# Discrete nonautonomous systems via rough paths

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**Abstract**

We develop a rough path version for discrete nonautonomous systems generated from a nonautonomous difference equation on a discrete time set (not necessarily regular), where the driving path is free from a realization of a specific stochastic process. Using a modified Davie's approach and the discrete version of sewing lemma, we derive a norm estimate for the solution. When applying to a dissipative system, we prove the existence and the upper semi-continuity of the global pullback attractor.

**Keywords:** stochastic differential equations (SDE), rough path theory, rough integrals, random dynamical systems, random attractors, stochastic perturbation, Euler scheme, numerical random attractors, random attractor approximation.

## 1 Introduction

Our project is motivated from the observations that

- In practice, one often works with discrete data on un-regular time step, for instance financial asset price in real time only shows regular time steps in minute or higher scales, while quite un-regular in second or lower scales.
- The residual noises from statistical estimates in general does not satisfy specific forms of distributions like normality, and often has long range dependence. Hence an attempt to model it as a realization of a given stochastic process only leads to generic results and not applicable to the empirical data itself.

These observations really pose a question on how one can model and study realistic systems from the start under discrete time sets without having to relate to some ideal limiting equation in the continuous time scale. In this work, we try to answer part of the question using rough path theory. Namely, we develop a discrete version for rough paths and try to relate it to the frame work of nonautonomous dynamical systems.

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# 2 Discrete framework

## 2.1 Discrete functions

Since we investigate discrete approximation of solutions of rough differential equations on $[0,T]$, we will have to deal with discrete functions on $[0,T]$: functions defined on a finite collection of points of $[0,T]$. In this subsection we present some basic notions of discrete functions.

Let $[a,b] \subset \mathbb{R}$ be a closed interval of $\mathbb{R}$, and $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be an arbitrary (finite) set of points of $[a,b]$, which actually make a partition of $[a,b]$ into $\cup_{i=0}^{n-1}[t_i,t_{i+1}] = [a,b]$. With an abuse of language we call $\Pi$ a (finite) partition of $[a,b]$. The number $|\Pi| := \max_{0\le i\le n-1}(t_{i+1}-t_i) > 0$ is called *mesh* of the finite partition $\Pi$. *A discrete $\mathcal{B}$-valued function* defined on $\Pi$, where $\mathcal{B}$ is some normed space, is simply a map

$$y : \Pi \to \mathcal{B}, \quad \Pi \ni t_i \mapsto y_{t_i} \in \mathcal{B}.$$

For discrete functions we introduce various norms which are natural discrete versions of the continuous ones. Namely,

$$\|y\|_{\infty,\Pi} := \sup_{t_i\in\Pi} |y_{t_i}|,$$

and

$$|\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi} := \sup_{t_i^*\in\Pi,\ 0\le i\le r,\ t_0^*<t_1^*\ldots<t_r^*} \left(\sum_{i=0}^{r-1} |y_{t_i^*} - y_{t_{i+1}^*}|^p\right)^{1/p},$$

$$\|y\|_{p-\mathrm{var},\Pi} = |y_a| + |\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi}.$$

Clearly $\|\cdot\|_{\infty,\Pi}$ and $\|\cdot\|_{p-\mathrm{var},\Pi}$ are norms on the space of discrete function determined on $\Pi$, whereas $|\!|\!|y_\cdot|\!|\!|_{p-\mathrm{var},\Pi}$ is a semi-norm. Now, let $c \in \Pi$ and $y$ be a discrete function defined on $\Pi$. Put $\Pi[a,c] := \{t \in \Pi : a \le t \le c\}$, $\Pi[c,b] := \{t \in \Pi : c \le t \le b\}$. Then we can consider the natural restriction of $y$ on $[a,c]$ and $[c,b]$ and we have

$$\begin{aligned}
|\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi[\mathrm{a,c}]} + |\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi[\mathrm{c,b}]} &\le 2^{1-1/p}\,|\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi},\\
|\!|\!|y|\!|\!|^p_{p-\mathrm{var},\Pi[\mathrm{a,c}]} + |\!|\!|y|\!|\!|^p_{p-\mathrm{var},\Pi[\mathrm{c,b}]} &\le |\!|\!|y|\!|\!|^p_{p-\mathrm{var},\Pi} \le 2^{p-1}[|\!|\!|y|\!|\!|^p_{p-\mathrm{var},\Pi[\mathrm{a,c}]} + |\!|\!|y|\!|\!|^p_{p-\mathrm{var},\Pi[\mathrm{c,b}]}].
\end{aligned}$$

The notion of a control function also has its discrete counterpart. For, let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a,b]$.

**Definition 2.1** *A non negative function $\omega$ defined on $\Delta\Pi := \{(s,t) \in \Pi^2 | s \le t\}$ is called a* discrete control function *on $\Pi$ if it vanishes on the diagonal, i.e. $\omega(s,s) = 0, \forall s \in \Pi$, and is superadditive, i.e. for all $s \le u \le t$ in $\Pi$*

$$\omega(s,u) + \omega(u,t) \le \omega(s,t).$$

If $\omega$ is a discrete control on $[a,b]$ and $|y_{t_k} - y_{t_l}| \le \omega^{1/p}(t_k,t_l)$ for all $t_k, t_l \in \Pi$, $p \ge 1$ then

$$|\!|\!|y|\!|\!|_{p-\mathrm{var},\Pi} \le \omega^{1/p}(a,b).$$

Furthermore, if $y_\cdot$ is a continuous function of bounded $p-$variation on $[a,b]$, and $\Pi[a,b]$ is a finite partition of $[a,b]$ then the function $y$ restricted on $\Pi[a,b]$ is a discrete function and we have the following relation between continuous and discrete norms of $y$:

$$\begin{aligned}
\|y\|_{\infty,\Pi[a,b]} &\le \|y\|_{\infty,[a,b]},\\
\|y\|_{p-\mathrm{var},\Pi[\mathrm{a,b}]} &\le \|y\|_{p-\mathrm{var},[\mathrm{a,b}]}.
\end{aligned}$$

The notion of discrete function and discrete control function can be easily generalized for the case of arbitrary (not necessarily finite) subset $\Pi \subset [a,b]$.

## 2.2 Discrete sewing lemma

In this section we fix $[a,b]$ and a finite partition $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ on $[a,b]$. The following lemma is the main result of this section. It is actually an algebraic result and provides us with an effective tool for investigation of discretized rough differential equations.

**Lemma 2.2 (Discrete sewing lemma)** *Let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a,b]$ and $F$ be an function defined on $\Pi \times \Pi$, vanished on the diagonal, i.e. $F(s,s) = 0, \forall s \in \Pi$. Put*

$$\begin{aligned} (\delta F)_{sut} &:= F_{s,t} - F_{s,u} - F_{u,t}, \quad s,u,t \in \Pi, \\ I_{k,l} &:= \sum_{k \le j \le l-1} F_{t_j,t_{j+1}} - F_{t_k,t_l}, \quad t_k \le t_l \in \Pi. \end{aligned}$$

*Assume that for a discrete control $\omega$ on $\Pi$ and a number $\lambda > 1$ we have for all $s \le u \le t$ in $\Pi$ the following inequality*

$$|(\delta F)_{sut}| \le \omega^\lambda(s,t). \tag{2.1}$$

*Then there exists a constant $\theta > 0$ depending only on $\lambda$ such that*

$$|I_{k,l}| \le K^* \omega^\lambda(t_k,t_l), \quad \forall\, t_k \le t_l \in \Pi. \tag{2.2}$$

*Proof:* We prove by induction on $l-k$. For $l = k, k+1$ the statement holds trivial. Assume that (2.2) hold for all $k,l$ such that $1 \le l-k < m-k$. We prove that (2.2) is true for $l = m$.

Define

$$l := \max\{k \le j \le m | \omega(t_k,t_j) \le \frac{1}{2}\omega(t_k,t_m)\}.$$

Then $\omega(t_k,t_l) \le \frac{1}{2}\omega(t_k,t_m)$ and $\omega(t_k,t_{l+1}) > \frac{1}{2}\omega(t_k,t_m)$. It follows that $\omega(t_{l+1},t_m) < \frac{1}{2}\omega(t_k,t_m)$ due to super additivity of $\omega$.

Now we have

$$\begin{aligned} I_{k,m} &= I_{k,l} + I_{l,m} - (\delta F)_{t_k,t_l,t_m} \\ &= I_{k,l} + I_{l,l+1} + I_{l+1,m} - (\delta F)_{t_l,t_{l+1},t_m} - (\delta F)_{t_k,t_l,t_m} \\ &= I_{k,l} + I_{l+1,m} - (\delta F)_{t_l,t_{l+1},t_m} - (\delta F)_{t_k,t_l,t_m} \quad \text{since} \quad I_{l,l+1} = 0 \end{aligned}$$

By inductive hypothesis and assumption of $\delta F$ we obtain

$$\begin{aligned} |I_{k,m}| &\le |I_{k,l}| + |I_{l+1,m}| + |(\delta F)_{t_l,t_{l+1},t_m}| + |(\delta F)_{t_k,t_l,t_m}| \\ &\le K^*\omega^\lambda(t_k,t_l) + K^*\omega^\lambda(t_{l+1},t_m) + \omega^\lambda(t_l,t_m) + \omega^\lambda(t_k,t_m) \\ &\le 2^{1-\lambda}K^*\omega^\lambda(t_k,t_m) + 2\omega^\lambda(t_k,t_m) \\ &\le (2^{1-\lambda}\theta + 2)\omega^\lambda(t_k,t_m) \\ &\le \theta\omega^\lambda(t_k,t_m) \end{aligned}$$

if we choose $K^* \ge \frac{2}{1-2^{1-\lambda}}$. This completes the proof. □

**Remark 2.3** 1. Take any discrete control function $\omega$ such that $\omega(s,t) \ne 0$ for $s \ne t$ in $\Pi$, for instance $\omega(s,t) = |t-s|$. Take any $\lambda > 1$. Put

$$C = \begin{cases} \sup_{s \le u \le t} \frac{|(\delta F)_{sut}|^{1/\lambda}}{\omega(s,t)}, & if \quad \omega(s,t) \ne 0 \\ 0, & if \quad \omega(s,t) = 0 \end{cases}$$

then $|(\delta F)_{sut}| \le [C\omega(s,t)]^\lambda$ for all $s \le u \le t$ in $\Pi$. Thus, there are abundant discrete control functions furnishing the condition (2.1) of the discrete sewing lemma.

