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Inertial algorithms for solving nonmonotone variational inequality problems

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Abstract This paper is devoted to presenting some new algorithms to find a solution of a nonmonotone variational inequality problem in the Euclidean space. These algorithms base upon the projection and inertial term which we hope to speed up the rate of convergence of the iteration process. The linesearch procedure is incorporated in algorithms to deal with the case the cost mapping is non Lipschitz, it becomes unnecessarily when it is Lipschitz. Moreover, we do not use the embedded projection methods as in methods used in literature to solve such a problem. The convergence of the sequences generated by these algorithms are obtained. Several numerical examples are also provided to illustrate the efficiency of proposed algorithms.

Keywords Variational inequality · nonmonotone · inertial algorithm · projection

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1 Introduction

Let $\mathbb{R}^n$ be the Euclidean space with the inner product $\langle\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex set in $\mathbb{R}^n$ and F be a continuous mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. The classical variational inequality problem associated with C and F (VIP (C,F) for short) consist of finding an element $x \in C$ such that

$$\langle F(x), y-x\rangle \geq 0,\ \forall y \in C.$$

We denote the set of solution of VIP(C,F) by S. The dual problem of this problem which is called Minty variational inequality problem (shortly, MVIP(C,F)) can be stated as follows: find $x \in C$ such that

$$\langle F(y), y-x\rangle \geq 0,\ \forall y \in C.$$

The solutuon set of MVIP(C,F) is denoted by S_M. It can be seen that $S_M = \bigcap_{x\in C} L(x)$ where $L(x) = \{y \in C : \langle F(x), x-y\rangle \geq 0\}$. It means that S_M is the intersection of the closed and convex halfspaces. Then S_M is a closed

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and convex set. It is well-known that $S_M \subset S$ if F is continuous on C, in addition, if F is pseudomonotone on C then $S \subset S_M$.

Variational inequality theory, initiated by Stampacchia [14,25], is an important tool in operation research and mathematical physics, economics, transportation, and so on. In fact, variational inequalities provide a unifying framework for the study of such diverse problems as boundary value problems, price equilibrium problems. Due to its vast applications, variational inequality theory has become a crucial field and has been extensively studied by many researchers [1,7,9,12,17,18] and the references quoted therein.

One of the most interesting and important problems in variational inequality theory is to build effective algorithms for finding a solution of VIPs among them the projection methods play an important role [4,10,16, 21,34]. It is well-know that [20] the projection method, in general, might not be convergent for the monotone variational inequality problems. Hence, Korpelevich [19] proposed the following extragradient algorithm

$$\begin{cases} x^0 \in C \\ y^k = P_C(x^k - \lambda F(x^k)), \\ x^{k+1} = P_C(x^k - \lambda F(y^k)), \end{cases}$$

where P_C is the metric projection onto C and $\lambda \in (0, \frac{1}{L})$ (L is the Lipschitz constant of mapping F). The sequence $\{x^k\}$ was proved to converge to a solution of VIPs under the main assumption that F is pseudomonotone and Lipschitz on C. However, when F is not pseudomonotonicity or is not Lipschitz continuous, the extragradient method may not be applied to solve VIPs directly.

To find a solution of nonmonotone variational inequality problem Ye and He [33] proposed to use the following shrinking projection algorithm:

Algorithm YH.

Choose $x^0 \in C$, $\sigma \in (0,1)$ and $\gamma \in (0,1)$. Set $k = 0$.

Step 1. Having x^k, Compute $z^k := P_C(x^k - F(x^k))$ and $r(x^k) = x^k - z^k$. If $r(x^k) = 0$, Stop. Otherwise, go to Step 2.
Step 2. Compute $z^k = x^k - \eta_k r(x^k)$, where $\eta_k = \gamma^{m_k}$, with m_k being the smallest nonnegative integer satisfying

$$\langle F(x^k) - F(x^k - \gamma^m r(x^k)), r(x^k)\rangle \le \sigma \|r(x^k)\|^2.$$

Step 3. Compute $x^{k+1} = P_{C\cap \widehat{H}_k}(x^k)$,
where $\widehat{H}_k := \cap_{j=0}^{j=k} H_j$ with $H_j := \{v : \langle F(z^j), v - y^j\rangle \le 0\}$. Let $k = k+1$ and return Step 1.

It was showed in [33] that if $S_M \neq \varnothing$ then the sequence $\{x^k\}$ converges to a solution $x^* \in S$. This method has been extended for solving nonmonotone multivalued variational inequalities [8] and equilibrium problems [11, 28]. One advantage of this algorithm is that it can be used to find a solution of VIPs when F is a nonmonotone and non-Lipschitz continuous function. However, at each iteration, it requires to solve an optimization problem in which its constraint set is the intersection of the constraint set in the previous iteration and a halfspace. Hence, the computation cost may increase, especially when n is large.

To overcome this drawback, very recently, Ye M [35] have proposed the following infeasible projection type algorithm:

Algorithm IPA.

Choose $x^0 \in \mathbb{R}^n$, $0 < \alpha_{min} < \alpha_{max}, \eta, \delta \in (0,1), \varepsilon > 0$. Set $k = 0$.

Step 1. Having x^k. Take $\alpha_0^k \in [\alpha_{min}, \alpha_{max}]$, $\alpha_k := \alpha_k^0 \eta^{m_k}$, where m_k is the smallest nonnegative integer m satisfying

$$\alpha_k^0 \eta^m \|F(x^k) - F(P_C(x^k - \alpha_k^0 \eta^m F(x^k)))\| \leq \delta \|x^k - P_C(x^k - \alpha_k^0 \eta^m F(x^k))\|.$$

Step 2. Compute $z^k := P_C(x^k - F(x^k))$, $r(x^k) = x^k - z^k$. If $r(x^k) = 0$, Stop. Otherwise, go to Step 3.
Step 3. Set $H_k = \{v \in \mathbb{R}^n : h_k(v) \leq 0\}$ with $h_k(v) := \langle x^k - z^k - \alpha_k(F(x^k) - F(z^k)), v - z^k \rangle$.
Let $t_k \in \arg\max\{dist(x^k, H_j) : 0 \leq j \leq k\}$ and $\widehat{H}_k := H_{t_k}$, and compute

$$x^{k+1} = P_{C \cap \widehat{H}_k}(x^k).$$

Let $k = k+1$ and return Step 1.

The author proved that sequence $\{x^k\}$ generated by this algorithm converge to a solution of VIPs if S_M is nonempty. One advantage of this algorithm is that it can be used to find a solution of nonmonotone and non-Lipschitz VIPs without using the embedded method as in [33].

On the other hand, an inertial type algorithm was first proposed by Polyak [24] as an acceleration process in solving a smooth convex minimization problem. An inertial-type algorithm can be consider as a two-step iterative method in which the next iterate is computed by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm may speed up the rate of convergence of the algorithm. Consequently, there are a vast of researchers have been investigating inertial type algorithms to solve their problems such as [3,5,22,30].

