Proper Efficiency in Linear Fractional Vector Optimization via Benson's Characterization*

N. T. T. Huong†, C.-F. Wen‡, J.-C. Yao§, and N. D. Yen¶

December 24, 2021

Abstract. Linear fractional vector optimization problems are special nonconvex vector optimization problems. They were introduced and first studied by E. U. Choo and D. R. Atkins in the period 1982–1984. This paper investigates the properness in the sense of Geoffrion of the efficient solutions of linear fractional vector optimization problems with unbounded constraint sets. Sufficient conditions for an efficient solution to be a Geoffrion's properly efficient solution are obtained via Benson's characterization (1979) of Geoffrion's proper efficiency.

Mathematics Subject Classification (2010). 90C31, 90C29, 90C33, 90C32, 90C47.

Key Words. Linear fractional vector optimization, efficient solution, Geoffrion's properly efficient solution, Benson's characterization, tangent cone.

1 Introduction

Introduced and firstly studied by Choo and Atkins [5, 6, 7], *linear fractional vector optimization problems* (LFVOPs) have many applications in management science and other fields. The problems have noteworthy properties and theoretical importance.

Topological properties of the solution sets of those problems and monotone affine vector variational inequalities have been studied by Choo and Atkins [6, 7], Benoist [1,

*The authors were supported respectively by Le Quy Don Technical University (Vietnam), Kaohsiung Medical University (Taiwan), China Medical University (Taiwan), and National Foundation for Science & Technology Development (Vietnam).

†Department of Mathematics, Faculty of Information Technology, Le Quy Don University, 236 Hoang Quoc Viet, Hanoi, Vietnam; email: nguyenhuong2308.mta@gmail.com.

‡Center for Fundamental Science; and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan; email: cfwen@kmu.edu.tw.

§Center for General Education, China Medical University, Taichung, Taiwan; email: yaojc@mail.cmu.edu.tw.

¶Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam; email: ndyen@math.ac.vn.

2], Yen and Phuong [26], Hoa et al. [10, 11, 12], Huong et al. [13, 15], and other authors. Necessary and sufficient conditions of the efficient solutions, stability properties, solution methods, and applications of this class of problems can be seen in [19, 20, 24, 25].

Geoffrion's proper efficiency concept [8], which was proposed for vector optimization problems with the standard ordering cone (the nonnegative orthant of an Euclidean space), has been extended for the case of problems with an arbitrary closed convex ordering cone by Borwein [4] and Benson [3]. Borwein's proper efficiency may differ from that of Geoffrion even if the ordering cone is the standard one. To rectify this situation, Benson's concept of proper efficiency [3] coincides with that of Geoffrion when the ordering cone is the standard one.

It is a well known that there is no difference between efficiency and Geoffrion's proper efficiency in linear vector optimization problem (see [23, Corollary 3.1.1 and Theorem 3.1.4] and [16, Remark 2.4]). By using necessary and sufficient conditions for efficiency in linear fractional vector optimization, Choo [5] has proved that the efficient solution set of a solution of a LFVOP with a *bounded constraint set* coincides with the Geoffrion's properly efficient solution set.

Recently, Huong, Yao, and Yen [16] have given sufficient conditions for an efficient solution of a LFVOP with *an unbounded constraint set* to be a Geoffrion's properly efficient solution via a direct approach. The recession cone of the constraint set and the derivatives of the scalar objective functions at the point in question are used in these sufficient conditions. Two new theorems on Geoffrion's properly efficient solutions of LFVOPs with unbounded constraint sets and seven illustrative examples can be found in a subsequent paper [17] of these authors. Provided that all the components of the objective function are properly fractional, Theorem 3.2 from [17] gives sufficient conditions for the efficient solution set to coincide with the Geoffrion properly efficient solution set. Allowing the objective function to have some affine components, Theorem 3.4 of [17] states sufficient conditions for an efficient solution to be a Geoffrion's properly efficient solution.

Verifiable sufficient conditions for an efficient point of a LFVOP to be a Borwein's properly efficient point have been obtained in [14].

In the present paper, sufficient conditions for an efficient solution of a LFVOP with an unbounded constraint set to belong to Geoffrion's properly efficient solution set are obtained via Benson's characterization of Geoffrion's proper efficiency. The conditions rely on the recession cone of the constraint set, the derivatives of the scalar objective functions, and the tangent cone of the constraint set at the efficient solution. Our result complements Theorems 3.1 and 3.2 of [16] and generalizes the theorem of Choo [5, p. 218] to the case of LFVOPs with arbitrary polyhedral convex constraint sets.

We would like to devote this paper to the 75th birthday of Prof. Phan Quoc Khanh, who has made remarkable research works on proper solutions of vector optimization problems [18] and approximate proper solutions of vector equilibrium problems [9].

The paper organization is as follows. Section 2 recalls some notations, definitions, and known results. Section 3 establishes the main result. Illustrative examples are given in Section 4.

2 Preliminaries

We denote by $\mathbb{N}$ the set of the positive integers. The scalar product and the norm in $\mathbb{R}^n$ are denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Vectors in $\mathbb{R}^n$ are represented by columns of real numbers in matrix calculations, but they are written as rows of real numbers in the text. If A is a matrix, then A^T stands for the transposed matrix of A. Thus, for any $x, y \in \mathbb{R}^n$, one has $\langle x, y \rangle = x^T y$.