2. Condition (2.1) is satisfied if

$$|(\delta F)_{sut}| \le \sum_{i=1}^{M} A_i \omega_{1i}^{1/q_i}(s,u)\omega_{2i}^{1/p_i}(u,t) \tag{2.3}$$

for all $s \le u \le t$ in $\Pi$, in which $\omega_{1i}, \omega_{2i}$ are some discrete controls on $\Pi$, and $p_i, q_i \ge 1, A_i \ge 0$ for $i = \overline{1,M}, M \ge 1$ such that $\min_i(\frac{1}{p_i} + \frac{1}{q_i}) > 1$.

**Corollary 2.4** *Let $\Pi = \{t_i : 0 \le i \le n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a,b]$. Let $y_\cdot : \Pi \to \mathbb{R}^{n\times d}$ and $x_{\cdot,\cdot} : \Pi^2 \to \mathbb{R}^d$ be discrete functions defined on $\Pi$ and $\Pi^2$, respectively. Assume that for some $p \ge 1$ and some discrete control function $\omega_x$ on $\Pi$ we have $|x_{s,t}| \le \omega_x^{1/p}(s,t)$ for all $s \le t \in \Pi$. Then for $q$ such that $1/p + 1/q > 1$ we have*

$$\Big|\sum_{j=0}^{n-1} y_{t_j} x_{t_j,t_{j+1}} - y_a x_{a,b}\Big| \le K^* \omega_y^{1/q}(a,b)\omega_x^{1/p}(a,b),$$

*where $\omega_y^{1/p}(s,t) := \|y\|^q_{q-\text{var},\Pi}$, $\theta = \frac{2}{1-2^{1-1/p-1/q}}$.*

*Proof:* Apply Lemma 2.2 with $\omega(s,t) = \omega_y^{1/q}(s,t).\omega_x^{1/p}(a,b)$ and $\lambda = 1/p + 1/q$. □

**Corollary 2.5** *Let $\Pi^j = \{t_k^j, 0 \le k \le n_j\}$, $j = 1, 2, \ldots$, be a sequence of finite partitions of $[a,b]$ satisfying $|\Pi^j| \to 0$ as $j \to \infty$. Let $F$ be a discrete function defined on $(\cup\Pi^j)^2$. Assume that there exists a discrete control function $\omega$ on $(\cup\Pi^j)$ such that the following conditions hold*
*(i) for all $s \le u \le t$ in $\cup\Pi^j$, $|F_{s,t} - F_{s,u} - F_{u,t}| \le \omega^\lambda(s,t)$,*
*(ii) $\sup_{0\le k\le n_j} \omega(t_k^j, t_{k+1}^j) \to 0$ as $j \to \infty$.*
*Then the sequence of real numbers*

$$S_{\Pi^j} := \sum_{0\le k\le n_j-1} F_{t_k^j, t_{k+1}^j}$$

*converges.*

*Proof:* Let $1 \le m \le l$ be arbitrary. First we consider the case $\Pi^m \subset \Pi^l$. Using Lemma 2.2, we have

$$\begin{aligned}
|S_{\Pi^l} - S_{\Pi^m}| &= |\sum_{0\le k\le n_l-1} F_{t_k^l,t_{k+1}^l} - \sum_{0\le k\le n_m-1} F_{t_k^m,t_{k+1}^m}| \\
&= \left|\sum_{0\le k\le n_m-1}\Big(\sum_{t_k^m\le t_i^l\le t_{k+1}^m} F_{t_i^l,t_{i+1}^l} - F_{t_k^m,t_{k+1}^m}\Big)\right| \\
&\le \sum_{0\le k\le n_m-1}\Big|\sum_{t_k^m\le t_i^l\le t_{k+1}^m} F_{t_i^l,t_{i+1}^l} - F_{t_k^m,t_{k+1}^m}\Big| \\
&\le K^* \sum_{0\le k\le n_m-1} \omega^\lambda(t_k^m, t_{k+1}^m) \\
&\le \theta \sup_{0\le k\le n_m} \omega^{\lambda-1}(t_k^m, t_{k+1-1}^m) \sum_{0\le k\le n_m-1} \omega(t_k^m, t_{k+1}^m) \\
&\le \theta \sup_{0\le k\le n_m} \omega^{\lambda-1}(t_k^m, t_{k+1}^m)\omega(a,b).
\end{aligned}$$

For the case $\Pi_m \not\subset \Pi_l$ we put $\hat{\Pi} = \Pi^m \cup \Pi^l$, then by the preceding result we have

$$\begin{aligned} |S_{\hat{\Pi}} - S_{\Pi^m}| &\leq \theta \sup_{0\leq k\leq n_m} \omega^{\lambda-1}(t_k^m, t_{k+1}^m)\omega(a,b), \\ |S_{\hat{\Pi}} - S_{\Pi^l}| &\leq \theta \sup_{0\leq k\leq n_l} \omega^{\lambda-1}(t_k^l, t_{k+1}^l)\omega(a,b). \end{aligned}$$

This implies

$$\begin{aligned} |S_{\Pi_l} - S_{\Pi^m}| &\leq |S_{\hat{\Pi}} - S_{\Pi^m}| + |S_{\hat{\Pi}} - S_{\Pi^l}| \\ &\leq K^* \left( \sup_{0\leq k\leq n_m} \omega^{\lambda-1}(t_k^m, t_{k+1}^m) + \sup_{0\leq k\leq n_l} \omega^{\lambda-1}(t_k^l, t_{k+1}^l) \right) \omega(a,b). \end{aligned}$$

Therefore, by assumption, $|S_{\Pi^l} - S_{\Pi^m}| \to 0$ as $m, l \to \infty$, hence the sequence $S_{\Pi^j}$, $j = 1, 2, \ldots$, is a Cauchy sequence. We conclude that the sequence $S_{\Pi^j}$ converges. □

Now we use the discrete sewing lemma above to derive estimate solutions of a discrete system of equations.

**Lemma 2.6** *Let $\Pi = \{t_i : 0 \leq i \leq n, a = t_0 < t_1 < \ldots < t_n = b\}$ be a finite partition of $[a, b]$. Consider a discrete system defined on $\Pi$*

$$y_{t_{j+1}} = y_{t_j} + F_{t_j,t_{j+1}} + \epsilon_{t_j,t_{j+1}}, \quad y_{t_0} = y^* \in \mathbb{R}^d, \quad j = 0, 1, \ldots, n-1, \tag{2.4}$$

*where $F : \Pi^2 \to \mathbb{R}^d$, $\epsilon : \Pi^2 \to \mathbb{R}^d$. Assume that*
*(i) There exists a discrete control function $\omega$ such that (2.1) is satisfied for $F$;*
*(ii) There exists a discrete control function $\omega_F$ such that for some $q \geq 1$ the following inequality holds*

$$|F_{s,t}| \leq \omega_F^{1/q}(s,t), \quad \forall s \leq t \in \Pi; \tag{2.5}$$

*(iii) There exists a discrete control function $\omega_0$ such that $|\epsilon_{t_j,t_{j+1}}| \leq \omega_0(t_j, t_{j+1})$.*
*Then we have*

$$\|y\|_{q-\text{var},\Pi} \leq K^*\omega^{\lambda}(a,b) + \omega_F^{1/q}(a,b) + \omega_0(a,b). \tag{2.6}$$

*Proof:* For any pair $t_k < t_l \in \Pi$ we have

$$\begin{aligned} |y_{t_l} - y_{t_k}| &= |\sum_{j=k}^{l-1} (y_{t_{j+1}} - y_{t_j})| \\ &= |\sum_{j=k}^{l-1} \left(F_{t_j,t_{j+1}} + \epsilon_{t_j,t_{j+1}}\right)| \\ &\leq |\sum_{j=k}^{l-1} F_{t_j,t_{j+1}}| + |\sum_{j=k}^{l-1} \epsilon_{t_j,t_{j+1}}| \\ &\leq |I_{k,l}| + |F_{t_k,t_l}| + \omega_0(t_k,t_l) \\ &\leq K^*\omega^{\lambda}(t_k,t_l) + |F_{t_k,t_l}| + \omega_0(t_k,t_l). \end{aligned}$$

Therefore, by (2.5), we get

$$|y_{t_l} - y_{t_k}| \leq K^*\omega^{\lambda}(t_k,t_l) + \omega_F^{1/q}(t_k,t_l) + \omega_0(t_k,t_l).$$

Consequently,

$$\|y\|_{q-\text{var},\Pi} \leq K^*\omega^{\lambda}(a,b) + \omega_F^{1/q}(a,b) + \omega_0(a,b).$$

□

### 2.3 Discrete greedy sequence

The original idea of a greedy sequence was introduced in [4, Definition 4.7]. Given $\alpha > 0$, a compact interval $I \in \mathbb{R}$ and a control $\overline{\omega} : \Delta(I) \to \mathbb{R}^+$, the construction of such a sequence aims to have a "greedy" approximation to the supremum in the definition of the so-called *accummulated $\alpha$-local $\overline{\omega}$-variation* (see [4, Definition 4.1])

$$M_{\alpha,I}(\overline{\omega}) = \sup_{\Pi(I),\overline{\omega}_{t_i,t_{i+1}} \leq \alpha} \sum_{t_i \in \Pi(I)} \overline{\omega}_{t_i,t_{i+1}}.$$

In particular, $\overline{\omega}_{s,t}$ is chosen to be $\left\|\cdot\right\|^p_{p\text{-var},[s,t]}$ in [4].
A similar version for stopping times was developed before in [17] and then has been studied further recently by [10] for stability of the system. A more detailed presentation of another version of greedy sequence of times which matches with the nonautonomous setting is given in [5].

The notion of greedy sequence of stopping times gives us a tool to estimate the growth of a function $x(t)$ in the arbitrary time intervals $(s,t)$ of the determination domain of $x(t)$ as it allows us to choose various "control" function $\overline{\omega}_{s,t}$ based on $x(t)$. Thus this notion gives us a tool to indirectly understand the growth of the solution $y(t)$ of a rough differential equation driven by a path $x(t)$ on each time intervals $(s,t)$ of the time interval of determination of the rough differential equation under consideration.