Motivated by this fact, in this paper, we introduce some new inertial type algorithms for finding a solution of the variational inequality problem in which the cost mapping F is nonmonotone in two cases F is Lipschitz or non-Lipschitz continuous, in addition, we do not use the embedded projection methods as in papers [8,11,28,33]. The rest of the paper is organized as follows. The next section contains some preliminaries on the metric projection and the natural residual mapping of the variational inequality problem. The proposed algorithms and their convergence are presented in the third section. The last section is devoted to present some numerical examples to illustrate the efficient of proposed algorithms.

2 Preliminaries

This section contains some basic results that will be used in our subsequent analysis.

Let C be a nonempty subset of n-dimensions Euclidean spaces $\mathbb{R}^n$. Let $d(x,C)$ denote the Euclidean distance from a vector x to C, i.e.,

$$d(x,C) := \inf\{\|x-y\|, y \in C\}.$$

If C is a nonempty and closed set, then it can be seen that

$$d(x,C) := \min\{\|x-y\|, y \in C\}.$$

In the rest of this paper, we consider the case C is a nonempty, closed and convex set. By P_C we denote the metric projection operator on C, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \leq \|y - x\|, \ \forall y \in C.$$

For a fixed $x \in \mathbb{R}^n$ and $\alpha \geq 0$, we denote the natural residual mapping

$$r(x,\alpha) := x - P_C(x - \alpha F(x)). \tag{1}$$

The following lemmas are well-known for the projection operator onto a closed convex set and the natural residual mapping.

Lemma 1 *([36]) Suppose that C is a nonempty closed convex subset in $\mathbb{R}^n$. Then, the following statements hold*

(a) $P_C(x)$ *is singleton and well defined for every x;*
(b) $z = P_C(x)$ *if and only if* $\langle x-z, y-z\rangle \leq 0, \forall y \in C$;
(c) $\|P_C(x) - P_C(y)\|^2 \leq \|x-y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2,\ \forall x, y \in C.$

Lemma 2 ([15]) *Let C be a nonempty closed and convex set in $\mathbb{R}^n$, t be a real-valued function on $\mathbb{R}^n$. We define* $\Omega := \{x \in C : t(x) \leq 0\}$. *If Ω is nonempty and t is Lipschitz continuous on C with modulus $M > 0$, then*

$$d(x,\Omega) \geq M^{-1}\max\{t(x), 0\},\ \forall x \in C,$$

where $d(x,\Omega)$ denotes the distance function from x to Ω.

Lemma 3 ([31]) *Let $C \subset \mathbb{R}^n$ be a nonempty closed and convex set, $r(x,\alpha)$ be defined in (1). Then x is a solution of VIP(F,C) if and only if $\|r(x,\alpha)\| = 0$ for each $\alpha > 0$.*

Lemma 4 ([13]) *For any $x \in \mathbb{R}^n$ and $\alpha \geq 0$, let $r(x,\alpha)$ be defined in (1). Then*

(a) function $\alpha \mapsto \|r(x,\alpha)\|$ *is nondecreasing whenever* $\alpha > 0$;
(b) function $\alpha \mapsto \frac{\|r(x,\alpha)\|}{\alpha}$ *is nonincreasing whenever* $\alpha > 0$.

Remark 1 From Lemma 4, we have the following inequality

$$\min\{1,\alpha\}\|r(x,1)\| \leq \|r(x,\alpha)\| \leq \max\{1,\alpha\}\|r(x,1)\| \text{ for any fixed } \alpha > 0. \tag{2}$$

Lemma 5 ([32]) *Let $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ be the nonnegative real number sequences satisfying*

$$\sum_{i=0}^{\infty} b_i < \infty \text{ and } a_{i+1} \leq a_i + b_i,\ \forall i.$$

Then, the sequence $\{a_k\}_{k=0}^{\infty}$ is convergent.

Lemma 6 ([23]) *Let $\{x^k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ and let C be a nonempty subset of $\mathbb{R}^n$. Suppose that, for every $x \in C, \{\|x^k - x\|\}_{k=0}^{\infty}$ converges and that every sequential cluster point of $\{x^k\}_{k=0}^{\infty}$ belongs to C. Then $\{x^k\}_{k=0}^{\infty}$ converges to a point in C.*

3 Inertial algorithms for solving nonmonotone variational inequality problems

Now, we assume that F is a continuous mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$ and C is a nonempty, closed and convex subset of $\mathbb{R}^n$ such that F not necessarily quasimonotone on C. The following algorithm give us a way to find a solution of VIP(C,F).

Algorithm 1.

Initialization. Let $\theta \in [0,1), \eta > 0$ and $\lambda, \delta \in (0,1)$. Choose positive sequence $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k \to 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} \mu_k < \infty$. Take $x^0, x^1 \in C$ and $k = 0$.
Iteration $k(k = 0,1,2,...)$. Having x^k do the following Steps.

Step 1. Compute $w^k := x^k + \theta_k(x^k - x^{k-1})$, where $0 \le \theta_k \le \overline{\theta}_k$, and

$$\overline{\theta}_k = \begin{cases} \min\{\theta, \frac{\mu_k}{\|x^k - x^{k-1}\|}\}, & x^k \ne x^{k-1} \\ \theta, & otherwise. \end{cases}$$

Step 2. Find m_k is the smallest nonnegative integer m satisfying

$$\langle F(w^k) - F(y^{k,m}), w^k - y^{k,m} \rangle \le \delta \left(\frac{\|w^k - y^{k,m}\|}{\eta \lambda^m} \right)^2, \tag{3}$$

where

$$y^{k,m} := P_C(w^k - \eta^2 \lambda^{2m} F(w^k)).$$

Set $\lambda_k := \eta^2 \lambda^{2m_k}, z^k := y^{k,m_k}$, $r(w^k, \lambda_k) = w^k - z^k$.
Step 3. Set $T_k = \{x \in \mathbb{R}^n : t_k(x) \le 0\}$ where

$$t_k(x) := \langle w^k - z^k - \lambda_k(F(w^k) - F(z^k)), x - z^k \rangle. \tag{4}$$

Select

$$j_k \in \arg\max\{d(w^k, T_j) : 1 \le j \le k\} \text{ and } \widehat{T}_k := T_{j_k}, \tag{5}$$

and compute

$$x^{k+1} := P_{\widehat{T}_k}(w^k). \tag{6}$$

and go to Iteration k with k is replaced by $k+1$.

Remark 2 (a) Observe that Algorithm 1 imply

$$\lim_{k \to \infty} \theta_k \|x^k - x^{k-1}\| = 0.$$

(b) We remark also here that the Step 1 in our Algorithm 1 is easily implemented in numerical computation since value of $\|x^k - x^{k-1}\|$ is a priori known before choosing θ_k.

We start our analysis of the algorithm covergence by proving the following lemmas.