Let $M \subset \mathbb{R}^n$ and $\bar{x} \in \overline{M}$, where $\overline{M}$ stands for the topological closure of M. The *Bouligand-Severi tangent cone* (see, e.g., [22]) of M at $\bar{x}$ is the set

$$T(\bar{x}; M) := \big\{ v \in \mathbb{R}^n \ : \ \ \exists \{t_k\} \subset \mathbb{R}_+ \setminus \{0\},\ t_k \to 0,\ \exists \{v^k\} \subset \mathbb{R}^n,\ v^k \to v, \\ \bar{x} + t_k v^k \in M \ \ \forall k \in \mathbb{N} \big\}.$$

It is well known that $T(\bar{x}; M)$ is a closed cone, which may be nonconvex if M is a nonconvex set. When M is convex, one has $T(\bar{x}; M) = \overline{\rm cone}(M - \bar{x})$ with

$$\operatorname{cone} Q = \{\lambda u \ : \ \lambda > 0,\ u \in Q\}$$

for any $Q \subset \mathbb{R}^n$ and $\overline{\rm cone}\, Q := \overline{\operatorname{cone} Q}$.

A nonzero vector $v \in \mathbb{R}^n$ (see [21, p. 61]) is said to be a *direction of recession* of a nonempty convex set $M \subset \mathbb{R}^n$ if $x + tv \in M$ for every $t \geq 0$ and every $x \in M$. The set composed by $0 \in \mathbb{R}^n$ and all the directions $v \in \mathbb{R}^n \setminus \{0\}$ satisfying the last condition, is called the *recession cone* of M and denoted by 0^+M. If M is closed and convex, then $0^+M = \{v \in \mathbb{R}^n \ : \ \exists x \in \Omega \ \text{s.t.}\ x + tv \in M \ \text{ for all } \ t > 0\}$.

Lemma 2.1 (See, e.g., [16, Lemma 2.10]) *Let $C \subset \mathbb{R}^n$ be closed and convex, $\bar{x} \in C$, and let $\{x^p\}$ be a sequence in $C \setminus \{\bar{x}\}$ with $\lim\limits_{p\to\infty} \|x^p\| = +\infty$. If $\lim\limits_{p\to\infty} \dfrac{x^p - \bar{x}}{\|x^p - \bar{x}\|} = v$, then $v \in 0^+C$.*

For any $\bar{x} \in K$, where K is a convex set, one has $0^+K \subset T_K(\bar{x})$. Consider *linear fractional functions* $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, m$, of the form

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i},$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let K be a *polyhedral convex set*, i.e., there exist $p \in \mathbb{N}$, a matrix $C = (c_{ij}) \in \mathbb{R}^{p\times n}$, and a vector $d = (d_i) \in \mathbb{R}^p$ such that $K = \big\{x \in \mathbb{R}^n \ : \ Cx \leq d\big\}$.

We assume that $b_i^T x + \beta_i > 0$ for all $i \in I$ and $x \in K$, where $I := \{1, \cdots, m\}$. Put $f(x) = (f_1(x), \ldots, f_m(x))$ and let

$$\Omega = \big\{x \in \mathbb{R}^n \ : \ b_i^T x + \beta_i > 0,\ \ \forall i \in I\big\}.$$

Clearly, Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. The *linear fractional vector optimization problem* (LFVOP) given by f and K is formally written as

$$\text{(VP)} \qquad\qquad \text{Minimize } \ f(x) \ \ \text{subject to } \ x \in K.$$

Definition 2.2 A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathbb{R}^m_+ \setminus \{0\}\big) = \emptyset$, where $\mathbb{R}^m_+$ denotes the nonnegative orthant in $\mathbb{R}^m$. One calls $x \in K$ a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\operatorname{int} \mathbb{R}^m_+\big) = \emptyset$, where $\operatorname{int} \mathbb{R}^m_+$ abbreviates the topological interior of $\mathbb{R}^m_+$.

The efficient solution set (resp., the weakly efficient solution set) of (VP) are denoted, respectively, by E and E^w.

Lemma 2.3 (See, e.g., [20] and [19, Lemma 8.1]) *Let* $\varphi(x) = \dfrac{a^T x + \alpha}{b^T x + \beta}$ *be a linear fractional function defined by* $a, b \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$. *Suppose that* $b^T x + \beta \neq 0$ *for every* $x \in K_0$, *where* $K_0 \subset \mathbb{R}^n$ *is an arbitrary polyhedral convex set. Then, one has*

$$\varphi(y) - \varphi(x) = \frac{b^T x + \beta}{b^T y + \beta} \langle \nabla \varphi(x), y - x \rangle,$$

for any $x, y \in K_0$, *where* $\nabla \varphi(x)$ *denotes the Fréchet derivative of* φ *at* x.

Definition 2.4 (See [8, p. 618]) One says that $\bar{x} \in E$ is a *Geoffrion's properly efficient solution* of (VP) if there exists a scalar $M > 0$ such that, for each $i \in I$, whenever $x \in K$ and $f_i(x) < f_i(\bar{x})$ one can find an index $j \in I$ such that $f_j(x) > f_j(\bar{x})$ and $A_{i,j}(\bar{x}, x) \le M$ with $A_{i,j}(\bar{x}, x) := \dfrac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})}$.

For LFVOPs, the ordering cone is the standard one. So, the notion of properly efficient solution in the sense of Benson [3] is as follows.

Definition 2.5 ([3, Def. 2.4]) An element $\bar{x} \in K$ is called a *Benson properly efficient solution* of (VP) if

$$\overline{\text{cone}}\,\big(f(K) + \mathbb{R}^m_+ - f(\bar{x})\big) \cap \big(-\mathbb{R}^m_+\big) = \{0\}. \tag{2.1}$$

The Benson properly efficient solution set of (VP) is denoted by E^{Be}. Since (2.1) surely yields $\big(f(K) - f(\bar{x})\big) \cap \big(-\mathbb{R}^m_+\big) = \{0\}$, property (2.1) implies that $\bar{x} \in E$. Applying [3, Theorem 3.2] to (VP), we get the following result.

