EQUIVARIANT MOTIVIC INTEGRATION ON SPECIAL FORMAL SCHEMES

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ABSTRACT. We construct, based on Nicaise's article in Math. Ann. in 2009, an equivariant geometric motivic integration for special formal schemes, such that when applying to algebraizable formal schemes, we can revisit our previous work in 2020 on equivariant motivic integration for algebraic varieties. We prove the change of variables formula for the integral by pointing out the existence of an equivariant Néron smoothening for a flat generically smooth special formal scheme. We also define the motivic Milnor fiber of a formal power series and predict that it is the right quantity to define the motivic Milnor fiber of a germ of complex analytic functions.

1. INTRODUCTION

In 1995, with the help of p-adic integration and the Weil conjectures (proved by Deligne), Batyrev [2] obtained an important result in birational geometry and mathematical physics that birational Calabi-Yau varieties have the same Betti numbers. Immediately after this event, Kontsevich in his seminar talk at Orsay on December 7, 1995 introduced a new idea which approaches directly to Batyrev's Theorem without using p-adic integration. Kontsevich's method involves arc spaces and the Grothendieck ring of varieties, which brings about the birth of geometric motivic integration. Nowadays, this kind of integration becomes one of the common central objects of algebraic geometry, singularity theory, mathematical physics. From algebraic varieties to formal schemes, the development of geometric motivic integration is contributed crucially by Denef-Loeser [16, 17], Sebag [38], Loeser-Sebag [29], Nicaise-Sebag [33, 34, 35, 37], Nicaise [32], and many others. Another point of view on motivic integration known as arithmetic motivic integration was also strongly developed due to the approaches of Denef-Loeser over p-adic fields [18], Cluckers-Loeser [12, 13, 14], Hrushovski-Kazhdan [25] and Hrushovski-Loeser [26] using model theory with respect to different languages. It is shown in [11] that arithmetic motivic integration has an important application to the fundamental lemma. For another theory of motivic integration that specializes to both of arithmetic and geometric points of view, we can also refer to more recent works such as [22], [10], [8].

It is natural to build an equivariant version of geometric motivic integration, which is very useful for applications to singularity theory. In fact, we can view the monodromy action on the classical Milnor fiber from natural actions of the group schemes of roots of unity on the contact loci of the singularity. Our previous work [28] developed the equivariant motivic integration in the inheritance of Denef-Loeser's classical motivic integration for stable semi-algebraic subsets of arc spaces of algebraic varieties, in which we work with good actions of finite and profinite group schemes. In a formal setting, Hartmann [24] recently has extended

the theory of Sebag and Loeser in [38] and [29] to an equivariant version with respect to abelian finite groups (other than group schemes).

Let R be a complete discrete valuation ring with fraction field K and residue field k, where k is a perfect field. Fix a uniformizing parameter $\varpi \in R$, denote $R_n = R/(\varpi^{n+1})$ and $R(n) = R[\tau]/(\tau^n - \varpi)$. We consider the present work as a continuation of the one in [28], which aims to reach a general theory of equivariant geometric motivic integration for special formal schemes endowed with good adic actions of finite and profinite group k-schemes. Important tools for our construction are Theorems 2.1 and 2.2, which extend [28, Thm. 2.2 and 2.3] to perfect ground fields in the finite group scheme setting. Namely, Theorems 2.1 and 2.2 provide practical criteria for the equality in the equivariant *modified* Grothendieck ring $\widetilde{\mathcal{M}}_k^G$ of k-varieties endowed with good action of a finite group k-scheme G, rather than in $\mathcal{M}_k^G$ (see Section 2.1 for definition of $\widetilde{\mathcal{M}}_S^G$).

We start with formal R-schemes topologically of finite type $\mathfrak{X}$ (stft formal schemes, for short) which is endowed with a good action of a finite group k-scheme G. Such a formal scheme $\mathfrak{X}$ corresponds to an inductive system of R_n-schemes X_n with induced G-action. In Proposition 3.7, we show that the Greenberg functor brings the G-action on $\mathfrak{X}$ to a good G-action on each k-variety $\mathrm{Gr}_n(X_n)$ (resp. on $\mathrm{Gr}(\mathfrak{X})$) such that the construction is functorial. The combination of Theorem 2.2 and Proposition 3.7 gives rise to a G-equivariant motivic measure $\mu_{\mathfrak{X}}^G$ with value in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$ on G-invariant stable cylinders of $\mathrm{Gr}(\mathfrak{X})$ (cf. Proposition 3.9). This fact allows us to define the motivic G-integral $\int_{\mathcal{A}} \widetilde{\mathbb{L}}^{-\alpha} d\mu_{\mathfrak{X}}^G \in \widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$ of a motivic function $\widetilde{\mathbb{L}}^{-\alpha}$ on such a cylinder $\mathcal{A}$, where $\widetilde{\mathbb{L}}$ is the class of $\mathbb{A}_k^1 \times_k \mathfrak{X}_s \to \mathfrak{X}_s$ in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$ (cf. Section 2.1) and $\alpha : \mathcal{A} \to \mathbb{Z} \cup \{\infty\}$ is a naively exponentially G-integrable function (cf. Definition 3.10). We revisit the change of variables formula of Sebag [38, Thm. 8.0.5] in the G-equivariant version, as mentioned below, in which the identity of integrals holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$ (see [37, Sec. 2.4] for a correction of [38, Thm. 8.0.5] - which is not correct in mixed characteristic).

Theorem 1 (Theorem 3.12). *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be quasi-compact flat stft formal R-schemes endowed with good G-actions, purely of the same relative dimension. Assume that $\mathfrak{X}$ is generically smooth and $\mathfrak{Y}$ is smooth over R. Let $\mathfrak{h}: \mathfrak{Y} \to \mathfrak{X}$ be a G-equivariant morphism of formal R-schemes such that $\mathfrak{h}_\eta$ is étale and $\mathfrak{Y}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$. Then, for any naively exponentially G-integrable function α on $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$, so is $\alpha \circ \mathrm{Gr}(\mathfrak{h}) + \mathrm{ord}_\varpi(\det \mathrm{Jac}_\mathfrak{h})$ on $\mathrm{Gr}(\mathfrak{Y})$, and moreover, the following holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$:*

$$\int_{\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})} \widetilde{\mathbb{L}}^{-\alpha} d\mu_{\mathfrak{X}}^G = (\mathfrak{h}_s)_! \int_{\mathrm{Gr}(\mathfrak{Y})} \widetilde{\mathbb{L}}^{-\alpha \circ \mathrm{Gr}(\mathfrak{h}) - \mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})} d\mu_{\mathfrak{Y}}^G.$$

Assume that $\mathfrak{X}$ is flat stft generically smooth. Then, for any gauge form ω on the generic fiber $\mathfrak{X}_\eta$ of $\mathfrak{X}$, we have a $\mathbb{Z}$-value function $\mathrm{ord}_{\varpi, \mathfrak{X}}(\omega)$ on $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$, which is naively exponentially G-integrable due to Lemma 3.14. Denote by $\int_{\mathfrak{X}} |\omega|$ the motivic G-integral of $\widetilde{\mathbb{L}}^{-\mathrm{ord}_{\varpi, \mathfrak{X}}(\omega)}$ on $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$. We also consider $\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{X}_s} \int_{\mathfrak{X}} |\omega|$ in $\widetilde{\mathcal{M}}_k^G$.