Lemma 7 *Let F be a continuous function on $\mathbb{R}^n$. Then there exists a nonnegative integer m such that, the condition (3) is satisfied.*

Proof If $w^k \in S$, then by Lemma 3, we have $\|r(w^k, \eta\lambda^m)\| = \|w^k - y^{k,m}\| = 0$. Therefore, (3) holds with $m = 0$. For $w^k \notin S$, we have $\|r(w^k, \eta^2\lambda^{2m})\| = \|w^k - y^{k,m}\| > 0$.
Since

$$\langle F(w^k) - F(y^{k,m}), w^k - y^{k,m} \rangle \le \|F(w^k) - F(y^{k,m})\| \|w^k - y^{k,m}\|,$$

therefore, to prove (3) we will prove that there exists a nonnegative m such that

$$\begin{aligned} \langle F(w^k) - F(y^{k,m}), w^k - y^{k,m} \rangle &\le \|F(w^k) - F(y^{k,m})\| \|w^k - y^{k,m}\| \\ &\le \frac{\delta}{(\eta\lambda^m)^2} \|w^k - y^{k,m}\|^2. \end{aligned}$$

It means that

$$\|F(w^k) - F(P_C(w^k - \eta^2\lambda^{2m}F(w^k)))\| \le \frac{\delta}{(\eta\lambda^m)^2} \|w^k - P_C(w^k - \eta^2\lambda^{2m}F(w^k))\|. \tag{7}$$

We prove by contradiction. Assume that, (7) is not satisfied for any m, i.e., that

$$F(w^k) - F(P_C(w^k - \eta^2\lambda^{2m}F(w^k)))\| \leq \frac{\delta}{(\eta\lambda^m)^2}\|w^k - P_C(w^k - \eta^2\lambda^{2m}F(w^k))\| \text{ for all } m. \tag{8}$$

If $w^k \in C$, then $w^k = P_C(w^k)$. Using the continuity of $F(\cdot)$ and $P_C(\cdot)$ we have

$$\lim_{m\to\infty}\|F(w^k) - F(P_C(w^k - \eta^2\lambda^{2m}F(w^k)))\| = 0. \tag{9}$$

On the other hand, using Lemma 3, $\lambda \in (0,1)$ and $w^k \notin S$, we get

$$\delta\frac{\|w^k - y^{k,m}\|}{(\eta\lambda^m)^2} = \delta\frac{\|r(w^k, \eta^2\lambda^{2m})\|}{(\eta\lambda^m)^2} \geq \frac{\|r(w^k,1)\|}{1} > 0. \tag{10}$$

Thus, (9) and (10) contradicts (8).

For $w^k \notin C$, we have

$$\lim_{m\to\infty}\|w^k - y^{k,m}\| = \lim_{m\to\infty}\|w^k - P_C(w^k - \eta^2\lambda^{2m}F(w^k))\| = \|w^k - P_C(w^k)\| > 0$$

and

$$\lim_{m\to\infty}\frac{\eta^2\lambda^{2m}}{\delta}\|F(w^k) - F(y^{k,m})\|$$
$$= \lim_{m\to\infty}\frac{\eta^2\lambda^{2m}}{\delta}\|F(w^k) - F(P_C(w^k - \eta^2\lambda^{2m}F(w^k)))\| = 0.$$

Combining this, we get another contradiction.
Consequently, (7) holds and the linesearch (3) is well-defined.

We proceed to prove the following lemma before proving the convergence of our Algorithm 1.

Lemma 8 *Let the solution set S_M of the Minty Problem be nonempty, $x^* \in S_M$ and t_k be defined in (23). Then the following results hold:*

(a) $t_k(x^*) \leq 0$ and $S_M \subseteq \bigcap_{k=1}^{\infty} T_k$.
(b) $t_k(w^k) \geq (1-\delta)\|r(w^k, \lambda_k)\|^2$. In particular, if $w^k \neq z^k$, then $t_k(w^k) > 0$ and $w^k \notin T_k$, $\forall k$.

Proof (a) Since $x^* \in S_M$, then $\langle F(z^k), z^k - x^*)\rangle \geq 0$. From (23), we have

$$\begin{aligned} t_k(x^*) &= \langle w^k - z^k - \lambda_k(F(w^k) - F(z^k)), x^* - z^k\rangle \\ &= \langle w^k - z^k - \lambda_k F(w^k), x^* - z^k\rangle + \lambda_k\langle F(z^k), x^* - z^k\rangle \\ &\leq \langle w^k - z^k - \lambda_k F(w^k), x^* - z^k\rangle \\ &= \langle w^k - \lambda_k F(w^k) - P_C(w^k - \lambda_k F(w^k)), x^* - P_C(w^k - \lambda_k F(w^k))\rangle \leq 0. \end{aligned}$$

Therefore, $x^* \in T_k$ for all k. This implies that $S_M \subseteq T_k$, $\forall k$. Moreover, $S_M \subseteq \bigcap_{k=1}^{\infty} T_k$.

(b) Similarly, using (3), we get

$$\begin{aligned} t_k(w^k) &= \langle w^k - z^k - \lambda_k(F(w^k) - F(z^k)), w^k - z^k\rangle \\ &= \|w^k - z^k\|^2 - \lambda_k\langle F(w^k) - F(z^k), w^k - z^k\rangle \\ &\geq \|w^k - z^k\|^2 - \delta\|w^k - z^k\|^2 \\ &\geq (1-\delta)\|r(w^k, \lambda_k)\|^2 \geq 0. \end{aligned}$$

If $w^k \neq z^k$ then $t_k(w^k) > 0$. Thus, $w^k \notin T_k$ for all k.

The following theorem established the convergence of Algorithm 1.

Theorem 1 *Suppose that F is a continuous function on $\mathbb{R}^n$ and S_M is nonempty. Then $\{x^k\}$ is generated by Algorithm 1 converges to a solution of VIP(C,F).*

Proof We break our proof into four steps.

Step (i) We first show that the sequence $\{x^k\}$ is bounded.
Indeed, for any fixed point $p \in \bigcap_{i=1}^{\infty} T_i$, it is clear that $p \in \widehat{T}_k$. Therefore, we have

$$\begin{aligned} \|x^{k+1} - p\| &= \|P_{\widehat{T}_k}(w^k) - P_{\widehat{T}_k}(p)\| \\ &\le \|w^k - p\| \\ &= \|x^k + \theta_k(x^k - x^{k-1}) - p\| \\ &\le \|x^k - p\| + \theta_k\|x^k - x^{k-1}\|. \end{aligned} \tag{11}$$

By Lemma 5, applied $a_i = \|x^k - p\|; b_i = \theta_k\|x^k - x^{k-1}\|$, we obtain that, for any fixed $p \in \bigcap_{i=1}^{\infty} T_i$, the sequence $\{\|x^k - p\|\}$ is convergent. Thus, we can deduce that the sequence $\{x^k\}$ is bounded.
In addition, we have

$$\|w^k\| = \|x^k + \theta_k(x^k - x^{k-1})\| \le \|x^k\| + \theta_k\|x^k - x^{k-1}\|.$$

Beside that $\lim\limits_{k\to\infty} \theta_k\|x^k - x^{k-1}\| = 0$ by Remark 2. This implies that $\{w^k\}$ is bounded. Because F is continuous on $\mathbb{R}^n$, then $\{F(w^k)\}$ is bounded. Combine this with definitions of z^k and the continuity of $P_C(\cdot)$, we further obtain that $\{z^k\}, \{F(z^k)\}$ and $\{w^k - z^k - \lambda_k(F(w^k) - F(z^k))\}$ are also bounded.