Proposition 2.6 *One has* $E^{Ge} = E^{Be}$, *i.e., the Benson properly efficient solution set of* (VP) *coincides with the Geoffrion properly efficient solution set of that problem.*

The equality $E^{Ge} = E^{Be}$ allows us to use the criterion (2.1) to verify whether $\bar{x}$ is a properly efficient solution of (VP) in the sense of Geoffrion, or not. Sometimes, checking (2.1) is easier than checking the condition in Definition 2.4. Next theorem is due to Choo [5].

Remark 2.7 (See [5, p. 218]) *If* K *is bounded, then* $E = E^{Ge}$.

The following lemma is straightforward but useful and interesting in itself. We thank the anonymous reviewer for providing us with this and so Section 3 will have the shorter proof in our scheme.

Lemma 2.8 *Let $A \subseteq \mathbb{R}^m$. One has*

$$\overline{\text{cone}}\,\big(A + \mathbb{R}^m_+\big) \cap \big(-\mathbb{R}^m_+\big) = \{0\} \Leftrightarrow \overline{\text{cone}}\,\big(A\big) \cap \big(-\mathbb{R}^m_+\big) = \{0\}.$$

Proof. The implication ($\Rightarrow$) is clear because $A \subset A + \mathbb{R}^m_+$. For ($\Leftarrow$), suppose to the contrary that $\overline{\text{cone}}\,\big(A\big) \cap \big(-\mathbb{R}^m_+\big) = \{0\}$, but there are some $v \in -\mathbb{R}^m_+$, $v \neq 0$, $t_k > 0$, $r^k \in \mathbb{R}^m_+$ and $a^k \in A$, $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \big[t_k(a^k + r^k)\big] = v.$$

Setting $u^k = t_k r^k \in \mathbb{R}^m_+$, we have

$$v = \lim_{k\to\infty} \big(t_k a^k + u^k\big).$$

If the sequence $\{u^k\}$ is bounded, we may assume it converge to some $u \in \mathbb{R}^m_+$. This implies $\lim\limits_{k\to\infty} t_k a^k = v - u \in -\mathbb{R}^m_+ \setminus \{0\} - \mathbb{R}^m_+ = -\mathbb{R}^m_+ \setminus \{0\}$ that contradicts the hypothesis. Consider the case $\{u^k\}$ is unbounded, by considering a subsequence (if necessary), we may assume that $\lim\limits_{k\to\infty} \|u^k\| = +\infty$ and $u^k \neq 0$ for all k. Furthermore, there is no loss of generality in assuming that $\lim\limits_{k\to\infty} \big(\|u^k\|^{-1} u^k\big) = z$, where $z \in \mathbb{R}^m_+ \setminus \{0\}$. Then

$$0 = \lim_{k\to\infty} \frac{v}{\|u^k\|} = \lim_{k\to\infty} \Big(\frac{t_k}{\|u^k\|} a^k + \frac{u^k}{\|u^k\|}\Big).$$

We arrive at a contradiction that $-z = \lim\limits_{k\to\infty} \dfrac{t_k}{\|u^k\|} a^k \in \overline{\text{cone}}\,\big(A\big)$.

3 Sufficient Conditions for the Geoffrion Proper Efficiency

In this section, we will establish a new theorem on the Geoffrion proper efficiency LFVOPs. It is proved by using the criterion of Benson for the Geoffrion proper efficiency, which has been recalled in Proposition 2.6.

Note that some objective functions of (VP) may be linear (affine, to be more precise), i.e., one may have $f_i(x) = a_i^T x + \alpha_i$ for some $i \in I$. Let $I_1 := \{i \in I \,:\, b_i \neq 0\}$. Then, $b_i = 0$ and $\beta_i = 1$ for all $i \in I_0$, where $I_0 := I \setminus I_1$.

Lemma 3.1 *If for some $u \in T(\bar{x}; K) \setminus \{0\}$ where $\bar{x} \in K$ one has $\langle \nabla f_i(\bar{x}), u\rangle \leq 0$ for all $i \in I$ and at least one inequality is strict, then $\bar{x}$ is not efficient.*

Proof. Let $u \in T(\bar{x}; K) \setminus \{0\}$, $\bar{x} \in K$. As K is a polyhedral convex set, there is a number $\tau > 0$ such that $[\bar{x}, \bar{x} + \tau u] \subset K$. Hence, for any fixed $t \in (0, \tau]$, by Lemma 2.3 one has

$$f_i(\bar{x} + tu) - f_i(\bar{x}) = \frac{b_i^T \bar{x} + \beta_i}{b_i^T(\bar{x} + tu) + \beta_i} \langle \nabla f_i(\bar{x}), tu\rangle \quad (i \in I). \tag{3.1}$$

Since $b_i^T x + \beta_i > 0$ for all $x \in K$, $i \in I$ and $\langle \nabla f_i(\bar{x}), u\rangle \leq 0$, for all i ∈ I, from (3.1) it follows that

$$f_i(\bar{x} + tu) \leq f_i(\bar{x}) \quad (\forall i \in I). \tag{3.2}$$

Since at least one inequality in $\langle \nabla f_i(\bar{x}), u\rangle \leq 0$, for all i ∈ I, is strict, we have $f_{i_0}(\bar{x} + t\bar{u}) < f_{i_0}(\bar{x})$. Combining the latter with (3.2) implies $\bar{x} \notin E$.

Theorem 3.2 *Assume that* $\bar{x} \in E$. *If* K *is bounded, then* $\bar{x} \in E^{Ge}$. *In the case where* K *is unbounded, if the regularity assumptions*

$$\begin{cases} \textit{There is no } z \in T(\bar{x}; K) \setminus \{0\} \textit{ such that} \\ \langle \nabla f_i(\bar{x}), z \rangle = 0 \textit{ for all } i \in I \end{cases} \tag{3.3}$$

and

$$\begin{cases} \textit{For any } z \in (0^+K) \setminus \{0\},\ a_i^T z > 0 \textit{ for all } i \in I_0 \\ \textit{and } b_i^T z > 0 \textit{ for all } i \in I_1, \end{cases} \tag{3.4}$$

are satisfied, then $\bar{x} \in E^{Ge}$.

