LONG TERM BEHAVIOR OF THREE-DIMENSIONAL RANDOM NAVIER-STOKES-VOIGT EQUATIONS DRIVEN BY COLORED NOISE IN UNBOUNDED DOMAINS

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Abstract. In this paper we study long term behavior of three-dimensional random Navier-Stokes-Voigt equations driven by colored noise defined in unbounded domains. We first prove the existence and uniqueness of pullback random attractors for the equations with Lipschitz diffusion terms. We then show the existence of a pullback random attractor for Navier-Stokes-Voigt equations with a multiplicative noise, and when the step length of the Winner shift approaches zero, we establish the convergence of solutions as well as the upper semicontinuity of pullback random attractors for Wong-Zakai approximation equations. We have to overcome some essential difficulties caused by the nonlinear noise, the pseudoparabolic regularization term $-\alpha^2\Delta u$ and the unboundedness of the considered domains.

1. Introduction

In this paper, we study the long-time behavior of solutions to three-dimensional random Navier-Stokes-Voigt equations (also known as the Navier-Stokes-Voight) driven by a general noise in an arbitrary (bounded or unbounded) domain $\mathcal{O}$ satisfying the Poincaré inequality

$$\int_{\mathcal{O}} |\nabla\phi|^2 dx \geq \lambda_1 \int_{\mathcal{O}} |\phi|^2 dx, \quad \forall \phi \in H_0^1(\mathcal{O}), \tag{1.1}$$

where λ_1 is a positive constant. More precisely, we consider the following system

$$\begin{cases} d(u-\alpha^2\Delta u) + [-\nu\Delta u + (u\cdot\nabla)u + \nabla p]dt = & f(x,t)dt \\ & +h(x,t,u)\circ dW(t),\ x\in\mathcal{O}, t>\tau, \\ \nabla\cdot u = 0, \quad x\in\mathcal{O}, t>\tau, & \\ u(x,t) = 0, \quad x\in\partial\mathcal{O}, t>\tau, & \\ u(x,\tau) = u_\tau(x), \quad x\in\mathcal{O}, & \end{cases} \tag{1.2}$$

where $u=(u_1,u_2,u_3)$ is the unknown velocity vector, $p=p(x,t)$ is the unknown pressure, u_0 is the initial velocity and $h(t,x,u)\circ dW(t)$ is a random field which will be specified later, the symbol $\circ$ indicates the equation is understood in sense of Stratonovich's integration. The function $f(x,t)$ is the density of volume forces and ν is the kinematic viscosity coefficient of the flows, and $\alpha>0$ is the spatial scale at which fluid motion is filtered.

In the present paper, we assume the nonlinear diffusion term $h(x,t,u)$ satisfies the following assumptions

$$h(x,t,u) = e^{\mu t}[\gamma(u-\alpha^2\Delta u) + S(u) + g(x)], \quad u\in V,$$

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where $\mu \geq 0, \gamma \geq 0, g \in D(A)$ and $S : V \to H$ is a continuous function satisfying

$$|S(u) - S(v)| \leq \gamma_1 \|u - v\|, \quad \forall u, v \in V, \tag{1.3}$$

$$|(S(u) - S(v), w)| \leq \gamma_2 |u - v| \|w\|, \quad \forall u, v, w \in V. \tag{1.4}$$

Here, as explained in [18], if $\mathcal{O}$ is unbounded we need to assume that

$$(S(u), u) = 0, \quad \forall u \in V, \tag{1.5}$$

while if $\mathcal{O}$ is bounded we can assume

$$|(S(u), u)| \leq \gamma_3 + \gamma_4 \|u\|^{1+\kappa}, \quad \forall u \in V, \tag{1.6}$$

where $\gamma_i \geq 0, i = \overline{1,4}$ and $\kappa \in [0, 1)$ are all constants. Clearly, for $\gamma_3 = \gamma_4 = 0$, condition in (1.6) becomes (1.5), thus (1.5) is weaker than (1.6). We can see that there are some continuous functions which satisfy the above assumptions. For example, let $S : V \to H$ be a nonlinear operator given by $S(u) = B(g_1, u)$ for all $u \in V$ where g_1 is a fixed function in $D(A)$ and B is the nonlinear operator defined in Subsection 2.1 below, then S fulfilles (1.3)-(1.5) (see [8, 9]).

When $h \equiv 0$, that is in the deterministic case, Navier-Stokes-Voigt model (1.2) was first investigated in [24] as a model for a viscoelastic fluid. It was reconsidered later in [10] as a relevant regularization of the Navier-Stokes equations. In fact, the Navier-Stokes-Voigt equations differ from the Navier-Stokes equations by the inclusion of the pseudoparabolic regularization term $-\alpha^2 \Delta u$ for velocity field u. It should be noted that when $\alpha = 0$ we recover the Navier-Stokes equations. The addition of the term $-\alpha^2 \Delta u_t$ regularizes the Navier-Stokes equations makes it globally well-posed and changes its parabolic character. In recent years, the long-time behavior of solutions in terms of existence of attractors to the 3D Navier-Stokes-Voigt equations have attracted the attention of many authors, in [22] the authors are studied the 3D Navier-Stokes-Voigt in a bounded domain with sufficiently smooth boundary and a time-independent external forcing term. They proved the existence of a global attractor in the space V via asymptotic compactness. They also give bounds on the number of determining modes in the attractor and the dimension of the attractor in terms of the system parameters. For more results in the deterministic case, we refer the interested reader to [2, 14, 22, 21] and references therein.

In the present paper we will study pullback random attractors for the random system (1.2). If the function f does not depend on time t, then system (1.2) becomes an autonomous stochastic equation. The definition of random attractor for autonomous stochastic systems was introduced in [11, 12, 13], and the existence of such attractors has been established for a variety of equations in [4, 7, 16] and the stochastic Navier-Stokes-Voigt equations in [6, 15, 25]. However, in these papers, only additive white noise was considered. When the stochastic term is a multiplicative noise or a general nonlinear function, the problem is more complicated (see [12]), and hence the classical random attractors theory does not apply in this case. Therefore, the main aim of the present paper is to prove the existence of a unique pullback random attractor for the stochastic Navier-Stokes-Voigt equations for a general noise. For more results on the existence and stability of solutions and large deviations principle of the 3D stochastic Navier-Stokes-Voigt equations, we refer the recent works in [1, 3, 23, 29].

Here we are concerned with the well-posedness as well as long-term dynamics of the non-autonomous random Navier-Stokes-Voigt equations (1.2). To do this, we will use some ideas in [17] where the authors used the Wong-Zakai approximations to analyze the pathwise random dynamical systems with nonlinear diffusion term. Note that in [35, 36], Wong and Zakai introduced the concept of approximating stochastic differential equations by deterministic differential equations, in which they studied both piecewise linear approximations and piecewise smooth approximations for one dimensional Brownian motions. Their work was later extended to stochastic differential equations of higher dimensions, see e.g. [19, 26] and recently in [20]. In the case of a nonlinear diffusion term, the idea of Wong-Zakai approximations for random pullback attractors is used for a wide class of stochastic partial differential equations in both bounded and unbounded domains [17, 18, 32, 33, 34].

In order to study the existence of a pullback random attractors for the stochastic Navier-Stokes-Voigt equations with a nonlinear diffusion term, we first define a continuous cocycle for the random Navier-Stokes-Voigt equations when the nonlinear noise in (1.2) is replaced by a Wong-Zakai approximation. Since the domain $\mathcal{O}$ is unbounded and the lack of parabolic character, i.e., if the initial datum $u_0 \in (H_0^1(\mathcal{O}))^3$ then the solution always belongs to $(H_0^1(\mathcal{O}))^3$ and has no higher regularity, we are unable to prove the pullback asymptotic compactness of the solution operator of random system (1.2) by using the standard techniques via compact Sobolev embeddings. To overcome this essential difficulty, we will use the so-called energy equation method, which was introduced by Ball [5]. However, to use this method we have to impose stronger restrictions on the diffusion term h (see condition (1.5) above) because, if not, the energy equation will involve the nonlinear part of h and make the idea of energy equations does not hold. Compare to [17, 18], in which the authors proved the existence random pullback attractors for 2D Navier-Stokes equations, the novelty of this work is that we consider the three dimensional case and different context of the equation because of the term $-\alpha^2\Delta u_t$.

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some results related to function spaces, operators related to Navier-Stokes-Voigt equations as well as some properties of Wong-Zakai approximations. In Section 3 we prove the existence and uniqueness of tempered random attractors for system (1.2) with Lipschitz diffusion terms. In Section 4 we first show the existence of a unique pullback random attractor for stochastic Navier-Stokes-Voigt equations driven by a multiplicative noise. Then we prove the convergence of solutions and upper semicontinuity of random pullback attractors for Wong-Zakai approximations of stochastic Navier-Stokes-Voigt equations as the correlation time $\delta \to 0$. It is worthy noticing that all results in the present paper are still true if Ω is a bounded domain, and in this case we can replace condition (1.5) by a much weaker one (1.6).

2. Preliminaries

2.1. **Function spaces and operators.** Denote by

$$\mathcal{V} = \{u \in (C_0^\infty(\mathcal{O}))^3 : \nabla \cdot u = 0\}$$

and

$$\begin{aligned} H &= \text{ the closuse of } \mathcal{V} \text{ in } [L^2(\mathcal{D})]^3, \\ V &= \text{ the closuse of } \mathcal{V} \text{ in } [H^1(\mathcal{D})]^3, \end{aligned}$$

with the respectively inner products are given by

$$(u,v) := \int_{\mathcal{D}} \sum_{i=1}^3 u_i v_i \, dx,$$

$$((u,v)) := \int_{\mathcal{D}} \sum_{i=1}^3 \nabla u_i \nabla v_i \, dx,$$

and the associated norms $|u|^2 := (u,u)$ and $\|u\|^2 := ((u,u))$, respectively. We have the following Poincaré inequality (see, e.g., [27, 28])

$$|u|^2 \leq \lambda_1^{-1}\|u\|^2 \qquad \forall u \in V.$$

We also consider another the scalar product in V defined by

$$(u,v)_V = (u,v) + \alpha^2((u,v)), \quad \text{ for } u,v \in V,$$

and the associated norm $\|\cdot\|_V$. Thus, for any $u \in V$ we can see that

$$\frac{\lambda_1}{1+\alpha^2\lambda_1}\|u\|_V^2 \leq \|u\|^2 \leq \frac{1}{\alpha^2}\|u\|_V^2$$

which shows that this norm is equivalent to the norm $\|\cdot\|$ on V.

Let $P : [L^2(\mathcal{O})]^3 \to H$ be the Leray-Helmholtz orthogonal projector, and the Stokes operator A subject to the Dirichlet boundary conditions with domain $D(A) = [H^2(\mathcal{O})]^3 \cap V$ is defined by $Au = -P\Delta u = -\Delta u$. The norm in $D(A)$ is $\|u\|_{D(A)} = |Au|$, $\forall u \in D(A)$.

We denote by V' denotes the dual space of V, and consider $B(\cdot,\cdot) : V \times V \to V'$ is a continuous bilinear form defined by

$$B(u,v) = P[(u\cdot\nabla)v] = (\nabla v)^T u.$$

Let $b(u,v,w)$ be the trilinear operator defined by

$$b(u,v,w) = \sum_{i,j=1}^{3}\int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u,v,w \in V,$$

then we have

$$b(u,v,w) = -b(u,w,v), \quad \forall u,v,w \in V,$$

and hence we get $b(u,v,v) = 0$. We have the following useful estimates which frequently use in the later.

Lemma 2.1 ([27, 28])**.** *We have*

$$|b(u,v,w)| \leq c\begin{cases} |u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|, \\ |u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\|w\|, \\ \|u\|\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \\ \lambda_1^{\frac{1}{4}}\|u\|\|v\|\|w\|, \end{cases} \qquad \forall u,v,w \in V.$$

In particular,

$$|b(u,v,u)| \leq c|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}\|v\|, \quad \forall u,v \in V.$$

Applying the Leray-Helmholtz orthogonal projector P to (1.2), we obtain the following functional evolution equation

$$\begin{cases} \dfrac{d}{dt}(u+\alpha^2 Au) + \nu Au + B(u,u) = f(t) + h(t,u)\circ dW(t), \\ u(\tau) = u_\tau, \end{cases} \tag{2.1}$$

where $u_\tau \in V$ and $f \in L^2_{loc}(\mathbb{R}; V')$.

2.2. **Cocycles for Navier-Stokes-Voigt equations.** We consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$, where Ω is given by

$$\Omega = \{\omega \in C(\mathbb{R};\mathbb{R}) : \omega(0) = 0\},$$

with the open compact topology, $\mathcal{F}$ is its Borel σ-algebra and $\mathbb{P}$ is the Winner measurable. The Brownian motion has the from $W(t,\omega) = \omega(t)$ and on $(\Omega,\mathcal{F},\mathbb{P})$ we consider Wiener shift operator $\{\theta_t\}_{t\in\mathbb{R}}$ defined by

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t), \quad t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

It is well known that the Gaussian measure $\mathbb{P}$ is ergodic and invariant for θ_t (see [4]). Thus $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical systems. Let us define the colored noise $\mathcal{W}_\delta : \Omega \to \mathbb{R}$ with the given correlation time $\delta \in \mathbb{R}\setminus\{0\}$ such that

$$\mathcal{W}_\delta(\omega) = \frac{\omega(\delta)}{\delta} \text{ or } \mathcal{W}_\delta(\theta_t\omega) = \frac{1}{\delta}\left(\omega(t+\delta) - \omega(t)\right).$$

Thanks to the properties of Wiener process, we find that $\mathcal{W}_\delta(\theta_t\omega)$ is a stationary with a normal distribution. Hence, the white noise can be approximated by $\mathcal{W}_\delta(\theta_t\omega)$ in the sense given in Lemma 2.2 below.