Step (ii) Next, we show that any cluster point x^* of the sequence $\{x^k\}$ belongs to T_k for all k.
Since $w^k \notin T_k$ for all k, then $d(w^k, T_k) > 0$. Combining Lemma 1, we get

$$\begin{aligned} d^2(w^k, \widehat{T}_k) &= \|w^k - P_{\widehat{T}_k}(w^k)\|^2 \\ &= \|w^k - x^{k+1}\|^2 \\ &\le \|w^k - p\|^2 - \|x^{k+1} - p\|^2 \\ &= \|x^k + \theta_k(x^k - x^{k-1}) - p\|^2 - \|x^{k+1} - p\|^2 \\ &= \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 + 2\theta_k\langle x^k - x^{k-1}, x^k - p\rangle. \end{aligned}$$

Then,

$$\begin{aligned} 0 < d^2(w^k, T_k) \le d^2(w^k, \widehat{T}_k) \le &\|x^k - p\|^2 - \|x^{k+1} - p\|^2 \\ &+ \theta_k^2\|x^k - x^{k-1}\|^2 + 2\theta_k\langle x^k - x^{k-1}, x^k - p\rangle. \end{aligned} \tag{12}$$

Substituting Remark 2 into (12), we get

$$\lim_{k\to\infty} d(w^k, T_k) = \lim_{k\to\infty} d(w^k, \widehat{T}_k) = 0. \tag{13}$$

Assume that x^* be a cluster point of $\{x^k\}$. Then there exist $\{x^{k_l}\} \subset \{x^k\}$ such that $\lim\limits_{l\to\infty} x^{k_l} = x^*$. From the definition of w^k, we get

$$\|w^k - x^k\| = \theta_k\|x^k - x^{k-1}\| \to 0 \text{ as } k \to \infty.$$

Since $x^{k_l} \to x^*$, then $w^{k_l} \to x^*$. By (23) and (24), we see that

$$0 \le d(w^k, T_i) \le d(w^k, \widehat{T}_k) \text{ for all } i \le k.$$

This together with (13) implies that

$$\lim_{k\to\infty} d(w^k, T_i) = 0 \text{ for any fixed } i.$$

Using the continuity of $d(\cdot,T_i)$ on $\mathbb{R}^n$, we obtain that

$$d(x^*,T_i)=0 \text{ for any fixed } i.$$

Therefore, $x^* \in \bigcap_{i=1}^{\infty} T_i$.

Step (iii) We now show that $x^* \in$ VIP(C,F).
The boundedness of the sequence $\{w^k - z^k - \lambda_k(F(w^k)-F(z^k))\}$ implies that there exists $M>0$ such that

$$\|w^k - z^k - \lambda_k(F(w^k)-F(z^k))\| \le M \ ,\forall k \in \mathbb{N}.$$

On the orther hand, we have

$$\begin{aligned}\|t_k(x)-t_k(y)\| &= \|\langle w^k - z^k - \lambda_k(F(w^k)-F(z^k)), x-y\rangle\| \\ &\le \|w^k - z^k - \lambda_k(F(w^k)-F(z^k))\|\|x-y\| \\ &\le M\|x-y\|.\end{aligned}$$

Combining Lemma 2 and Lemma 8, we get

$$d(w^k,T_k) \ge M^{-1}t_k(w^k) \ge M^{-1}(1-\delta)\|r(w^k,\lambda_k)\|^2 > 0.$$

This with (13) established $\lim_{k\to\infty}\|r(w^k,\lambda_k)\|=0$.
We may assume without loss of generality that $\widetilde{\lambda} := \lim_{l\to\infty}\lambda_{k_l}$. We show that $\lim_{k\to\infty}\|r(w^k,1)\|=0$.
We now consider two distinct cases.
Case 1. If $\widetilde{\lambda}>0$, then $\widetilde{\lambda} \le \lambda_i$ for all i. Using (2), we get

$$0 \le \|r(w^i,1)\| \le \frac{\|r(w^i,\lambda_i)\|}{\min\{\lambda_i,1\}} \le \frac{\|r(w^i,\lambda_i)\|}{\min\{\widetilde{\lambda},1\}}.$$

Hence, $\lim_{i\to\infty}\|r(w^i,\lambda_i)\|=0$. Thus, $\lim_{i\to\infty}\|r(w^i,1)\|=0$.
Consequently, $\lim_{l\to\infty}\|r(w^{k_l},1)\|=0$.
Case 2. If $\widetilde{\lambda}=0$, then $\lim_{l\to\infty}\lambda_{k_l}=0$. It means that, $\lim_{l\to\infty}\eta^2\lambda^{2m_{k_l}}=0$.
By the linesearch rule (3), for $m_{k_l}-1$, we have

$$\langle F(w^{k_l})-F(y^{k_l,m_{k_l}-1})\rangle > \delta\left(\frac{\|w^{k_l}-y^{k_l,m_{k_l}-1}\|}{\eta\lambda^{m_{k_l}-1}}\right)^2. \tag{14}$$

It follows that

$$\langle F(w^{k_l})-F(y^{k_l,m_{k_l}-1})\rangle > \delta\left(\frac{\|w^{k_l}-y^{k_l,m_{k_l}-1}\|}{\eta\lambda^{m_{k_l}-1}}\right)^2 \ge \frac{\|r(w^{k_l},1)\|^2}{1} > 0. \tag{15}$$

Furthermore, $P_C(\cdot)$ is nonexpansive, then

$$\begin{aligned}\|w^{k_l}-y^{k_l,m_{k_l}-1}\| \le \|w^{k_l}-y^{k_l,m_{k_l}}\| + \|y^{k_l,m_{k_l}}-y^{k_l,m_{k_l}-1}\| &= \|r(w^{k_l},\lambda_{k_l})\| \\ &+ \|P_C(w^{k_l}-\eta^2\lambda^{2m_{k_l}}F(w^{k_l})) - P_C(w^{k_l}-\eta^2\lambda^{2(m_{k_l}-1)}F(w^{k_l}))\| \\ &\le \|r(w^{k_l},\lambda_{k_l})\| + \eta^2\lambda^{2m_{k_l}}(\lambda^{-2}-1)\|F(w^{k_l})\| \to 0 \ as\ k\to\infty.\end{aligned}$$

Since $\lim_{l\to\infty} w^{k_l}=x^*$ and $F(\cdot)$ is continuous, we obtain that

$$\lim_{l\to\infty}\left\|F(w^{k_l})-F(y^{k_l,m_{k_l}-1})\right\|=0.$$

This together with (14), we have $\lim_{l\to\infty}\|r(w^{k_l},1)\|=0$. Using the continuity of $\|r(\cdot,1)\|$, we have $\|r(x^*,1)\|=0$. Combining Lemma 3, we see that any cluster point of $\{x^k\}$ is a solution of VIP(C,F).