Proof. If K is bounded, then by Remark 2.7 one has $\bar{x} \in E^{Ge}$. Now, consider the situation where K is unbounded. Suppose to the contrary that $\bar{x} \notin E^{Ge}$, that is, due to Lemma 2.8 and Proposition 2.6, there are some $v = (v_1, \ldots v_m) \in -\mathbb{R}^m_+ \setminus \{0\}$, $v^k \in \operatorname{cone}\big(f(K) - f(\bar{x})\big)$ for all $k \in \mathbb{N}$ with $\lim\limits_{k\to\infty} v^k = v \leq 0$ and there exists $i_0 \in I$ such that $v_{i_0} < 0$. Then, there exist $x^k \in K$, $\tau_k > 0$ such that $v_i^k = \tau_k\big(f(x^k) - f(\bar{x})\big)$. By Lemma 2.3 one has

$$v_i^k = \frac{b_i^T \bar{x} + \beta_i}{b_i^T x^k + \beta_i} \langle \nabla f_i(\bar{x}), \tau_k (x^k - \bar{x}) \rangle \quad (i \in I). \tag{3.5}$$

Case 1: *The sequence* $\{x^k\}$ *is bounded.* In this case, we may assume that $\{x^k\}$ converges to a point $\hat{x} \in K$. Then, we have $\lim\limits_{k\to\infty} \big(b_i^T x^k + \beta_i\big) = b_i^T \hat{x} + \beta_i > 0$.

If the sequence $\{\tau_k(x^k - \bar{x})\}$ is bounded, we may assume that $\lim\limits_{k\to\infty} \big[\tau_k(x^k - \bar{x})\big] = \bar{u}$, where $\bar{u} \in \mathbb{R}^n$. If $\bar{u} = 0$ then, passing (3.5) to limit as $k \to \infty$, we get $v_i = 0$ for all $i \in I$, which contradict the property $v \in -\mathbb{R}^m_+ \setminus \{0\}$. If $\bar{u} \neq 0$, then passing (3.5) to the limit as $k \to \infty$ gives

$$v_i = \frac{b_i^T \bar{x} + \beta_i}{b_i^T \hat{x} + \beta_i} \langle \nabla f_i(\bar{x}), \bar{u} \rangle \quad (i \in I). \tag{3.6}$$

This implies that

$$\langle \nabla f_i(\bar{x}), \bar{u} \rangle = \frac{b_i^T \hat{x} + \beta_i}{b_i^T \bar{x} + \beta_i} v_i \leq 0 \quad (i \in I) \tag{3.7}$$

Since $v_{i_0} < 0$ from (3.6) it follows that $\langle \nabla f_{i_0}(\bar{x}), \bar{u} \rangle < 0$. Combining this with (3.7), one has $\bar{x} \notin E$ by Lemma 3.1.

If $\{\tau_k(x^k - \bar{x})\}$ is unbounded, we may assume that $\lim\limits_{k\to\infty} \tau_k \|(x^k - \bar{x})\| = +\infty$. Put $z^k = \|x^k - \bar{x}\|^{-1}\big(x^k - \bar{x}\big)$. Hence, by the closedness of the Bouligand-Severi tangent cone, $\lim\limits_{k\to\infty} z^k = \bar{z} \in T(\bar{x}; K)$. For every k, from (3.5) it follows that

$$\frac{v_i^k}{\tau_k \|(x^k - \bar{x})\|} = \frac{b_i^T \bar{x} + \beta_i}{b_i^T x^k + \beta_i} \left\langle \nabla f_i(\bar{x}), z^k \right\rangle \quad (i \in I). \tag{3.8}$$

Passing the inequalities in (3.8) to limit as $k \to \infty$ and note that $b_i^T x + \beta_i > 0$ for all $x \in K$, $i \in I$, one has

$$\langle \nabla f_i(\bar{x}), \bar{w} \rangle = 0 \quad (i \in I). \tag{3.9}$$

This is in contradiction with assumption (3.3).

CASE 2: $\{x^k\}$ *is unbounded.* By taking a subsequence if necessary we may assume that $\lim_{k\to\infty} \|x^k\| = +\infty$ and $x^k \neq \bar{x}$ for all $k \in \mathbb{N}$. By (3.5), one has

$$v_i^k = \frac{b_i^T \bar{x} + \beta_i}{b_i^T x^k + \beta_i} \langle \nabla f_i(\bar{x}), \tau_k(x^k - \bar{x}) \rangle \quad (i \in I_1). \tag{3.10}$$

Since $b_i = 0$, $\beta_i = 1$, and $\nabla f_i(\bar{x}) = a_i$ for all $i \in I_0$, by (3.5) one has

$$v_i^k = a_i^T [\tau_k(x^k - \bar{x})] \quad (i \in I_0). \tag{3.11}$$

From (3.10) it follows that

$$v_i^k = \frac{b_i^T \bar{x} + \beta_i}{\frac{b_i^T (x^k - \bar{x})}{\|x^k - \bar{x}\|} + \frac{\beta_i}{\|x^k - \bar{x}\|} + \frac{b_i^T \bar{x}}{\|x^k - \bar{x}\|}} \langle \nabla f_i(\bar{x}), \tau_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \rangle \tag{3.12}$$

for every $i \in I_1$.

If $\{\tau_k\}$ is bounded, we may assume that $\lim_{k\to\infty} \tau_k = \bar{\tau}$. Clearly, $\bar{\tau} \geq 0$. Then $\lim_{k\to\infty} \tau_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = \lim_{k\to\infty} \tau_k z^k = \bar{\tau}\bar{z}$ with $\|z\| = 1$, and $z \in (0^+K) \setminus \{0\}$.

