t_n to $t\in[t_n,t_{n+1})$, we obtain that

$$
\begin{aligned}
\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds &\le \int_{t_n}^t \Big(\nu\|w(s)\| + 2c_0(M_0M_1)^{1/2}|w(s)|^{1/2}\|w(s)\|^{1/2} \\
&\quad + c_0|w(s)|\|w(s)\| + \frac{\mu}{\lambda_1^{1/2}}|w(s)|\Big)ds + \frac{\mu\kappa}{\lambda_1^{1/2}}E_1 \\
&\quad + \frac{2c_0\kappa M_0^{1/2}M_1^2}{\lambda_N^{1/4}} + \mu\kappa\int_{t_n}^t \left\|\frac{dw}{d\tau}(\tau)\right\|_{V'} d\tau.
\end{aligned}
$$

Using condition on κ we deduce

$$
\begin{aligned}
\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds &\le c\int_{t_n}^t \Big(\nu\|w(s)\| + 2c_0(M_0M_1)^{1/2}|w(s)|^{1/2}\|w(s)\|^{1/2} \\
&\quad + c_0|w(s)|\|w(s)\| + \frac{\mu}{\lambda_1^{1/2}}|w(s)|\Big)ds + c\frac{\mu\kappa}{\lambda_1^{1/2}}E_1 + \frac{cc_0\kappa M_0^{1/2}M_1^2}{\lambda_N^{1/4}}.
\end{aligned}
$$

By the Hölder inequality it implies that

$$
\left(\int_{t_n}^t \left\|\frac{dw}{ds}(s)\right\|_{V'} ds\right)^2 \le c\kappa\int_{t_n}^t \varphi(s)ds + \frac{c\mu^2\kappa^2}{\lambda_1}E_1^2 + \frac{cc_0\kappa M_0M_1^4}{\lambda_N^{1/2}}, \qquad (4.16)
$$

where $\varphi(s)$ is the same as in (3.15). Substituting estimates (4.12), (4.13), (4.14), (4.15) and (4.16) into (4.11) we have

$$
\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le -\left(2\mu - \frac{4c_0^2M_1^2C_\Omega e^{C_\Omega\sqrt{N}}}{\nu}\right)\|w\|^2_{L^2(\Omega)^2} + \frac{2c_0M_0M_1^3(c_0+c\mu\kappa M_1)}{\lambda_N^{1/2}}
$$

$$+\frac{c\mu^2\kappa}{\nu}\sum_{n=0}^{\infty}\chi_n\int_{t_n}^{t}\varphi(s)ds+\mu^2\left(\frac{4}{\nu\lambda_1}+\frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2. \tag{4.17}$$

We denote $R = 2M_0 + \frac{3\sqrt{2}\mu}{\nu\lambda_1}E_1$ then since $w \in C([t_0,\infty); H)$, and

$$|w(t_0)| \le |v(t_0)| + |u(t_0)| \le 2M_0 \le R,$$

there exists $\tau \in (t_0,\infty)$ such that

$$|w(t)| \le 2R, \ \ \forall t \in [t_0,\tau].$$

Define

$$\tilde{t} = \sup\left\{\tau \in [t_0,\infty) \ : \ \sup_{t\in[t_0,\tau]}|w(t)| \le 2R\right\}.$$

Suppose that $\tilde{t} < t_1$. Then, integrating (4.17) from t_0 to $t \le \tilde{t}$, we obtain that

$$\begin{aligned}
&|w(t)|^2-|w(t_0)|^2+\nu\int_{t_0}^{t}\|w(s)\|^2ds\\
&\le-\left(2\mu-\frac{4c_0^2M_1^2C_\Omega e^{C_\Omega\sqrt{N}}}{\nu}\right)\int_{t_0}^{t}\|w(s)\|^2_{L^2(\Omega)^2}ds+\frac{2\mu\kappa c_0M_0M_1^3(c_0+c\mu\kappa M_1)}{\lambda_N^{1/2}}\\
&\quad+\frac{c\mu^2\kappa^2}{\nu}\int_{t_0}^{t}\varphi(s)ds+\mu^2\kappa\left(\frac{4}{\nu\lambda_1}+\frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2.
\end{aligned}\tag{4.18}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty}\chi_n\int_{t_n}^{t}\varphi(s)ds \le \int_{t_0}^{t}\varphi(s)ds, \ \forall t\in[t_0,\tilde{t}].$$