Step (iv) Finally, we prove that $\{x^k\}$ is globally convergent to a solution of VIP(C,F).
Since x^* is a cluster point of $\{x^k\}$, then from (11), we have $x^*\in\bigcap_{i=1}^{\infty}T_i$ and

$$\|x^{k+1}-x^*\|\le\|x^k-x^*\|+\theta_k\|x^k-x^{k-1}\|.$$

This implies that the sequence $\{\|x^k-x^*\|\}$ is convergent and there exits $a\ge 0$ such that $\lim_{k\to\infty}\|x^k-x^*\|=a$. By definition of x^*, there exits the subsequence $\{x^{k_l}\}\subset\{x^k\}$ such that $\lim\limits_{l\to\infty}\|x^{k_l}-x^*\|=0$. Using Lemma 6, we have

$$a=0 \text{ and } \lim_{k\to\infty}\|x^k-x^*\|=\lim_{l\to\infty}\|x^{k_l}-x^*\|=0.$$

i.e., $\lim\limits_{k\to\infty}x^k=x^*$.
Consequently, $\{x^k\}$ is globally convergent to a solution of VIP Problem .

Next, we propose a modification of Algorithm 1 with another linesearch.

Algorithm 2.

Initialization. Choose $\theta\in[0,1),\eta>0$ and $\lambda,\delta\in(0,1)$. Choose positive sequence $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k\to 0$ as $k\to\infty$ and $\sum_{k=1}^{\infty}\mu_k<\infty$. Take $x^0,x^1\in C$ and $k=0$.
Iteration $k(k=0,1,2,...)$. Having x^k do the following Steps.
Step 1. Compute $w^k:=x^k+\theta_k(x^k-x^{k-1})$, where $0\le\theta_k\le\overline{\theta}_k$, and

$$\overline{\theta}_k=\begin{cases}\min\{\theta,\frac{\mu_k}{\|x^k-x^{k-1}\|}\}, & x^k\ne x^{k-1}\\ \theta, & otherwise.\end{cases}$$

Step 2. Find m_k is the smallest nonnegative integer m satisfying

$$\eta\lambda^m\|F(w^k)-F(y^{k,m})\|\le\delta\|w^k-y^{k,m}\|. \tag{16}$$

where

$$y^{k,m}:=P_C(w^k-\eta\lambda^mF(w^k)).$$

Set $\lambda_k:=\eta^2\lambda^{2m_k}, z^k:=y^{k,m_k}$, $r(w^k,\lambda_k)=w^k-z^k$.
Step 3. Set $T_k=\{x\in\mathbb{R}^n:t_k(x)\le 0\}$ where

$$t_k(x):=\langle w^k-z^k-\lambda_k(F(w^k)-F(z^k)),x-z^k\rangle. \tag{17}$$

Select

$$j_k\in\arg\max\{d(w^k,T_j):1\le j\le k\} \text{ and } \widehat{T}_k:=T_{j_k}, \tag{18}$$

and compute

$$x^{k+1}:=P_{\widehat{T}_k}(w^k). \tag{19}$$

and go to Iteration k with k is replaced by $k+1$.

Theorem 2 *Suppose that F is a continuous function on $\mathbb{R}^n$ and $S_M\ne\varnothing$. Then $\{x^k\}$ is generated by Algorithm 2 converges to a solution of VIP(C,F).*

Proof From (16), we have

$$\|F(w^k) - F(y^{k,m})\| \le \frac{\delta}{\eta \lambda^m} \|w^k - y^{k,m}\|. \tag{20}$$

Using this inequality and the same idea as in the proof of Lemma 7, we can prove that the linesearch rule (16) is well-defined.

For $x^* \in S_M$, similarly to Lemma 8, we have $t_k(x^*) \le 0$ and $S_M \subseteq \bigcap_{k=1}^{\infty} T_k$. Moreover, using (16) we obtain

$$\begin{aligned} t_k(w^k) &= \langle w^k - z^k - \lambda_k(F(w^k) - F(z^k)), w^k - z^k \rangle \\ &= \|w^k - z^k\|^2 - \lambda_k \langle F(w^k) - F(z^k), w^k - z^k \rangle \\ &\ge \|w^k - z^k\|^2 - \lambda_k \|F(w^k) - F(z^k)\| \|w^k - z^k\| \\ &\ge \|w^k - z^k\|^2 - \delta \|w^k - z^k\|^2 \\ &\ge (1-\delta) \|r(w^k, \lambda_k)\|^2 > 0. \end{aligned}$$

Therefore, $w^k \notin T_k$ for all k.
The proof of convergence can be done by the similar way as in Theorem 1, so we omit it. For instance, instead of using (14), we use the following inequality is

$$\eta \lambda^{m_{k_l}-1} \|F(w^{k_l}) - F(y^{k_l, m_{k_l}-1})\| > \delta \|w^{k_l} - y^{k_l, m_{k_l}-1}\|. \tag{21}$$

It follows that

$$\|F(w^{k_l}) - F(y^{k_l, m_{k_l}-1})\| > \frac{\delta \|w^{k_l} - y^{k_l, m_{k_l}-1}\|}{\eta \lambda^{m_{k_l}-1}} \ge \frac{\delta \|r(w^{k_l}, 1)\|}{1} > 0. \tag{22}$$

The proof is completed.

In the following Algorithm, we will show that if F is L-Lipschitz continuous mapping, then the linesearch in (3) and (16) can be removed.

Algorithm 3.

Initialization. Choose $\theta \in [0,1), \eta > 0$ and $\lambda, \delta \in (0,1)$. Choose positive sequence $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k \to 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} \mu_k < \infty$. Take $x^0, x^1 \in C$ and $k = 0$.
Iteration $k (k = 0, 1, 2, \ldots)$. Having x^k do the following Steps.
Step 1. Compute $z^k := P_C(w^k - \alpha F(w^k))$, where $w^k := x^k + \theta_k(x^k - x^{k-1})$, $0 \le \theta_k \le \overline{\theta}_k$, and

$$\overline{\theta}_k = \begin{cases} \min\{\theta, \frac{\mu_k}{\|x^k - x^{k-1}\|}\}, & x^k \ne x^{k-1} \\ \theta, & otherwise. \end{cases}$$

Step 2. Set $T_k = \{x \in \mathbb{R}^n : t_k(x) \le 0\}$, where

$$t_k(x) := \langle w^k - z^k - \lambda_k(F(w^k) - F(z^k)), x - z^k \rangle. \tag{23}$$

Select

$$j_k \in \arg\max\{d(w^k, T_j) : 1 \le j \le k\} \text{ and } \widehat{T}_k := T_{j_k}, \tag{24}$$

and compute

$$x^{k+1} := P_{\widehat{T}_k}(w^k), \tag{25}$$

and go to Iteration k with k is replaced by $k+1$.

In this next theorem, we establish the convergence analysis of the sequence of iterates generated by our proposed Algorithm 3 to the solution of VIP (C, F).