First, suppose that $\bar{\tau} \neq 0$. By the regularity condition (3.4), we have $b_i^T \bar{z} > 0$ for every $i \in I_1$. Taking the limits in (3.12) as $k \to \infty$, we get

$$v_i = \bar{\tau} \frac{b_i^T \bar{x} + \beta_i}{b_i^T \bar{z}} \langle \nabla f_i(\bar{x}), \bar{z} \rangle \quad (i \in I_1). \tag{3.13}$$

This means that

$$\langle \nabla f_i(\bar{x}), \bar{z} \rangle = \bar{\tau} \frac{b_i^T \bar{z}}{b_i^T \bar{x} + \beta_i} v_i \leq 0 \quad (i \in I_1). \tag{3.14}$$

From (3.11) it follows that

$$\frac{v_i^k}{\|x^k - \bar{x}\|} = a_i^T \tau_k \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \quad (i \in I_0). \tag{3.15}$$

Taking the limits in (3.15) as $k \to \infty$, we get

$$0 = a_i^T \bar{\tau} \bar{z} \quad (i \in I_0). \tag{3.16}$$

This means that $\langle \nabla f_i(\bar{x}), \bar{z} \rangle = 0$ for all $i \in I_0$. Since $v_{i_0} < 0$ from (3.14) and (3.16) it follows that $i_0 \in I_1$ and $\langle \nabla f_{i_0}(\bar{x}), \bar{u} \rangle < 0$. Then, by Lemma 3.1, $\bar{x} \notin E$, a contradiction.

Now, suppose that $\bar{\tau} = 0$. Letting $k \to \infty$, from (3.12) we get

$$v_i = 0 \quad (i \in I_1). \tag{3.17}$$

By (3.11), for every $i \in I_0$, one has

$$v_i^k = \tau_k \|x^k - \bar{x}\| a_i^T \left(\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right) = \tau_k \|x^k - \bar{x}\| \langle \nabla f_i(\bar{x}), z^k \rangle. \tag{3.18}$$

Since $\lim_{k\to\infty} z^k = \lim_{k\to\infty} \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = \bar{z}$, where $\bar{z}$ is a unit vector belonging to the recession cone 0^+K, by the regularity condition (3.4) we have $\langle \nabla f_i(\bar{x}), \bar{z}\rangle = a_i^T \bar{z} > 0$ for every $i \in I_0$. Then, there exists an integer k_0 such that $\tau_k \langle \nabla f_i(\bar{x}), z^k\rangle > 0$ for all $k \ge k_0$. For each $i \in I_0$, combining this with (3.18) we get $v_i^k \ge 0$ for all $k \ge k_0$. Hence, passing the inequality $v_i^k \ge 0$ to limit as $k \to \infty$ gives

$$v_i \ge 0 \quad (i \in I_0). \tag{3.19}$$

The inequalities in (3.17) and (3.19) mean that $v \ge 0$. We have thus arrived at a contradiction, because $v \le 0$ and $v_{i_0} < 0$.

If $\{\tau_k\}$ is unbounded, we may assume that $\lim_{k\to\infty} \tau_k = +\infty$. From (3.12) it follows that

$$\frac{v_i^k}{\tau_k} = \frac{b_i^T \bar{x} + \beta_i}{\dfrac{b_i^T (x^k - \bar{x})}{\|x^k - \bar{x}\|} + \dfrac{\beta_i}{\|x^k - \bar{x}\|} + \dfrac{b_i^T \bar{x}}{\|x^k - \bar{x}\|}} \langle \nabla f_i(\bar{x}), \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \rangle \tag{3.20}$$

for every $i \in I_1$. By (3.11), one has

$$\frac{v_i^k}{\tau_k \|x^k - \bar{x}\|} = a_i^T \left(\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \right) \qquad (i \in I_0). \tag{3.21}$$

By condition (3.4), $b_i^T \bar{z} > 0$ for every $i \in I_1$. Since $\lim_{k\to\infty} \|x^k\| = +\infty$, one has $\lim_{k\to\infty} \|x^k - \bar{x}\| = +\infty$. Not that $\bar{z} \in (0^+K) \setminus \{0\} \subset T(\bar{x}; K) \setminus \{0\}$. Passing (3.20) to limit as $k \to \infty$, we get

$$\langle \nabla f_i(\bar{x}), \bar{z}\rangle = 0 \quad (i \in I_1). \tag{3.22}$$

Passing (3.21) to limit as $k \to \infty$, we get $0 = a_i^T \bar{z} \quad (i \in I_0)$. Hence

$$\langle \nabla f_i(\bar{x}), \bar{z}\rangle = 0 \quad (i \in I_0). \tag{3.23}$$

(3.22) and (3.23) give a contradiction with (3.3). □

4 Illustrative Examples

To show the usefulness of Theorem 3.2, we will apply it to some examples, which were analyzed in [16] by other results and methods.