Since $|w(t)| \le 2R$ for all $t \in [t_0,\tilde{t}]$, and using the spectral inequality (2.11) we have

$$\varphi(s) \le (2\nu^2+4R^2)\|w(s)\|^2+\left(\frac{(M_0M_1)^2}{\nu^2}+\frac{\mu^2}{\lambda_1}\right)C_\Omega e^{C_\Omega\sqrt{N}}\|w(s)\|^2_{L^2(\Omega)^2}.$$

Here we have used (2.11). Hence, (4.18) becomes

$$\begin{aligned}
&|w(t)|^2-|w(t_0)|^2+\left(\nu-(2\nu^2+4R^2)\frac{c\mu^2\kappa^2}{\nu}\right)\int_{t_0}^{t}\|w(s)\|^2ds\\
&\le-\Bigg\{2\mu-\frac{4c_0^2M_1^2C_\Omega e^{C_\Omega\sqrt{N}}}{\nu}\\
&\qquad-\frac{c\mu^2\kappa^2}{\nu}\left(\frac{(M_0M_1)^2}{\nu^2}+\frac{\mu^2}{\lambda_1}\right)C_\Omega e^{C_\Omega\sqrt{N}}\Bigg\}\int_{t_0}^{t}\|w\|^2_{L^2(\Omega)^2}ds\\
&\quad+\frac{2\mu\kappa c_0M_0M_1^3(c_0+c\mu\kappa M_1)}{\lambda_N^{1/2}}+\mu^2\kappa\left(\frac{4}{\nu\lambda_1}+\frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2.
\end{aligned}\tag{4.19}$$

From condition (4.7) on κ and condition (4.5) on μ, we deduce from (4.19) that

$$|w(t)|^2-|w(t_0)|^2+\frac{\nu}{2}\int_{t_0}^{t}\|w(s)\|^2ds \le cE_1^2+\overline{\varepsilon},$$

which implies in particular that

$$\int_{t_0}^{t}\|w(s)\|^2ds \le \frac{1}{\nu}\left(2|w(t_0)|^2+cE_1^2+2\overline{\varepsilon}\right), \ \forall t\in\left[t_0,\tilde{t}\right]. \tag{4.20}$$

Using the Poincaré inequality (2.1), we obtain as same as (3.23) that

$$\varphi(s) \le \left(2\nu^2+4R^2+\lambda_1^{-1}\left(\frac{(M_0M_1)^2}{\nu^2}+\frac{\mu^2}{\lambda_1}\right)\right)\|w(s)\|^2.$$

Then we have from (4.17) that for all $t \in \big[t_0, \tilde{t}\big]$:

$$\begin{aligned}\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le &- \left(2\mu - \frac{4c_0^2 M_1^2 C_\Omega e^{C_\Omega \sqrt{N}}}{\nu}\right)\|w\|^2_{L^2(\Omega)^2} + \frac{2c_0 M_0 M_1^3 (c_0 + c\mu\kappa M_1)}{\lambda_N^{1/2}} \\ &+ \frac{c\mu^2\kappa}{\nu}\left(2\nu^2 + 4R^2 + \lambda_1^{-1}\left(\frac{(M_0M_1)^2}{\nu^2} + \frac{\mu^2}{\lambda_1}\right)\right)\|w(s)\|^2 \\ &+ \mu^2\left(\frac{4}{\nu\lambda_1} + \frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2. \end{aligned} \tag{4.21}$$