The major purpose of the present article is to develop a theory of G-equivariant motivic integration of gauge forms on special formal R-schemes, which is an equivariant version of Nicaise's integration [32], also a natural upgrade of our previous work [28]. It is important to notice that we shall only consider *adic* G-actions on special formal R-schemes, and only consider G-equivariant *adic* morphisms of special formal R-schemes, which guarantee the existence of induced G-equivariant k-morphisms at the reduction level; we also note that any morphism of stft formal R-schemes is automatically an adic morphism. To construct motivic integration on a special formal scheme $\mathfrak{X}$ Nicaise [32] uses its dilatation that always exists with

respect to a coherent ideal sheaf $\mathcal{I}$ on $\mathfrak{X}$ containing ϖ. For any flat special formal R-scheme endowed with a good adic G-action, we prove in Proposition 4.7 that under certain conditions on a coherent ideal sheaf $\mathcal{I}$ on $\mathfrak{X}$, it admits a so-called G-dilatation $\pi : \mathfrak{U} \to \mathfrak{X}$. When $\mathcal{I}$ defines $\mathfrak{X}_0$, $\mathfrak{U}$ is a flat stft formal R-scheme, and for any gauge form ω on $\mathfrak{X}_\eta$, the differential form $\pi_\eta^*\omega$ is also a gauge form on $\mathfrak{U}_\eta$. It is natural to define the motivic G-integrals of ω on $\mathfrak{X}$ and on $\mathfrak{X}_\eta$ as follows

$$\int_{\mathfrak{X}} |\omega| := \pi_{s!} \int_{\mathfrak{U}} |\pi_\eta^*\omega| \in \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G \ \text{ and } \ \int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{U}_\eta} |\omega| \in \widetilde{\mathcal{M}}_k^G.$$

The following proposition is the most important technical result, which asserts the existence of a G-Néron smoothening for a special formal scheme.

Proposition 2 (Proposition 4.9). *Let G be a smooth finite group k-scheme. Then every flat generically smooth special formal R-scheme $\mathfrak{X}$ endowed with a good adic G-action admits a G-Néron smoothening $\mathfrak{h}\colon \mathfrak{Y} \to \mathfrak{X}$.*

This tool guarantees through Proposition 4.10 that the notion of G-equivariant motivic integration for special formal schemes is an obvious extension of the one for stft formal schemes. It also promotes its crucial effects on the main results of the present article. Indeed, by Proposition 4.10, we realize some basic properties of the G-equivariant motivic integration, such as the compatibility with formal completion (Proposition 4.15) and the additivity (Corollary 4.16). Moreover, if ω is an $\mathfrak{X}$-*bounded* gauge form, we obtain in Proposition 4.17 a G-equivariant version of [32, Prop. 5.14] on the expression of the motivic integral $\int_{\mathfrak{X}} |\omega|$ via the connected components of $\mathfrak{X}_0$.

The most significant result on the G-equivariant motivic integration for special formal schemes that we obtain, also under the support of Proposition 4.9, is the special G-equivariant change of variables formula (see Theorem 4.14).

Theorem 3 (Theorem 4.14). *Let G be a smooth finite group k-scheme. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be generically smooth special formal R-schemes endowed with good adic actions of G, and let $\mathfrak{h}\colon \mathfrak{Y} \to \mathfrak{X}$ be an adic G-equivariant morphism of formal R-schemes such that the induced morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is an open embedding and $\mathfrak{Y}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$. If ω is a gauge form on $\mathfrak{X}_\eta$, then*

$$\int_{\mathfrak{X}} |\omega| = \mathfrak{h}_{0!} \int_{\mathfrak{Y}} |\mathfrak{h}_\eta^*\omega| \quad in \ \ \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G.$$

An important application of Theorem 4.14 and Proposition 4.17 is the proof of rationality of the (monodromic) motivic volume Poincaré series. Assume that $\mathfrak{X}$ is generically smooth. Then, for any $n \in \mathbb{N}^*$, $\mathfrak{X}(n) := \mathfrak{X} \times_R R(n)$ is a generically smooth special formal $R(n)$-scheme and it is naturally endowed with an adic good μ_n-action (see Lemma 4.21). Let $\omega(n)$ be the gauge form induced by ω. We apply the μ_n-equivariant motivic integration developed in this article to the formal schemes $\mathfrak{X}(n)$ and consider the motivic volume Poincaré series with respect to a gauge form ω on $\mathfrak{X}_\eta$:

$$P(\mathfrak{X}, \omega; T) := \sum_{n \geq 1} \Big(\int_{\mathfrak{X}(n)} |\omega(n)| \Big) T^n \ \in \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^{\hat{\mu}}[[T]].$$

Assume, in addition, that $\mathfrak{X}$ is flat and admits a tame resolution of singularities $\mathfrak{h}$, and that ω is $\mathfrak{X}$-bounded. Then we obtain in Theorem 4.23 a description of $\int_{\mathfrak{X}(n)} |\omega(n)|$ in terms of $\mathfrak{h}$ provided n is prime to the characteristic exponent of k; this result is an equivariant version of [32, Thm. 7.12]. If k is of characteristic zero, then $\mathfrak{X}$ has a (tame) resolution of singularities,

therefore together with Theorem 4.23 we prove the rationality of $P(\mathfrak{X}, \omega; T)$ (Corollary 4.26). Note that when k has characteristic zero we have $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^{\hat{\mu}} = \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$. Using the morphism $\lim_{T\to\infty}$ in [16] we define $\mathrm{MV}(\mathfrak{X}; \widehat{K^s}) := -\lim_{T\to\infty} \mathbb{L}^{\dim_R \mathfrak{X}} P(\mathfrak{X}, \omega; T) \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$ with $\dim_R \mathfrak{X}$ the relative dimension of $\mathfrak{X}$, which is called the motivic volume of $\mathfrak{X}$.

Let f be a formal power series in $R\{x\}[[y]]$ such that the series $f(x, 0)$ is non-constant, with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_{m'})$. Denote by $\mathfrak{X}_f$ the formal completion of $\mathrm{Spf}(k\{x\}[[y]])$ along (f), which is a special formal R-scheme of pure relative dimension $m + m' - 1$ with the reduction $(\mathfrak{X}_f)_0 = \mathrm{Spec}k[x]/(f(x, 0))$. Denote by ω/df the Gelfand-Leray form associated to a top differential form ω on $\mathfrak{X}_f$ considered as a formal scheme over k (we may take $\omega = dx_1 \wedge \cdots \wedge dx_d$). Let k be of characteristic zero. Using tame resolution of singularity we obtain the formulas for $\int_{\mathfrak{X}_f(n)} |(\omega/df)(n)|$ and $\mathrm{MV}(\mathfrak{X}_f; \widehat{K^s})$ (cf. Proposition 4.30), which extend Denef-Loeser's formulas for the contact locus $[\mathfrak{X}_n(f)]$ and the motivic nearby cycles $\mathcal{S}_f$ of a polynomial f (cf. Corollary 4.31). This result allows us to define the so-called motivic nearby cycles and motivic Milnor fiber of a series in $k\{x\}[[y]]$.