Theorem 3 *Let F be L-Lipschitz continuous function on $\mathbb{R}^n$ and $S_M \neq \varnothing$. Then the sequence $\{x^k\}$ is generated by Algorithm 3 converges to a solution of VIP(C,F).*

Proof Following the same reasoning as in Lemma 8, we obtain that $t_k(x^*) \leq 0$ and $S_M \subseteq T_k$ for all k. Futhermore, since F is L-Lipschitz continuous function on $\mathbb{R}^n$ and $0 < \alpha < \frac{1}{L}$, we let

$$\begin{aligned} t_k(w^k) &= \left\langle w^k - z^k - \alpha(F(w^k) - F(z^k)), w^k - z^k \right\rangle \\ &= \left\| w^k - z^k \right\|^2 - \alpha \left\langle F(w^k) - F(z^k), w^k - z^k \right\rangle \\ &\geq \left\| w^k - z^k \right\|^2 - \alpha L \left\| w^k - z^k \right\|^2 \\ &\geq (1 - \alpha L) \left\| r(w^k, \alpha) \right\|^2 > 0. \end{aligned}$$

Hence, $w^k \notin T_k$ for all k. Consequently, the rest of the convergence proof is identical to that of Theorem 1 with $\delta = \alpha L$. Therefore, $\{x^k\}$ conveges to a solution x^* of VIP(C,F).

4 Numerical examples

In this section, we provide five numerical examples to show the practicability and the advantage of proposed algorithms by comparing them with the embedded projection Algorithm of M. Ye and Y. He ([33]) and Algorithm IPA of M. Ye ([35]).

All the programs are written in Mathlab R2015 and performed on a Laptop DELL Intel (R), Core (TM) i7-9700 CPU, 3.00 Ghz, Ram 16.0 GB.

In IPA, they take $\alpha_{min} = 10^{-10}, \alpha_{max} = 10^{10}$. Inspired by the renowned Barzilai-Borwein step-size and self-adaptive step-size, they take parameter α_k^0 as follows: set $\alpha_0^0 = 1$ and take, for $k \geq 1$,

$$\alpha_k^0 = \begin{cases} P_{[10^{-10},10^{10}]}\left(\frac{\|x^k - x^{k-1}\|^2}{\langle x^k - x^{k-1}, F(x^k) - F(x^{k-1}) \rangle} \right) & if \left\langle x^k - x^{k-1}, F(x^k) - F(x^{k-1}) \right\rangle > 10^{-12}, \\ P_{[10^{-10},10^{10}]}(1.5\alpha_{k-1}) & otherwise. \end{cases}$$

Example 1 Consider a problem of the form (see Exercise 4.7 in [6])

$$\min\{g(x) = \frac{f_0(x)}{cx + d} : x \in C\},$$

with $C = \{x \in \mathbb{R}^n : f_i(x) \leq 0, i = 1, ..., m, Ax = b\}$, where $f_0, f_1, ..., f_m$ are convex functions, A is a matrix of order $m \times n$, $b \in \mathbb{R}^m$.

This is a quasiconvex optimization problem. Choose $C = \{x = (x_1, x_2, ..., x_5)^T \in \mathbb{R}^5 : x_i \geq 0, i = 1, ..., 5, \sum_{i=1}^{5} x_i = a\}, a > 0$ and $g(x) = \frac{\frac{1}{2}x^T Hx + q^T x + r}{\sum_{i=1}^{5} x_i}$. Then g is a smooth quasiconvex function and can attain its minimum value on C, where $q = (-1, ..., -1)^T, r = 1$ and $H = hI$ is a positive diagonal matrix with $h \in (0.1, 1.6)$. Denoting $F(x) = (F_1(x), ..., F_5(x))^T$ is the derivative of $g(x)$, where $F_i(x) = \frac{hx_i \sum_{j=1}^{5} x_j - \frac{1}{2}h \sum_{j=1}^{5} {x_j}^2 - 1}{\left(\sum_{j=1}^{5} x_j\right)^2}$. Then the problem $VIP(C,F)$ is a quasimonotone variational inequality with $S_M = \{(\frac{1}{5}a, ..., \frac{1}{5}a)^T\}$ (see [33]). We choose $h = 1.2, \sigma = 0.4, \gamma = 0.99$ in Algorithm YH, $\sigma = 0.4, \eta = 0.99$ in Algorithm IPA, $\theta = 0.1, \eta = 0.99, \lambda = 0.99, \delta = 0.8, \mu_k = \frac{1}{(k+1)^{1.8}}$ in Algorithm 1 and Algorithm 2. The initial point is x^0 in Algorithm YH, Algorithm IPA, and $x^0 = x^1$ in Algorithm 1,2. We make comparison of algorithms with $\varepsilon = \|r(x^k)\| \leq 10^{-4}$ and report the result in Table 1, Figure 1.

x^0	a	Alg. YH		Alg. IPA		Alg. 1		Alg. 2	
		CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
$(0,0,0,0,5)^T$	5	0.55	35	15.47	24	0.41	32	0.44	32
$(2,1,0,0,2)^T$	-	0.45	31	11.31	30	0.31	30	0.36	30
$(1.5,1.2,1.3,0.3,0.7)^T$	-	0.44	29	10.20	19	0.3	28	0.29	28
$(5,0,0,0,5)^T$	10	0.88	70	13.19	23	0.59	66	0.51	66
$(1,3,2,3,1)^T$	-	0.72	57	11.72	21	0.50	58	0.46	58
$(1.7,1.8,1.9,3.5,1.1)^T$	-	0.72	56	10.86	20	0.50	57	0.47	57

Table 1 Experiment for Example 1.

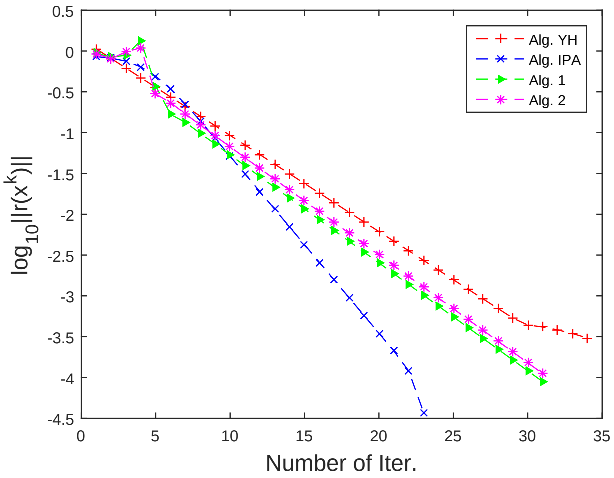

Fig. 1 Number of iterations in Example 1.

Example 2 In this example, we consider the affine variational inequality proplem

$$C = [0,1]^n \text{ and } F(x) = Mx + d,$$

where M is a $n \times n$ tridiagonal matrix given by

$$M = \begin{bmatrix} 4 & -2 & & & \\ 1 & 4 & -2 & & \\ & 1 & 4 & -2 & \\ & & \cdots & \cdots & \cdots \\ & & & 1 & 4 \end{bmatrix}$$

and $d = (-1, ..., -1)^T$ (see [1], [29], [33]).