Example 4.1 (See [7, Example 2]) Consider problem (VP) with

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \ge 2,\ 0 \le x_2 \le 4\},$$

$$f_1(x) = \frac{-x_1}{x_1 + x_2 - 1}, \quad f_2(x) = \frac{-x_1}{x_1 - x_2 + 3}.$$

It is well known that $E = E^w = \{(x_1, 0) \,:\, x_1 \ge 2\} \cup \{(x_1, 4) \,:\, x_1 \ge 2\}$. Since $I_1 = I$ and $0^+K = \{v = (v_1, 0) \,:\, v_1 \ge 0\}$, condition (3.4) is fulfilled. For any $x = (x_1, x_2) \in K$, one has

$$\nabla f_1(x) = \begin{pmatrix} \dfrac{-x_2 + 1}{(x_1 + x_2 - 1)^2} \\ \dfrac{x_1}{(x_1 + x_2 - 1)^2} \end{pmatrix}, \quad \nabla f_2(x) = \begin{pmatrix} \dfrac{x_2 - 3}{(x_1 - x_2 + 3)^2} \\ \dfrac{-x_1}{(x_1 - x_2 + 3)^2} \end{pmatrix}.$$

So, for any $\bar{x} \in \{(\bar{x}_1, 0) : \bar{x}_1 \geq 2\} \cup \big\{(x_1, 4) : x_1 \geq 2\big\}$ and $v = (v_1, v_2) \in \mathbb{R}^2$, one sees that

$$\begin{cases} \langle \nabla f_1(\bar{x}), v\rangle = 0 \\ \langle \nabla f_2(\bar{x}), v\rangle = 0 \end{cases} \iff \begin{cases} v_1 = 0 \\ v_2 = 0. \end{cases}$$

Hence, condition (3.3) is satisfied for any $\bar{x} \in E$. Thus, by Theorem 3.2 we can assert that $E^{Ge} = E$.

Example 4.2 (See [11, p. 483]) Consider problem (VP) where $n = m = 3$,

$$K = \big\{x \in \mathbb{R}^3 : \ x_1 + x_2 - 2x_3 \leq 1,\ x_1 - 2x_2 + x_3 \leq 1, \\ -2x_1 + x_2 + x_3 \leq 1,\ x_1 + x_2 + x_3 \geq 1\big\},$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{x_1 + x_2 + x_3 - \dfrac{3}{4}} \qquad (i = 1, 2, 3).$$

According to [11], one has

$$\begin{aligned} E = E^w = \ & \{(x_1, x_2, x_3) : x_1 \geq 1,\ x_3 = x_2 = x_1 - 1\} \\ & \cup \{(x_1, x_2, x_3) : x_2 \geq 1,\ x_3 = x_1 = x_2 - 1\} \\ & \cup \{(x_1, x_2, x_3) : x_3 \geq 1,\ x_2 = x_1 = x_3 - 1\}. \end{aligned} \tag{4.1}$$

Since $0^+K = \{v = (\tau, \tau, \tau) : \tau \geq 0\}$ and $I_1 = I$, it is easy to verify that condition (3.4) is satisfied. Now, setting $p(x) = \big(x_1 + x_2 + x_3 - \dfrac{3}{4}\big)^2$, one has

$$\nabla f_1(x) = \frac{1}{p(x)} \left(-x_2 - x_3 + \frac{1}{4}, x_1 - \frac{1}{2}, x_1 - \frac{1}{2}\right),$$

$$\nabla f_2(x) = \frac{1}{p(x)} \left(x_2 - \frac{1}{2}, -x_1 - x_3 + \frac{1}{4}, x_2 - \frac{1}{2}\right),$$

$$\nabla f_3(x) = \frac{1}{p(x)} \left(x_3 - \frac{1}{2}, x_3 - \frac{1}{2}, -x_1 - x_2 + \frac{1}{4}\right).$$

Given any $\bar{x} \in E$ and $v = (\tau, \tau, \tau) \in 0^+K$, by (4.1) we see that one of the following situations must occur: (i) $x_1 \geq 1,\ x_3 = x_2 = x_1 - 1$; (ii) $x_2 \geq 1,\ x_3 = x_1 = x_2 - 1$; (iii) $x_3 \geq 1,\ x_2 = x_1 = x_3 - 1$. If (i) occurs (resp., (ii), or (iii) occurs), then the equality $\langle \nabla f_1(\bar{x}), v\rangle = 0$ (resp., $\langle \nabla f_2(\bar{x}), v\rangle = 0$, or $\langle \nabla f_3(\bar{x}), v\rangle = 0$) means that $\frac{1}{4}\tau = 0$. Thus, condition (3.3) is fulfilled for any $\bar{x} \in E$, and we have $E^{Ge} = E$ by Theorem 3.2.

Example 4.3 (See [11, pp. 479–480]) Consider problem (VP) where $n = m$, $m \geq 2$,

$$K = \Big\{x \in \mathbb{R}^m : x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_m \geq 0,\ \sum_{k=1}^{m} x_k \geq 1\Big\},$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{\displaystyle\sum_{k=1}^{m} x_k - \dfrac{3}{4}} \qquad (i = 1, \ldots, m).$$

Note that $0^+K = \mathbb{R}^m_+$. Setting $q(x) = \left(\sum_{k=1}^m x_k - \frac{3}{4}\right)^2$, we have

$$\nabla f_i(x) = \frac{1}{q(x)}\left(x_i - \frac{1}{2}, ..., -\sum_{k\neq i} x_k + \frac{1}{4}, ..., x_i - \frac{1}{2}\right)$$

where the expression $-\sum_{k\neq i} x_k + \frac{1}{4}$ is the $i-$th component of $\nabla f_i(x)$. Hence, the equality $E^{Ge} = E$ can be proved by using Theorem 3.2 similarly as it has been done in the preceding example.

Example 4.4 (See [16, Example 2.6]) Consider the problem (VP) where

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0,\ x_2 \geq 0\},$$
$$f_1(x) = -x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}.$$

As it has been shown in [16], $E = \{(x_1, 0) \,:\, x_1 \geq 0\}$ and $E^{Ge} = \emptyset$. To check the conditions in Theorem 3.2, note that $I_0 = \{1\}$, $I_1 = \{2\}$, $a_1 = (0, -1)^T$, $b_2 = (1, 1)^T$, and $0^+K = K$. For every efficient solution $\bar{x} = (\bar{x}_1, 0)$, $\bar{x}_1 > 0$, one has

$$\nabla f_1(\bar{x}) = (0, -1)^T, \quad \nabla f_2(\bar{x}) = \begin{pmatrix} 0 \\ \dfrac{1}{\bar{x}_1 + 1} \end{pmatrix},$$

and $T_K(\bar{x}) = \{v = (v_1, v_2) \,:\, v_1 \in \mathbb{R},\ v_2 \geq 0\}$. Hence (3.3) and (3.4) are violated if one choses $v = (1, 0) \in (0^+K) \setminus \{0\} \subset T_K(\bar{x}) \setminus \{0\}$. For $\bar{x} = (0, 0)$ we have $T_K(\bar{x}) = \mathbb{R}^2_+$. Conditions (3.3) and (3.4) are violated if one choses $v = (1, 0)$. The violation of the regularity conditions in Theorem 3.2 is a reason for $\bar{x} \notin E^{Ge}$.