We notice that an abstract definition of a greedy sequence of stopping times need not be born by a rough differential equation and not only for the probabilistic setting (hence the stopping time here is not necessarily stopping time in the probabilistic setting although in the stochastic setting they are actually stopping times in probabilistic sense): all we need is just a function $\overline{\omega}_{s,t}$ which must not always be a control function. Let us gives here our rigorous abstract definition of greedy sequence of stopping time.

First we deal with the continuous time case. Let $[a,b] \subset \mathbb{R}$ be an arbitrary time interval, and $\Delta_{[a,b]} := \{(s,t) : a \leq s \leq t \leq b\} \subset [a,b] \times [a,b]$. Let $\omega : \Delta_{[a,b]} \to [0,\infty)$ be a continuous function of two variables such that

$$\begin{array}{lll} (i) & \text{(zero on diagonal)} \quad \omega(t,t) = 0 \quad \text{for all} \quad t \in [a,b] \quad \text{and} \\ (ii) & \text{(monotonity)} \quad \omega(s,u) \leq \omega(s,t) \quad \text{for all} \quad a \leq s \leq u \leq t \leq b. \end{array} \tag{2.7}$$

For any given parameter $\mu > 0$, based on $\omega$ and $\mu$ we construct a nondecreasing sequence of times $G_{[a,b],\omega,\mu} = \{\tau_0, \tau_1, \tau_2, \tau_3, \dots\}$ such that $\tau_0 \equiv a$ and

$$\tau_{i+1} := \inf\{t \in [\tau_i, b] : \omega(\tau_i, t) \geq \mu\} \wedge b. \tag{2.8}$$

The sequence $G_{[a,b],\omega,\mu}$ is called the *greedy sequence of stopping times on* $[a,b]$ (constructed from $\omega$ and $\mu$). Notice that once $\tau_i$ is determined then if $\omega(\tau_i, b) < \mu$ then $\tau_{i+1} = b$, and if $\omega(\tau_i, b) > \mu$ then $\tau_{i+1} \in (\tau_i, b)$ is determined uniquely since the function $\omega(\tau_i, \cdot)$ is continuous, nondecreasing and $\omega(\tau_i, \tau_i) = 0$, $\omega(\tau_i, b) > \mu$. Note that in case $\tau_i < b$ and $\omega(\tau_i, b) \geq \mu$, we see that $\tau_{i+1}$ is intuitively the first time $\omega(\tau_i, \cdot)$ reaches $\mu$. Clearly any control function (see [16, Definition 1.6]) satisfies the condition (2.7) above, hence we can construct a greedy sequence based on it. Further, any positive power of a control function, which is not necessarily a control, also satisfies (2.7).

Now let $x$ be a given continuous-time function of finite $p$-variation on $[a,b]$. Consider the function $\omega(s,t) := \left\|x\right\|_{p\text{-var},[s,t]}$. It is easily seen that this function $\omega$ satisfies the condition (2.7) above, hence for any parameter $\mu > 0$ we can construct the greedy sequence $G_{[a,b],\omega,\mu} = \{\tau_0, \tau_1, \tau_2, \tau_3, \dots\}$ of stopping times. We have $\tau_0 \equiv a$, and by definition of $\omega$

$$\tau_{i+1} = \inf\{t \in [\tau_i, b] : \left\|x\right\|_{p-\text{var},[\tau_i,t]} \geq \mu\} \wedge b. \tag{2.9}$$

Assign

$$N = N_{[a,b],\mu}(x) := \sup\{n : \tau_n < b\} + 1. \tag{2.10}$$

Since the function $\|x\|_{p\text{-var},[t_0,t]}$ is continuous and nondecreasing w.r.t. $t$ with $\kappa(t_0) = 0$ (see [16]) we obtain

$$\|x\|_{p-\text{var},[\tau_i,\tau_{i+1}]} = \mu, \quad i = 0, \cdots, N-2. \tag{2.11}$$

Then the following estimate holds

$$N_{[a,b],\mu}(x) \leq 1 + \frac{1}{\mu^p} \|x\|^p_{p\text{-var},[\text{a,b}]}. \tag{2.12}$$

Now we turn to the discrete framework. Let $[a, b] \subset \mathbb{R}$ be a closed interval of $\mathbb{R}$, and $\Pi = \{t_i : 0 \leq i \leq n, a = t_0 < t_1 < \ldots < t_n = b\}$ be an arbitrary finite partition of $[a, b]$. Let $\omega$ be a discrete control function on $\Pi$ and $\beta > 0$ be arbitrary. Suppose we are given a fixed $\delta > 0$. We define a *discrete sequence of greedy times* $G_{\Pi,\beta,\delta} := \{\tau_0^*, \tau_1^*, \ldots\}$ based on the discrete control $\omega$, and the parameters $\beta, \delta$ and taking values in $\Pi$ as follows:
1. Set $\tau_0^* = t_0 = a$,
2. Suppose that the point $\tau_i^* \in G_{\Pi,\beta,\delta}$ is already constructed, thus $\tau_i^* = t_k \in \Pi$. If $\tau_i^* = t_m = b$ then we stop and the sequence $G_{\Pi,\beta,\delta}$ has been constructed. If $\tau_i^* < t_m = b$, hence $k < m$, then we define $\tau_{i+1}^* \in G_{\Pi,\beta,\delta}$ as follows
(i) if $\omega^\beta(t_k, t_{k+1}) > \delta$ then we set $\tau_{i+1}^* = t_{k+1}$;
(ii) otherwise we set $\tau_{i+1}^* = \max\{t_l \in \Pi : k < l \leq m, \omega^\beta(t_k, t_l) \leq \delta\}$.
Note that the discrete sequence $G_{\Pi,\beta,\delta}$ is finite and it can be constructed for any partition of any time interval $[a, b]$.

Put $N^* = N^*_{\Pi[a,b],\beta,\delta} := \#G_{\Pi,\beta,\delta}$ the number of points $t_i^*$ of $G_{\Pi,\beta,\delta}$ in $[a, b]$. From the definition of the sequence $G_{\Pi,\beta,\delta}$ it follows that for any $0 \leq i \leq N^* - 2$ we have $\omega^\beta(\tau_i^*, \tau_{i+2}^*) > \delta$. Therefore, since $\omega$ is a control function we have

$$N^* - 2 \leq \sum_{i=0}^{N^*-3} \frac{1}{\delta^{1/\beta}} \omega(\tau_i^*, \tau_{i+2}^*) \leq \frac{1}{\delta^{1/\beta}} \big(\omega(\tau_0^*, \tau_{N^*}^*) + \omega(\tau_1^*, \tau_{N^*}^*)\big) \leq \frac{2}{\delta^{1/\beta}} \omega(a, b).$$

Hence,

$$N^*_{\Pi[a,b],\beta,\delta} \leq 2 + \frac{2}{(\delta)^{1/\beta}} \omega(a, b). \tag{2.13}$$

# 3 Discrete Young systems

In this section we consider a fixed partition $\Pi := \{a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b\}$ with $|\Pi| := \sup_k |t_{k+1} - t_k|$. Let $x : \Pi \to \mathbb{R}$ be a discrete function defined on $\Pi$. We consider the discrete system on $\Pi$ driven by $x$, defined by

$$\begin{aligned} & y_{t_0} \in \mathbb{R}^d, \\ & y_{t_{k+1}} = y_{t_k} + f(y_{t_k})(t_{k+1} - t_k) + g(y_{t_k})[x(t_{k+1}) - x(t_k)] + \varepsilon^*_{t_k,t_{k+1}}, \quad k = 0, 1, \ldots, n-1, \end{aligned} \tag{3.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function of linear growth, i.e. there exists a constant $C_f > 0$ such that

$$|f(y)| \leq C_f|y| + |f(0)|, \quad \forall y \in \mathbb{R}^d,$$

where $f(0)$ is the value of $f$ evaluated at the vector $0 \in \mathbb{R}^d$; $g : \mathbb{R}^d \to \mathbb{R}^d$ is Lipchitz continuous functions with Lipschitz constant $C_g$; $\varepsilon^* : \Pi^2 \to \mathbb{R}^d$ is a noise which is assumed to satisfied the condition

$$|\varepsilon^*_{t_k,t_{k+1}}| \le H^*\omega^*(t_k,t_{k+1})$$

with $H^*$ being a positive constant and $\omega^*$ being a discrete control on $\Delta\Pi$. This system is a discrete form of the (contnuous-time) Young differential equation

$$dy = f(y(t))dt + g(y(t))dx(t),\ t \in [a,b], \tag{3.2}$$

where $x : [a,b] \to \mathbb{R}$ is a continuous function defined on $[a,b]$ and not necessary of bounded variation.

Clearly, for any initial value $y_{t_0}$, the discrete system (3.1) has unique solution. We are interested in the relation between the norm of the solution (output) $y$ and the norm of the driver $x$. The main result of this Section is Theorem 3.4 where we estimate the discrete norm of $y$ in term of $p$-variation norm of $x$ and other input factors. As an application, the estimation for the Euler scheme of (3.2) is derived at the end of the Section.

Let $q \ge p \ge 1$ be arbitrary numbers satisfying $\lambda := 1/p + 1/q > 1$. From now on we fix these constants $p, q, \lambda$ and the constants

$$K^* := \frac{2}{1-2^{1-1/p-1/q}}, \quad \delta_1 := \frac{1}{2C_g(1+K^*)}, \quad \delta_2 := \frac{1}{4C_f}.$$

Firstly, we have the following Lemma as a direct consequence of Lemma 2.6.