Finally, we borrow Section 5 of the present article to discuss an observation concerning motivic Milnor fiber of complex analytic function germs. The original approach of Denef-Loeser to motivic Milnor fiber (cf. [16, 20]) is only for regular function germs since their theory of motivic integration works only with algebraic varieties and usual schemes. Hence, it is surprising there were unproven declarations that the concept of motivic Milnor fiber for complex analytic function germs is only an obvious extension of that of Denef-Loeser. To provide a solution to this extension we recommend Conjectures 5.1 and 5.2, which may be expected as a bridge connecting equivariant motivic integration for special formal schemes and singularity theory.

2. Preliminaries

2.1. Equivariant Grothendieck rings of varieties.
Let k be a perfect field, and let S be a k-variety. By an S-variety we mean a k-variety X together with a morphism $X \to S$. As in [16] and [17], let Var_S denote the category of S-varieties and $K_0(\mathrm{Var}_S)$ its Grothendieck ring. By definition, $K_0(\mathrm{Var}_S)$ is the quotient of the free abelian group generated by all S-isomorphism classes $[X \to S]$ in Var_S modulo the relations

$$[X \to S] = [X_{\mathrm{red}} \hookrightarrow X \to S]$$

and

$$[X \to S] = [Y \to S] + [X \setminus Y \to S],$$

where X_{red} is the reduced subscheme of X, and Y is a Zariski closed subvariety of X. Together with fiber product over S, $K_0(\mathrm{Var}_S)$ is a commutative ring with unity $1 = [\mathrm{Id}\colon S \to S]$. Put $\mathbb{L} = [\mathbb{A}_k^1 \times_k S \to S]$. Denote by $\mathcal{M}_S$ the localization of $K_0(\mathrm{Var}_S)$ which makes $\mathbb{L}$ invertible.

Let X be a k-variety, and let G be an algebraic group. An action of G on X is said to be *good* if every G-orbit is contained in an affine open subset of X. We fix a good action of G on the k-variety S (where we may choose the trivial action). By definition, the G-equivariant Grothendieck group $K_0^G(\mathrm{Var}_S)$ of G-equivariant morphisms of k-varieties $X \to S$, where X is endowed with a good G-action, is the quotient of the free abelian group generated by the G-equivariant isomorphism classes $[X \to S, \sigma]$, where X is a k-variety endowed with a good G-action σ and $X \to S$ is a G-equivariant morphism of k-varieties, modulo the relations

$$[X \to S, \sigma] = [Y \to S, \sigma|_Y] + [X \setminus Y \to S, \sigma|_{X \setminus Y}]$$

for Y being σ-invariant Zariski closed in X, and

$$[X \times_k \mathbb{A}_k^n \to S, \sigma] = [X \times_k \mathbb{A}_k^n \to S, \sigma']$$

if σ and σ' lift the same good G-action on X. As above, we have the commutative ring with unity structure on $K_0^G(\mathrm{Var}_S)$ by fiber product, where G-action on the fiber product is through the diagonal G-action. Denote by $\mathcal{M}_S^G$ the localization ring $K_0^G(\mathrm{Var}_S)[\mathbb{L}^{-1}]$, where we view $\mathbb{L}$ as the class of $\mathbb{A}_k^1 \times_k S \to S$ endowed with the trivial action of G.

Denote by $\hat{\mu}$ the limit of the projective system of group schemes $\mu_n = \mathrm{Spec}\,(k[\xi]/(\xi^n - 1))$ with transition morphisms $\mu_{mn} \to \mu_n$ sending λ to λ^m. Define $K_0^{\hat{\mu}}(\mathrm{Var}_S) = \varinjlim K_0^{\mu_n}(\mathrm{Var}_S)$ and $\mathcal{M}_S^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_S)[\mathbb{L}^{-1}]$.

Let $f \colon S \to S'$ be a morphism of algebraic k-varieties. For any group k-scheme G, denote by f^* the ring homomorphism $K_0^G(\mathrm{Var}_{S'}) \to K_0^G(\mathrm{Var}_S)$ induced from the fiber product (the pullback morphism), and by $f_!$ the $K_0(\mathrm{Var}_k)$-linear homomorphism $K_0^G(\mathrm{Var}_S) \to K_0^G(\mathrm{Var}_{S'})$ defined by the composition with f (the push-forward morphism). The pullback morphism induces a unique morphism of localizations $f^* \colon \mathcal{M}_{S'}^G \to \mathcal{M}_S^G$, the push-forward morphism induces a unique $\mathcal{M}_k$-linear morphism $f_! \colon \mathcal{M}_S^G \to \mathcal{M}_{S'}^G$ by sending $a\mathbb{L}_S^{-n}$ to $(f_!a)\mathbb{L}_{S'}^{-n}$ for any a in $K_0^G(\mathrm{Var}_S)$ and any $n \in \mathbb{N}$. When S' is $\mathrm{Spec}\,k$, we usually write $\int_S$ instead of $f_!$.

Now, we mention the so-called modified Grothendieck ring of S-varieties. A morphism of schemes $X \to Y$ is said to be *universal homeomorphism* if for every morphism of schemes $Y' \to Y$, the base change morphism $X \times_Y Y' \to Y'$ is a homeomorphism. Two S-schemes X and Y are *universally S-homeomorphic* if there is a universal homeomorphism of S-schemes from X to Y or from Y to X. Let I_S^{uh} the ideal in $K_0(\mathrm{Var}_S)$ generated by elements of the form $[X \to S] - [Y \to S]$ with X, Y being universally S-homeomorphic. The quotient ring $K_0\langle \mathrm{Var}_S \rangle := K_0(\mathrm{Var}_S)/I_S^{uh}$ is called the *modified Grothendieck ring* of S-varieties. Define $\langle X \to S \rangle := [X \to S] + I_S^{uh}$ and $\widetilde{\mathcal{M}}_S := K_0\langle \mathrm{Var}_S \rangle[\widetilde{\mathbb{L}}^{-1}]$, where $\widetilde{\mathbb{L}} := \langle \mathbb{A}_k^1 \times_k S \to S \rangle$. We can define the G-equivariant version in an obvious way to get $K_0^G\langle \mathrm{Var}_S \rangle$, $\widetilde{\mathcal{M}}_S^G$, $K_0^{\hat{\mu}}\langle \mathrm{Var}_S \rangle$, $\widetilde{\mathcal{M}}_S^{\hat{\mu}}$, f^*, $f_!$, and so on. This definition of modified Grothendieck ring of varieties is only important in the case where the characteristic of k is positive, because if the characteristic of k is zero, then $I_S^{uh} = 0$, thus $\widetilde{\mathbb{L}} = \mathbb{L}$, $K_0\langle \mathrm{Var}_S \rangle = K_0(\mathrm{Var}_S)$ and $\widetilde{\mathcal{M}}_S^G = \mathcal{M}_S^G$ (see [36]).