Lemma 2.2. ([18, Lemma 2.1]) *Let the correlation time $\delta \in (0,1]$. There exists a $\{\delta_t\}_{t\in\mathbb{R}}$-invariant subset (still denoted by) Ω of full measure, such that for $\omega \in \Omega$:*

i)
$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0; \tag{2.2}$$

ii) *The mapping*
$$(t,\omega) \mapsto \mathcal{W}_\delta(\theta_t\omega) = -\frac{1}{\delta^2}\int_{-\infty}^{0} e^{\frac{s}{\delta}}\theta_t\omega(s)ds \tag{2.3}$$
is a stationary solution (also called a colored noise or Ornstein-Uhlenbeck process) of one-dimensional stochastic differential equation
$$d\mathcal{W}_\delta + \frac{1}{\delta}\mathcal{W}_\delta dt = \frac{1}{\delta}dW$$
with continuous trajectories satisfying
$$\lim_{t\to\pm\infty} \frac{|\mathcal{W}_\delta(\theta_t\omega)|}{t} = 0, \quad \textit{for every } 0 < \delta \le 1, \tag{2.4}$$
and
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \mathcal{W}_\delta(\theta_t\omega)ds = \mathbb{E}(\mathcal{W}_\delta) = 0, \quad \textit{uniformly for } 0<\delta\le 1; \tag{2.5}$$

iii) *For arbitrary $T > 0$,*
$$\lim_{\delta\to 0}\int_0^t \mathcal{W}_\delta(\theta_t\omega)ds = \omega(t) \quad \textit{uniformly for } t\in[\tau,\tau+T].$$

This leads us to consider the following random Navier-Stokes-Voigt equations as Wong-Zakai approximations of (1.2) for $\tau \in \mathbb{R}$ and $\delta \in \mathbb{R}\setminus\{0\}, t > \tau$:

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \alpha^2\Delta u) - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t) + h(x,t,u)\mathcal{W}_\delta(\theta_t\omega), x\in\mathcal{O}, t>\tau,\\ u(x,\tau) = u_\tau(x), \quad x\in\mathcal{O}, \end{cases} \tag{2.6}$$

which is a random partial differential equation driven a stationary process.

Applying the Leray-Helmholtz orthogonal projector P, we can rewrite system (2.6) in the abstract form

$$\begin{cases} \dfrac{\partial}{\partial t}(u + \alpha^2 Au) + \nu Au + B(u,u) = f(t) + h(t,u)\mathcal{W}_\delta(\theta_t\omega), t>\tau,\\ u(\tau) = u_\tau, \end{cases} \tag{2.7}$$

for $\tau\in\mathbb{R}$ and $u_\tau \in V$.

Let us recall the definition of weak solutions to problem (2.7).

Definition 2.1. *Let $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and $\tau \in \mathbb{R}, \omega \in \Omega$ and let $u_\tau \in V$ be given. A mapping $u(\cdot, \tau, \omega, u_\tau) : [\tau, \infty) \to V$ is called a weak solution of problem (2.7) if*

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty); V)$$

and satisfies

$$\begin{aligned} &u(t) + \alpha^2 Au(t) + \nu \int_\tau^t Au(s)ds + \int_\tau^t B(u(s), u(s))ds \\ &= u_\tau + \int_\tau^t f(s)ds + \int_\tau^t h(s, u)\mathcal{W}_\delta(\theta_s \omega)ds \end{aligned}$$

for any $t \geq \tau$.

By a standard Galerkin method [28], one can prove that if all assumptions (1.1) and (1.3)-(1.6) are satisfied, then for all $t > \tau, \tau \in \mathbb{R}$ and for every $u_\tau \in V$ and $\omega \in \Omega$, problem (2.7) has a unique solution in the sense of Definition 2.1. Moreover, $u(t, \tau, \omega, u_\tau)$ is continuous with respect to initial data u_τ and $(\mathcal{F}, \mathcal{B}(V))$-measurable in $\omega \in \Omega$.

Therefore, we can define a cocycle (or Random Dynamical System) $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times V \to V$ for system (2.7) such that for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in V$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau).$$

Then Φ is a continuous cocycle on V over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ in the following sense

$$\Phi(t + \tau, s, \omega, u_\tau) = \Phi(t, \tau + s, \theta_\tau \omega, \Phi(\tau, s, \omega, u_\tau)).$$

Let us denote by $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ the family of nonempty subsets of V satisfying for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-ct} \|D(\tau - t, \theta_{-t}\omega)\|_V = 0, \tag{2.8}$$

where $\|D\|_V = \sup_{u\in V} \|u\|_V$. And denote $\mathcal{D}$ is a collection of all tempered families of bounded nonempty subsets of V, i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfying (2.8)}\}.$$

3. Existence and uniqueness of pullback attractors for random Navier-Stokes-Voigt equations driven by colored noise

In this section, we will prove the existence and uniqueness of random $\mathcal{D}$-pullback attractor for system (2.7) with nonlinear diffusion term.

We assume $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ and there exists a number $\eta \in (0, \nu d_0)$ with $d_0 = \frac{\lambda_1}{1+\alpha^2\lambda_1}$ such that

$$\int_{-\infty}^\tau e^{\eta s} \|f(s, \cdot)\|^2_{V'} ds < +\infty, \quad \forall \tau \in \mathbb{R}, \tag{3.1}$$

and for every positive number $c > 0$,

$$\lim_{\tau\to-\infty} e^{c\tau} \int_{-\infty}^0 e^{\eta s} \|f(s + r, \cdot)\|^2_{V'} ds = 0. \tag{3.2}$$

We first show uniform estimates on solutions of (2.7).

Lemma 3.1. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ satisfies (3.1). Then for every $0 < \delta \leq 1, \tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\delta, \tau, \omega, D) > 0$ such that for all $t \geq T$ and $\sigma \geq \tau - t$, the solution u of problem (2.7) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\begin{aligned} &\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_V \\ &\leq M e^{\int_{\sigma-\tau}^0 (\nu d_0 - 2\gamma e^{\mu(s+\tau)})\mathcal{W}_\delta(\theta_r \omega)dr} \end{aligned}$$

$$+ M\Big(\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s}(\nu d_0 - 2\gamma e^{\mu(r+\tau)}\mathcal{W}_\delta(\theta_r\omega))dr} \|f(s+\tau,\cdot)\|_{V'}^2 ds$$
$$+ \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s}(\nu d_0 - 2\gamma e^{\mu(r+\tau)}\mathcal{W}_\delta(\theta_r\omega))dr} e^{2\mu(s+\tau)}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds\Big),$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ *and* M *is a positive constant independent of* τ, ω, D, σ *and* δ.

Proof. Multiplying the first equation in (2.7) by u, we have

$$\frac{1}{2}\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu\|u\|^2 = \langle f, u\rangle + e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)(h(t,u), u)$$
$$= \langle f, u\rangle + e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)(\gamma(u + \alpha^2 Au) + S(u) + g, u). \tag{3.3}$$

We have

$$2\langle f, u\rangle \le \frac{\nu}{4}\|u\|^2 + \frac{4}{\nu\lambda_1}\|f(t,\cdot)\|_{V'}^2,$$

and

$$2e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)|(g,u)| \le \frac{\nu}{4}\|u\|^2 + \frac{4}{\nu\lambda_1}|g|^2 e^{2\mu t}|\mathcal{W}_\delta(\theta_t\omega)|^2.$$

Hence, using (1.5) and (3.3) we obtain

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \frac{3\nu}{2}\|u\|^2 \le 2\gamma e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)\|u\|_V^2 + \frac{4}{\nu\lambda_1}\|f(t,\cdot)\|_{V'}^2$$
$$+ \frac{4}{\nu\lambda_1}|g|^2 e^{2\mu t}|\mathcal{W}_\delta(\theta_t\omega)|^2,$$

since $d_0\|u\|_V^2 \le \|u\|^2$, then we obtain that

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + (\nu d_0 - 2\gamma e^{\mu t}\mathcal{W}_\delta(\theta_t\omega))\|u\|_V^2 + \frac{1}{2}\|u\|^2$$
$$\le \frac{4}{\nu\lambda_1}\|f(t,\cdot)\|_{V'}^2 + \frac{4}{\nu\lambda_1}|g|^2 e^{2\mu t}|\mathcal{W}_\delta(\theta_t\omega)|^2. \tag{3.4}$$

Removing the term $\frac{1}{2}\|u\|^2$, then multiplying (3.4) by $e^{\int_0^t(\nu d_0 - 2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_t\omega))dr}$ and integrating on $[\tau - t, \sigma]$, we get

$$\|u(\sigma, \tau - t, \omega, u_{\tau-t})\|_V^2 \le e^{\int_\sigma^{\tau-t}(\nu d_0 - 2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|u_{\tau-t}\|_V^2$$
$$+ \frac{4}{\nu\lambda_1}\int_{\tau-t}^{\sigma} e^{\int_\sigma^{\tau-t}(\nu d_0 - 2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\big(\|f(s,\cdot)\|_{V'}^2 + |g|^2 e^{2\mu s}|\mathcal{W}_\delta(\theta_s\omega)|^2\big)ds. \tag{3.5}$$

We now estimate each term on the right hand-side of (3.5).

- If $\mu = 0$, then by (2.5) and the ergodic theory we have

$$\lim_{s\to-\infty}\frac{1}{s}\int_0^s(\nu d_0 - 2\gamma\mathcal{W}_\delta(\theta_r\omega))dr = (\nu d_0 - 2\gamma\mathbb{E}(\mathcal{W}_\delta)) = \nu d_0 > \eta.$$

It follows that, there exists $s_0 = s_0(\omega, \delta) < 0$ such that for all $s \le s_0$,

$$\int_0^s(\nu d_0 - 2\gamma\mathcal{W}_\delta(\theta_r\omega))dr < \eta s. \tag{3.6}$$

Hence, by (3.6) and (3.1) we obtain

$$\int_{-\infty}^{s_0} e^{\int_0^s(\nu d_0 - 2\gamma\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds \le \int_{-\infty}^{s_0} e^{\eta s}\|f(s+\tau,\cdot)\|_{V'}^2 ds < +\infty,$$

which implies that

$$\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^{s}(\nu d_0 - 2\gamma\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds < +\infty, \tag{3.7}$$

and similarly from (2.2) and (3.6), we get

$$\int_{-\infty}^{s_0} e^{\int_0^s (\nu d_0 - 2\gamma \mathcal{W}_\delta(\theta_r\omega))dr} |g|^2 e^{2\mu s} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds < +\infty.$$

- If $\mu > 0$, by (2.4) we obtain

$$\lim_{r\to-\infty} (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega)) = \nu d_0 > \eta,$$

thus there exists $s_1 = s_1(\tau,\omega,\delta) < 0$ such that for all $s \le s_1$,

$$(\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega)) > \eta. \tag{3.8}$$

By (3.8) and (3.7), it follows that

$$\begin{aligned}&\int_{-\infty}^{s_1} e^{\int_{s_1}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|f(s+\tau,\cdot)\|_{V'}^2 ds \\ &\qquad \le e^{-\eta s_1} \int_{-\infty}^{s_1} e^{\eta s} \|f(s+\tau,\cdot)\|_{V'}^2 ds,\end{aligned}$$

which implies that

$$\int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|f(s+\tau,\cdot)\|_{V'}^2 ds < +\infty. \tag{3.9}$$

Similarly, by (2.2) and (3.8) we obtain

$$\frac{4}{\nu\lambda_1} \int_{-\infty}^{s_1} e^{\int_{s_1}^s k_1(\nu\lambda_1 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} |g|^2 e^{2\mu s} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds < +\infty,$$

thus,

$$\frac{4}{\nu\lambda_1} \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s k_1(\nu\lambda_1 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} |g|^2 e^{2\mu(s+\tau)} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds < +\infty. \tag{3.10}$$

Since $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, using (3.6) and (3.8) we get for $\mu \ge 0$,

$$\begin{aligned}&e^{\int_0^{-t} (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega)dr)} \|u_{\tau-t}\|_V^2 \\ &\le e^{\int_0^{-t} (\nu\lambda_1 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega)dr)} \|D(\tau-t, \theta_{-t}\omega)\|_V^2 \to 0\end{aligned}$$

as $t \to \infty$. Thus, there exists $T = T(\tau,\omega,D,\delta) > 0$ such that for all $t \ge T$,

$$e^{\int_0^{-t} (\nu\lambda_1 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|u_{\tau-t}\|_V^2 \le 1,$$

and hence for all $t \ge T$ and $\mu \ge 0$, we get

$$e^{\int_{\sigma-\tau}^{-t} (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|u_{\tau-t}\|_V^2 \le e^{\int_{\sigma-\tau}^0 (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr}.$$

This together with (3.5), (3.9) and (3.10) yields the proof. □

From Lemma 3.1 we have the following result.

Lemma 3.2. *Assume that* $f \in L^2_{\rm loc}(\mathbb{R}; V')$ *satisfies* (3.1) *and* (3.2). *Then for every* $0 < \delta \le 1, \tau \in \mathbb{R}, \omega \in \Omega$ *and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,D,\delta) > 0$ *such that for every* $k \ge 0$ *and for all* $t \ge T + k$, *the solution* u *of system* (2.7) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned}&\|u(\tau-k, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|_V^2 \\ \le& M\Bigg(e^{\int_{\sigma-\tau}^0 (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \\ &+ \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|f(s+\tau,\cdot)\|_{V'}^2 ds \\ &+ \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} e^{2\mu(s+\tau)} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds \Bigg),\end{aligned} \tag{3.11}$$

where $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ *and* M *is a positive constant independent of* τ, ω, D, k *and* δ.