**Lemma 3.1** *The solution $y$ of the discrete system* (3.1) *satisfies*

$$\begin{aligned}\left\vert\!\left\vert\!\left\vert y\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},\Pi} \le &(C_g|y_a| + g_0)\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi} + [C_f\|y\|_{\infty,\Pi} + f_0](b-a)\\ &+ C_g(K^*+1)\left\vert\!\left\vert\!\left\vert y\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},\Pi}\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi} + H^*\omega^*(a,b),\end{aligned} \tag{3.3}$$

*where $f_0 = |f(0)|$, $g_0 = |g(0)|$ with $f(0), g(0)$ being the values of $f, g$ evaluated at the vector $0 \in \mathbb{R}^d$.*

*Proof:* To apply Lemma 2.6, we rewrite (3.1) in the form of (2.4)

$$y_{t_{k+1}} = y_{t_k} + F_{t_k,t_{k+1}} + \varepsilon_{t_k,t_{k+1}}, \quad k = 0,1,\dots,n-1,$$

where

$$\begin{aligned}F_{s,t} &:= g(y_s)(x(t)-x(s)), \quad s,t \in \Pi,\\ \epsilon_{t_j,t_{j+1}} &:= f(y_{t_j})(t_{j+1}-t_j) + \epsilon^*_{t_j,t_{j+1}}.\end{aligned}$$

Put

$$\begin{aligned}\omega(s,t) &:= C_g^{1/\lambda}(\left\vert\!\left\vert\!\left\vert y\right\vert\!\right\vert\!\right\vert^q_{q-\mathrm{var},\Pi[\mathrm{s,t}]})^{p/(p+q)}(\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert^p_{p-\mathrm{var},\Pi[\mathrm{s,t}]})^{q/(p+q)};\\ \omega_F^{1/q}(s,t) &:= (g_0 + C_g\|y\|_{q-\mathrm{var},\Pi})\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[\mathrm{s,t}]} \ge |F_{a,t}-F_{a,s}| + C_g\left\vert\!\left\vert\!\left\vert y\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi}\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[\mathrm{s,t}]};\\ \omega_0(s,t) &:= [C_f\|y\|_{\infty,\Pi} + f_0](t-s) + H^*\omega^*(s,t),\ (s,t) \in \Delta\Pi.\end{aligned}$$

Then, one can easily prove that $\omega$, $\omega_F$ and $\omega_0$ are discrete control functions. Now, we verify the assumptions of Lemma 2.6:

$$
\begin{aligned}
|(\delta F)_{sut}| &= |(g(y_u)-g(y_s))(x_t-x_u)| \\
&\le C_g \left\|y\right\|_{q-\mathrm{var},\Pi[\mathrm{s,t}]} \left\|x\right\|_{p-\mathrm{var},\Pi[\mathrm{s,t}]} \\
&= C_g[(\left\|y\right\|^q_{q-\mathrm{var},\Pi[\mathrm{s,t}]})^{p/(p+q)}(\left\|x\right\|^p_{p-\mathrm{var},\Pi[\mathrm{s,t}]})^{q/(p+q)}]^\lambda \\
&= \omega^\lambda(s,t); \\
|F_{s,t}| &\le |F_{a,t}-F_{a,s}| + |(g(y_a)-g(y_s))(x_t-x_s)| \\
&\le |F_{a,t}-F_{a,s}| + C_g \left\|y\right\|_{p-\mathrm{var},\Pi} |(x_t-x_s)| \\
&\le \omega_F^{1/p}(s,t); \\
|\epsilon_{t_j,t_{j+1}}| &\le \omega_0(t_j,t_{j+1}).
\end{aligned}
$$

Thus, Lemma 2.6 is applicable and gives

$$
\begin{aligned}
\left\|y\right\|_{q-\mathrm{var},\Pi} &\le K^*\omega^\lambda(a,b) + \omega_F^{1/q}(a,b) + \omega_0(a,b) \\
&= K^*C_g \left\|y\right\|_{q-\mathrm{var},\Pi[\mathrm{a,b}]} \left\|x\right\|_{p-\mathrm{var},\Pi[\mathrm{a,b}]} + (g_0 + C_g\|y\|_{q-\mathrm{var},\Pi}) \left\|x\right\|_{p-\mathrm{var},\Pi[\mathrm{a,b}]} \\
&\quad +[C_f\|y\|_{\infty,\Pi} + f_0](b-a) + H^*\omega^*(a,b) \\
&= (C_g|y_a| + g_0) \left\|x\right\|_{p-\mathrm{var},\Pi} + [C_f\|y\|_{\infty,\Pi} + f_0](b-a) \\
&\quad +(K^*+1)C_g \left\|y\right\|_{q-\mathrm{var},\Pi[\mathrm{a,b}]} \left\|x\right\|_{p-\mathrm{var},\Pi[\mathrm{a,b}]} + H^*\omega^*(a,b).
\end{aligned}
$$

The proof is completed □

Now we aim to get estimates for the norms of the discrete function $y$ on $\Pi$. For this, first we need to make estimate the norms of $y$ on appropriate small parts of $\Pi$ determined by constructions of discrete greedy sequences, and then using the properties of greedy sequences to derive estimate for norms of the function $y$ on $\Pi$. To do this, we construct two greedy sequences as follows. Choose two sets of parameters to construct two discrete greedy sequences:

1. $\omega_1(s,t) = \left\|x\right\|^p_{p-\mathrm{var},[\mathrm{s,t}]}$, $\beta_1 = 1/p$, $\delta_1$, and construct the first sequence $G_{\Pi,\beta_1,\delta_1} = \{t_i^*\} \subset \Pi$.
2. $\omega_2(s,t) = |t-s|$, $\beta_2 = 1$, $\delta_2$, and construct the second sequence $G_{\Pi,\beta_2,\delta_2} = \{t_i^{**}\} \subset \Pi$.

Denote by $\hat{G} := \{\hat{\tau}_i : \hat{\tau}_0 < \hat{\tau}_1 < \ldots < \hat{\tau}_{\hat{N}-1}\}$ the ordered increasingly combined sequence $\{t_i^*\} \cup \{t_i^{**}\}$. Then, by (2.13), the number $\hat{N} := \#\hat{G}$ of points $\hat{\tau}_i$ of $\hat{G}$ satisfies

$$
\hat{N} \le 4 + \frac{2}{\delta_2}(b-a) + \frac{2}{\delta_1^p} \left\|x\right\|_{p-\mathrm{var},[\mathrm{a,b}]}. \tag{3.4}
$$

Furthermore, for an arbitrary pair of two consecutive points $[\hat{\tau}_i, \hat{\tau}_{i+1}]$ of the combined sequence $\hat{G}$, by the construction we have two alternative cases

$$
1. \qquad (\hat{\tau}_i, \hat{\tau}_{i+1}) \cap \Pi = \emptyset, \quad \text{or} \tag{3.5}
$$

$$
2. \qquad (\hat{\tau}_i, \hat{\tau}_{i+1}) \cap \Pi \neq \emptyset, \quad \text{in this case} \quad \begin{cases} |\hat{\tau}_{i+1} - \hat{\tau}_i| \le \delta_2, \\ \left\|x\right\|^p_{p-\mathrm{var},[\hat{\tau}_i,\hat{\tau}_{i+1}]} \le \delta_1 \end{cases} \tag{3.6}
$$

Note that if $\Pi = \{kh : 0 \le k \le n\}$ then the number $\#G_{\Pi,\beta_2,\delta_2}$ of point $t_i^{**}$ of $G_{\Pi,\beta_2,\delta_2}$ is equal to the number of points of continuous greedy sequence and is equal to $1 + \frac{b-a}{\delta_2}$. In this case we have a better estimate for $\hat{N}$ as

$$
\hat{N} \le 3 + \frac{1}{\delta_2}(b-a) + \frac{2}{\delta_1^p} \left\|x\right\|^p_{p-\mathrm{var},[\mathrm{a,b}]}. \tag{3.7}
$$

Next, in two lemmas below, we evaluate $y$ on each small portion of $\Pi$, which are in fact determined by consecutive points of the combined sequence $\hat{G}$. For $\hat{a}, \hat{b} \in \Pi, \hat{a} < \hat{b}$, we denote $\hat{\Pi} = \Pi[\hat{a}, \hat{b}] := \{t \in \Pi | \hat{a} \leq t \leq \hat{b}\}$.

**Lemma 3.2** *Let $\hat{a}, \hat{b} \in \Pi$ be such that $\|x\|_{p-\mathrm{var},\Pi[\hat{a},\hat{b}]} \leq \delta_1, |\hat{b} - \hat{a}| \leq \delta_2$. Then for the solution $y$ of the discrete system* (3.1) *the following inequalities hold*

$$\|y\|_{\infty,\Pi[\hat{a},\hat{b}]} \leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big]e^{\kappa} - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})] \tag{3.8}$$

*and*

$$\|y\|_{q-\mathrm{var},\Pi[\hat{a},\hat{b}]} \leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big] e^{\kappa} - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})], \tag{3.9}$$

*where* $\kappa = \ln(2 + \frac{2}{1+K^*})$, $M = \frac{1}{C_f} + \frac{2}{C_g(1+K^*)}$.

*Proof:* Apply Lemma 3.1 with $\Pi[a, b]$ replaced by $\Pi[\hat{a}, \hat{b}]$. Since by assumption $\|x\|_{p-\mathrm{var},\Pi[\hat{a},\hat{b}]} \leq \delta_1 = \frac{1}{2C_g(1+K^*)}$ and $|\hat{b} - \hat{a}| \leq \delta_2 = \frac{1}{4C_f}$ we have

$$\begin{aligned}\|y\|_{q-\mathrm{var},\hat{\Pi}} \leq &(C_g|y_{\hat{a}}| + g_0) \|x\|_{p-\mathrm{var},\hat{\Pi}} + [C_f\|y\|_{\infty,\hat{\Pi}} + f_0](\hat{b} - \hat{a}) \\ &+ C_g(K^* + 1) \|y\|_{q-\mathrm{var},\hat{\Pi}} \|x\|_{p-\mathrm{var},\hat{\Pi}} + H^*\omega^*(\hat{a}, \hat{b}) \\ \leq &(C_g|y_{\hat{a}}| + g_0)\frac{1}{2C_g(K^* + 1)} + [C_f\|y\|_{\infty,\hat{\Pi}} + f_0]\frac{1}{4C_f} + \frac{1}{2} \|y\|_{p-\mathrm{var},\hat{\Pi}} + H^*\omega^*(\hat{a}, \hat{b}),\end{aligned}$$

which implies

$$\|y\|_{q-\mathrm{var},\hat{\Pi}} \leq (C_g|y_{\hat{a}}| + g_0)\frac{1}{C_g(K^* + 1)} + [C_f\|y\|_{\infty,\hat{\Pi}} + f_0]\frac{1}{2C_f} + 2H^*\omega^*(\hat{a}, \hat{b}). \tag{3.10}$$

From this it follows that

$$\sup_{a \leq t_i \leq b} |y_{t_j} - y_{\hat{a}}| \leq \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{1}{2}\|y\|_{\infty,\hat{\Pi}} + \frac{f_0}{2C_F} + \frac{g_0}{(1 + K^*)C_g} + 2H^*\omega^*(a, b).$$