We take $\sigma = 0.4, \gamma = 0.1$ in Algorithm YH, $\sigma = 0.5, \eta = 0.99$ in Algorithm IPA. And take $\theta = 0.5, \lambda = 0.6, \delta = 0.4$, $\eta = 0.9, \mu_k = \frac{1}{(k+2)^{1.8}}$ in Algorithm 1, $\lambda = 0.1, \delta = 0.5, \theta = 0.2, \eta = 0.99, \mu_k = \frac{1}{(k+1)^{1.5}}$ in Algorithm 2. The starting point is $x^0 = (0,...,0)^T$ in Algorithm YH, Algorithm IPA, $x^0 = x^1 = (0,...,0)^T$ in Algorithm 1,2. We terminate the iteration if $\varepsilon = \|r(x^k)\| \leq 10^{-4}$. The results are shown in Table 2 and Figure 2.

n	Alg. YH		Alg. IPA		Alg. 1		Alg. 2	
	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
50	24.95	456	9.69	18	0.17	22	0.09	21
100	659.41	894	10.80	18	0.23	23	0.16	22
150	3556.47	1773	52.09	18	1.61	23	0.87	22
200	3986.81	1628	63.66	18	1.95	23	1.09	23
500	50530.97	3889	269.20	18	6.95	21	10.25	33

Table 2 Experiment for Example 2.

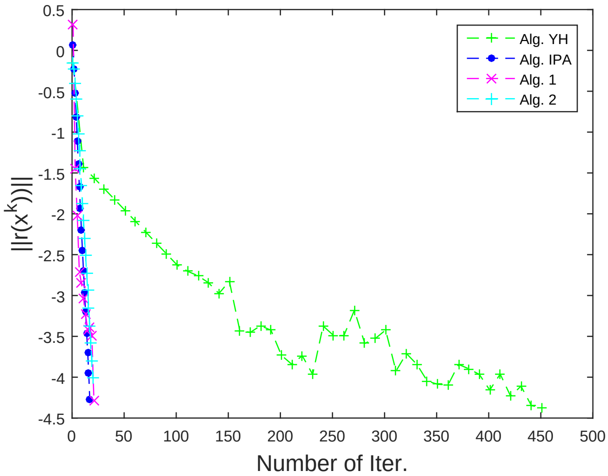

Fig. 2 Number of iterations of Algorithms in Example 2 ($n = 50, x^0 = x^1 = (0,...,0)^T$).

Example 3 In this example, we let

$$C = [-1,1]^n \text{ and } F(x) = (x_1^2,...,x_n^2)^T.$$

Here, the mapping is not quasimonotone but it is easy to check that $S_M = \{(-1,...,-1)^T\}$ (see [33]). Futhermore, it can be see that F satisfies the Lipschitz condition with $L = 2\sqrt{n}$. We use Algorithm YH with parameters $\sigma = 0.5, \gamma = 0.99$, Algorithm IPA with $\sigma = 0.5, \eta = 0.99$, Algorithm 1 with $\theta = 0.8, \eta = 0.99, \lambda = 0.99, \delta = 0.4, \mu_k = \frac{1}{(k+2)^{1.3}}$ and Algorithm 2 with $\theta = 0.5, \eta = 0.99, \lambda = 0.99, \delta = 0.4, \mu_k = \frac{1}{(k+2)^{1.3}}$. The starting point is $x_0 = (-\frac{3}{4},...,-\frac{3}{4})^T$ in Algorithm YH, Algorithm IPA and $x^0 = x^1 = (-\frac{3}{4},...,-\frac{3}{4})^T$ in Algorithm 1,2. The numerical result is shown in Table 3, Figure 3. The stopping criterion used is $\varepsilon = \|r(x^k)\| \leq 10^{-4}$.

Example 4 Let $C = [0,1]^n$ and $F(x) = (x_1^2 - x_1,...,x_n^2 - x_n)^T$.

Here, the mapping is not quasimonotone and $S_M = \{(1,...,1)^T\}$ (see [33]). We take $\sigma = 0.95; \gamma = 0.99$ in Algorithm YH, $\sigma = 0.95, \eta = 0.99$ in Algorithm IPA , $\theta = 0.1, \eta = 0.99, \lambda = 0.99, \delta = 0.99, \mu_k = \frac{1}{(k+2)^{1.7}}$ in

n	Alg. YH		Alg. IPA		Alg. 1		Alg. 2	
	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
100	0.27	5	2.17	4	0.25	4	0.15	4
500	1.11	5	30.20	5	0.59	4	0.50	4
1000	1.78	5	61.88	5	0.86	4	0.72	4
5000	10.92	5	399.41	5	5.72	5	5.50	5
10000	33.77	5	1180.38	5	12.81	4	12.16	4

Table 3 Experiment for Example 3.

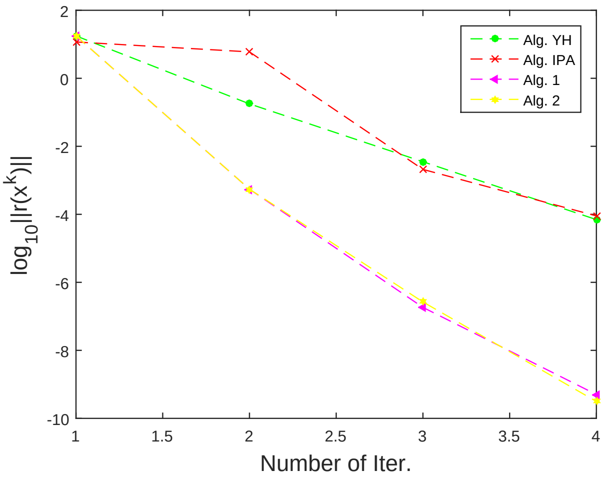

Fig. 3 Number of iterations of Algorithms in Example 3 ($n = 5000, x^0 = x^1 = (-\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \ldots, -\frac{3}{4})^T$).

Algorithm 1 and $\theta = 0.9, \eta = 0.8, \lambda = 0.9, \delta = 0.9, \mu_k = \frac{1}{(k+1)^3}$ in Algorithm 2. Let $x^0 = (\frac{1}{6}, \ldots, \frac{1}{6})^T$ be initial point in Algorithm YH, Algorithm IPA, $x^0 = x^1 = (\frac{1}{6}, \ldots, \frac{1}{6})^T$ be initial points in Algorithm 1, Algorithm 2. To terminate the Algorithms, we use the stopping criteria $\varepsilon = \|r(x^k)\| \leq 10^{-4}$. The resurts are listed in Table 4 and the corresponding figures are displayed in Figure 4 .

n	Alg. YH		Alg. IPA		Alg. 1		Alg. 2	
	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
100	0.31	9	1.62	7	0.23	9	0.23	12
500	2.59	10	19.07	7	1.37	10	1.82	13
1000	4.05	9	39.13	7	2.34	10	5.06	14
5000	27.82	10	253.15	8	13.25	10	77.12	17
10000	82.00	10	705.14	8	36.73	10	316.98	19

Table 4 Experiment for Example 4.

In the last example, we present to numerical experiments to illustrate the performance of algorithms when F is L-Lipschitz continuous mapping. We compare the convergence behavior of Algorithm YH and Algorithm 1 with Algorithm 3 and the following algorithm (denoted by Algorithm IPAL) which is proposed by Minglu Ye ([35]).

Algorithm IPAL.