Proof. Given $\tau \in \mathbb{R}$ and $k \geq 0$, take $\sigma = \tau - k$. Let $T = T(\tau, \omega, D, \delta) > 0$ as in Lemma 3.1. For $t \geq T + k$, we have $t \geq T$ and $\sigma \geq \tau - t$. Thus, by Lemma 3.1 we get (3.11). □

We now show the weak continuity of solutions to (2.7).

Lemma 3.3. *Let* $0 < \delta \leq 1, \tau \in \mathbb{R}, \omega \in \Omega$ *and* $u_\tau, u_{\tau,n} \in V$ *for all* $n \in \mathbb{N}$. *If* $u_{\tau,n} \rightharpoonup u_\tau$ *in* V, *then the solution* u *of systems* (2.7) *satisfies:*
i) $u(r, \tau, \omega, u_{\tau,n}) \rightharpoonup u(r, \tau, \omega, u_\tau)$ *in* V *for all* $r \geq \tau$;
ii) $u(\cdot, \tau, \omega, u_{\tau,n}) \rightharpoonup u(\cdot, \tau, \omega, u_\tau)$ *in* $L^2(\tau, \tau + T; V)$ *for all* $T > 0$.

Proof. Convergences in $i)$ and $ii)$ can be obtained almost the same way as in [2, Lemma 4.1] (see also [6, Lemma 3.3]), thus we omit it here. □

We next prove the existence of a $\mathcal{D}$-pullback absorbing set in V for system (2.7).

Lemma 3.4. *Suppose* (1.3)-(1.5) *and* (2.8) *are satisfied. Then there exists a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K = \{K(\tau, \omega) : \tau \in R, \omega \in \Omega\} \in \mathcal{D}$ *in* V *for the continuous cocycle* Φ *associated to problem* (2.7).

Proof. Let us denote, for given $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$K(\tau, \omega) = \{u \in V : \|u\|_V^2 \leq R(\tau, \omega)\},$$

where

$$R(\tau, \omega) = M\bigg(1 + \int_{-\infty}^0 e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \|f(s+\tau, \cdot)\|_{V'}^2 ds + \int_{-\infty}^0 e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} e^{2\mu(s+\tau)} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds\bigg).$$

Since $R(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $\tau \in \mathbb{R}$ and $K(\tau, \cdot) : \Omega \to 2^V$ is a measurable set-valued mapping, it follows from Lemma 3.1 that for each $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(s-\tau, \theta_{-t}\omega)) = u(\tau, \tau - t, \theta_{-\tau}\omega, D(s-\tau, \theta_{-t}\omega)) \subset K(\tau, \omega). \quad (3.12)$$

We now show that $K \in \mathcal{D}$, i.e., for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{t \to -\infty} e^{ct} \|K(\tau + t, \theta_t\omega)\|_V^2 = 0.$$

Indeed, for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned} &\lim_{t\to-\infty} e^{ct} \|K(\tau + t, \theta_t\omega)\|_V^2 = \lim_{t\to-\infty} e^{ct} R(\tau + t, \theta_t\omega) \\ &= M \lim_{t\to-\infty} e^{ct} \int_{-\infty}^0 e^{\int_0^s (\nu d_0 - 2\gamma e^{\mu(r+\tau+t)} \mathcal{W}_\delta(\theta_{r+t}\omega))dr} \|f(s+\tau+t, \cdot)\|_{V'}^2 ds \\ &\quad + M \lim_{t\to-\infty} e^{ct} \int_{-\infty}^0 e^{\int_0^s (\nu d_0 - 2\gamma e^{\mu(r+\tau+t)} \mathcal{W}_\delta(\theta_{r+t}\omega))dr} e^{2\mu(s+\tau+t)} |\mathcal{W}_\delta(\theta_{s+t}\omega)|^2 ds. \end{aligned} \quad (3.13)$$

We need to estimate the following term

$$e^{\int_0^s (\nu d_0 - 2\gamma e^{\mu(r+\tau+t)} \mathcal{W}_\delta(\theta_{r+t}\omega))dr}.$$

From (2.3) we get

$$\mathcal{W}_\delta(\theta_{r+t}\omega) = -\frac{1}{\delta^2} \int_{-\infty}^0 e^{\frac{\sigma}{\delta}} \theta_{r+t}\omega(\sigma) d\sigma$$

$$= -\frac{1}{\delta}\int_{-\infty}^{0} e^{\sigma}\omega(r+t+\delta\sigma)d\sigma + \frac{1}{\delta}\omega(r+t),$$

and therefore for $\delta\mu \neq 1$, by Fubini's theorem and the integration by parts formula, we get

$$\begin{aligned}
&\int_0^s e^{\mu(r+\tau+t)}\mathcal{W}_\delta(\theta_{r+t}\omega)dr \\
&= -\frac{1}{\delta}\int_0^s \left(\int_{-\infty}^0 e^{\mu(r+\tau+t)}e^{\sigma}\omega(r+t+\delta\sigma)d\sigma\right)dr + \frac{1}{\delta}\int_0^s e^{\mu(r+\tau+t)}\omega(r+t)dr \\
&= -\frac{1}{\delta}\int_{-\infty}^0 e^{\sigma}\left(\int_0^s e^{\mu(r+\tau+t)}\omega(r+t+\delta\sigma)dr\right)d\sigma + \frac{1}{\delta}\int_0^s e^{\mu(r+\tau+t)}\omega(r+t)dr \\
&= -\frac{1}{\delta}\int_{-\infty}^0 e^{(1-\mu\delta)\sigma}\left(\int_{t+\delta\sigma}^{s+t+\delta\sigma} e^{\mu(r+\tau)}\omega(r)dr\right)d\sigma + \frac{1}{\delta}\int_t^{s+t} e^{\mu(r+\tau)}\omega(r)dr \\
&= -\frac{1}{\delta}\frac{e^{\mu\tau}}{1-\mu\delta}\int_t^{s+t} e^{\mu r}\omega(r)dr + \frac{1}{1-\mu\delta}\int_{-\infty}^0 e^{\sigma}e^{\mu(s+t+\tau)}\omega(s+t+\delta\sigma)d\sigma \\
&\quad - \frac{1}{1-\mu\delta}\int_{-\infty}^0 e^{\sigma}e^{\mu(t+\tau)}\omega(t+\delta\sigma)d\sigma + \frac{1}{\delta}\int_t^{s+t} e^{\mu(r+\tau)}\omega(r)dr \\
&= -\frac{1}{\delta}\frac{\mu e^{\mu\tau}}{1-\mu\delta}\int_t^{s+t} e^{\mu r}\omega(r)dr + \frac{1}{1-\mu\delta}\int_{-\infty}^0 e^{\sigma}e^{\mu(s+t+\tau)}\omega(s+t+\delta\sigma)d\sigma \\
&\quad - \frac{1}{1-\mu\delta}\int_{-\infty}^0 e^{\sigma}e^{\mu(t+\tau)}\omega(t+\delta\sigma)d\sigma + \frac{1}{\delta}\int_t^{s+t} e^{\mu(r+\tau)}\omega(r)dr \\
&= \frac{\mu e^{\mu\tau}}{1-\mu\delta}\int_t^{s+t} e^{\mu r}\omega(r)dr + \frac{e^{\mu(s+t+\tau)}}{1-\mu\delta}\int_{-\infty}^0 e^{r}\omega(s+t+\delta r)dr \\
&\quad - \frac{e^{\mu(t+\tau)}}{1-\mu\delta}\int_{-\infty}^0 e^{r}\omega(t+\delta r)dr. \qquad (3.14)
\end{aligned}$$

For $\gamma > 0, \delta\mu \neq 1$ and $\tau \in \mathbb{R}$, let

$$\gamma_0 = e^{-\mu\tau}|\delta\mu - 1| \cdot \min\{\frac{\nu d_0 - \eta}{2\gamma}, \frac{c}{8\gamma}\} > 0.$$

By (2.2), we find that there exists $T_1 = T_1(\omega) > 0$ such that for all $|t| \geq T_1$,

$$|\omega(t)| < \gamma_0|t|. \qquad (3.15)$$

By (3.15), we note that for $\mu > 0, t \leq -T_1$ and $s \leq 0$,

$$\begin{aligned}
\left|\int_t^{t+s} e^{\mu r}\omega(r)dr\right| &\leq \int_{t+s}^t e^{\mu r}|\omega(r)|dr \\
&\leq \gamma_0\int_{t+s}^t e^{\mu r}|r|dr \leq \gamma_0\int_{-\infty}^0 e^{\mu r}|r|dr \\
&= \frac{\gamma_0}{\mu^2}. \qquad (3.16)
\end{aligned}$$

Also for $t \leq -T_1, r, s \leq 0$ and $\delta \in (0, 1]$, we have $s + t + r\delta \leq -T_1$, and thus

$$|\omega(r\delta + s + t)| < \gamma_0|r\delta + s + t| \leq \gamma_0(|r| + |s| + |t|). \qquad (3.17)$$

By (3.17), for $t \leq -T_1$ and $s \leq 0$, we find that

$$\begin{aligned}
\left|e^{\mu(s+t)}\int_{-\infty}^0 e^{r}\omega(r\delta+s+t)dr\right| &\leq \gamma_0\int_{-\infty}^0 e^{r}(|r|+|s|+|t|)dr \\
&\leq \gamma_0(|s|+|t|+1) = \gamma_0 - \gamma_0 s - \gamma_0 t. \qquad (3.18)
\end{aligned}$$

Similarly, for $t \le -T_1$ and $r \le 0$, we have $t + r\delta \le -T_1$, and hence

$$|\omega(t + r\delta)| < \gamma_0 |t + r\delta| \le \gamma_0(|t| + |r|).$$

This shows that for all $t \le -T_1$,

$$\begin{aligned} \left| e^{\mu t} \int_{-\infty}^0 e^r \omega(r\delta + t) dr \right| &\le \gamma_0 \int_{-\infty}^0 e^r(|r| + |t|) dr \\ &\le \gamma_0 - \gamma_0 t, \end{aligned}$$

which along with (3.14), (3.15) and (3.18) implies that for $\mu > 0, \gamma > 0, \delta \in (0, 1]$ with $\mu\delta \ne 1$ and $t \le -T_1$,

$$-2\gamma \int_0^s e^{\mu(r+\tau+t)} \mathcal{W}_\delta(\theta_{r+t}\omega) dr \le \frac{2\gamma\gamma_0}{|\delta\mu - 1|} e^{\mu\tau} (\frac{1}{\mu} + 2 - s - 2t). \tag{3.19}$$

Now let $c_1 = \min\{\eta + 2\mu, \frac{c}{2}\}$. We deduce from (3.13) and (3.19) that for $\mu > 0, \gamma > 0, \delta \in (0, 1]$ with $\mu\delta \ne 1$ and $t \le -T_1$,

$$\begin{aligned} &\lim_{t\to-\infty} e^{ct} \|K(\tau + t, \theta_t \omega)\|_V \\ &\le M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{\mu\tau}}{\mu|\delta\mu-1|}} \lim_{t\to-\infty} e^{\frac{c}{2}t} \int_{-\infty}^0 e^{\eta s} \Big(\|f(s + \tau + t, \cdot)\|_{V'}^2 + e^{2\mu(s+\tau+t)} |\mathcal{W}_\delta(\theta_{s+t}\omega)|^2 \Big) ds \\ &\le M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{\mu\tau}}{\mu|\delta\mu-1|} - \frac{c}{2}\tau} \lim_{t\to-\infty} e^{\frac{c}{2}t} \int_{-\infty}^0 e^{\eta s} \|f(s + t, \cdot)\|_{V'}^2 ds \\ &+ M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{\mu\tau}}{\mu|\delta\mu-1|} - \frac{c}{2}\tau} \lim_{t\to-\infty} \int_{-\infty}^0 e^{c_1 t} e^{(\eta+2\mu)s} e^{2\mu(\tau+t)} |\mathcal{W}_\delta(\theta_{s+t}\omega)|^2 ds \\ &\le M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{\mu\tau}}{\mu|\delta\mu-1|} - \frac{c}{2}\tau} \lim_{t\to-\infty} e^{\frac{c}{2}t} \int_{-\infty}^0 e^{\eta s} \|f(s + t, \cdot)\|_{V'}^2 ds \\ &+ M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{2\mu\tau}}{\mu|\delta\mu-1|}} \lim_{t\to-\infty} e^{2(\mu+\tau)} \int_{-\infty}^0 e^{c_1(s+t)} |\mathcal{W}_\delta(\theta_{s+t}\omega)|^2 ds \\ &\le M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{\mu\tau}}{\mu|\delta\mu-1|} - \frac{c}{2}\tau} \lim_{t\to-\infty} e^{\frac{c}{2}t} \int_{-\infty}^0 e^{\eta s} \|f(s + t, \cdot)\|_{V'}^2 ds \\ &+ M e^{\frac{2\gamma\gamma_0(2\mu+1)e^{2\mu\tau}}{\mu|\delta\mu-1|}} \lim_{t\to-\infty} e^{2\mu(t+\tau)} \int_{-\infty}^0 e^{c_1 s} |\mathcal{W}_\delta(\theta_s\omega)|^2 ds. \end{aligned} \tag{3.20}$$

Moreover, since

$$\int_{-\infty}^0 e^{c_1 s} |\mathcal{W}_\delta(\theta_s \omega)|^2 ds < +\infty,$$

thus from (3.20) and (3.2) we obtain

$$\lim_{t\to-\infty} e^{ct} \|K(\tau + t, \theta_t \omega)\|_V^2 = 0.$$

The lemma is proved. □

We now prove $\mathcal{D}$-pullback asymptotic compactness of the solutions by using the method of energy equations introduced by Ball [5]. Then we show the existence of a unique random $\mathcal{D}$-pullback attractor for the system (2.7).