Consequently, we have

$$\frac{1}{2}\|y\|_{\infty,\hat{\Pi}} - |y_{\hat{a}}| \leq \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{f_0}{2C_F} + \frac{g_0}{(1 + K^*)C_g} + 2H^*\omega^*(\hat{a}, \hat{b}).$$

Therefore,

$$\begin{aligned}\|y\|_{\infty,\hat{\Pi}} &\leq (2 + \frac{2}{1 + K^*})|y_{\hat{a}}| + \frac{f_0}{C_F} + \frac{2g_0}{(1 + K^*)C_g} + 4H^*\omega^*(\hat{a}, \hat{b}) \\ &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big]e^{\kappa} - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})].\end{aligned}$$

Thus, (3.8) is proved. Combining this with inequality (3.10) we obtain

$$\begin{aligned}\|y\|_{q-\mathrm{var},\hat{\Pi}} &\leq \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{1}{2}\|y\|_{\infty,\hat{\Pi}} + \frac{f_0}{2C_F} + \frac{g_0}{(1 + K^*)C_g} + H^*2\omega^*(\hat{a}, \hat{b}) \\ &= \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{1}{2}\left(\|y\|_{\infty,\hat{\Pi}} + \frac{f_0}{C_F} + \frac{2g_0}{(1 + K^*)C_g} + 4H^*\omega^*(\hat{a}, \hat{b})\right) \\ &\leq \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{1}{2}\Big[\|y\|_{\infty,\hat{\Pi}} + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big] \\ &\leq \frac{1}{1 + K^*}|y_{\hat{a}}| + \frac{1}{2}\Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big]e^{\kappa} \\ &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})\Big] e^{\kappa} - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a}, \hat{b})] - |y_{\hat{a}}|.\end{aligned}$$

This implies (3.9) immediately. The proof is complete. □

**Lemma 3.3** *Let $\hat{a}, \hat{b}$ be two consecutive points in $\Pi$ such that $|\hat{b}-\hat{a}| \leq \delta_2$. Then for the solution $y$ of the discrete system* (3.1) *the following inequalities hold*

$$\begin{aligned} \|y\|_{\infty,\Pi[\hat{a},\hat{b}]} &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a},\hat{b})\Big] e^{\kappa}[1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}] \\ &\qquad -[M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a},\hat{b})], \end{aligned} \tag{3.11}$$

*and*

$$\begin{aligned} \|y\|_{q-\text{var},\Pi[\hat{\text{a}},\hat{\text{b}}]} &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a},\hat{b})\Big] e^{\kappa}[1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}] \\ &\qquad -[M(f_0 \vee g_0) + 4H^*\omega^*(\hat{a},\hat{b})]. \end{aligned} \tag{3.12}$$

*Proof:* By definition of the system (3.1) we have

$$\begin{aligned} |y_{\hat{b}}| &\leq |y_{\hat{a}} + f(y_{\hat{a}})(\hat{b}-\hat{a}) + g(y_{\hat{a}})(x(\hat{b}) - x(\hat{a})) + \varepsilon^*_{\hat{a},\hat{b}}| \\ &\leq |y_{\hat{a}}| + C_f(\hat{b}-\hat{a})|y_{\hat{a}}| + f_0(\hat{b}-\hat{a}) + C_g|y_{\hat{a}}|\,|x(\hat{b}) - x(\hat{a})| + g_0|x(\hat{b}) - x(\hat{a})| + H^*\omega^*(\hat{a},\hat{b}). \end{aligned}$$

We now estimate two terms in this inequality. Notice that, by virtue of the Young inequality, for all $0 \leq \alpha \leq 1, \beta \geq 0$ and $p \geq 1$ we have $\alpha\beta < \alpha + \beta^p$. Therefore, we have

$$C_g|x(\hat{b}) - x(\hat{a})| = \frac{1}{(1+K^*)} \cdot \Big(C_g(1+K^*)\left\|x\right\|_{p-\text{var},\hat{\Pi}}\Big) \leq \frac{1}{(1+K^*)} + [C_g(1+K^*)\left\|x\right\|_{p-\text{var},\hat{\Pi}}]^p$$

and

$$g_0|x(\hat{b}) - x(\hat{a})| = \frac{g_0}{C_g(1+K^*)}[C_g(1+K^*)\left\|x\right\|_{p-\text{var},\hat{\Pi}}] \leq \frac{g_0}{C_g(1+K^*)}\Big[1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big].$$

As a result we get

$$\begin{aligned} |y_{\hat{b}}| &\leq |y_{\hat{a}}| + \frac{1}{2}|y_{\hat{a}}| + \frac{f_0}{2C_F} + H^*\omega^*(\hat{a},\hat{b}) + |y_{\hat{a}}|\left[\frac{1}{1+K^*} + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\right] \\ &\quad + \frac{g_0}{(1+K^*)C_g}\Big[1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big]. \end{aligned}$$

Consequently,

$$\begin{aligned} |y_{\hat{b}}| &\leq e^{\kappa}|y_{\hat{a}}|\Big(1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big) + M(f_0 \vee g_0)\Big(1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big) \\ &\quad + H^*\omega^*(\hat{a},\hat{b}) \\ &\leq [|y_{\hat{a}}| + M(f_0 \vee g_0)]\, e^{\kappa}\Big(1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big) \\ &\quad - M(f_0 \vee g_0)\Big(1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big) + H^*\omega^*(\hat{a},\hat{b}) \\ &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})\Big] e^{\kappa}\Big[1 + (1+K^*)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}\Big] \\ &\qquad - [M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})]. \end{aligned} \tag{3.13}$$

Then we get

$$\begin{aligned} \|y\|_{\infty,\hat{\Pi}} &= \max\{|y_{\hat{a}}|, |y_{\hat{b}}|\} \\ &\leq \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})\Big] e^{\kappa}[1 + (1+K)^p C_g^p \left\|x\right\|^p_{p-\text{var},\hat{\Pi}}] \\ &\quad - [M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})]. \end{aligned}$$

Now, since $\hat{a}$ and $\hat{b}$ are two consecutive points of $\Pi$ we have

$$||y||_{q-\mathrm{var},\hat{\Pi}} = |y_{\hat{a}}| + |||y|||_{q-\mathrm{var},\hat{\Pi}} = |y_{\hat{a}}| + |y_{\hat{b}} - y_{\hat{a}}|.$$

An inspection of the estimation of $|y_{\hat{b}}|$ in (3.13) above shows immediately that

$$\begin{aligned} ||y||_{q-\mathrm{var},\hat{\Pi}} \quad \leq \quad & \Big[|y_{\hat{a}}| + M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})\Big] e^{\kappa} \Big[1 + (1+K^*)^p C_g^p \, |||x|||^p_{p-\mathrm{var},\hat{\Pi}}\Big] \\ & - [M(f_0 \vee g_0) + H^*\omega^*(\hat{a},\hat{b})]. \end{aligned}$$

The proof is completed. □

Now we state the main result of this Section.

**Theorem 3.4** *If $|\Pi| \leq \delta_2$, there exists a positive constant $D$ depending on $f_0, g_0, \kappa$ such that*

$$\|y\|_{\infty,\Pi[a,b]} \leq [|y_a| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{8\kappa C_f(b-a) + D[1+(1+K^*)^p C_g^p |||x|||^p_{p-\mathrm{var},\Pi[\mathrm{a,b}]}]} \tag{3.14}$$

*and*

$$\|y\|_{q-\mathrm{var},\Pi[\mathrm{a,b}]} \quad \leq \quad [|y_a| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{8\kappa C_f(b-a) + D[1+(1+K^*)^p C_g^p |||x|||^p_{p-\mathrm{var},\Pi[\mathrm{a,b}]}]} \hat{N}^{1-1/p} \tag{3.15}$$

*in which $\hat{N}$ is estimated by* (3.4)*. Furthermore, in case $[a,b]$ is partitioned equally, i.e if $\Pi = \{kh : 0 \leq k \leq n\}$, then $\hat{N}$ is estimated by* (3.7) *and the multiplier by $\kappa$ in the exponents of* (3.14)–(3.15) *can be changed from 8 to 4.*

*Proof:* Recall the combined sequence $\hat{G} = \{\hat{\tau}_i : \hat{\tau}_0 < \hat{\tau}_1 < \ldots < \hat{\tau}_{\hat{N}-1}\} = G_{\Pi,\beta_1,\delta_1} \cup G_{\Pi,\beta_2,\delta_2}$ of two discrete greedy sequences $G_{\Pi,\beta_1,\delta_1}$ and $G_{\Pi,\beta_2,\delta_2}$. Note that since $|\Pi| \leq \delta_2$, for every $i$, $|\hat{\tau}_{i+1} - \hat{\tau}_i| \leq \delta_2$ and $[\hat{\tau}_i, \hat{\tau}_{i+1}]$ satisfies either (3.5) or (3.6). Therefore, by Lemmas 3.2–3.3, the following inequalities hold for all $0 \leq i \leq \hat{N} - 2$

$$\begin{aligned} \|y\|_{\infty,\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]} \quad \leq \quad & [|y_{\hat{\tau}_i}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})]\, e^{\kappa} \Big[1 + (1+K^*)^p C_g^p \, |||x|||^p_{p-\mathrm{var},\Pi[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]}\Big] \\ & - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})] \end{aligned} \tag{3.16}$$

$$\begin{aligned} \|y\|_{q-\mathrm{var},\Pi[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]} \quad \leq \quad & [|y_{\hat{\tau}_i}| + M(f_0 \vee g_0) + 4H^*\omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})]\, e^{\kappa} \Big[1 + (1+K^*)^p C_g^p \, |||x|||^p_{p-\mathrm{var},[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]}\Big] \\ & - [M(f_0 \vee g_0) + 4H^*\omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})]. \end{aligned} \tag{3.17}$$

Note that if we replace $\omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})$ by $\omega^*(a,b) \geq \omega^*(\hat{\tau}_i,\hat{\tau}_{i+1})$ then the right-hand sides of (3.16) and (3.17) increase. Hence we have

$$\begin{aligned} \|y\|_{\infty,\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]} \quad \leq \quad & [|y_{\hat{\tau}_i}| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa} \Big[1 + (1+K^*)^p C_g^p \, |||x|||^p_{p-\mathrm{var},\Pi[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]}\Big] \\ & - [M(f_0 \vee g_0) + 4H^*\omega^*(a,b)] \end{aligned} \tag{3.18}$$

$$\begin{aligned} \|y\|_{q-\mathrm{var},\Pi[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]} \quad \leq \quad & [|y_{\hat{\tau}_i}| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa} \Big[1 + (1+K^*)^p C_g^p \, |||x|||^p_{p-\mathrm{var},[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]}\Big] \\ & - [M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]. \end{aligned} \tag{3.19}$$