2.2. **Equivariant piecewise trivial fibrations.** Let X, Y and F be algebraic k-varieties endowed with a good action of an algebraic group G. Let A and B be G-invariant constructible subsets of X and Y, respectively. Let $f \colon X \to Y$ be a G-equivariant morphism such that $f(A) \subseteq B$. The restriction $f \colon A \to B$ is called a *G-equivariant piecewise trivial fibration with fiber F* if there exists a stratification of B into finitely many G-invariant locally closed subsets B_i such that $f^{-1}(B_i)$ is a G-invariant constructible subset of A and $f^{-1}(B_i)$ is G-equivariant isomorphic to $B_i \times_k F$ with respect to the diagonal action of G on $B_i \times_k F$, and such that, over B_i, f equals the projection $B_i \times_k F \to B_i$.

For a morphism of algebraic k-varieties $X \to Y$ and an immersion $S \to Y$, we write X_S for the fiber product $X \times_Y S$. If Y is endowed with a good G-action, then for y in Y, the *stabilizer subgroup G_y* of G over y is the subgroup of elements in G fixing y.

Theorem 2.1. *Let k be a perfect field and let G be a finite group k-scheme. Suppose that X and Y are algebraic k-varieties endowed with a good G-action and that $f \colon X \to Y$ is a G-equivariant morphism. Then f is a G-equivariant piecewise trivial fibration if and only if there is an algebraic k-variety F endowed with a good G-action such that for every y in Y, there is a G_y-equivariant isomorphism of algebraic $\kappa(y)$-varieties $X_y \xrightarrow{\cong} F \times_k \kappa(y)$.*

Proof. Similarly as in the proof of [28, Thm. 2.2]. □

Theorem 2.2. *Let k be a perfect field and let G be a finite group k-scheme. Suppose that X and Y are algebraic k-varieties endowed with a good G-action and that $f\colon X \to Y$ is a G-equivariant morphism. If there is an n in $\mathbb{N}$ such that for every y in Y, there is an isomorphism of algebraic $\kappa(y)$-varieties $X_y \cong \mathbb{A}^n_{\kappa(y)}$, then $\langle X \rangle = \langle Y \rangle \widetilde{\mathbb{L}}^n$ in $K_0^G \langle \mathrm{Var}_k \rangle$.*

Proof. Similarly as in the proof of [28, Thm. 2.3]. □

2.3. **Rational series.** Let $\mathcal{M}$ be a commutative ring with unity containing $\mathbb{L}$ and $\mathbb{L}^{-1}$, and let $\mathcal{M}[[T]]$ be the set of formal power series in one variate T with coefficients in $\mathcal{M}$, which is a ring and also a $\mathcal{M}$-module with respect to usual operations for series. Denote by $\mathcal{M}[[T]]_{\mathrm{sr}}$ the submodule of $\mathcal{M}[[T]]$ generated by 1 and by finite products of terms $\frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b}$ for (a, b) in $\mathbb{Z} \times \mathbb{N}^*$. Any element of $\mathcal{M}[[T]]_{\mathrm{sr}}$ is called a *rational* series. It is proved in [16] that there exists a unique $\mathcal{M}$-linear morphism

$$\lim_{T \to \infty} : \mathcal{M}[[T]]_{\mathrm{sr}} \to \mathcal{M}$$

such that

$$\lim_{T \to \infty} \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} = -1$$

for any (a, b) in $\mathbb{Z} \times \mathbb{N}^*$.

The Hadamard product of two formal power series $p = \sum_{n \geq 1} p_n T^n$ and $q = \sum_{n \geq 1} q_n T^n$ in $\mathcal{M}[[T]]$ is a formal power series in $\mathcal{M}[[T]]$ defined as follows

$$p * q := \sum_{n \geq 1} p_n q_n T^n.$$

This product is commutative, associative, and has the unity $\sum_{n \geq 1} T^n$. It also preserves the rationality as seen in the following lemma.

Lemma 2.3 (Looijenga [30]). *If $p(T)$ and $q(T)$ are rational series in $\mathcal{M}[[T]]$, then $p(T) * q(T)$ is also a rational series, and in this case,*

$$\lim_{T \to \infty} p(T) * q(T) = - \lim_{T \to \infty} p(T) \cdot \lim_{T \to \infty} q(T).$$

3. Equivariant motivic integration on stft formal schemes

Let R be a complete discrete valuation ring with fraction field K and residue field k, in which k is perfect. Let $\varpi \in R$ be a uniformizing parameter, which will be fixed throughout this article, and let $R_n = R/(\varpi^{n+1})$. We also fix a separable closure K^s of K, denote respectively by K^t and K^{sh} the tame closure and strict henselization of K in K^s.

3.1. **Formal schemes topologically of finite type with action.** It was mentioned in EGA1, Ch. 0, 7.5.3, that an adic R-algebra A is *of finite type* if $(\varpi)A$ is an ideal of definition of A and $A/(\varpi)A$ is a k-algebra of finite type. For any $n \in \mathbb{N}$, let $R\{x_1, \ldots, x_n\}$ denote the R-algebra of restricted power series in n variables, namely,

$$R\{x_1, \ldots, x_n\} = \varprojlim_{\ell} (R/(\varpi^{\ell}))[x_1, \ldots, x_n].$$

Clearly, $R\{x_1, \ldots, x_n\}$ is a Noetherian ring and the definition of $R\{x_1, \ldots, x_n\}$ is independent of the choice of ϖ. It is a fact that A is of finite type if it is topologically R-isomorphic to a quotient algebra of $R\{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$.

For any adic ring A, we denote by $\mathrm{Spf}\,A$ the set of all open prime ideals of A, which has a structure of a locally ringed space and is called *the formal spectrum of A*. A Noetherian adic formal scheme is a locally ringed space which is locally isomorphic to the formal spectrum of a Noetherian adic ring.

Definition 3.1. A *formal R-scheme topologically of finite type* is a Noetherian adic formal scheme which is a finite union of affine formal schemes of the form $\mathrm{Spf}\,A$ with A an adic R-algebra of finite type.

If $\mathfrak{X}$ is a separated formal R-scheme topologically of finite type over R, it will be abbreviated as *stft* formal R-scheme. Such an $\mathfrak{X}$ is nothing but the inductive limit of the R_n-schemes

$$(3.1) \qquad\qquad X_n = (\mathfrak{X}, \mathcal{O}_\mathfrak{X} \otimes_R R_n),$$

and the transition morphisms $X_n \to X_m$ as R_m-schemes (for $n \le m$) are induced from the truncated map $R_m \to R_n$. Using the morphism $X_n \to X_m$ we have

$$(3.2) \qquad\qquad X_n \cong X_m \times_{\mathrm{Spec}\,R_m} \mathrm{Spec}\,R_n.$$

Clearly, the category of formal R-schemes topologically of finite type admits fiber products.