Example 4.5 (See [16, Example 4.7]) Consider problem (VP) with $m = 3$, $n = 2$,

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0,\ x_2 \geq 0\},$$
$$f_1(x) = -x_1 - x_2, \quad f_2(x) = \frac{x_2}{x_1 + x_2 + 1}, \quad f_3(x) = x_1 - x_2.$$

According to [16], $E = \{x = (x_1, x_2) \,:\, x_1 \geq 0,\ x_2 \geq 0,\ x_2 < x_1 + 1\}$, while

$$E^w = \{x = (x_1, x_2) \,:\, x_1 \geq 0,\ x_2 \geq 0,\ x_2 \leq x_1 + 1\}.$$

Let us prove that $E^{Ge} = \emptyset$. Taking any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$, one has $\bar{x}_1 \geq 0$, $\bar{x}_2 \geq 0$ and $\bar{x}_2 < \bar{x}_1 + 1$. Since $(1, 1) \in 0^+K$, we see that $x^p := \bar{x} + p(1, 1)$ belongs to K for any $p \in \mathbb{N}$. One has $f_1(x^p) < f_1(\bar{x})$ and $f_2(x^p) > f_2(\bar{x})$, while $f_3(x^p) = f_3(\bar{x})$. As observed in Section 2, we will have $\bar{x} \notin E^{Ge}$ if for every scalar $M > 0$ there exist $x \in K$ and $i \in I$ with $f_i(x) < f_i(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x) > f_j(\bar{x})$, one has $A_{i,j}(\bar{x}, x) > M$. For each $p \in N$, we choose $i = 1$. Then, $f_i(x^p) < f_i(\bar{x})$ and $j = 2$ is the unique index in I satisfying $f_j(x^p) > f_j(\bar{x})$. Moreover, for $(i, j) = (1, 2)$, we have

$$\begin{aligned} A_{i,j}(\bar{x}, x^p) = A_{1,2}(\bar{x}, x^p) &= \frac{f_1(\bar{x}) - f_1(x^p)}{f_2(x^p) - f_2(\bar{x})} \\ &= \frac{-\bar{x}_1 - \bar{x}_2 - (-\bar{x}_1 - \bar{x}_2 - 2p)}{\dfrac{\bar{x}_2 + p}{\bar{x}_1 + \bar{x}_2 + 1 + 2p} - \dfrac{\bar{x}_2}{\bar{x}_1 + \bar{x}_2 + 1}} \\ &= \frac{2(\bar{x}_1 + \bar{x}_2 + 1 + 2p)(\bar{x}_1 + \bar{x}_2 + 1)}{\bar{x}_1 + 1 - \bar{x}_2}. \end{aligned}$$

Since $\bar{x}_1 \geq 0$, $\bar{x}_2 \geq 0$ and $\bar{x}_2 < \bar{x}_1 + 1$, one has $\lim\limits_{p\to\infty} A_{1,2}(\bar{x}, x^p) = +\infty$. So, for every $M > 0$, there exist $p \in N$ and $i \in I$ with $f_i(x^p) < f_i(\bar{x})$ such that, for all $j \in I$ satisfying $f_j(x^p) > f_j(\bar{x})$, one has $A_{i,j}(\bar{x}, x^p) > M$. This proves that $\bar{x} \notin E^{Ge}$.

The fact that $E^{Ge} = \emptyset$ can also be proved by using Proposition 2.6. Indeed, take an element $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$ and construct the sequence $\{x^p\} \subset K$ as above. We need to show that (2.1) is not satisfied. For every $p \in \mathbb{N}$, choosing $u^p = (0,0,0) \in \mathbb{R}^3_+$ and $t_p = \dfrac{1}{p}$, one has

$$\lim_{p\to\infty} t_p(f(x^p) + u^p - f(\bar{x})) = \lim_{p\to\infty} \frac{1}{p} \begin{pmatrix} f_1(\bar{x} + p(1,1)) - f_1(\bar{x}) \\ f_2(\bar{x} + p(1,1)) - f_2(\bar{x}) \\ f_3(\bar{x} + p(1,1)) - f_3(\bar{x}) \end{pmatrix}$$

$$= \lim_{p\to\infty} \frac{1}{p} \begin{pmatrix} -2p \\ \dfrac{p(\bar{x}_1 + 1 - \bar{x}_2)}{(\bar{x}_1 + \bar{x}_2 + 1 + 2p)(\bar{x}_1 + \bar{x}_2 + 1)} \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \in -\mathbb{R}^3_+.$$

This means that $\overline{\text{cone}}\,\big(f(K) + \mathbb{R}^m_+ - f(\bar{x})\big) \cap \big(-\mathbb{R}^m_+\big) \neq \{0\}$. Thus $\bar{x}$ is not a Benson's properly efficient solution of (VP). So, by Proposition 2.6, $\bar{x} \notin E^{Ge}$. Since $\bar{x} \in E$ can be chosen arbitrarily, we can assert that $E^{Ge} = \emptyset$.