From (3.18) it follows that for all $0 \leq i \leq \hat{N} - 2$ we have

$$|y_{\hat{\tau}_{i+1}}| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b) \leq [|y_{\hat{\tau}_i}| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa + (1+K^*)^p C_g^p |||x|||^p_{p-\mathrm{var},\Pi[\hat{\tau}_\mathrm{i},\hat{\tau}_\mathrm{i+1}]}}.$$

Moreover, by induction, for $0 \le m \le \hat{N}-1$, we have

$$
\begin{aligned}
& \sup_{a\le t_j\le \hat{\tau}_m} |y_{t_j}| + M(f_0\vee g_0) + 4H^*\omega^*(a,b) \\
\le\ & [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)] \prod_{i=0}^{m-1} e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]}} \\
\le\ & [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa m + (1+K^*)^p C_g^p \sum_{i=0}^{m-1} \|x\|^p_{p-\mathrm{var},\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]}}. \qquad (3.20)
\end{aligned}
$$

Put $m = \hat{N}-1$ in (3.20), taking into account the estimation (3.4) and the choice of parameters we get

$$
\begin{aligned}
& \sup_{a\le t_j\le b} |y_{t_j}| + M(f_0\vee g_0) + 4H^*\omega^*(a,b) \\
\le\ & [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa(\hat{N}-1) + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}} \\
\le\ & [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa(3+8C_f(b-a)+2(2C_g(1+K^*))^p\|x\|^p_{p-\mathrm{var},\Pi[a,b]}) + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}},
\end{aligned}
$$

from which we obtain (3.14). Moreover, it is easily seen that in case $[a,b]$ is partitioned equally, i.e if $\Pi = \{kh : 0 \le k \le n\}$, then we can estimate $\hat{N}$ by (3.7) and the multiplier by $\kappa$ in the exponent of (3.14) can be changed from 8 to 4.

Now, to prove the remaining part of the theorem, we use property of $p$-var seminorm and (3.17) to deduce the following inequalities

$$
\begin{aligned}
\|y\|_{q-\mathrm{var},\Pi[a,b]} \le\ & (\hat{N}-1)^{1-1/p} \sum_{i=0}^{\hat{N}-2} \|y\|_{q-\mathrm{var},\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]} \\
\le\ & (\hat{N}-1)^{1-1/p} \Big[ \sum_{i=0}^{\hat{N}-2} |y_{\hat{\tau}_i}| \left( e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},[\hat{\tau}_i,\hat{\tau}_{i+1}]}} - 1 \right) \\
& \qquad + (M(f_0\vee g_0) + 4H^*\omega^*(a,b))\, e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]}} \Big]
\end{aligned}
$$

Using the estimate of $|y_{\hat{\tau}_i}|$ just derived above applied to each interval $[a,\hat{\tau}_i]$ we get

$$
\begin{aligned}
& \|y\|_{q-\mathrm{var},\Pi[a,b]} \\
\le\ & (\hat{N}-1)^{1-1/p} [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)] \times \\
& \times \sum_{i=0}^{\hat{N}-2} \left( e^{(i+1)\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,\hat{\tau}_{i+1}]}} - e^{i\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,\hat{\tau}_i]}} \right) \\
& + (\hat{N}-1)^{1-1/p} \sum_{i=0}^{\hat{N}-2} (M(f_0\vee g_0) + 4H^*\omega^*(a,b))\, e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\hat{\tau}_i,\hat{\tau}_{i+1}]}} \\
\le\ & (\hat{N}-1)^{1-1/p} [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)] \times \\
& \times \left( e^{(\hat{N}-1)\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}} - e^{0\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,\hat{\tau}_0]}} \right) \\
& + (\hat{N}-1)^{1-1/p}(\hat{N}-1) [M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}} \\
\le\ & (\hat{N}-1)^{1-1/p} [|y_a| + M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{(\hat{N}-1)\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}} - |y_a| \\
& + (\hat{N}-1)^{2-1/p} [M(f_0\vee g_0) + 4H^*\omega^*(a,b)]\, e^{\kappa + (1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[a,b]}}
\end{aligned}
$$

Therefore, due to (3.4), we can find a constant $D$ depending on $C_f, C_g, \kappa$ such that

$$||y||_{q-\mathrm{var},\Pi[\mathrm{a,b}]} \leq [|y_a| + M(f_0 \vee g_0) + 4H^*\omega^*(a,b)]\, e^{8\kappa C_f(b-a)+D[1+(1+K^*)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\mathrm{a,b}]}]}(\hat{N}-1)^{1-1/p}.$$

Thus (3.15) is proved. Moreover, it is easily seen that in case $[a,b]$ is partitioned equally, i.e if $\Pi = \{kh : 0 \leq k \leq n\}$, then we can estimate $\hat{N}$ by (3.7) and the multiplier by $\kappa$ in the exponent of (3.15) can be changed from 8 to 4. The theorem is proved. □

**Remark 3.5** If $|\varepsilon^*_{t_k,t_{k+1}}| \leq \omega^\eta(t_k, t_{k+1})$ for some $\eta > 1$, and control $\omega$. One may take $H^* = \sup_{t,s\in\Pi,|t-s|\leq|\Pi|} \omega^{\eta-1}(s,t)$. If, additionally, $\omega$ is a continuous control on $[a,b]$, then $H^* \to 0$ as $|\Pi| \to 0$.

# 4 Application

Now we consider the stochastic differential equation

$$dy(t) = f(y(t))dt + g(y(t))dZ(t) \tag{4.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is dissipative, i.e. there exist $c, d > 0$ so that for all $y \in \mathbb{R}^d$,

$$\langle y, f(y)\rangle \leq \bar{c} - \bar{d}|y|^2$$

and global Lipschitz continuous functions, $g : \mathbb{R}^d \to \mathbb{R}^d$ is linear of the form $g(y) = Cy + g(0)$, and $Z$ is a two-sided stochastic process with stationary increments such that almost sure all realizations of $Z$ are in the space $\mathcal{C}^{p-\mathrm{var}}(\mathbb{R}, \mathbb{R})$ of continuous paths with finite $p$ - variation norm, for some $1 \leq p$. By assumption, $g$ is Lipchitz continuous with constant $C_g = |C|$. An example for such a process $Z$ is a fractional Brownian motion $B^H$ ([26]) with Hurst index $H \in (0,1)$.

It is well known that equation (4.1) can be solved in the path-wise approach by taking a realization $x \in \mathcal{C}^{p-\mathrm{var}}(\mathbb{R}, \mathbb{R})$ (which is also called a driving path) and considering the rough differential equation

$$dy_t = f(y_t)dt + g(y_t)dx_t, \quad y_0 \in \mathbb{R}^d. \tag{4.2}$$

Here we restrict to the case $p \in (1,2)$ for simplicity where we make use Young integral to to define the integral w.r.t. $x$. Less regular cases are treated in the forth coming paper.

By assumption on $Z$ one can construct a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ (see for instant [11]) and work on this space. It is proved that (4.1) generates a RDS $\varphi(t,x)y_0$. Moreover we recall the following result from [8].

**Theorem 4.1** *There exists $\varepsilon > 0$ such that if $|C| < \epsilon$, the generated RDS $\varphi$ of (4.1) possesses a pullback random attractor $\mathcal{A}(x)$.*

In the following we consider the Euler scheme of the system. We define for each realization $x \in (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ the scheme

$$\begin{aligned} y_{t_0} &\in \mathbb{R}^d, \\ y_{t_{k+1}} &= y_{t_k} + f(y_{t_k})(t_{k+1} - t_k) + g(y_{t_k})[x(t_{k+1} - x(t_k)] \end{aligned} \tag{4.3}$$

in which $t_k \in \Pi = \{t_k = kh,\ k \in \mathbb{N}\}$ with $h = 1/m$ for some $m \in \mathbb{N}^*$ for convenience. We prove that (4.3) generates a RDS $\varphi^h(t,x)y_0$ on time set $\{kh, k \in \mathbb{N}\}$ and then the existence and the convergence of the pullback attractor $\mathcal{A}^h$ to $\mathcal{A}$ in the pathwise sense as $h$ tends to 0.

We denote by $\Delta_k$ the closed interval $[k, k+1]$.

## 4.1 Existence of pullback attractors

Given a random dynamical system $\varphi$ on $\mathbb{R}^d$ with time set $\mathbb{T}$ is $\mathbb{Z}$ or $\mathbb{R}$, we follow [19], [2, Chapter 9] to present the notion of random pullback attractor. Recall that a random variable $\rho(x) > 0$ is called *tempered* if it satisfies

$$\lim_{t\to\pm\infty} \frac{1}{t}\log^+ \rho(\theta_t x) = 0, \quad \text{a.s.} \tag{4.4}$$

(see e.g. [2, pp. 164, 386]) which, by [21, p. 220]), is equivalent to the sub-exponential growth

$$\lim_{t\to\pm\infty} e^{-c|t|}\rho(\theta_t x) = 0 \quad \text{a.s.} \quad \forall c > 0.$$

A random set $D(x)$ is called *tempered* if it is contained in a ball $B(0,\rho(x))$ a.s., where the radius $\rho(x)$ is a tempered random variable.
A random subset $A$ is called invariant, if $\varphi(t,x)A(x) = A(\theta_t x)$ for all $t \in \mathbb{R},\ x \in \Omega$. An invariant random compact set $\mathcal{A} \in \mathcal{D}$ is called a *pullback random attractor* in $\mathcal{D}$, if $\mathcal{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}x)\hat{D}(\theta_{-t}x)|\mathcal{A}(x)) = 0. \tag{4.5}$$

$\mathcal{A}$ is called a *forward random attractor* in $\mathcal{D}$, if $\mathcal{A}$ is invariant and attracts any closed random set $\hat{D} \in \mathcal{D}$ in the forward sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t,x)\hat{D}(x)|\mathcal{A}(\theta_t x)) = 0. \tag{4.6}$$

The existence of a random pullback attractor follows from the existence of a random pullback absorbing set (see [19, Theorem 3]). A random set $\mathcal{B} \in \mathcal{D}$ is called *pullback absorbing* in a universe $\mathcal{D}$ if $\mathcal{B}$ absorbs all sets in $\mathcal{D}$, i.e. for any $\hat{D} \in \mathcal{D}$, there exists a time $t_0 = t_0(x,\hat{D})$ such that

$$\varphi(t,\theta_{-t}x)\hat{D}(\theta_{-t}x) \subset \mathcal{B}(x), \text{ for all } t \geq t_0. \tag{4.7}$$

Given a universe $\mathcal{D}$ and a random compact pullback absorbing set $\mathcal{B} \in \mathcal{D}$, there exists a unique random pullback attractor in $\mathcal{D}$, given by

$$\mathcal{A}(x) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t,\theta_{-t}x)\mathcal{B}(\theta_{-t}x)}. \tag{4.8}$$

The following is crucial to the main theorem Theorem 4.3.