Substituting (4.20) into (4.21) to deduce

$$\begin{aligned}\frac{d}{dt}|w|^2 + \frac{\nu}{2}\|w\|^2 + &\left(2\mu - \frac{4c_0^2 M_1^2 C_\Omega e^{C_\Omega \sqrt{N}}}{\nu}\right)\|w\|^2_{L^2(\Omega)^2} \\ &\le \left(2\nu^2 + 4R^2 + \frac{(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\frac{c\mu^2\kappa}{\nu}\left(|w(t_0)|^2 + E_1^2 + \bar{\varepsilon}\right) \\ &\quad + \frac{2c_0 M_0 M_1^3 (c_0 + c\mu\kappa M_1)}{\lambda_N^{1/2}} + \mu^2\left(\frac{4}{\nu\lambda_1} + \frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2. \end{aligned} \tag{4.22}$$

From condition (4.7) on κ and condition (4.5) on N, we have

$$\frac{2c_0 M_0 M_1^3 (c_0 + c\mu\kappa M_1)}{\lambda_N^{1/2}} \le \frac{\bar{\varepsilon}}{2}.$$

From condition (4.6) on μ, then from (4.22) we have

$$\begin{aligned}\frac{d}{dt}|w|^2 + \frac{\nu}{2}\|w\|^2 \le &\left(2\nu^2 + 4R^2 + \frac{(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\frac{c\mu^2\kappa}{\nu}\left(|w(t_0)|^2 + E_1^2 + \bar{\varepsilon}\right) \\ &+ \bar{\varepsilon} + \mu^2\left(\frac{4}{\nu\lambda_1} + \frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2. \end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}|w|^2 + \frac{\nu\lambda_1}{2}|w|^2 \le &\left(2\nu^2 + 4R^2 + \frac{(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right)\frac{c\mu^2\kappa}{\nu}\left(|w(t_0)|^2 + E_1^2 + \bar{\varepsilon}\right) \\ &+ \bar{\varepsilon} + \mu^2\left(\frac{4}{\nu\lambda_1} + \frac{c\mu^2\kappa^2}{\nu\lambda_1}\right)E_1^2. \end{aligned} \tag{4.23}$$

Using the Gronwall inequality to (4.23) in $[t_0, t], t < \tilde{t}$

$$\begin{aligned}|w(t)|^2 \le &\left(1 - e^{-\frac{\nu\lambda_1}{2}(t-t_0)}\right)\left(\gamma_1|w(t_0)|^2 + \gamma_2 E_1^2 + \frac{8\mu^2}{(\nu\lambda_1)^2}E_1^2 + (\gamma_2+1)\frac{\bar{\varepsilon}}{\nu\lambda_1}\right) \\ &+ |w(t_0)|^2 e^{-\frac{\nu\lambda_1}{2}(t-t_0)} \end{aligned} \tag{4.24}$$

where

$$\gamma_1 = \frac{c\mu^2\kappa}{\nu^2\lambda_1}\left(\nu^2 + M_0^2 + \frac{\mu^2}{(\nu\lambda_1)^2}E_1^2 + \frac{(M_0M_1)^2}{\nu^2\lambda_1} + \frac{\mu^2}{\lambda_1^2}\right),$$

and

$$\gamma_2 = \gamma_1 + \frac{c\mu^4\kappa^2}{\nu^2\lambda_1^2}.$$

Since $|w(t_0)| \le R$, then (4.24) becomes

$$\begin{aligned}|w(t)|^2 \le &\left(1 - e^{-\frac{\nu\lambda_1}{2}(t-t_0)}\right)\left(\gamma_1 R^2 + \gamma_2 E_1^2 + \frac{8\mu^2}{(\nu\lambda_1)^2}E_1^2 + (\gamma_2+1)\frac{\bar{\varepsilon}}{\nu\lambda_1}\right) \\ &+ R^2 e^{-\frac{\nu\lambda_1}{2}(t-t_0)}. \end{aligned}$$

Using the choice of κ in (4.7) with suitable constant c, we get

$$\gamma_1 \leq 1/2, \gamma_2 \leq \min\left\{\frac{\mu^2}{(\nu\lambda_1)^2}, 1\right\}$$

so that

$$\gamma_1 R^2 + \gamma_2 E_1^2 + \frac{8\mu^2}{(\nu\lambda_1)^2} E_1^2 \leq R^2.$$