Now, let us check the regularity conditions (3.3) and (3.4) in Theorem 3.2. One has $I_0 = \{1,3\}$, $I_1 = \{2\}$, and $0^+K = K = \mathbb{R}^2_+$. Since $\nabla f_1 \bar{x}) = (-1,-1)$ and $\nabla f_3(\bar{x}) = (1,-1)$ for every $\bar{x} = (\bar{x}_1, \bar{x}_2) \in E$, one simultaneously has $\langle \nabla f_1(\bar{x}), z\rangle = 0$ and $\langle \nabla f_3(\bar{x}), z\rangle = 0$ for $z = (z_1, z_2) \in T(\bar{x}; K) \setminus \{0\}$ only if $z = (0,0)$. So, (3.3) is fulfilled for all $\bar{x} \in E$. However, choosing $i = 3$ and $z = (1,1) \in (0^+K) \setminus \{0\}$, one has $i \in I_0$ and $a_i^T z = 0$. Hence, (3.4) is violated.

We thank the anonymous reviewers for their valuable suggestions and great help.

References

[1] J. Benoist, *Connectedness of the efficient set for strictly quasiconcave sets*, J. Optim. Theory Appl. 96 (1998), pp. 627–654.

[2] J. Benoist, *Contractibility of the efficient set in strictly quasiconcave vector maximization*, J. Optim. Theory Appl. 110 (2001), pp. 325–336.

[3] B. Benson, *An improved definition of proper efficiency for vector maximization with respect to cones*, J. Math. Anal. Appl. 71 (1979), pp. 232–241.

[4] J. M. Borwein, *Proper efficient points for maximizations with respect to cones*, SIAM J. Control Optim. 15 (1977), pp. 57–63.

[5] E. U. Choo, *Proper efficiency and the linear fractional vector maximum problem*, Oper. Res. 32 (1984), pp. 216–220 (http://www.jstor.org/stable/170530).

[6] E. U. Choo and D. R. Atkins, *Bicriteria linear fractional programming*, J. Optim. Theory Appl. 36 (1982), pp. 203–220.

[7] E. U. Choo and D. R. Atkins, *Connectedness in multiple linear fractional programming*, Management Science 29 (1983), pp. 250–255.

[8] A. M. Geoffrion, *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl. 22 (1968), pp. 613–630.

[9] L. P. Hai, L. Huerga, P. Q. Khanh, and V. Novo, *Variants of the Ekeland variational principle for approximate proper solutions of vector equilibrium problems*, J. Global Optim. 74 (2019), pp. 361—382.

[10] T. N. Hoa, N. Q. Huy, T. D. Phuong, and N. D. Yen, *Unbounded components in the solution sets of strictly quasiconcave vector maximization problems*, J. Global Optim. 37 (2007), pp. 1–10.

[11] T. N. Hoa, T. D. Phuong, and N. D. Yen, *Linear fractional vector optimization problems with many components in the solution sets*, J. Industr. Manag. Optim. 1 (2005), pp. 477–486.

[12] T. N. Hoa, T. D. Phuong, and N. D. Yen, *On the parametric affine variational inequality approach to linear fractional vector optimization problems*, Vietnam J. Math. 33 (2005), pp. 477–489.

[13] N. T. T. Huong, T. N. Hoa, T. D. Phuong, and N. D. Yen, *A property of bicriteria affine vector variational inequalities*, Appl. Anal. 10 (2012), pp. 1867–1879.

[14] N. T. T. Huong, N. N. Luan, N. D. Yen, and X. P. Zhao, *The Borwein proper efficiency in linear fractional vector optimization*, J. Nonlinear Convex Anal. 20 (2019), pp. 2579–2595.

[15] N. T. T. Huong, J.-C. Yao, and N. D. Yen, *Connectedness structure of the solution sets of vector variational inequalities*, Optimization 66 (2017), pp. 889–901.

[16] N. T. T. Huong, J.-C. Yao, and N. D. Yen, *Geoffrion's proper efficiency in linear fractional vector optimization with unbounded constraint sets*, J. Optim. Global Optim. 78 (2020), pp. 545–562.

[17] N. T. T. Huong, J.-C. Yao, and N. D. Yen, *New results on proper efficiency for a class of vector optimization problems*, Applicable Analysis, First Online [DOI: 10.1080/00036811.2020.1712373] (2020).

[18] P. Q. Khanh, *Proper solutions of vector optimization problems*, J. Optim. Theory Appl. 74 (1992), pp. 105–130.

[19] G. M. Lee, N. N. Tam, and N. D. Yen, *Quadratic Programming and Affine Variational Inequalities: A Qualitative Study*, Series: "Nonconvex Optimization and its Applications", Vol. 78, Springer Verlag, New York, 2005.

[20] C. Malivert, *Multicriteria fractional programming*, in "Proceedings of the 2nd catalan days on applied mathematics" (M. Sofonea and J. N. Corvellec, Eds.), Presses Universitaires de Perpinan, 1995, pp. 189–198.

[21] R. T. Rockafellar, *Convex Analysis,* Princeton University Press, Princeton, New Jersey, 1970.

[22] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[23] Y. Sawaragi, H. Nakayama, Hirotaka, and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, New York, 1985.

[24] R. E. Steuer, *Multiple Criteria Optimization: Theory, computation and application*, John Wiley & Sons, New York, 1986.

[25] N. D. Yen, *Linear fractional and convex quadratic vector optimization problems*, in "Recent Developments in Vector Optimization" (Q. H. Ansari and J.-C. Yao, Eds.), Springer Verlag, 2012, pp. 297–328.

[26] N. D. Yen and T. D. Phuong, *Connectedness and stability of the solution set in linear fractional vector optimization problems*, in "Vector Variational Inequalities and Vector Equilibria" (F. Giannessi, Ed.), Kluwer Academic Publishers, Dordrecht, 2000, pp. 479–489.