Definition 3.2. A *morphism* of stft formal R-schemes $\mathfrak{Y} \to \mathfrak{X}$ is a morphism between the underlying locally topologically ringed spaces over R. This morphism is said to be *locally of finite type* if locally it is isomorphic to a morphism of the form $\mathrm{Spf}\,B \to \mathrm{Spf}\,A$, where the corresponding R-morphism $A \to B$ is of finite type.

Notation 3.3. In general, for any Noetherian adic formal scheme $\mathfrak{X}$, we denote by $\mathfrak{X}_s$ the *special fiber* $\mathfrak{X} \times_R k$ of $\mathfrak{X}$, which is a formal k-scheme, and denote by $\mathfrak{X}_0$ the closed subscheme of $\mathfrak{X}$ defined by the largest ideal of definition of $\mathfrak{X}$, called the *reduction* of $\mathfrak{X}$, which is a reduced noetherian scheme. If $\mathfrak{X}$ is a stft formal R-scheme, then $\mathfrak{X}_s = X_0$ (cf. eq. (3.1)), which is a separated k-scheme of finite type, and has the property $\mathfrak{X}_0 = (\mathfrak{X}_s)_{\mathrm{red}}$.

Assume that $\mathfrak{X} = \mathrm{Spf}\,A$, where A is an R-algebra of finite type. The tensor product $A \otimes_R K$ is then a K-affinoid algebra in the sense of Tate [43]. The rigid K-variety $\mathrm{Spm}(A \otimes_R K)$ is called *the generic fiber of* $\mathfrak{X}$ and denoted by $\mathfrak{X}_\eta$. It is shown that the correspondence

$$\mathrm{Spf}\,A \mapsto \mathrm{Spm}(A \otimes_R K)$$

is functorial, and that it can be extended to any stft formal R-schemes $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ by glueing procedure along open coverings of $\mathfrak{X}$ (see [7], [15]). The rigid K-variety $\mathfrak{X}_\eta$ is separated and quasi-compact. The formal scheme $\mathfrak{X}$ is called *generically smooth* if its generic fiber $\mathfrak{X}_\eta$ is a smooth rigid K-variety.

Definition 3.4. Let G be a finite group k-scheme with $m_G \colon G \times_k G \to G$ the multiplication and $e_G \in G(k)$ the neutral element. A *(left) G-action* on a formal k-scheme $\mathfrak{X}$ is a morphism of formal k-schemes

$$\theta \colon G \times_k \mathfrak{X} \to \mathfrak{X}, \quad (g,x) \mapsto g \cdot x$$

such that

- The composite map $\mathfrak{X} \cong \mathrm{Spec}\,k \times \mathfrak{X} \xrightarrow{e_G \times \mathrm{Id}_\mathfrak{X}} G \times \mathfrak{X} \xrightarrow{\theta} \mathfrak{X}$ is the identity, and
- The following diagram commutes:

$$
\begin{array}{ccc}
G \times_k G \times_k \mathfrak{X} & \xrightarrow{\mathrm{Id}_G \times \theta} & G \times_k \mathfrak{X} \\
{\scriptstyle m_G \times \mathrm{Id}_\mathfrak{X}}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\
G \times \mathfrak{X} & \xrightarrow{\quad\theta\quad} & \mathfrak{X}.
\end{array}
$$

Assume that $(\mathfrak{X}, \theta)$ and $(\mathfrak{Y}, \theta')$ are formal k-schemes endowed with G-action. Then a morphism $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ is called G-*equivariant* if the following diagram commutes:

$$
\begin{array}{ccc}
G \times_k \mathfrak{X} & \xrightarrow{\ \theta\ } & \mathfrak{X} \\
{\scriptstyle \mathrm{Id}_G \times \mathfrak{f}} \big\downarrow & & \big\downarrow {\scriptstyle \mathfrak{f}} \\
G \times_k \mathfrak{Y} & \xrightarrow{\ \theta'\ } & \mathfrak{Y}.
\end{array}
$$

Observe that $\mathrm{Spf}\,R = \varinjlim \mathrm{Spec}\,R_n$. Then any G-action σ on $\mathrm{Spf}\,R$ induces a unique G-action σ_n on $\mathrm{Spec}\,R_n$ such that the natural k-morphism $\iota_n: \mathrm{Spec}\,R_n \to \mathrm{Spf}\,R$ and the transitions $\iota_{n,m}: \mathrm{Spec}\,R_n \to \mathrm{Spec}\,R_m$ are G-invariant for $n \le m$ in $\mathbb{N}$. From now on, we fix a G-action σ on $\mathrm{Spf}\,R$.

Definition 3.5. Let $\mathfrak{X}$ be a formal R-scheme, with structural morphism $\mathfrak{X} \to \mathrm{Spf}\,R$ viewed as a morphism of formal k-scheme. A G-*action* on $\mathfrak{X}$ is a G-action on the formal k-scheme $\mathfrak{X}$ (with the k-scheme structure induced from $k \hookrightarrow R$) such that $\mathfrak{f}$ is a G-equivariant k-morphism. A G-action on $\mathfrak{X}$ is called *good* if any orbit of it is contained in an affine open formal subscheme of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a stft formal R-scheme, which is the inductive limit of R_n-scheme X_n mentioned above. Then a G-action θ on $\mathfrak{X}$ induces a unique G-action θ_n on X_n such that the natural morphism $\rho_n: X_n \to \mathfrak{X}$ and the transition morphisms $\rho_{n,m}: X_n \to X_m$ are G-equivariant as k-morphisms for $n \le m$ in $\mathbb{N}$. If $\mathfrak{f}: \mathfrak{Y} \to \mathfrak{X}$ is a morphism of stft formal R-schemes, then the G-equivariance of $\mathfrak{f}$ induces G-equivariant morphisms $f_n: Y_n \to X_n$ for $n \in \mathbb{N}$ which are compatible with the natural morphism and transition morphisms. In particular, θ_0 is the G-action on the special fiber $\mathfrak{X}_s = X_0$ induced from θ, and the G-equivariant morphism of k-schemes

$$
\mathfrak{f}_s := f_0: \mathfrak{Y}_s \to \mathfrak{X}_s
$$

is induced from the G-equivariant morphism of formal R-schemes $\mathfrak{f}$.

3.2. **Greenberg spaces of stft formal schemes.** Let A be a k-algebra. If R has equal (resp. unequal) characteristic we put $L(A) = A$ (resp. $L(A) = W(A)$), where $W(A)$ is the ring of Witt vectors over A. Define $R_A := R \widehat{\otimes}_{L(k)} L(A)$.