Lemma 3.5. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ satisfies (3.1) and (1.3)-(1.5) are fulfilled. Then the cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in V, i.e., for every $0 < \delta \le 1, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and $t_n \to \infty, u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in V.*

Proof. We first observe that for $\sigma = \tau$ in Lemma 3.1, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_V^2 \le R(\tau, \omega) \quad \forall t \ge T, \tag{3.21}$$

where $R(\tau, \omega)$ as in Lemma 3.4 and $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. Since $t_n \to \infty$, then there exists $N_0 \in \mathbb{N}$ such that $t_n \ge T$ for all $n \ge N_0$. Moreover, since $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, we get from (3.21) that

$$\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{0,n})\|_V^2 \le R(\tau, \omega) \quad \forall n \ge N_0,$$

this shows that the sequence $\{u(\tau, \tau - t, \theta_{-\tau}\omega, u_{0,n})\}_{n \ge N_0}$ is bounded in V, hence there exists $\bar{u} \in V$ such that (up to a subsequence)

$$u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) \rightharpoonup \bar{u} \quad \text{ in } V. \tag{3.22}$$

By the weak lower semicontinuous property of norms, we get from (3.22) that

$$\liminf_{n\to\infty} \|u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\|_V \ge \|\bar{u}\|_V. \tag{3.23}$$

To get the strong convergence of $u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) \to \bar{u}$ in V, by (3.23) we only need to prove that

$$\limsup_{n\to\infty} \|u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\|_V \le \|\bar{u}\|_V. \tag{3.24}$$

Indeed, for given $k \in \mathbb{N}$, we have

$$u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) = u(\tau, \tau - k, \theta_{-\tau}\omega, u(\tau - k, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})). \tag{3.25}$$

For each $k \in \mathbb{N}$, let N_k be sufficiently large such that $t_n \ge T + k$ for all $n \ge N_k$. By Lemma 3.4 we have

$$\|u(\tau - k, \tau - t_n, \theta_{-\tau}\omega, u_{\tau-t})\|_V^2 \le M\Bigg(e^{\int_{\sigma-\tau}^0 (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} + \int_{-\infty}^{\sigma-\tau} e^{\int_{\sigma-\tau}^s (\nu d_0 - 2\gamma e^{\mu(r+\tau)} \mathcal{W}_\delta(\theta_r\omega))dr} \big(\|f(s+\tau, \cdot)\|_{V'}^2 ds + e^{2\mu(s+\tau)} |\mathcal{W}_\delta(\theta_s\omega)|^2\big) ds \Bigg), \tag{3.26}$$

which shows that for each $k \in \mathbb{N}$, the sequence $\{u(\tau - k, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n \ge N_k}$ is bounded in V. By a diagonal process, we can find a subsequence (not relabelled) and an element $\bar{u}_k \in V$ for each $k \in \mathbb{N}$ such that

$$u(\tau - k, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) \rightharpoonup \bar{u}_k \text{ in } V. \tag{3.27}$$

This together with (3.25) and Lemma 3.4, we obtain for each $k \in \mathbb{N}$,

$$u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) \rightharpoonup u(\tau, \tau - k, \theta_{-\tau}\omega, \bar{u}_k) \text{ in } V, \tag{3.28}$$

and

$$u(\cdot, \tau - k, \theta_{-\tau}\omega, u(\tau - k, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})) \rightharpoonup u(\cdot, \tau - k, \theta_{-\tau}\omega, \bar{u}_k) \text{ in } L^2(\tau - k, \tau; V).$$

By (3.22) and (3.28) we have

$$u(\tau, \tau - k, \theta_{-\tau}\omega, \bar{u}_k) = \bar{u}. \tag{3.29}$$

On the other hand, from (3.3) we have

$$\frac{d}{dt}(\|u\|_V^2) + 2\nu\|u\|^2 = 2\langle f(t, \cdot), u\rangle + 2e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)(\gamma(u + \alpha^2 Au) + g, u). \tag{3.30}$$

Multiplying (3.30) by $e^{\eta(t-\tau)}$, we get that

$$\begin{aligned} &\frac{d}{dt}(e^{\eta(t-\tau)}\|u\|_V^2) + 2\nu e^{\eta(t-\tau)}\|u\|^2 - \eta e^{\eta(t-\tau)}\|u\|_V^2 \\ &= 2e^{\eta(t-\tau)}\langle f(t, \cdot), u\rangle + 2e^{\eta(t-\tau)}e^{\mu t}\mathcal{W}_\delta(\theta_t\omega)(\gamma(u + \alpha^2 Au) + g, u), \end{aligned}$$

and integrating from σ to τ we have

$$e^{\eta(\tau-\tau)}\|u(\tau,\sigma,\omega,u_\sigma)\|_V^2 = e^{\eta(\sigma-\tau)}\|u_\sigma\|_V^2 + 2\int_\sigma^\tau e^{\eta(s-\tau)}\langle f(s,\cdot),u(s,\sigma,\omega,u_\sigma)\rangle ds$$
$$- 2\int_\sigma^\tau e^{\eta(s-\tau)}e^{\mu s}\mathcal{W}_\delta(\theta_s\omega)(\gamma(u+\alpha^2Au)(s,\sigma,\omega,u_\sigma)+g,u(s,\sigma,\omega,u_\sigma))ds$$
$$+ 2\int_\sigma^\tau e^{\eta(s-\tau)}(\nu\|u(s,\sigma,\omega,u_\sigma)\|^2 - \frac{\eta}{2}\|u(s,\sigma,\omega,u_\sigma)\|_V^2)ds.$$

Choose $\sigma = \tau - k$ and by (3.29) we get

$$\|\bar{u}\|_V^2 = \|u(\tau,\tau-k,\omega,\bar{u}_k)\|_V^2$$
$$= e^{-\eta k}\|\bar{u}_k\|_V^2 + 2\int_{\tau-k}^\tau e^{\eta(s-\tau)}\langle f(s,\cdot),u(s,\tau-k,\omega,\bar{u}_k)\rangle ds$$
$$- 2\int_{\tau-k}^\tau e^{\eta(s-\tau)}e^{\mu s}\mathcal{W}_\delta(\theta_s\omega)(\gamma(u+\alpha^2Au)(s,\tau-k,\omega,\bar{u}_k)+g,u(s,\tau-k,\omega,\bar{u}_k))ds$$
$$+ 2\int_{\tau-k}^\tau e^{\eta(s-\tau)}(\nu\|u(s,\tau-k,\omega,\bar{u}_k)\|^2 - \frac{\eta}{2}\|u(s,\tau-k,\omega,\bar{u}_k)\|_V^2)ds. \quad (3.31)$$

Hence, we obtain

$$\|\bar{u}\|_V^2 = e^{-\eta k}\|\bar{u}_k\|_V^2 + 2\int_{-k}^0 e^{\eta(s-\tau)}\langle f(s,\cdot),u(s,\tau-k,\omega,\bar{u}_k)\rangle ds$$
$$+ 2\int_{\tau-k}^\tau e^{\eta(s-\tau)}e^{\mu s}\mathcal{W}_\delta(\theta_s\omega)(\gamma(u+\alpha^2Au)(s,\tau-k,\omega,\bar{u}_k)+g,u(s,\tau-k,\omega,\bar{u}_k))ds$$
$$- 2\int_{\tau-k}^\tau e^{\eta(s-\tau)}(\nu\|u(s,\tau-k,\omega,\bar{u}_k)\|^2 - \frac{\eta}{2}\|u(s,\tau-k,\omega,\bar{u}_k)\|_V^2)ds.$$

Denote by

$$[u]_1^2 = 2\nu\|u\|^2 - \eta\|u\|_V^2 \quad \forall u \in V, \quad (3.32)$$

then $[\cdot]_1$ is a new norm on V which is equivalent to the norm $\|\cdot\|_V$ on V. Indeed, by $\eta \in (0, d_0\nu)$ and $d_0\|u\|_V^2 \le \|u\|^2$ one can see that

$$[u]_1^2 = 2\nu\|u\|^2 - \eta\|u\|_V^2 \ge \nu\|u\|^2 - \eta\|u\|_V^2 \ge \left(\nu - \frac{\eta}{d_0}\right)\|u\|^2,$$

thus we get

$$\left(\nu - \frac{\eta}{d_0}\right)\|u\|^2 \le [u]_1^2 \le 2\nu\|u\|^2, \quad \forall u \in V.$$

Thus, we have

$$\begin{aligned}\|\bar{u}\|_V^2 =& e^{-\eta k}\|\bar{u}_k\|_V^2 - 2\int_{-k}^0 e^{\eta s}[u(s,\tau-k,\omega,\bar{u}_k)]_1^2 ds\\ &+ 2\int_{-k}^0 e^{\eta s}\langle f(s,\cdot),u(s,\tau-k,\omega,\bar{u}_k)\rangle ds\\ &+ 2\gamma\int_{-k}^0 e^{\eta(s+\tau)}e^{\mu s}\mathcal{W}_\delta(\theta_s\omega)\|u(s,\tau-k,\omega,\bar{u}_k)\|_V^2 ds\\ &+ 2\int_{-k}^0 e^{\eta(s+\tau)}e^{\mu s}\mathcal{W}_\delta(\theta_s\omega)(g,u(s,\tau-k,\omega,\bar{u}_k))ds. \quad (3.33)\end{aligned}$$

On the other hand, from (3.25) and (3.31) we obtain

$$\|u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})\|_V^2$$
$$= \|u(\tau,\tau-k,\theta_{-\tau}\omega,u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))\|_V^2$$
$$= e^{-\eta k}\|u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})\|_V^2$$

$$
\begin{aligned}
&-2\int_{-k}^{0} e^{\eta s}[u(s+\tau,\tau-k,\theta_{-\tau}\omega,u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))]_1^2 ds\\
&+2\int_{-k}^{0} e^{\eta s}\langle f(s+\tau,\cdot),u(s+\tau,\tau-k,\theta_{-\tau}\omega,u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))\rangle ds\\
&+2\gamma\int_{-k}^{0} e^{\mu(s+\tau)}e^{\eta s}\mathcal{W}_\delta(\theta_s\omega)\|u(s+\tau,\tau-k,\theta_{-\tau}\omega,u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))\|_V^2 ds\\
&+2\int_{-k}^{0} e^{\mu(s+\tau)}e^{\eta s}\mathcal{W}_\delta(\theta_s\omega)(g,u(s+\tau,\tau-k,\theta_{-\tau}\omega,u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})))ds.
\end{aligned} \tag{3.34}
$$

Using $\sigma=\tau-k$ and $t=t_n$ in (3.5) we get

$$
\begin{aligned}
&e^{-\eta k}\|u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})\|_V^2\\
&\le e^{\int_{\tau-k}^{\tau-t_n}(\nu d_0-2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|u_{\tau-t_n}\|_V^2\\
&+\frac{4}{\nu\lambda_1}\int_{\tau-t_n}^{\tau-k} e^{\int_{\tau-k}^{\tau-t_n}(\nu d_0-2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}|g|^2 e^{2\mu t}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds.
\end{aligned} \tag{3.35}
$$

Moreover, since $u_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$, we get that

$$
\begin{aligned}
&e^{\int_0^{-t_n}(\nu d_0-2\gamma e^{\mu(r+\tau)}\mathcal{W}_\delta(\theta_r\omega))dr}\|u_{0,n}\|_V^2\\
&\le e^{\int_0^{-t_n}(\nu d_0-2\gamma e^{\mu(r+\tau)}\mathcal{W}_\delta(\theta_r\omega))dr}\|D(\tau-t_n,\theta_{-t_n}\omega)\|_V^2\to 0
\end{aligned}
$$

as $n\to\infty$. Then we obtain from (3.35) that

$$
\begin{aligned}
&\limsup_{n\to\infty} e^{-\eta k}\|u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})\|_V^2\\
&\le \frac{4}{\nu\lambda_1}\int_{-\infty}^{-k} e^{\int_0^s(\nu d_0-2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds\\
&+M_1\int_{-\infty}^{-k} e^{2\mu(s+\tau)}e^{\int_0^s(\nu d_0-2\gamma e^{\mu(r+\tau)})ds}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds.
\end{aligned}
$$

Next, using the weak convergence in (3.27) and let $n\to\infty$, we get from (3.32) that

$$
\begin{aligned}
&\limsup_{n\to\infty}\left(\int_{-k}^{0} e^{\eta s}[u(s+\tau,\tau-k,\theta_{-\tau\omega},u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))]_1^2 ds\right)\\
&\le -\liminf_{n\to\infty}\int_{-k}^{0} e^{\eta s}[u(s+\tau,\tau-k,\theta_{-\tau\omega},u(\tau-k,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}))]_1^2 ds\\
&\le -\int_{-k}^{0} e^{\eta s}[u(s+\tau,\tau-k,\theta_{-\tau\omega},\bar{u}_k)]_1^2 ds,
\end{aligned}
$$

and therefore by (3.34) it implies that

$$
\begin{aligned}
&\limsup_{n\to\infty}\|u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})\|_V^2\\
&\le \frac{4}{\nu\lambda_1}\int_{-\infty}^{-k} e^{\int_0^s(\nu d_0-2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds\\
&+M_1\int_{-\infty}^{-k} e^{2\mu(s+\tau)}e^{\int_0^s(\nu d_0-2\gamma e^{\mu(r+\tau)})ds}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds\\
&-2\int_{-k}^{0} e^{\eta s}[u(s+\tau,\tau-k,\theta_{-\tau\omega},\bar{u}_k)]_1^2 ds\\
&+2\int_{-k}^{0} e^{\eta s}\langle f(s+\tau,\cdot),u(s+\tau,\tau-k,\theta_{-\tau}\omega,\bar{u}_k)\rangle ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\int_{-k}^{0} e^{\eta s} e^{\mu(s+\tau)} \mathcal{W}_\delta(\theta_s\omega)\|u(s+\tau,\tau-k,\theta_{-\tau}\omega,\bar{u}_k)\|_V^2 ds\\
&+ 2\int_{-k}^{0} e^{\eta s} e^{\mu(s+\tau)} \mathcal{W}_\delta(\theta_s\omega)(g, u(s+\tau,\tau-k,\theta_{-\tau}\omega,\bar{u}_k))ds.
\end{aligned}
$$