In particular, $|w(\tilde{t})| \leq R$, and from the definition of $\tilde{t}$ we conclude that $\tilde{t} \geq t_1$. Therefore, we also have $|w(t_1)| \leq R$ and we can apply the same previous arguments to obtain that $\tilde{t} \geq t_2$ and $|w(t_2)| \leq R$. Continuing inductively, we obtain that $\tilde{t} \geq t_n$, for all $n \geq 0$. Furthermore, we get the same as (4.24) that

$$\begin{aligned}|w(t)|^2 \leq &\left(1 - e^{-\frac{\nu\lambda_1}{2}(t-t_0)}\right)\left(\gamma_1 |w(t_n)|^2 + \gamma_2 E_1^2 + \frac{4\mu^2}{(\nu\lambda_1)^2} E_1^2 + (\gamma_2+1)\frac{\overline{\varepsilon}}{\nu\lambda_1}\right) \\ &+ |w(t_n)|^2 e^{-\frac{\nu\lambda_1}{2}(t-t_0)}, \qquad (4.25)\end{aligned}$$

for all $t \in [t_n, t_{n+1}]$ and for all $n \in \mathbb{N}$. Therefore,

$$|w(t_{n+1})|^2 \leq \theta |w(t_n)|^2 + c\frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + 2\frac{\overline{\varepsilon}}{\nu\lambda_1}, \ \forall n \geq 0,$$

where

$$\theta = e^{-\frac{\nu\lambda_1}{2}\kappa} + \gamma_1\left(1 - e^{-\frac{\nu\lambda_1}{2}\kappa}\right) < 1.$$

Thus,

$$|w(t_n)|^2 \leq \theta^n |w(t_0)|^2 + \left(c\frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + 2\frac{\overline{\varepsilon}}{\nu\lambda_1}\right)\sum_{j=0}^{n-1}\theta^j, \ \forall n \geq 1. \qquad (4.26)$$

Combining (4.25) and (4.26) we deduce that

$$|w(t)|^2 \leq \theta^n |w(t_0)|^2 + \left(c\frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + 2\frac{\overline{\varepsilon}}{\nu\lambda_1}\right)\left(1 + \sum_{j=0}^{n-1}\theta^j\right), \forall t \in [t_n, t_{n+1}], \ \forall n \geq 1.$$

Now we have

$$\begin{aligned}|v_N(t) - u(t)|^2 &\leq |w(t)|^2 + |Q_N u(t)|^2 \\ &\leq \theta^n |w(t_0)|^2 + \left(c\frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + 2\frac{\overline{\varepsilon}}{\nu\lambda_1}\right)\left(1 + \sum_{j=0}^{n-1}\theta^j\right) + \frac{\varepsilon}{4},\end{aligned}$$

for all $t \in [t_n, t_{n+1}]$, $\forall n \geq 1$, provided by N is large enough satisfying (4.5), i.e.,

$$|Q_N u|^2 \leq \frac{1}{\lambda_N}\|Q_N u\|^2 \leq \frac{1}{\lambda_N}\|u\|^2 \leq \frac{1}{\lambda_N} M_1^2 \leq \frac{\varepsilon}{4}.$$

Hence

$$|v_N(t) - u(t)|^2 \leq c\frac{\mu^2}{(\nu\lambda_1)^2} E_1^2 + \varepsilon,$$

for t sufficiently large. Moreover, if $E_1 = 0$, we have

$$|v_N(t) - u(t)|^2 \leq \varepsilon$$

for t sufficiently large. □

Remark 4.2. In the previous work [8], the authors studied finite-dimensional continuous data assimilation for the two-dimensional Navier-Stokes equations with local observables. In Theorem 4.1 we have studied the finite-dimensional discrete data assimilation for the two-dimensional Navier-Stokes equations in a special case of the local interpolant operators, namely the low Fourier modes projector. It is worthy noticing that our results are not only hold for the periodic case but also for the no-slip boundary case, and without assumption on the suitable Gevrey regularity of solutions in the periodic case or the fast enough decay condition of $\hat{u}_N = (u, \phi_N)$ in the no-slip boundary case as in the continuous data assimilation problem in [8].

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