Let $\mathfrak{X}$ be a stft formal R-scheme, which is the inductive limit of the R_n-schemes X_n described in eq. (3.1). In [23], Greenberg shows that the functor defined locally by

$$
\mathrm{Spec}\,A \mapsto \mathrm{Hom}_{R_n}\!\left(\mathrm{Spec}\left(R_n \otimes_{L(k)} L(A)\right), X_n\right)
$$

from the category of k-schemes to the category of sets is presented by a k-scheme $\mathrm{Gr}_n(X_n)$ of finite type such that, for any k-algebra A,

$$
\mathrm{Gr}_n(X_n)(A) = X_n(R_n \otimes_{L(k)} L(A)).
$$

For $n \le m$ and for $\gamma: \mathrm{Spec}(R_m \otimes_{L(k)} L(A)) \to X_m$ in $\mathrm{Gr}_m(X_m)(A)$, tensoring with $\mathrm{Spec}\,R_n$ over $\mathrm{Spec}\,R_m$ we get an element

$$
(3.3) \qquad \widetilde{\gamma} := \gamma \times_{R_m} \mathrm{Id}_{R_n}: \mathrm{Spec}(R_n \otimes_{L(k)} L(A)) \to X_m \times_{\mathrm{Spec}\,R_m} \mathrm{Spec}\,R_n \cong X_n
$$

in $\mathrm{Gr}_n(X_n)(A)$, where $\gamma \times_{R_m} \mathrm{Id}_{R_n}$ stands for $\gamma \times_{\mathrm{Spec}\,R_m} \mathrm{Id}_{\mathrm{Spec}\,R_n}$. Thus, the correspondence $\gamma \mapsto \widetilde{\gamma}$ gives a map $\mathrm{Gr}_m(X_m)(A) \to \mathrm{Gr}_n(X_n)(A)$, and by the functoriality on A we have a canonical morphism of k-schemes

$$
\pi_n^m = (\pi_{\mathfrak{X}})_n^m : \mathrm{Gr}_m(X_m) \to \mathrm{Gr}_n(X_n).
$$

We obtain a projective system $\left\{ \big(\mathrm{Gr}_n(X_n); \pi_n^m\big) \mid n \leq m \right\}$ in the category of separated k-schemes of finite type. Put

$$\mathrm{Gr}(\mathfrak{X}) := \varprojlim_n \mathrm{Gr}_n(X_n),$$

which exists in the category of k-schemes.

In the sequel, we shall use the fact that every point $x \in \mathrm{Gr}(\mathfrak{X})$ with residue field $\kappa(x) \supseteq k$ corresponds a morphism $\gamma \colon \mathrm{Spf}\, R_{\kappa(x)} \to \mathfrak{X}$.

Definition 3.6. For any stft formal R-scheme $\mathfrak{X}$, the k-scheme $\mathrm{Gr}(\mathfrak{X})$ defined previously is called *the Greenberg space of $\mathfrak{X}$.*

Denote by $\pi_n = \pi_{\mathfrak{X},n}$ the canonical projection $\mathrm{Gr}(\mathfrak{X}) \to \mathrm{Gr}_n(X_n)$. We have the following commutative diagram, for all $n \leq m$ in $\mathbb{N}$,

$$
\begin{array}{ccc}
\mathrm{Gr}(\mathfrak{X}) & \xrightarrow{\;\;\pi_m\;\;} & \mathrm{Gr}_m(X_m) \\
& \searrow{\scriptstyle \pi_n} & \downarrow{\scriptstyle \pi_n^m} \\
& & \mathrm{Gr}_n(X_n).
\end{array}
$$

The properties of the space $\mathrm{Gr}(\mathfrak{X})$ are shown in the initial work by Sebag [38].

Let $\mathfrak{f} \colon \mathfrak{Y} \to \mathfrak{X}$ be a morphism of stft formal R-schemes. Then $\mathfrak{f}$ is the injective limit of morphisms of compatible R_n-schemes $f_n \colon Y_n \to X_n$. For any k-algebra A, the morphisms f_n induce by composition compatible maps

$$Y_n(R_n \otimes_{L(k)} L(A)) \to X_n(R_n \otimes_{L(k)} L(A)).$$

Since this construction is functorial in variable A, it defines k-morphisms of schemes

$$\mathrm{Gr}_n(f_n) \colon \mathrm{Gr}_n(Y_n) \to \mathrm{Gr}_n(X_n),$$

which are compatible, i.e. the diagram

$$
\begin{array}{ccc}
\mathrm{Gr}_m(Y_m) & \xrightarrow{\;\mathrm{Gr}_m(f_m)\;} & \mathrm{Gr}_m(X_m) \\
{\scriptstyle (\pi_{\mathfrak{Y}})_n^m}\downarrow & & \downarrow{\scriptstyle (\pi_{\mathfrak{X}})_n^m} \\
\mathrm{Gr}_n(Y_n) & \xrightarrow{\;\mathrm{Gr}_n(f_n)\;} & \mathrm{Gr}_n(X_n)
\end{array}
$$

commutes for $n \leq m$ in $\mathbb{N}^*$. Taking projective limit we get a unique morphism of k-scheme

$$\mathrm{Gr}(\mathfrak{f}) \colon \mathrm{Gr}(\mathfrak{Y}) \to \mathrm{Gr}(\mathfrak{X})$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Gr}(\mathfrak{Y}) & \xrightarrow{\;\mathrm{Gr}(\mathfrak{f})\;} & \mathrm{Gr}(\mathfrak{X}) \\
{\scriptstyle \pi_{\mathfrak{Y},n}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{X},n}} \\
\mathrm{Gr}_n(Y_n) & \xrightarrow{\;\mathrm{Gr}_n(f_n)\;} & \mathrm{Gr}_n(X_n).
\end{array}
$$

3.3. G-actions on Greenberg spaces. As above, we consider a stft formal R-scheme $\mathfrak{X}$, with $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} \otimes_R R_n)$. Let G be a finite group k-scheme. Let θ be a good G-action on $\mathfrak{X}$, i.e. a G-action $\theta \colon G \times_k \mathfrak{X} \to \mathfrak{X}$ on formal k-scheme $\mathfrak{X}$ (the k-scheme structure on $\mathfrak{X}$ is induced from the inclusion $k \hookrightarrow R$) with the structural morphism $\mathfrak{X} \to \mathrm{Spf}\, R$ being G-equivariant.