This together with (3.33) give us

$$
\begin{aligned}
&\limsup_{n\to\infty}\|u(\tau,\tau-t_n,\theta_{-\tau},u_{0,n})\|_V^2 \le \|\bar{u}\|_V^2 + e^{-\eta k}\|\bar{u}\|_V^2\\
&+ \frac{4}{\nu\lambda_1}\int_{-\infty}^{-k} e^{\int_0^s(\nu d_0 - 2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds\\
&+ M_1\int_{-\infty}^{-k} e^{2\mu(s+\tau)} e^{\int_0^s(\nu d_0 - 2\gamma e^{\mu(r+\tau)})ds}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds\\
&\le \|\bar{u}\|_V^2 + \frac{4}{\nu\lambda_1}\int_{-\infty}^{-k} e^{\int_0^s(\nu d_0 - 2\gamma e^{\mu r}\mathcal{W}_\delta(\theta_r\omega))dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds\\
&+ M_1\int_{-\infty}^{-k} e^{2\mu(s+\tau)} e^{\int_0^s(\nu d_0 - 2\gamma e^{\mu(r+\tau)})ds}|\mathcal{W}_\delta(\theta_s\omega)|^2 ds,
\end{aligned}
$$

and let $k\to\infty$ we get

$$
\limsup_{n\to\infty}\|u(\tau,\tau-t_n,\theta_{-\tau},u_{0,n})\|_V^2 \le \|\bar{u}\|_V^2.
$$

From this and by (3.23) we get

$$
\lim_{n\to\infty}\|u(\tau,\tau-t_n,\theta_{-\tau},u_{0,n})\|_V^2 = \|\bar{u}\|_V^2.
$$

This completes the proof. □

We now prove the main result of this section, i.e., the existence of $\mathcal{D}$-pullback attractors of Φ.

Theorem 3.1. *Let* (1.3)-(1.5) *hold and let* $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ *satisfy* (3.1)-(3.2). *Then the cocycle* Φ *associated with problem* (2.7) *has a* $\mathcal{D}$*-pullback attractor* $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau\in\mathbb{R}, \omega\in\Omega\} \in \mathcal{D}$ *in* V*. Moreover, if* f *is* T*-periodic, then the attractor* $\mathcal{A}$ *is also* T*-periodic, i.e.,* $\mathcal{A}(\tau+T,\omega) = \mathcal{A}(\tau,\omega)$ *for every* $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$.

Proof. From Lemma 3.4 we have that Φ has a closed measurable $\mathcal{D}$-pullback absorbing set $K\in\mathcal{D}$, and by Lemma 3.5 we see that Φ is $\mathcal{D}$-pullback asymptotically compact in V. Thus, by [31, Proposition 2.10], the process Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$.

If f is T-periodic, then the cocycle Φ is also T-periodic, that is, $\Phi(t,\tau+T,\omega,\cdot) = \Phi(t,\tau,\omega,\cdot)$ for all $t\in\mathbb{R}^+, \tau\in\mathbb{R}$ and $\omega\in\Omega$. Hence, by Lemma 3.4, we have that $K(\tau+T,\omega) = K(\tau,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$. Therefore, by [31, Proposition 2.11], the random attractor $\mathcal{A}$ for Φ is also T-periodic.

The proof is completed. □

4. Convergence of random attractors for multiplicative noise

4.1. **Existence of random attractors.** In this subsection, for $\tau\in\mathbb{R}$ is given, we consider the following stochastic Navier-Stokes-Voigt equations with multiplicative noise for $t>\tau$ and $x\in\mathcal{O}$:

$$
\begin{cases}
\dfrac{\partial}{\partial t}(u-\alpha^2\Delta u) - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t) + (u-\alpha^2\Delta u)\circ\dfrac{dW}{dt},\\
u(x,\tau) = u_\tau(x),
\end{cases}
\tag{4.1}
$$

where $u_\tau\in V$.

Applying the Leray-Helmholtz projector P to (4.1), we rewrite equation (4.1) in an abstract form

$$\begin{cases} \frac{d}{dt}(u+\alpha^2 Au) + \nu Au + B(u,u) = f(t) + (u+\alpha^2 Au)\circ \frac{dW}{dt}, t>\tau, \\ u(\tau) = u_\tau. \end{cases} \tag{4.2}$$

We now transform equation (4.2) into a pathwise deterministic equation by using the change of variables

$$v(t,\tau,\omega) = e^{-\omega(t)}u(t,\tau,\omega).$$

Thus (4.2) becomes

$$\begin{cases} \frac{d}{dt}(v+\alpha^2 Av) + \nu Av + e^{\omega(t)}B(v,v) = e^{-\omega(t)}f(t), \\ v(\tau) = v_\tau \text{ in } V, \end{cases} \tag{4.3}$$

where $v_\tau(x) = e^{-\omega(\tau)}u_\tau(x)$. By the Galerkin approximations method we can prove the existence and uniqueness of solutions for (4.3). Therefore, the solution v is continuous in v_τ in V and generates a cocycle (or Random Dynamical System) $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times V \to V$ such that for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $v_\tau \in W$

$$\Phi_0(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = e^{\omega(t)-\omega(-\tau)}v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau).$$

Then Φ_0 is a continuous cocycle on V over $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$.

We next drive the uniform estimates on the solutions of system (4.3).

Lemma 4.1. *Let $f \in L^2_{\rm loc}(\mathbb{R};V')$ satisfy (3.1)-(3.2). Then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$ and $\tau \geq \tau - t$, the solutions of the system (4.3) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\begin{aligned} &\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_V^2 + \frac{\nu}{2}\int_{\tau-t}^{\tau}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds \\ &\leq e^{-\nu d_0(s-\tau)} + \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau} e^{\nu d_0(s-\tau)}e^{-2\theta_\tau\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds. \end{aligned}$$

Proof. Multiplying the first equation of the system (4.3) by v in V, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_V^2 + \nu\|v\|^2 &= e^{-\omega(t)}\langle f, v\rangle \\ &\leq \frac{\nu}{4}\|v\|^2 + \frac{1}{\nu\lambda_1}e^{-2\omega(t)}\|f\|_{V'}^2. \end{aligned} \tag{4.4}$$

This implies that

$$\frac{d}{dt}\|v\|_V^2 + \nu d_0\|v\|_V^2 + \frac{\nu}{2}\|v\|^2 = \frac{2}{\nu\lambda_1}e^{-2\omega(t)}\|f\|_{V'}^2, \tag{4.5}$$

and multiplying (4.5) by $e^{\nu d_0}$ and integrating from $\tau - t$ to r, we get

$$\begin{aligned} &\|v(\tau,\tau-t,\omega,v_{\tau-t})\|_V^2 + \frac{\nu}{2}\int_{\tau-t}^{r} e^{\nu d_0(s-r)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2 ds \\ &\leq e^{-\nu d_0(r-\tau+t)}\|v_{\tau-t}\|_V^2 + \frac{2}{\nu\lambda_1}\int_{\tau-t}^{r} e^{\nu d_0(s-r)}e^{-2\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds. \end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ we get

$$\begin{aligned} &\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_V^2 + \frac{\nu}{2}\int_{\tau-t}^{r} e^{\nu d_0(s-r)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds \\ &\leq e^{-\nu d_0(r-\tau+t)}\|v_{\tau-t}\|_V^2 + \frac{2}{\nu\lambda_1}\int_{\tau-t}^{r} e^{\nu d_0(s-r)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds. \end{aligned} \tag{4.6}$$

Since $v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, we obtain

$$e^{-\nu d_0 t}\|v_{\tau-t}\|_V^2 \le e^{-\nu d_0 t}\|D(\tau - t, \theta_{-t}\omega)\|_V^2 \to 0$$

as $t \to \infty$. Thus, there exists $T = T(\tau, \omega, D) > 0$ such that

$$e^{-\nu d_0 t}\|v_{\tau-t}\|_V^2 \le 1 \quad \forall t \ge T,$$

this implies that

$$e^{-\nu d_0(r-\tau+t)}\|v_{\tau-t}\|_V^2 \le e^{-\nu d_0(r-\tau)} \quad \forall t \ge T. \tag{4.7}$$

On the other hand, since (2.2) we have $\frac{\omega(t)}{t} \to 0$ as $t \to \pm\infty$, there exists $r_0 < 0$ such that for all $s \le r_0$,

$$-2\theta_{-\tau}\omega(s) \le -(\nu d_0 - \eta)s,$$

this implies that

$$e^{-2\theta_\tau\omega(s)} \le e^{-(\nu d_0-\eta)s}.$$

Therefore, we get for all $s \le r_0$,

$$e^{\nu d_0 s}e^{-2\theta_\tau\omega(s)}\|f(s,\cdot)\|_{V'}^2 = e^{(\nu d_0-\eta)s}e^{-2\theta_\tau\omega(s)}e^{\eta s}\|f(s,\cdot)\|_{V'}^2 \le e^{\eta s}\|f(s,\cdot)\|_{V'}^2.$$

By (3.1) we get for every $s \in \mathbb{R}, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\int_{-\infty}^{r} e^{\nu d_0 s}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds \le \int_{-\infty}^{s} e^{\eta s}\|f(s,\cdot)\|_{V'}^2 ds < +\infty.$$

This together with (4.6) and (4.7) we get the conclusion of the lemma. □

Lemma 4.2. *For every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for every $k \ge 0$ and for all $t \ge T + k$, the solution v of the system* (4.3) *with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\begin{aligned}&\|v(\tau - k, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_V^2 \\ &\le e^{\nu d_0 k} + \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{-\nu d_0(\tau-k-s)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds,\end{aligned} \tag{4.8}$$

where $v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Given $\tau \in \mathbb{R}$ and $k \ge 0$, let $s = \tau - k$. Let $T > 0$ be the constant as in Lemma 4.1. Also, $t \ge T + k$ implies $t \ge T$ and $r \ge \tau - t$, we get the inequality (4.8). □

Similarly to Lemma 3.3, we have the weak convergence of solutions of system (4.3), which will help us to show the asymptotic compactness of these solutions.

Lemma 4.3. *Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $v_\tau, v_{\tau,n} \in V$ for all $n \in \mathbb{N}$. If $v_{\tau,n} \rightharpoonup v_\tau$ in V, then the solution v of system* (4.3) *has the following convergences as $n \to \infty$:*

i) *$v(s, \tau, \omega, v_{\tau,n}) \rightharpoonup v(s, \tau, \omega, v_\tau)$ in V for all $s \ge \tau$;*
ii) *$v(\cdot, \tau, \omega, v_{\tau,n}) \rightharpoonup v(\cdot, \tau, \omega, v_\tau)$ in $L^2(\tau, \tau + T; V)$ for every $T > 0$.*

We now prove the pullback asymptotic compactness of solutions of system (4.3).

Lemma 4.4. *Assume $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and* (3.1) *holds. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and $t_n \to \infty, v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, the sequence $v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})$ of solutions of system* (4.3) *has a convergent subsequence in V.*

Proof. From Lemma 4.2 with $k = 0$, it follows that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \ge T$,

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{\tau-t})\|_V^2 \le 1 + \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau} e^{\nu d_0(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds, \tag{4.9}$$

where $v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$. Since $v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$ and by $t_n \to +\infty$, there exists $N_0 \in \mathbb{N}$ such that $t_n \geq T$ for all $n \geq N_0$. Then, for all $n \geq N_0$ we have from (4.9) that

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|_V^2 \leq 1 + \frac{2}{\nu\lambda_1} \int_{-\infty}^{\tau} e^{\nu d_0(s-\tau)} e^{-2\theta_{-\tau}\omega(s)} \|f(s, \cdot)\|_{V'}^2 ds.$$

This shows that the sequence $\{v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\}$ is bounded in V for all $n \geq N_0$. Thus, there is a function $\bar{v} \in V$ such that (up to a subsequence)

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \rightharpoonup \bar{v} \quad \text{in } V. \tag{4.10}$$

By property of the weak convergence, we have

$$\liminf_{n\to\infty} \|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|_V \geq \|\bar{v}\|_V. \tag{4.11}$$

We now prove that

$$\limsup_{n\to\infty} \|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|_V \leq \|\bar{v}\|_V$$

and this together with (4.11) give us the strong convergence of (4.10). As in Lemma 3.5, we use the method of energy equations. For a given $k \in \mathbb{N}$, we have

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) = v(\tau, \tau - k, \theta_{-\tau}\omega, v(\tau - k, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})). \tag{4.12}$$

For each k, let N_k be sufficiently large such that $t_n \geq T + k$ for all $n \geq N_k$. From Lemma 4.2, we get that