**Lemma 4.2** *There exist $\Lambda_1, \Lambda_2$ of the form $e^{D[1+(1+K)^p C_g^p \|x\|^p_{p-\mathrm{var},\Delta_n}]}$ such that*

$$|y_{n+1}| \quad \leq \quad e^{-\bar{d}}|y_n|[1 + (h^{1-1/p} + C_g)\Lambda_1(x,\Delta_n)] + (f_0 \vee g_0)\Lambda_2(x,\Delta_n)$$

*for all $n \in \mathbb{N}$.*

*Proof:* We have

$$\begin{aligned}
|y^2_{t_{k+1}}| &= |y_{t_k}|^2 + |f(y_{t_k})|^2h^2 + |g(y_{t_k})|^2|x_{t_k,t_{k+1}}|^2 \\
&\quad +2h\langle y_{t_k}, f(y_{t_k})\rangle + 2\langle y_{t_k}, g(y_{t_k})x_{t_k,t_{k+1}}\rangle + 2h\langle f(y_{t_k}), g(y_{t_k})x_{t_k,t_{k+1}}\rangle \\
&\leq |y_{t_k}|^2 + 2h^2(C_f^2|y_{t_k}|^2 + f_0^2) + 2(C_g^2|y_{t_k}|^2 + g_0^2)|x_{t_k,t_{k+1}}|^2 \\
&\quad +2h(\bar{c} - \bar{d}|y_{t_k}|^2) + 2h(C_f|y_{t_k}| + f_0)(C_g|y_{t_k}| + g_0)|x_{t_k,t_{k+1}}| + 2\langle y_{t_k}, g(y_{t_k})x_{t_k,t_{k+1}}\rangle \\
&\leq |y_{t_k}|^2(1 - 2h\bar{d}) + 2\langle y_{t_k}, g(y_{t_k})x_{t_k,t_{k+1}}\rangle \\
&\quad +2|y_{t_k}|^2(h^2C_f^2 + C_g^2|x_{t_k,t_{k+1}}|^2 + 2C_fC_gh|x_{t_k,t_{k+1}}|) \\
&\quad +M(h + h^2 + |x_{t_k,t_{k+1}}|h + |x_{t_k,t_{k+1}}|^2) \\
&\leq |y_{t_k}|^2e^{-2\bar{d}h} + 2\langle y_{t_k}, g(y_{t_k})x_{t_k,t_{k+1}}\rangle \\
&\quad +2|y_{t_k}|^2(h^2C_f^2 + C_g^2|x_{t_k,t_{k+1}}|^2 + 2C_fC_gh|x_{t_k,t_{k+1}}|) \\
&\quad +(f_0 \vee g_0).M(h + h^2 + |x_{t_k,t_{k+1}}|h + |x_{t_k,t_{k+1}}|^2)
\end{aligned}$$

in which $M$ is a generic constsnt, depends on $C_f, C_g, c$. By induction

$$\begin{aligned}
|y^2_{t_k}| &\leq |y_n|^2e^{-2\bar{d}(t_k-n)} + 2\sum_{i=0}^{k-1} e^{-2\bar{d}(t_k-t_i)}\langle y_{t_i}, g(y_{t_i})x_{t_i,t_{i+1}}\rangle \\
&\quad +2|y|^2_{\infty,\Pi[n,n+1]}\sum_{i=0}^{k-1}(h^2C_f^2 + C_g^2|x_{t_i,t_{i+1}}|^2 + 2C_fC_gh|x_{t_i,t_{i+1}}|) \\
&\quad +M\sum_{i=0}^{k-1}(h + h^2 + |x_{t_i,t_{i+1}}|h + |x_{t_i,t_{i+1}}|^2) \\
&\leq |y_n|^2e^{-2\bar{d}(t_k-n)} + 2|\sum_{i=0}^{k-1} e^{-2d(t_k-t_i)}\langle y_{t_i}, g(y_{t_i})x_{t_i,t_{i+1}}\rangle| \\
&\quad +2|y|^2_{\infty,\Pi[n,n+1]}(hC_f^2 + C_g^2 \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},\Pi[n,n+1]} + 2C_fC_gh^{1-1/p} \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[n,n+1]}) \\
&\quad +(f_0 \vee g_0)M(1 + h + \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[n,n+1]} + \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},\Pi[n,n+1]}).
\end{aligned}$$

Particulaly,

$$\begin{aligned}
|y^2_{n+1}| &\leq |y_n|^2e^{-2\bar{d}} + 2|\sum_{i=0}^{m-1} e^{-2d(n+1-t_i)}\langle y_{t_i}, g(y_{t_i})x_{t_i,t_{i+1}}\rangle| \\
&\quad +2|y|^2_{\infty,\Pi[n,n+1]}(hC_f^2 + C_g^2 \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},\Pi[n,n+1]} + 2C_fC_gh^{1-1/p} \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[n,n+1]}) \\
&\quad +(f_0 \vee g_0)M(1 + h + \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Pi[n,n+1]} + \left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},\Pi[n,n+1]}). \qquad (4.9)
\end{aligned}$$

We are now going to estimate $|\sum_{i=0}^{m-1} e^{-2\bar{d}(t_k-t_i)}\langle y_{t_i}, g(y_{t_i})x_{t_i,t_{i+1}}\rangle|$. Put

$$F_{s,t} = e^{-2\bar{d}(n+1-s)}\langle y_s, g(y_s)x_{s,t} \; s,t \in \Pi[n, n+1].$$

We then have for $n \le s \le u \le t \le n+1$

$$
\begin{aligned}
|(\delta F)_{sut}| &= |e^{-2d(n+1-s)}\langle y_s, g(y_s)\rangle(x_t - x_s) - e^{-2\bar{d}(n+1-s)}\langle y_s, g(y_s)\rangle(x_u - x_s) \\
&\quad -e^{-2\bar{d}(n+1-u)}\langle y_u, g(y_u)\rangle(x_t - x_u)| \\
&= |[e^{-2\bar{d}(n+1-u)}\langle y_u, g(y_u)\rangle - e^{-2\bar{d}(n+1-s)}\langle y_s, g(y_s)\rangle][(x_t - x_u)]| \\
&= |[e^{-2\bar{d}(n+1-u)} - e^{-2d(n+1-s)}]\langle y_u, g(y_u)\rangle - e^{-2\bar{d}(n+1-s)}[\langle y_u, g(y_u)\rangle - \langle y_s, g(y_s)\rangle]|.|(x_t - x_u)| \\
&\le \|x\|_{p-\mathrm{var},\Pi[\mathrm{u,t}]} \Big[2\bar{d}e^{-2\bar{d}(n+1-u)}(u-s)(C_g\|y\|_{\infty,\Pi[n,n+1]} + g_0) + \\
&\qquad + e^{-2\bar{d}(n+1-s)} \|y\|_{p-\mathrm{var},\Pi[s,u]} (2C_g\|y\|_{\infty,\Pi[s,u]} + g_0)\Big] \\
&\le \|x\|_{p-\mathrm{var},\Pi[\mathrm{s,t}]} \Big[2\bar{d}(t-s) + \|y\|_{p-\mathrm{var},\Pi[\mathrm{s,t}]}\Big] (2C_g\|y\|_{\infty,\Pi[n,n+1]} + g_0)
\end{aligned}
$$

in which we use the following estimate

$$
\begin{aligned}
|\langle y_u, g(y_u)\rangle - \langle y_s, g(y_s)\rangle| &= |\langle y_u, g(y_u)\rangle - \langle y_u, g(y_s)\rangle + \langle y_u, g(y_s)\rangle - \langle y_s, g(y_s)\rangle| \\
&\le |\langle y_u, g(y_u) - g(y_s)\rangle| + |\langle y_u - y_s, g(y_s)\rangle| \\
&\le |y_u|C_g|y_u - y_s| + (C_g|y_s| + g_0)|y_u - y_s| \\
&\le \|y\|_{p-\mathrm{var},\Pi[s,u]} (2C_g\|y\|_{\infty,\Pi[s,u]} + g_0).
\end{aligned}
$$

It means that (2.1) is satisfied. We can apply Lemma 2.2 to obtain

$$
\begin{aligned}
&\left|\sum_{i=0}^{m-1} e^{-2\bar{d}(t_k - t_i)}\langle y_{t_i}, g(y_{t_i})x_{t_i,t_{i+1}}\rangle\right| \\
&\le e^{-2\bar{d}}\langle y_n, g(y_n)x_{n,n+1}\rangle \\
&+K^* \|x\|_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]} \Big[2\bar{d} + \|y\|_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]}\Big] (2C_g\|y\|_{\infty,\Pi[n,n+1]} + g_0) \\
&\le D(p,d) \|x\|_{p-\mathrm{var},[\mathrm{n,n+1}]} \left[C_g\|y\|^2_{p-\mathrm{var},\Pi[n,(n+1)]} + \frac{g_0^2}{C_g^2}\right]
\end{aligned}
$$