Proposition 3.7. *With the previous notation and hypotheses, for any stft formal R-scheme $\mathfrak{X}$, there exist good G-actions*

$$\mathrm{Gr}(\theta)\colon G \times_k \mathrm{Gr}(\mathfrak{X}) \to \mathrm{Gr}(\mathfrak{X})$$

and

$$\mathrm{Gr}_n(\theta_n)\colon G \times_k \mathrm{Gr}_n(X_n) \to \mathrm{Gr}_n(X_n),$$

for all $n \in \mathbb{N}$, satisfying the following conditions:

  (i) $\mathrm{Gr}_0(\theta_0) = \theta_0$, *i.e.* $\mathrm{Gr}(\theta)$ *and* θ *induce the same action on* $\mathfrak{X}_s = X_0$;
  (ii) *The k-morphisms π_n and π_n^m are G-equivalent for every $n \leq m$ in $\mathbb{N}$;*
  (iii) *If $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is a G-equivariant morphism of stft formal R-schemes endowed with G-action, then the induced morphisms $\mathrm{Gr}(\mathfrak{f})$ and $\mathrm{Gr}_n(f_n)$ are G-equivariant.*

Proof. Let A be an arbitrary k-algebra. Then, for every $n \in \mathbb{N}^*$, the G-action σ_n on $\mathrm{Spec}R_n$ induces naturally a G-action on $\mathrm{Spec}(R_n \otimes_{L(k)} L(A))$, which is by abuse of notation also denoted by σ_n. Consider the following map

$$\mathrm{Gr}_n(\theta_n)(A) := \mathrm{Gr}_n(\sigma_n, \theta_n)(A)\colon G(A) \times X_n(R_n \otimes_{L(k)} L(A)) \to X_n(R_n \otimes_{L(k)} L(A))$$

in which $\mathrm{Gr}_n(\theta_n)(g, \gamma)$, for $g \in G(A)$ and $\gamma \in X_n(R_n \otimes_{L(k)} L(A))$, is given by the commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}(R_n \otimes_{L(k)} L(A)) & \xrightarrow{\ i\ } & \mathrm{Spec}A \times_k \mathrm{Spec}(R_n \otimes_{L(k)} L(A)) \\
{\scriptstyle g\cdot\gamma := \mathrm{Gr}_n(\theta_n)(g,\gamma)}\downarrow & & \downarrow{\scriptstyle g\times\gamma} \\
X_n & \xleftarrow{\ \theta_n\ } & G \times_k X_n.
\end{array}$$

If $e\colon \mathrm{Spec}A \to G$ is the neutral element of $G(A)$, then it follows from the construction that $e \cdot \gamma = \gamma$ for every $\gamma \in X_n(R_n \otimes_{L(k)} L(A))$. For any $g, h \in G(A)$ and $\gamma \in X_n(R_n \otimes_{L(k)} L(A))$, we have

$$h \cdot (g \cdot \gamma) = \theta_n \circ (h \times g \cdot \gamma) \circ i,$$
$$(hg) \cdot \gamma = \theta_n \circ (hg \times \gamma) \circ i.$$

Since θ_n is a G-action on X_n we have $\theta_n \circ (h \times g \cdot \gamma) = \theta_n \circ (hg \times \gamma)$. Hence $h \cdot (g \cdot \gamma) = (hg) \cdot \gamma$, from which $\mathrm{Gr}_n(\theta_n)(A)$ is an action of the abstract group $G(A)$ on $\mathrm{Gr}_n(X_n)(A)$.

The construction is functorial because for every homomorphism of k-algebras $A \to B$ the diagram

$$\begin{array}{ccc}
\mathrm{Spec}(R_n \otimes_{L(k)} L(B)) & \longrightarrow & \mathrm{Spec}B \times_k \mathrm{Spec}(R_n \otimes_{L(k)} L(B)) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(R_n \otimes_{L(k)} L(A)) & \longrightarrow & \mathrm{Spec}A \times_k \mathrm{Spec}(R_n \otimes_{L(k)} L(A))
\end{array}$$

commutes. Therefore, by the Yoneda lemma, we obtain a G-action

$$\mathrm{Gr}_n(\theta_n) := \mathrm{Gr}_n(\sigma_n, \theta_n)\colon G \times_k \mathrm{Gr}_n(X_n) \to \mathrm{Gr}_n(X_n)$$

on $\mathrm{Gr}_n(X_n)$, which relates to σ_n and θ_n. It is obvious from the construction that $\mathrm{Gr}_0(\theta_0) = \theta_0$. Moreover, because for every $n \leq m$ in $\mathbb{N}$ the diagram

$$\begin{array}{ccc}
G \times_k X_n & \xrightarrow{\ \theta_n\ } & X_n \\
{\scriptstyle \mathrm{Id}\times\rho_{n,m}}\downarrow & & \downarrow{\scriptstyle \rho_{n,m}} \\
G \times_k X_m & \xrightarrow{\ \theta_m\ } & X_m
\end{array}$$

commutes, it follows by the construction that $\pi_n^m \colon \mathrm{Gr}_m(X_m) \to \mathrm{Gr}_n(X_n)$ are G-equivariant. Clearly, if θ_n is good, so is $\mathrm{Gr}_n(\theta_n)$.

Now, putting $\mathrm{Gr}(\theta) := \varprojlim \mathrm{Gr}_n(\theta_n)$ we get a good G-action $\mathrm{Gr}(\theta) \colon G \times_k \mathrm{Gr}(\mathfrak{X}) \to \mathrm{Gr}(\mathfrak{X})$, which guarantees that the canonical morphisms $\pi_n \colon \mathrm{Gr}(\mathfrak{X}) \to \mathrm{Gr}_n(X_n)$ are G-equivariant. More concretely, for any g in G, any k-algebra A, $g \in G(A)$ and γ in $\mathrm{Gr}(\mathfrak{X})(A)$, we have

$$g \cdot \gamma = \mathrm{Gr}(\theta)(g, \gamma) = \theta \circ (g \times \gamma) \circ i.$$

The part (iii) is straightforward by the construction of G-action. $\qquad\square$

Assume $\mathfrak{X}$ is a quasi-compact stft formal R-scheme. By [38], the image $\pi_n(\mathrm{Gr}(\mathfrak{X}))$ of $\mathrm{Gr}(\mathfrak{X})$ in $\mathrm{Gr}_n(X_n)$ is a constructible subset of $\mathrm{Gr}_n(X_n)$. By the construction of G-actions on $\mathrm{Gr}(\mathfrak{X})$ and $\mathrm{Gr}_n(X_n)$, the semi-algebraic set $\pi_n(\mathrm{Gr}(\mathfrak{X}))$ is G-invariant in $\mathrm{Gr}_n(X_n)$. If, in addition, $\mathfrak{X}$ is smooth of pure relative dimension d, we can extend [38, Lem. 3.4.2] to the equivariant setting in which the $\mathfrak{X}_s$-morphism π_n is G-equivariant surjective and the $\mathfrak{X}_s$-morphism π_n^{n+1} is a G-equivariant locally trivial fibration in the Zariski topology with fiber $\mathbb{A}_{\mathfrak{X}_s}^d$ for any $n \in \mathbb{N}$.

Furthermore, by using [38, Lem. 4.3.25] and the construction of G-action $\mathrm{Gr}_n(X_n)$ here we get the following proposition.

Proposition 3.8. *Let $\mathfrak{X}$ be a relatively d-dimensional flat quasi-compact stft formal R-scheme which is endowed with a good G-action. Put*

$$\mathrm{Gr}^{(e)}(\mathfrak{X}) = \mathrm{Gr}(\mathfrak{X}) \setminus \pi_e^{-1}(\mathrm{Gr}_e((X_e)_{\mathrm{sing}})),$$

where $(X_e)_{\mathrm{sing}}$ is the closed subscheme of non-smooth points of X_e. Then, there is an integer $c \geq 1$ such that, for any e and n in $\mathbb{N}$ with $n \geq ce$, the $\mathfrak{X}_s$-morphism projection

$$\pi_{n+1}(\mathrm{Gr}(\mathfrak{X})) \to \pi_n(\mathrm{Gr}(\mathfrak{X}))$$

is a G-equivariant piecewise trivial fibration over $\pi_n(\mathrm{Gr}^{(e)}(\mathfrak{X}))$ with fiber $\mathbb{A}_{\mathfrak{X}_s}^d$.