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|_V^2 \leq e^{\nu d_0 k} + \frac{2}{\nu\lambda_1} \int_{-\infty}^{\tau-k} e^{-\nu d_0(s+\tau-k)} e^{-2\theta_{-\tau}\omega(s)} \|f(s, \cdot)\|_{V'}^2 ds$$

for $k \geq N_k$. Hence, for each fixed $k \in \mathbb{N}$, the sequence $\{v(\tau - k, \tau - t_n, \theta_{-\tau}, v_{0,n})\}$ is bounded in V. Thus, for each $k \in \mathbb{N}$, there is a function $\bar{v}_k \in V$ such that (up to a subsequence)

$$v(\tau - k, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \rightharpoonup \bar{v}_k \text{ in } V \text{ as } n \to \infty. \tag{4.13}$$

By Lemma 4.3 and from (4.12)-(4.13), we get that

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \rightharpoonup v(\tau, \tau - k, \theta_{-\tau}\omega, \bar{v}_k) \text{ in } V, \tag{4.14}$$

$$v(\cdot, \tau - k, \theta_{-\tau}\omega, v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \rightharpoonup v(\tau, \tau - k, \theta_{-\tau}\omega, \bar{v}_k) \text{ in } L^2(\tau - k, \tau; V). \tag{4.15}$$

By the uniqueness of the weak limits we obtain from (4.10) and (4.14) that

$$v(\tau, \tau - k, \theta_{-\tau}\omega, \bar{v}_k) = \bar{v}. \tag{4.16}$$

We recall from (4.4) that,

$$\frac{d}{dt}\|v\|_V^2 + 2\nu\|v\|^2 = 2e^{-\omega(t)}\langle f, v\rangle.$$

For each $\omega \in \Omega$, and $\tau \in \mathbb{R}$, multiplying the above equation by $e^{\eta t}$ integrating the above equation from $\tau - k$ to τ we get that

$$\int_{\tau-k}^{\tau} d(e^{\eta s}\|v(s, \tau - k, \omega, v_{\tau-k})\|_V^2) ds = 2\int_{\tau-k}^{\tau} e^{\eta s}\langle f(s, \cdot), v(s, \tau - k, \omega, v_{\tau-k})\rangle ds$$
$$- 2\int_{\tau-k}^{\tau} e^{\eta s}[\nu\|v(s, \tau - k, \omega, v_{\tau-k})\|^2 - \frac{\eta}{2}\|v(s, \tau - k, \omega, v_{\tau-k})\|_V^2] ds. \tag{4.17}$$

As in (3.32), we denote by

$$[v]_1^2 = 2\nu\|v\|^2 - \eta\|v\|_V^2, \quad \forall v \in V, \tag{4.18}$$

then $[\cdot]_1$ define a new norm on V which is equivalent to the usual norm $\|\cdot\|_V$ on V. Thus we get from (4.17) and (4.18) that

$$\|v(\tau,\tau-k,\omega,v_{\tau-k})\|_V^2 = e^{-\eta k}\|v_{\tau-k}\|_V^2 - 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\omega,v_{\tau-k})]_1^2 ds + \frac{2}{\nu\lambda_1}\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(s,\tau-k,\omega,v_{\tau-k})\rangle ds. \tag{4.19}$$

Moreover, since $v(\tau,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)=\bar{v}$ in (4.16), we obtain from (4.19) that

$$\begin{aligned}\|\bar{v}\|_V^2 &= \|v(\tau,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)\|_V^2 \\ &= e^{-\eta k}\|\bar{v}_k\|_V^2 - 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\omega,\bar{v}_k)]_1^2 ds \\ &+ \frac{2}{\nu\lambda_1}\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(s,\tau-k,\omega,\bar{v}_k)\rangle ds. \end{aligned} \tag{4.20}$$

Next, using (4.12) and (4.20) we get that

$$\begin{aligned}&\|v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2 \\ &= \|v(\tau,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))\|_V^2 \\ &= e^{-\eta k}\|v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2 \\ &- 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))]_1^2 ds \\ &+ 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(\tau,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))\rangle ds. \end{aligned} \tag{4.21}$$

We are now passing to the limit in each term on the right-hand side of (4.21) as $n\to\infty$. For the first term, we use (4.6) for $r=\tau-k$ and $t=t_n$ to get

$$\begin{aligned}&e^{-\eta k}\|v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2 \\ &\le e^{-\eta t_n}\|v_{0,n}\|_V^2 + \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{\eta(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds. \end{aligned} \tag{4.22}$$

Note that $v_{0,n}\in D(\tau-t_n,\theta_{-t_n}\omega)$, we have

$$e^{-\eta t_n}\|v_{0,n}\|_V^2 \le e^{-\eta t_n}\|D(\tau-t_n,\theta_{-t_n}\omega)\|_V^2 \to 0 \text{ as } n\to\infty. \tag{4.23}$$

From (4.22)-(4.23) we obtain

$$\begin{aligned}&\limsup_{n\to\infty} e^{-\eta k}\|v(\tau-k,\tau-t_n,\theta_{-\tau},\omega,v_{0,n})\|_V^2 \\ &\le \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{\eta(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds. \end{aligned} \tag{4.24}$$

For the second term in (4.21), we get from the weak lower semicontinuity property of norm $[\cdot]_1$ in (4.18) and (4.15) that

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))]_1^2 ds \\ &\le -\liminf_{n\to\infty}\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))]_1^2 ds \\ &\le -\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)]_1^2 ds. \end{aligned} \tag{4.25}$$

For the last term, we use the weak convergence in (4.14) that

$$2\lim_{n\to\infty}\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(s,\tau-k,\theta_{-\tau}\omega, v(\tau-k,\tau-t_n,\theta_{-\tau}\omega,v_{0,n}))\rangle ds$$
$$= 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(s,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)\rangle ds. \tag{4.26}$$

Thus, we get from (4.24)-(4.26) and (4.21) that

$$\begin{aligned}
&\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2\\
&\le \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{\eta(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds\\
&- \int_{\tau-k}^{\tau} e^{\eta(s-\tau)}[v(s,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)]_1^2 ds\\
&+ 2\int_{\tau-k}^{\tau} e^{\eta(s-\tau)}e^{-\omega(s)}\langle f(s,\cdot), v(s,\tau-k,\theta_{-\tau}\omega,\bar{v}_k)\rangle ds.
\end{aligned} \tag{4.27}$$

Replacing (4.20) into (4.27) we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2\\
&\le \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{\eta(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds + \|\bar{v}\|_V^2 - e^{-\eta k}\|\bar{v}_k\|_V^2\\
&\le \frac{2}{\nu\lambda_1}\int_{-\infty}^{\tau-k} e^{\eta(s-\tau)}e^{-2\theta_{-\tau}\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds + \|\bar{v}\|_V^2.
\end{aligned} \tag{4.28}$$

From this let $k\to\infty$ we get

$$\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\|_V^2 \le \|\bar{v}\|_V^2$$

which together with (4.11) give us the conclusion of the lemma. □

From Lemma 4.1, we immediately have the existence of $\mathcal{D}$-pullback absorbing set for the continuous cocycle Φ_0.

Lemma 4.5. *Let $f\in L^2_{\mathrm{loc}}(\mathbb{R};V')$ such that (3.1) holds. Then the continuous cocycle Φ_0 associated with the system (4.3) possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K_0=\{K_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ defined by*

$$K_0(\tau,\omega)=\{u\in V:\|u\|_V^2\le R_0(\tau,\omega)\}$$

where

$$R_0(\tau,\omega)=\frac{4}{\nu\lambda_1}\int_{-\infty}^{\tau} e^{\nu d_0(s-\tau)}e^{-2\omega(s)}\|f(s,\cdot)\|_{V'}^2 ds.$$

Theorem 4.1. *Let $f\in L^2_{\mathrm{loc}}(\mathbb{R};V')$ such that (3.1) holds. Then the continuous cocycle Φ associated with the system (4.3) has a unique random $\mathcal{D}$-pullback attractor*

$$\mathcal{A}_0=\{\mathcal{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D} \text{ in } V.$$

Moreover, if f is T-periodic with $T>0$, then the random attractor $\mathcal{A}$ is also T-periodic, i.e.,

$$\mathcal{A}_0(\tau+T,\omega)=\mathcal{A}_0(\tau,\omega)\quad \text{for every } \tau\in\mathbb{R},\omega\in\Omega.$$

Proof. By Lemma 4.5, the continuous cocycle Φ_0 has a closed measurable $\mathcal{D}$-pullback absorbing set $K_0\in\mathcal{D}$ and by Lemma 4.3 Φ_0 is $\mathcal{D}$-pullback asymptotically compact in V, then by [30, Proposition 2.10] we get the existence and uniqueness of $\mathcal{D}$-pullback attractor $\mathcal{A}_0$ of Φ_0.

In addition, if f is T-periodic, then the cocycle Φ_0 is also T-periodic, i.e.,

$$\Phi_0(t,\tau+T,\omega,\cdot)=\Phi_0(t,\tau,\omega,\cdot) \quad \forall t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega.$$

By Lemma 4.5 we also have $K_0(\tau+T,\omega)=K_0(\tau,\omega)$ for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$, and therefore using [30, Proposition 2.11] we get the T-periodicity of the random attractor $\mathcal{A}_0$. □

4.2. **Existence of approximation attractors and convergence of random attractors.** In this subsection, for each $\delta>0$ we consider the pathwise random equations

$$\begin{cases} d(u_\delta+\alpha^2 Au_\delta)+\nu Au_\delta+B(u_\delta,u_\delta)=f(t)+\mathcal{W}_\delta(\theta_t\omega)(u_\delta+\alpha^2Au_\delta), \\ u_\delta(\tau)=u_{\delta,\tau}. \end{cases} \tag{4.29}$$

Define

$$v_\delta(t,\tau,\omega,v_{\delta,\tau})=e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}u_\delta(t,\tau,\omega,v_{\delta,\tau}),$$

then (4.29) becomes

$$\begin{cases} \dfrac{dv_\delta}{dt}+\nu Av_\delta+e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}B(v_\delta,v_\delta)=e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}f(t) \\ v_\delta(\tau)=v_{\delta,\tau}, \end{cases} \tag{4.30}$$

where $v_{\delta,\tau}=e^{-\int_0^\tau \mathcal{W}_\delta(\theta_r\omega)dr}u_{\delta,\tau}\in V$.

As a special case of results in Section 3, problem (4.30) defines a continuous cocycle Φ_δ in V which has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\delta$ for every $\delta\neq 0$.

Next, we will prove that solutions of (4.30) converge to the coresponding solution of equation (4.3) as $\delta\to 0$. To do this, we first establish the uniform estimates for solutions of system (4.30) via the following lemma.

Lemma 4.6. *For every $\tau\in\mathbb{R},\omega\in\Omega$ and $T>0$, there exist $\delta_0=\delta_0(\tau,\omega,T)>0$ and $M_0=M_0(\tau,\omega,T)>0$ such that solutions of (4.30) satisfies the following estimate*

$$\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|_V^2+\frac{\nu}{2}\int_\tau^t\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2ds\le M_0\left(\|v_{\delta,\tau}\|_V^2+\int_\tau^t\|f(s,\cdot)\|_{V'}^2ds\right), \tag{4.31}$$

for all $0<|\delta|<\delta_0$ and $t\in[\tau,\tau+T]$.

Proof. From (4.30), it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_\delta\|_V^2+\nu\|v_\delta\|^2 &= e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\langle f,v_\delta\rangle \\ &\le \frac{\nu}{4}\|v_\delta\|^2+\frac{1}{\nu\lambda_1}e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s,\cdot)\|_{V'}^2. \end{aligned} \tag{4.32}$$

Since $d_0\|v_\delta\|_V^2\le\|v_\delta\|^2$, we obtain from (4.32)

$$\frac{d}{dt}\|v_\delta\|_V^2+\nu d_0\|v_\delta\|_V^2+\frac{1}{2}\|v_\delta\|^2\le\frac{2}{\nu\lambda_1}e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s,\cdot)\|_{V'}^2, \tag{4.33}$$

thus for all $\tau\in\mathbb{R}$ and $t\ge\tau$ and ω, by Gronwall's inequality we get

$$\begin{aligned} &\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|_V^2+\frac{\nu}{2}\int_\tau^t\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2ds \\ &\le e^{\nu d_0(\tau-t)}\|v_{\delta,\tau}\|_V^2+\frac{2}{\nu\lambda_1}\int_\tau^t e^{\nu d_0(s-t)-2\int_0^s \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s,\cdot)\|_{V'}^2ds. \end{aligned} \tag{4.34}$$

By continuity of $\omega(t)$ on $[\tau,\tau+T]$, we infer that there exist $\delta_0=\delta_0(\tau,\omega,T)>0$ and $c=(\tau,\omega,T)>0$ such that

$$\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr\le c$$

for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$. Therefore, from (4.34) we can conclude that there exists $M_0 = M_0(\tau, \omega, T) > 0$ such that

$$\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|_V^2+\frac{\nu}{2}\int_\tau^t \|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2 ds \le M_0\left(\|v_{\delta,\tau}\|_V^2+\int_\tau^t \|f(s,\cdot)\|_{V'}^2 ds\right). \tag{4.35}$$

This shows that (4.31) as claimed. □

The following result give us the uniform estimates of solutions of (4.30) when time t is sufficiently large.