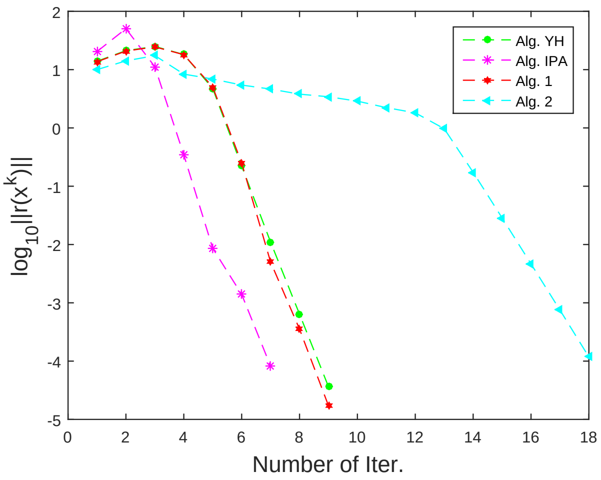

Fig. 4 Number of iterations of Algorithms in Example 4 ($n=10000, x^0=(\frac{1}{6},\frac{1}{6},\frac{1}{6},...,\frac{1}{6})^T$).

Initialization. Choose $0<\lambda<\frac{1}{L}, \varepsilon>0$ and $x^0\in\mathbb{R}^n$.
Iteration $k(k=0,1,2,...)$. Having x^k do the following Steps.
Step 1. Compute $z^k := P_C(x^k-\lambda F(x^k))$.
Step 2. Set $H_k=\{v\in\mathbb{R}^n : h_k(v)\le 0\}$ with $h_k(v) := \langle x^k-z^k-\lambda(F(x^k)-F(z^k)), v-z^k\rangle$. Select $t_k\in \arg\max\{dist(x^k,H_j) : 0\le j\le k\}$ and $\widehat{H}_k := H_{t_k}$, and compute $x^{k+1} := P_{\widehat{H}_k}(x^k)$. and go to Iteration k with k is replaced by $k+1$.

Example 5 Consider $C=[-\frac{n\pi}{2},\frac{n\pi}{2}]^n$. Let $F : C\to\mathbb{R}^n$ be defined by

$$F(x)=(\cos\frac{x_1}{n},\cos\frac{x_2}{n},...,\cos\frac{x_n}{n})^T.$$

It is clear that, F is not quasimonotone on C. Since, for $x=(-\frac{n\pi}{3},\frac{n\pi}{2},\frac{n\pi}{2},...,\frac{n\pi}{2})^T$ and $y=(-\frac{n\pi}{4},\frac{n\pi}{3},\frac{n\pi}{2},...,\frac{n\pi}{2})^T$, we have

$$\langle F(x), y-x\rangle = -\frac{n\pi}{24}<0 \text{ and } \langle F(y), y-x\rangle = \frac{(\sqrt{2}-1)n\pi}{12\sqrt{2}}>0.$$

In addition,

$$\|F(x)-F(y)\|\le\frac{1}{\sqrt{n}}\|x-y\|, \forall x,y\in C,$$

so F is Lipschitz continuous with constant $L=\frac{1}{\sqrt{n}}$. On the other hand,

$$S=\{(x_1,x_2,...,x_n)^T : x_i\in\{-\frac{n\pi}{2};\frac{n\pi}{2}\}\}$$

and

$$S_M=\{(-\frac{n\pi}{2},-\frac{n\pi}{2},...,-\frac{n\pi}{2})^T\}.$$

We use Algorithm YH with $\sigma=0.3,\gamma=0.99$, Algorithm IPAL with parameters $L=\frac{1}{\sqrt{n}},\lambda=\frac{1-\sigma}{L}$, where $\sigma=0.01$, Algorithm 1 with $\mu_k=\frac{1}{(k+1)^{1.5}},\theta=0.99,\eta=0.99,\lambda=0.8,\delta=0.8$ and Algorithm 3 with parameters $\mu_k=\frac{1}{(k+3)^{1.5}}, L=\frac{1}{\sqrt{n}},\theta=0.01,\alpha=\frac{1-\sigma}{L}$, where $\sigma=0.01$, to find a solution of VIP(C,F). The starting point is $x^0=(-\frac{n\pi}{8},-\frac{n\pi}{8},-\frac{n\pi}{8},...,-\frac{n\pi}{8})^T$ in all algorithms and $x^1=(-\frac{n\pi}{8},-\frac{n\pi}{8},-\frac{n\pi}{8},...,-\frac{n\pi}{8})^T$ in Algorithm 1, $x^1=(-\frac{n\pi}{16},-\frac{n\pi}{16},-\frac{n\pi}{16},...,-\frac{n\pi}{16})^T$ in Algorithm 3. To terminate the Algorithms, we use the stopping criteria

n	Alg. YH		Alg. IPAL		Alg. 1		Alg. 3	
	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.
10	1.35	92	0.53	31	0.81	100	0.34	31
50	62.66	462	4.03	75	7.27	620	0.77	78
100	635.09	952	19.03	114	19.61	1224	1.38	117
150	2265.48	1353	44.34	138	206.42	1988	10.23	143
200	5493.66	1835	92.83	175	306.91	2619	12.81	177

Table 5 Experiment for Example 5.

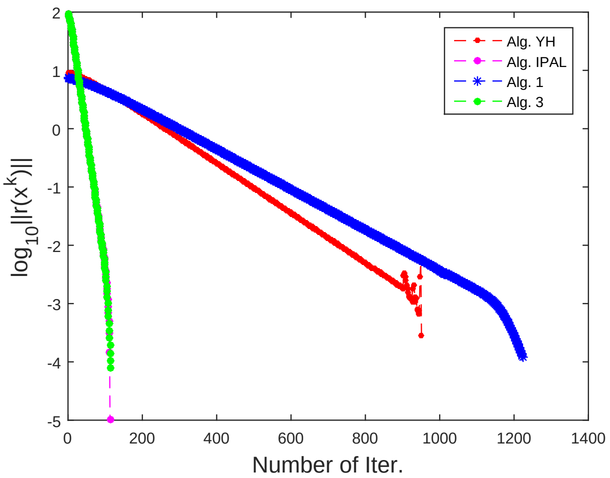

Fig. 5 Number of iterations of Algorithms in Example 5 ($n = 100, x^0 = (-\frac{n\pi}{8}, -\frac{n\pi}{8}, -\frac{n\pi}{8}, \ldots, -\frac{n\pi}{8})^T$).

$\varepsilon = \|r(x^k)\| \leq 10^{-4}$. The numerical results are described in Figure 5, which shows that Algorithm IPAL and Algorithm 3 behave better then Algorithm YH, Algorithm 1. Table 5 show that Algorithm IPAL, Algorithm 3 are better than Algorithm YH, Algorithm 1 in number of steps and CPU time.

Remark 3 It is clearly seen from the results of Example 5 that the inertial Algorithm 3 outperformed the non-inertial Algorithm IPAL in term of time taken for computation.

Conclusion. We have introduced some inertial type algorithms for for finding a solution of a nonmonotone variational inequality problem with or without linesearch. All the proposed algorithms do not use the embedding projections and their convergence is obtained. Some numerical examples are reported to illustrate the convergence and also to show the advantage of the new algorithms over the existing method for solving these problems.

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