Combining this with (4.9) we obtain

$$
\begin{aligned}
|y^2_{n+1}| &\le |y_n|^2 e^{-2\bar{d}} \\
&\quad +2D\|y\|^2_{p-\mathrm{var},\Pi\Delta_n}(hC_f^2 + C_g^2 \|x\|^2_{p-\mathrm{var},\Pi[n,n+1]} + 2C_fC_gh^{1-1/p} \|x\|_{p-\mathrm{var},\Pi[n,n+1]} + \\
&\qquad +C_g \|x\|_{p-\mathrm{var},\Pi[n,n+1]}) \\
&\quad +M(1 + h + \|x\|_{p-\mathrm{var},\Pi[n,n+1]} + \|x\|^2_{p-\mathrm{var},\Pi[n,n+1]}) \\
&\le |y_n|^2 e^{-2\bar{d}} + 2D\|y\|^2_{p-\mathrm{var},\Pi[n,n+1]}(h^{1-1/p} + C_g \|x\|_{p-\mathrm{var},\Pi[n,n+1]})(1 + C_g \|x\|_{p-\mathrm{var},\Pi[n,n+1]}) \\
&\quad +(f_0 \vee g_0)M(1 + h + \|x\|_{p-\mathrm{var},\Pi[n,n+1]} + \|x\|^2_{p-\mathrm{var},\Pi[n,n+1]}).
\end{aligned} \tag{4.10}
$$

Next we make use Proposition 3.4

$$
\begin{aligned}
\|y\|_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]} &\le [|y_n| + M(f_0 \vee g_0)]\, e^{4\kappa C_f + D[1+(1+\theta)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]}]} \hat{N}^{1-1/p} \\
&\text{hence} \\
\|y\|^2_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]} &\le \big[|y_n|^2 + M^2(f_0 \vee g_0)^2\big]\, e^{\ln 2 + 8\kappa C_f + D[1+(1+\theta)^p C_g^p \|x\|^p_{p-\mathrm{var},\Pi[\mathrm{n,n+1}]}]}
\end{aligned}
$$

to obtain the conclusion. □

**Theorem 4.3** *Consider (4.3) where $x \in (\Omega, \mathcal{F}, \mathbb{P}, \theta)$, $y_0 \in \mathbb{R}^d$. (4.3) generates a discrete random dynamical system $\varphi^{(h)}$. Moreover, there exists $\epsilon > 0$ not depending on $h$ such that for $|C| < \epsilon$, there exists $h_0 > 0$ such that for all $h \leq h_0$ the discrete RDS $\Phi^h$ possesses a random pullback attractor $\mathcal{A}^h(x)$.*

*Proof:* It follows from [9] that the discrete scheme (3.1) generates a discrete random dynamical system $\varphi^{(h)}$. Namely, put

$$\bar{\varphi}(x)u_0 = u_0 + [Au_0 + f(u_0)]h + g(u_0)(x(h) - x(0)).$$

Then

$$\varphi^h(n, u_0, x) = \bar{\varphi}(\theta_{nh}x) \circ \bar{\varphi}(\theta_{(n-1)h}x) \cdots \circ \bar{\varphi}(x)u_0.$$

$\varphi^h$ is an RDS on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with discrete time set $\mathbb{T} = \{nh, n \in \mathbb{N}\}$. The measurability of $\bar{\varphi}(x)$ w.r.t. $(u_0, x)$ and continuity of $\bar{\varphi}(x)$ w.r.t. $u_0$ follows from the fact that for $(x, u_0), (x, u_0') \in \Omega \times V$,

$$\begin{aligned}
\|\bar{\varphi}(x)u_0 - \bar{\varphi}(x')u_0'\| &\leq \|u_0 - u_0'\| + \|[A_\lambda(u_0 - u_0') + f(u_0) - f(u_0')]h\| + \|[g(u_0) - g(u_0')]x(h)\| \\
&\quad + \|g(u_0')[x(h) - x'(h)]\| \\
&\leq D(u_0, x, h)(\|u_0 - u_0'\| + \|x - x'\|_{p-\mathrm{var},[0,h]}) \\
&\leq D(u_0, x, h)(\|u_0 - u_0'\| + d_p(x, x')).
\end{aligned}$$

Using similar arguments in [11] and [6], we can choose $\epsilon > 0$ depending on $\bar{d}$ and $\mathbb{E}\,\|\!|Z|\!\|^p_{p-\mathrm{var},[-1,1]}$ such that if $|C| < \epsilon$ there exists tempered r.v. $\tilde{R}(x)$, $h_0 > 0$, such that for $h < h_0$

$$|y_{t_k}(\theta_{-t_k}x, y_0)| \leq \tilde{R}(x)$$

for $t_k$ large enough. This proves the existence of a random pullback attractor $\mathcal{A}^h$ of $\varphi^h$. □

**Remark 4.4** $\tilde{R}(x)$ does not depend on $h$.

## 4.2 Convergence of numeric attractor

The following Proposition proves the convergence of numerical solution to the solution of (4.2). We denote by $z(\cdot, 0, z_0)$ the solution of (4.2) on arbitrary $[a, b]$. Recall from [11] that $z$ is bounded by a constant depending on $z_0, \|\!|x|\!\|_{p-\mathrm{var},[a,b]}$.

**Theorem 4.5** *The following limit holds*

$$\lim_{|\Pi[a,b]| \to 0} \sup_{0 \leq k \leq m} |z(t_k, 0, y_{t_0}) - y_{t_k}| = 0. \tag{4.11}$$

*Proof:* Denote by $z_{t_k}$ the value of $z$ at time $t_k$. Then

$$\begin{aligned}
|z_{t_{k+1}} - y_{t_{k+1}}| &= \Big| z_{t_k} - y_{t_k} + \int_{t_k}^{t_{k+1}} [f(z(u)) - f(z_{t_k})]du + \int_{t_k}^{t_{k+1}} [g(z(u)) - g(z_{t_k})]dx(u) \\
&\quad + [f(z_{t_k}) - f(y_{t_k})](t_{k+1} - t_k) + [g(z_{t_k}) - g(y_{t_k})](x(t_{k+1}) - x(t_k)) \Big| \\
|z_{t_l} - y_{t_l} - z_{t_k} + y_{t_k}| &\leq |\sum_{j=k}^{l-1} (f(z_{t_j}) - f(y_{t_j})(t_{j+1} - t_j) + (g(z_{t_j}) - g(y_{t_j})(x(t_{j+1}) - x(t_j)) + \\
&\quad + \sum_{j=k}^{l-1} [C_f \, |||z|||_{p-\mathrm{var},[t_j,t_{j+1}]} (t_{j+1} - t_j) + K C_g \, |||x|||_{p-\mathrm{var},[t_k,t_{k+1}]} \, |||z|||_{p-\mathrm{var},[t_k,t_{k+1}]}] \\
&\leq |\sum_{j=k}^{l-1} (f(z_{t_j}) - f(y_{t_j})(t_{j+1} - t_j) + (g(z_{t_j}) - g(y_{t_j})(x(t_{j+1}) - x(t_j))| \\
&\quad + H^* \omega^*(t_k, t_l)
\end{aligned}$$

in which $H^* \to 0$ as $|\Pi| \to 0$, $\omega^*(s,t) = |||x|||^{\epsilon}_{p-\mathrm{var},[s,t]} \, |||z|||_{p-\mathrm{var},[s,t]}]$ with suitable $\epsilon < 1$ so that $\frac{\epsilon+1}{p} > 1$ (see Remark 3.5). $H^*, \omega^*$ depend on $z, y$. Now we repeat arguments in Proposition 3.4 to obtain

$$\begin{aligned}
\sup_{t_i \in \Pi[a,b]} |z_{t_i} - y_{t_i}| &\leq [|z_0 - y_0| + 4H^* \omega^*(a,b)] \, e^{4\kappa C_f (b-a) + D[1+1+\theta)^p C_g^p (1 + |||x|||^p_{p-\mathrm{var},[a,b]})]} \\
&\leq D^* H^* \to 0 \text{ as } |\Pi| \to 0
\end{aligned}$$

since $z_0 = y_0$, in which $D^*$ is a constant depends on $A, f, g, x, b-a, z_0$. This completes the proof. □

**Remark 4.6** In [9], the freezing technique is used to prove the convergence of Euler scheme to the solution. Here, our proof is direct since we can make use of the Lipchitz continuity of $f$.

**Theorem 4.7** *(Convergence of numerical attractor) Assume that $|C|$ is small enough so that $\mathcal{A}$ and $\mathcal{A}^h$ exist. Then the numeric attractor $\mathcal{A}^h$ converges to the attractor $\mathcal{A}$ in the Hausdorf semi-distance, i.e. $d(\mathcal{A}^h, \mathcal{A}) \to 0$ as $h \to 0^+$, a.s.*

*Proof:* We proceed a contradiction arguments. Namely, assume the assertion is false, then there exists an $\varepsilon_0 > 0$ and a sequence $h_j \downarrow 0^+$ such that $d(\mathcal{A}^{h_j}, \mathcal{A}) > \varepsilon_0$ for all $j \in \mathbb{N}$. Since these attractors are compact sets, there exists $a_j \in \mathcal{A}^{h_j}$ such that $d(a_j, \mathcal{A}) > \varepsilon_0$. Due to the invariance, there exists for each $m_j \in \mathbb{N}$ a point $b_j \in \mathcal{A}^{h_j}(\theta_{-m_j h_j} x)$ such that $\varphi^{h_j}(m_j, \theta_{-m_j h_j} x) b_j = y^{h_j}_{m_j}(b_j) = a_j$. Respectively one considers the continuous solution $\varphi(m_j h, \theta_{-m_j h_j} x, b_j) = z(m_j h_j, \theta_{-m_j h_j} x, b_j)$ and applies the triangle inequality to obtain

$$\varepsilon_0 < d(a_j, \mathcal{A}) \leq \|z(m_j h_j, \theta_{-m_j h_j} x, b_j) - y^{h_j}_{m_j}(b_j)\| + d(\varphi(m_j h_j, \theta_{-m_j h_j} x, b_j), \mathcal{A}). \tag{4.12}$$

On the other hand, since $\mathcal{A}$ is the pullback attractor of $\varphi$ there exists a fixed $T(\varepsilon_0)$ such that for any $m_j h_j \in [T(\varepsilon_0), T(\varepsilon_0) + 1]$

$$d(\varphi(m_j h_j, \theta_{-m_j h_j} x, b_j), \mathcal{A}) \leq \frac{\varepsilon_0}{2}. \tag{4.13}$$

In addition, due to Proposition 4.5 we have

$$\|z(m_j h_j, \theta_{-m_j h_j} x, b_j) - y^{h_j}_{m_j}(b_j)\| \leq \frac{\varepsilon_0}{2} \tag{4.14}$$

by choosing $h_j$ small enough. (4.13) and (4.14) contradict to (4.12), which completes the proof. □

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