Lemma 4.7. *For every $\delta \neq 0, \tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution of (4.29) satisfies the following estimate*

$$\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|_V^2 \le \frac{2}{\nu\lambda_1}\int_{-\infty}^0 e^{\nu d_0 s+2\int_s^0 \mathcal{W}_\delta(\theta_r)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds$$

with $u_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. It follows from (4.33) that

$$\frac{d}{dt}\|v_\delta\|_V^2+\nu d_0\|v_\delta\|_V^2 \le \frac{2}{\nu\lambda_1}e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s,\cdot)\|_{V'}^2,$$

and by Gronwall's inequality we get for every $\tau \in \mathbb{R}, t \in \mathbb{R}^+$ and $\omega \in \Omega$ that

$$\begin{aligned}&\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|_V^2\\ &\le e^{-k_2 t}\|v_{\delta,\tau-t}\|_V^2+\frac{2}{\nu\lambda_1}\int_{\tau-t}^{\tau} e^{k_2(s-\tau)}e^{-2\int_0^s \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|f(s,\cdot)\|_{V'}^2 ds.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|_V^2\\ &= e^{2\int_0^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,e^{-\int_0^{\tau-t}\mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}u_{\delta,\tau-t})\|_V^2\\ &\le e^{-\nu d_0 t}e^{2\int_{\tau-t}^{\tau}\mathcal{W}_\delta(\theta_{r-\tau})dr}\|u_{\delta,\tau-t}\|_V^2\\ &\quad+\frac{2}{\nu\lambda_1}\int_{\tau-t}^{\tau} e^{\nu d_0(s-\tau)}e^{2\int_s^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|f(s,\cdot)\|_{V'}^2 ds\\ &\le e^{-\nu d_0 t+2\int_{-t}^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|_V^2\\ &\quad+\frac{2}{\nu\lambda_1}\int_{-t}^0 e^{\nu d_0 s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds.\end{aligned}$$

Moreover, by the assumption on f we have

$$\int_{-\infty}^0 e^{\nu d_0 s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds < +\infty, \tag{4.36}$$

and by $u_{\delta,\tau-t} \in D(\tau - t, \theta_t\omega)$ and $D \in \mathcal{D}$, by the ergodic theory we have

$$\lim_{s\to-\infty}\frac{1}{s}\int_0^s (\nu d_0-2\gamma\mathcal{W}_\delta(\theta_r\omega))dr = \nu d_0-2\gamma\mathbb{E}(\mathcal{W}_\delta(\theta_r\omega)) = \nu d_0,$$

thus there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,

$$\begin{aligned}e^{-\nu d_0 t+2\int_{-t}^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|_V^2 &\le e^{-\frac{\nu d_0}{2}t}\|D(\tau-t,\theta_t\omega)\|_V^2\\ &\le \frac{2}{\nu\lambda_1}\int_{-\infty}^0 e^{\nu d_0 s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds.\end{aligned} \tag{4.37}$$

Therefore, from (4.36) and (4.37) we get (4.35), this completes the proof. □

We now prove the cocycle Φ_0 has a $\mathcal{D}$-pullback absorbing set in V.

Lemma 4.8. *For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the continuous cocycle Φ_δ has a $\mathcal{D}$-pullback absorbing set $K_\delta \in \mathcal{D}$ given by*

$$K_\delta(\tau,\omega) = \{u_\delta \in V : \|u_\delta\|_V^2 \le R_\delta(\tau,\omega)\}, \tag{4.38}$$

where

$$R_\delta(\tau,\omega) = \frac{4}{\nu\lambda_1}\int_{-\infty}^0 e^{\nu d_0 s + 2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds.$$

Moreover, if $\tau \in \mathbb{R}$ and $\omega \in \Omega$ are fixed then

$$\lim_{\delta\to 0} R_\delta(\tau,\omega) = \frac{2}{\nu\lambda_1}\int_{-\infty}^0 e^{\nu d_0 s - 2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds. \tag{4.39}$$

Proof. By Lemma 4.7, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T_0 = T_0(\tau,\omega,D) > 0$ such that

$$\Phi_\delta(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_\delta(\tau,\omega), \quad \forall t \ge T_0.$$

It is clear that $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in (4.38) is a closed measurable random set in V. Thus, to prove K_δ is a $\mathcal{D}$-pullback absorbing set for Φ_δ, we need to prove K_δ is tempered. Indeed, for arbitrary constant $c > 0$, we have

$$\begin{aligned} e^{ct}\|K_\delta(\tau,\omega)\|_V^2 &= e^{ct}R_\delta(\tau+t,\theta_t\omega) \\ &= \frac{4}{\nu\lambda_1}e^{ct}\int_{-\infty}^0 e^{\nu d_0 - 2\int_0^s \mathcal{W}_\delta(\theta_{r+t}\omega)dr}\|f(s+\tau+t,\cdot)\|_{V'}^2 ds \\ &= \frac{4}{\nu\lambda_1}e^{ct}\int_{-\infty}^0 e^{\nu d_0 - \frac{2}{\delta}\int_s^{s+\delta}\omega(r+t)dr - \frac{2}{\delta}\int_\delta^0 \omega(r+t)dr}\|f(s+\tau+t,\cdot)\|_{V'}^2 ds. \end{aligned} \tag{4.40}$$

Since $\frac{|\omega(t)|}{t} \to 0$ as $t \to \pm\infty$, then there exists $T_1 = T_1(\omega) \le 0$ such that

$$|\omega(t)| \le -c_1 t \quad \forall t \ge T_1, \tag{4.41}$$

and by the mean value theorem, we infer that for every $s \le 0$, there exists $s_0 \in [s, s+\delta]$ such that

$$-\frac{2}{\delta}\int_s^{s+\delta}\omega(r+t)dr = -2\omega(s_0+t).$$

Clearly $s_0 \le |\delta|$, hence by (4.41) we obtain

$$|\omega(s_0+t)| \le -c_1(s_0+t), \quad \forall t \le T - |\delta|. \tag{4.42}$$

and by $s - s_0 \le |\delta|$ and (4.42), it follows that

$$e^{-\frac{2}{\delta}\int_s^{s+\delta}\omega(r+t)dr} = e^{-2\omega(s_0+t)} \le e^{-2c_1(s+t)+2c_1|\delta|}, \tag{4.43}$$

for all $t \le T_1 - |\delta|$. And by similarly as in (4.43) we also get

$$e^{-\frac{2}{\delta}\int_\delta^0 \omega(r+t)dr} \le e^{-2c_1 t + 2c_1|\delta|} \quad \forall t \le T_1 - |\delta|. \tag{4.44}$$

Hence, by (4.43)-(4.44) and (4.40) we have

$$e^{ct}R_\delta(\tau,\omega) \le \frac{4}{\nu\lambda_1}e^{4c_1|\delta| - (\nu d_0 - 2c_1)\tau}e^{(c-4c_1)t}\int_{-\infty}^0 e^{(\nu d_0 - 2c_1)s}\|f(s+t,\cdot)\|_{V'}^2 ds. \tag{4.45}$$

Choose $c_1 = \min\{\frac{\nu d_0 - \eta}{2}, \frac{c}{8}\}$, we obtain from (4.45)

$$\begin{aligned} e^{ct}R_\delta(\tau,\omega) &\le \frac{4}{\nu\lambda_1}e^{4c_1|\delta| - (\nu d_0 - 2c_1)\tau}e^{(c-4c_1)t}\int_{-\infty}^0 e^{(\nu d_0 - 2c_1)s}\|f(s+t,\cdot)\|_{V'}^2 ds \\ &\le \frac{2}{\nu\lambda_1}e^{4c_1|\delta| - (\nu d_0 - 2c_1)\tau}e^{\frac{c}{2}t}\int_{-\infty}^0 e^{\eta s}\|f(s+t,\cdot)\|_{V'}^2 ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}\lim_{t\to-\infty} e^{ct}\|K_\delta(\tau,\omega)\|_V^2 &= \lim_{t\to-\infty} e^{ct} R_\delta(\tau,\omega)\\ &\le \frac{4}{\nu\lambda_1}\lim_{t\to-\infty} e^{4c_1|\delta|-(\nu d_0-2c_1)\tau} e^{\frac{c}{2}t}\int_{-\infty}^0 e^{\eta s}\|f(s+t,\cdot)\|_{V'}^2 ds\\ &= 0,\end{aligned}$$

and hence K_δ is tempered, therefore $K_\delta \in \mathcal{D}$.

In order to get (4.39), we write

$$2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr = -2\int_s^{s+\delta}\frac{\omega(r)}{\delta}dr + 2\int_0^\delta \frac{\omega(r)}{\delta}dr, \tag{4.46}$$

by $\lim\limits_{\delta\to 0}\frac{\omega(r)}{\delta}dr = 0$, there exists $\delta_1 = \delta_1(\omega)$ such that

$$\left|2\int_0^\delta \frac{\omega(r)}{\delta}dr\right| \le 1 \quad \forall 0 < |\delta| < \delta_1. \tag{4.47}$$

According to the mean value theorem there is a $r_0 \in [s, s+\delta]$ such that

$$-2\int_s^{s+\delta}\frac{\omega(r)}{\delta}dr = -2\omega(r_0),$$

Since $|s-r_0| < |\delta|$, then for $|\delta| \le 1$ and $s \le T_1 - 1$ we have $r_0 \le s + |\delta| \le T_1$. Hence, using (4.42) with $c \le (k_1-\eta)/2$ we get

$$|\omega(r_0)| \le -\frac{\nu d_0-\eta}{2}r_0 \le \frac{\nu d_0-\eta}{2}|\delta| - \frac{\nu d_0-\eta}{2}s \le \frac{\nu d_0-\eta}{2} - \frac{\nu d_0-\eta}{2}s. \tag{4.48}$$

Combining (4.46)-(4.48) we get for $0 < |\delta| < \delta_2 = \min\{1, \delta_1\}$ and all $s \le T_1 - 1$,

$$2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr \le (\nu d_0-\eta) - (\nu d_0-\eta)s + 1. \tag{4.49}$$

For $s\in\mathbb{R}, \delta\ne 0, \tau\in\mathbb{R}$ and $\omega\in\Omega$, let

$$\tilde{R}_\delta(\tau,\omega,s) = \frac{4}{\nu\lambda_1}e^{\nu d_0 s + 2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|f(s+\tau,\cdot)\|_{V'}^2 ds. \tag{4.50}$$

By Lemma 2.2 iii), we have

$$\lim_{\delta\to 0}\tilde{R}_\delta(\tau,\omega,s) = \frac{4}{\nu\lambda_1}e^{\nu d_0 s - 2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2. \tag{4.51}$$

Thus, by (4.49) and (4.50) for all $0 < |\delta| < \delta_2$ and $s \le T_1 - 1$, we have

$$\tilde{R}_\delta(\tau,\omega,s) \le \frac{4}{\nu\lambda_1}e^{1+(\nu d_0-\eta)}e^{\eta s}\|f(s+\tau,\cdot)\|_{V'}^2.$$

By the dominated convergence theorem we have from (4.51) that

$$\lim_{\delta\to 0}\int_{-\infty}^{T_1-1}\tilde{R}_\delta(\tau,\omega,s)ds = \frac{4}{\nu\lambda_1}\int_{-\infty}^{T_1-1} e^{\nu d_0 s-2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds, \tag{4.52}$$

and since $2\int_s^0 \mathcal{W}_\delta(\theta_\omega)dr \to -2\omega(s)$ uniformly on $[T_1-1, 0]$, thus

$$\lim_{\delta\to 0}\int_{T_1-1}^0 \tilde{R}_\delta(\tau,\omega,s)ds = \frac{4}{\nu\lambda_1}\int_{T_1-1}^0 e^{\nu d_0 s-2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds. \tag{4.53}$$

Therefore, by (4.52) and (4.53), we obtain

$$\begin{aligned}&\lim_{\delta\to 0} R_\delta(\tau,\omega,s) = \lim_{\delta\to 0}\int_{-\infty}^0 \tilde{R}_\delta(\tau,\omega,s)ds\\ &= \lim_{\delta\to 0}\int_{-\infty}^{T_1-1}\tilde{R}_\delta(\tau,\omega,s)ds + \lim_{\delta\to 0}\int_{T_1-1}^0 \tilde{R}_\delta(\tau,\omega,s)ds\end{aligned}$$

$$= \frac{4}{\nu\lambda_1}\int_{-\infty}^{T_1-1} e^{\nu d_0 s-2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds + \frac{4}{\nu\lambda_1}\int_{T_1-1}^{0} e^{\nu d_0 s-2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds$$
$$= \frac{4}{\nu\lambda_1}\int_{-\infty}^{0} e^{\nu d_0 s-2\omega(s)}\|f(s+\tau,\cdot)\|_{V'}^2 ds,$$

and this completes the proof. □

We are now ready to prove the convergence of solutions of equation (4.29) as $\delta \to 0$.

Lemma 4.9. *Suppose that $\{\delta_n\}_{n=1}^{\infty}$ is a sequence such that $\delta_n \to 0$ as $n \to \infty$. Let u_{δ_n} and u be the solutions of* (4.29) *and* (4.2) *with initial data $u_{\delta_n,\tau}$ and u_τ, respectively. If $\|u_{\delta_n,\tau} - u_\tau\|_V \to 0$ then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $t > \tau$,*

$$\|u_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau}) - u(t,\tau,\omega,u_\tau)\|_V \to 0 \text{ as } n \to \infty. \tag{4.54}$$

Proof. Denote $w = v_{\delta_n} - v$, then by (4.30) and (4.3) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_V^2 + \nu\|w\|^2 &= \langle e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}B(v_\delta,v_\delta) - e^{\omega(t)}B(v,v), w\rangle \\ &= (e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr})\langle f, w\rangle.\end{aligned} \tag{4.55}$$

By Lemma 2.1 we have

$$\begin{aligned}&\langle e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}B(v_\delta,v_\delta) - e^{\omega(t)}B(v,v), w\rangle \\ &= e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\langle B(v_\delta,v_\delta) - B(v,v), w\rangle + \left(e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\right)\langle B(v,v), w\rangle \\ &= -e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\langle B(w,v), w\rangle - \left(e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}-\right)\langle B(v,v), w\rangle \\ &\le e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|w\|\|v\|\|w\| - \left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\right|\|v\|\|v\|\|w\| \\ &\le \frac{\nu}{4}\|w\|_V^2 + ce^{2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|w\|_V^2\|v\|^2 \\ &\quad + c\|w\|_V^2\|v\|^2 + c\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\right|^2\|v\|^2,\end{aligned} \tag{4.56}$$

and

$$\begin{aligned}&(e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr})\langle f, w\rangle \\ &\le \left|e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|\|f\|_{V'}|w| \\ &\le \frac{\nu}{4}\|w\|^2 + \frac{1}{\nu\lambda_1}\left|e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|^2\|f(t,\cdot)\|_{V'}^2.\end{aligned} \tag{4.57}$$

Thus, from (4.56)-(4.57) and (4.55) we get

$$\begin{aligned}\frac{d}{dt}\|w\|_V^2 + \nu\|w\|^2 &\le 2c\left(e^{2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} + 1\right)\|v\|^2\|w\|_V^2 \\ &\quad + c\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\right|^2\|v\|^2 \\ &\quad + c\left|e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|^2\|f(t,\cdot)\|_{V'}^2.\end{aligned}$$

Next, by the uniformly consequence of $\mathcal{W}_\delta(t)$ to $\omega(t)$ as $\delta \to 0$, for every $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon,\tau,\omega,T) > 0$ such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau+T]$,

$$\left|e^{\omega(t)} - e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|^2 \le \epsilon,$$

and

$$\left|e^{-\omega(t)} - e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|^2 = \left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right|^2\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr-\omega(t)} - 1\right|^2 \le c\epsilon,$$

where we have used the fact that $\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\right| < c$.

Summing up, we obtain

$$\frac{d}{dt}\|w\|_V^2 \le c\|v\|^2\|w\|_V^2 + c\epsilon\|v\|^2 + c\epsilon\|f(t,\cdot)\|_{V'}^2.$$

Multiplying by $\xi(t) = e^{-c\int_\tau^t \|v(s)\|^2 ds}$ we obtain

$$\begin{aligned}\frac{d}{dt}(\xi(t)\|w\|_V^2) \le{}& c\xi(t)\|v\|^2\|w\|_V^2 - c\xi(t)\|v\|^2\|w\|_V^2\\ &+ c\epsilon\xi(t)\|v\|^2 + c\epsilon\xi(t)\|f(t,\cdot)\|_{V'}^2.\end{aligned}$$

Thus, integrating over $[\tau,t]$ we have

$$\begin{aligned}\xi(t)\|w(t)\|_V^2 \le{}& \|w(\tau)\|_V^2 + c\epsilon\int_\tau^t \xi(s)\|v(s)\|^2\|v(s)\|_V^2 ds\\ &+ c\epsilon\int_\tau^t \xi(s)\|f(s,\cdot)\|_{V'}^2 ds.\end{aligned}$$

for all $0<|\delta|<\delta_1$, and by Gronwall's inequality we obtain

$$\begin{aligned}\xi(t)\|w(t)\|_V^2 \le{}& \|w(\tau)\|_V^2 + \epsilon c\int_\tau^t \xi(s)\|v(s)\|^2 ds\\ &+ \epsilon c\int_\tau^t \xi(s)\|f(s,\cdot)\|_{V'}^2 ds,\end{aligned}$$

thus there exist $\delta_2 \in (0,\delta_1)$ and $c_1 = c_1(\tau,\omega,T) > 0$ such that for all $0<|\delta|<\delta_2$ and $t\in[\tau,\tau+T]$,

$$\|w\|_V^2 \le c_1\|w(\tau)\|_V^2 + \epsilon c_1.$$

This implies that

$$\|v_\delta(t) - v(t)\|_V^2 \le c_1\|v_{\delta,\tau} - v_\tau\|_V^2 + \epsilon c_1.$$

Moreover, we have

$$\begin{aligned}u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau) &= e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}v_\delta(t,\tau,\omega,v_{\delta,\tau}) - e^{\omega(t)}v(t,\tau,\omega,v_\tau)\\ &= e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}(v_\delta(t,\tau,\omega,v_{\delta,\tau}) - v(t,\tau,\omega,v_\tau))\\ &\quad + (e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)})v(t,\tau,\omega,v_\tau), \end{aligned} \tag{4.58}$$

where $v_{\delta,\tau} = e^{-\int_0^\tau W_\delta(\theta_r\omega)dr}u_{\delta,\tau}$ and $v_\tau = e^{-\omega(\tau)}u_\tau$. Hence, by property of $\mathcal{W}_\delta(\theta_t\omega)$ and by (4.58), there exist $\delta_3 \in (0,\delta_2)$ and $c_3 = c_3(\tau,\omega,T) > 0$ such that for all $0<|\delta|<\delta_3$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}\|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\|_V^2 &\le c_3\|v_\delta(t,\tau,\omega,v_{\delta,\tau}) - v(t,\tau,\omega,v_\tau)\|_V^2\\ &\quad + c_3\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr - \omega(t)} - 1\right|^2\|v(t,\tau,\omega,v_\tau)\|_V^2\\ &\le c\|v_{\delta,\tau} - v_\tau\|_V^2 + \epsilon c,\end{aligned}$$

this proves (4.54) as desired. □

We note that Lemma 4.3 can also be written in the following form, which will help us to prove the asymptotic compactness of solutions of system (4.30).

Lemma 4.10. *Assume that $f \in L^2_{\text{loc}}(\mathbb{R};V')$ and $\{\delta_n\}_{n=1}^\infty$ is a sequence such that $\delta_n \to 0$. Let v_{δ_n} and v be the solutions of (4.30) and (4.3) with initial data $v_{\delta_n,\tau}$ and v_τ, respectively. If $v_{\delta_n,\tau} \to v_\tau$ in V as $n\to\infty$, then for every $\tau\in\mathbb{R},\omega\in\Omega$ and $t>\tau$, we have*

$$\begin{aligned}v_{\delta_n}(r,\tau,\omega,v_{\delta_n,\tau}) &\rightharpoonup v(r,\tau,\omega,v_\tau) \text{ in } V, \quad \forall r\ge\tau,\\ v_{\delta_n}(\cdot,\tau,\omega,v_{\delta_n,\tau}) &\rightharpoonup v(\cdot,\tau,\omega,v_\tau) \text{ in } L^2(\tau,\tau+T;V), \quad \forall T>0,\end{aligned}$$

and

$$v_{\delta_n}(\cdot,\tau,\omega,v_{\delta_n,\tau}) \to v(\cdot,\tau,\omega,v_\tau) \text{ in } L^2(\tau,\tau+T;L^2(\mathcal{O}_R)), \forall T>0, R>0,$$

where $\mathcal{O}_R = \{x \in \mathcal{O} : |x| < R\}$.

Proof. For simplicity, we write $v_n(r) = v_{\delta_n}(r,\tau,\omega,v_{\delta_n,\tau})$ and $v(r) = v(r,\tau,\omega,v_\tau)$ for $r \geq \tau, \tau \in \mathbb{R}$ and $\omega \in \Omega$.

We first observe that from Lemma 4.8 that the sequence $\{v_n\}_{n=1}^\infty$ is bounded in $C([\tau,\tau+T];V)$. Since $\delta_n \to 0$ and by

$$\lim_{\delta_n\to 0}\int_0^t \mathcal{W}_{\delta_n}(\theta_r\omega)dr = \omega(t) \text{ uniformly for } t \in [\tau,\tau+T],$$

there exists $N_1 = N_1(\tau,T,\omega) \geq 1$ such that

$$\left|\int_0^t \mathcal{W}_{\delta_n}(\theta_r\omega)dr - \omega(t)\right| \leq 1 \quad \forall n \geq N_1, t \in [\tau,\tau+T]. \tag{4.59}$$

By the continuity of $\omega(t)$ on the interval $[\tau,\tau+T]$ we get that

$$|\omega(t)| \leq c_1 \quad \forall t \in [\tau,\tau+T], \tag{4.60}$$

for some $c_1 = c_1(\tau,T,\omega) > 0$. Hence, by (4.59) and (4.60) we obtain

$$\left|\int_0^t \mathcal{W}_{\delta_n}(\theta_r\omega)dr\right| \leq 1 + c_1 \quad \forall n \geq N_1, t \in [\tau,\tau+T]. \tag{4.61}$$

By (4.61), we have for all $n \geq N_1$ and $t \in [\tau,\tau+T]$,

$$e^{\int_0^t \mathcal{W}_{\delta_n}(\theta_r\omega)dr}\|B(v_n(t),v_n(t))\|_{V'} \leq c\|v_n(t)\|_V^2,$$

this implies that

$$\left\{e^{\int_0^t \mathcal{W}_{\delta_n}(\theta_r\omega)dr}B(v_n(t),v_n(t))\right\} \text{ is bounded in } L^2(\tau,\tau+T;V'). \tag{4.62}$$

Using the fact that the sequence $\{v_n\}_{n=1}^\infty$ is bounded in $C([\tau,\tau+T];V)$, and (4.62), by similar arguments in [2], from (4.30) we can see that

$$\{\frac{dv_n}{dt}\} \text{ is bounded in } L^2(\tau,\tau+T;V').$$

Hence for any $R > 0$, we have

$$H^1(\mathcal{O}_R) \hookrightarrow\hookrightarrow L^2(\mathcal{O}_R)$$

is compact and

$$\{v_n\} \text{ is bounded in } L^\infty(\tau,\tau+T;H^1(\mathcal{O}_R)),$$
$$\{\frac{dv_n}{dt}\} \text{ is bounded in } L^2(\tau,\tau+T;L^2(\mathcal{O}_R)).$$

This shows that $\{v_n\}$ is relatively compact in $L^2(\tau,\tau+T;L^2(\mathcal{O}_R))$ for all $R > 0$. Then there exist $\bar{\xi} \in V, \tilde{v} \in L^\infty(\tau,\tau+T;V)$ such that (up to a subsequence)

$$v_n(r) \rightharpoonup \bar{\xi} \text{ in } V, \tag{4.63}$$
$$v_n \rightharpoonup \tilde{v} \text{ weakly-star in } L^\infty(\tau,\tau+T;V), \tag{4.64}$$
$$v_n \rightharpoonup \tilde{v} \text{ in } L^2(\tau,\tau+T;V), \tag{4.65}$$
$$v_n \to \tilde{v} \text{ in } L^2(\tau,\tau+T;L^2(\mathcal{O}_R)), \tag{4.66}$$

for all $R > 0$. Let $n \to \infty$, we see that $\tilde{v}$ is the solution of (4.3) with the initial condition $\tilde{v}(\tau) = v_\tau$ and $\tilde{v} = \bar{\xi}$. By the uniqueness of solutions, we have to $\tilde{v} = v$, and thus the sequence $\{v_n\}$ converges to v in the sense of (4.63)-(4.66).

The lemma is proved. □

We next show the uniform compactness of the family of random attractors $\mathcal{A}_\delta$, which help us prove the upper semicontinuity of the attractor as $\delta \to 0$.

Lemma 4.11. *Let $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ satisfy (3.1)-(3.2). Let $\tau \in \mathbb{R}$ and $\omega \in \Omega$ be fixed. If $u_n \in \mathcal{A}_{\delta_n}(\tau, \omega)$, then the sequence $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence in V whenever $\delta_n \to 0$ as $n \to \infty$.*

Proof. For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ we know that $R_{\delta_n}(\tau, \omega) \to R_0(\tau, \omega)$ as $\delta_n \to 0$, where R_0 in Lemma 4.5. Thus, there exists $N_1 = N_1(\tau, \omega) > 0$ such that

$$R_{\delta_n}(\tau, \omega) \leq 2R_0(\tau, \omega) \quad \forall n \geq N_1.$$

From this and by $u_n \in \mathcal{A}_{\delta_n}(\tau, \omega) \subset B_{\delta_n}(\tau, \omega)$, we have

$$\|u_n\|_V^2 \leq 2R_0(\tau, \omega) \quad \forall n \geq N_1.$$

Hence $\{u_n\}$ is bounded in V, thus there exists $u_0 \in V$ such that (up to a subsequence)

$$u_n \rightharpoonup u_0 \text{ weakly in } V \text{ as } n \to \infty. \tag{4.67}$$

Using Lemma 4.10 and repeating the same arguments as in Lemma 4.4 from (4.17) to (4.28), we can conclude that

$$\|u_n\|_V \to \|u_0\|_V,$$

this together with (4.67) shows that

$$u_n \to u_0 \text{ strongly in } V \text{ as } n \to \infty.$$

Therefore we get the conclusion of the lemma. □

Finally, we prove the upper semicontinuity of random attractors for the systems (4.29) via the following theorem.

Theorem 4.2. *Let $f \in L^2_{\text{loc}}(\mathbb{R}; V')$ satisfy (3.1)-(3.2). Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ we have*

$$\lim_{\delta \to 0} \text{dist}_V(\mathcal{A}_\delta(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0.$$

Proof. By Lemma 4.8, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\limsup_{\delta \to 0} \|K_\delta(\tau, \omega)\|_V^2 \leq \limsup_{\delta \to 0} R_\delta(\tau, \omega) = R_0(\tau, \omega). \tag{4.68}$$

We consider a sequence $\delta_n \to 0$ and since $u_{n,\tau} \to u_\tau$ in V, by Lemma 4.9 we obtain for every $t \geq 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi_\delta(t, \tau, \omega, u_{n,\tau}) \to \Phi_0(t, \tau, \omega, u_\tau) \quad \text{ in } V. \tag{4.69}$$

Thus, by (4.68)-(4.69) and Lemma 4.11, we can apply [31, Theorem 3.2] to obtain that

$$\lim_{\delta \to 0} \text{dist}_V(\mathcal{A}_\delta(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0.$$

This completes the proof. □

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