Let n be in $\mathbb{N}$. Recall that a subset $\mathcal{A}$ of $\mathrm{Gr}(\mathfrak{X})$ is *cylindrical of level n* if $\mathcal{A} = \pi_n^{-1}(C)$ with C a constructible subset of $\mathrm{Gr}_n(X_n)$. Assume that $\mathfrak{X}$ is a stft formal R-scheme endowed with a good G-action. If C is a G-invariant constructible subset of $\mathrm{Gr}_n(X_n)$, then $\pi_n^{-1}(C)$ is a G-invariant cylinder of $\mathrm{Gr}(\mathfrak{X})$. If $\mathfrak{X}$ is as in Proposition 3.8, then a G-invariant cylinder $\mathcal{A}$ of $\mathrm{Gr}(\mathfrak{X})$ is said to be *stable of level n* if it is G-invariant cylindrical of level n, and for every $n \leq m$ in $\mathbb{N}$, the $\mathfrak{X}_s$-morphism

$$\pi_m(\mathrm{Gr}(\mathfrak{X})) \to \pi_n(\mathrm{Gr}(\mathfrak{X}))$$

is a G-equivariant piecewise trivial fibration over $\pi_m(\mathcal{A})$ with fiber $\mathbb{A}_{\mathfrak{X}_s}^{(m-n)d}$.

Assume that G is a finite group k-scheme. Let $\mathbf{C}_{\mathfrak{X}}^G$ be the set of G-invariant cylinders of $\mathrm{Gr}(\mathfrak{X})$ which are stable of some level. For simplicity of notation, we shall write $\mathbb{L}$ for $\mathbb{L}_{\mathfrak{X}_s} = \left[\mathbb{A}_k^1 \times_k \mathfrak{X}_s \to \mathfrak{X}_s\right]$ and $\widetilde{\mathbb{L}}$ for $\mathbb{L} + I_{\mathfrak{X}_s}^{uh} \in K_0^G \langle \mathrm{Var}_{\mathfrak{X}_s} \rangle$.

Proposition 3.9. *Let $\mathfrak{X}$ be a relatively d-dimensional flat quasi-compact stft formal R-scheme endowed with a good G-action. Then there exists a unique additive mapping*

$$\mu_{\mathfrak{X}}^G \colon \mathbf{C}_{\mathfrak{X}}^G \to \widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$$

such that for any G-invariant stable cylinder $\mathcal{A}$ of level n of $\mathrm{Gr}(\mathfrak{X})$,

$$\mu_{\mathfrak{X}}^G(\mathcal{A}) = \langle \pi_n(\mathcal{A}) \to \mathfrak{X}_s \rangle \widetilde{\mathbb{L}}^{-(n+1)d}$$

Proof. For $\mathcal{A} \subseteq \mathrm{Gr}(\mathfrak{X})$ being a G-invariant stable cylinder of level n, it is a G-invariant stable cylinder of any level $m \geq n$. Since the hypothesis of Theorem 2.2 is satisfied, we have

$$\langle \pi_m(\mathcal{A}) \to \mathfrak{X}_s \rangle = \langle \pi_n(\mathcal{A}) \to \mathfrak{X}_s \rangle \widetilde{\mathbb{L}}^{(m-n)d}$$

holding true in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$. It implies that $\langle \pi_m(\mathcal{A}) \to \mathfrak{X}_s \rangle \widetilde{\mathbb{L}}^{-(m+1)d}$ is constant in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$ independent of $m \geq n$, and we define

$$\mu_{\mathfrak{X}}^G(\mathcal{A}) := \langle \pi_n(\mathcal{A}) \to \mathfrak{X}_s \rangle \widetilde{\mathbb{L}}^{-(n+1)d}.$$

Forgetting the action it was shown in [38, Proposition 4.3.13] (see also [37]) the additivity of $\mu_{\mathfrak{X}}^G$. Now, for such an $\mathcal{A}$ as previous, if there exist G-invariant stable cylinders $\mathcal{A}'$ and $\mathcal{A}''$ of level $\geq n$ such that $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$, then $\mathcal{A}' \cap \mathcal{A}''$ is also a G-invariant stable cylinder of level $\geq n$. Therefore, $\mu_{\mathfrak{X}}^G$ is also additive in the G-action setting. $\square$

Definition 3.10. Let G be a finite group k-scheme. For any $\mathcal{A}$ in $\mathbf{C}_{\mathfrak{X}}^G$ and any simple function $\alpha \colon \mathcal{A} \to \mathbb{Z} \cup \{\infty\}$, we say that $\widetilde{\mathbb{L}}^{-\alpha}$ is *naively G-integrable*, or that α is *naively exponentially G-integrable*, if α takes only finitely many values in $\mathbb{Z}$ and if all the fibers of α are in $\mathbf{C}_{\mathfrak{X}}^G$. We define the *motivic G-integral* of $\widetilde{\mathbb{L}}^{-\alpha}$ on $\mathcal{A}$ as follows

$$\int_{\mathcal{A}} \widetilde{\mathbb{L}}^{-\alpha} d\mu_{\mathfrak{X}}^G := \sum_{n \in \mathbb{Z}} \mu_{\mathfrak{X}}^G(\alpha^{-1}(n)) \widetilde{\mathbb{L}}^{-n} \ \in \widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G.$$

Remark 3.11. When considering the version without G-action, i.e., $\mathfrak{X}$ is endowed with trivial G-action, we shall write simply μ (resp. $\int_{\mathcal{A}} \widetilde{\mathbb{L}}^{-\alpha} d\mu$) in stead of $\mu_{\mathfrak{X}}^G$ (resp. $\int_{\mathcal{A}} \widetilde{\mathbb{L}}^{-\alpha} d\mu_{\mathfrak{X}}^G$).

3.4. **Change of variables formula.** Let $\mathfrak{X}$ be a quasi-compact generically smooth flat stft formal R-schemes. By [38, Thm. 8.0.4], there exist a smooth quasi-compact stft formal R-schemes $\mathfrak{Y}$ and an R-morphism $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ such that $\mathrm{Gr}(\mathfrak{Y}) \to \mathrm{Gr}(\mathfrak{X})$ is a bijection. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be endowed with good G-actions, and let $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ be G-equivariant. By Proposition 3.7, the induced morphisms $\mathrm{Gr}(\mathfrak{h})$ and $\mathrm{Gr}_n(h_n)$ are G-equivariant.

In the following theorem, we use the function $\mathrm{ord}_\varpi(\det \mathrm{Jac}_\mathfrak{h}) \colon \mathrm{Gr}(\mathfrak{X}) \backslash \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}}) \to \mathbb{N} \cup \{\infty\}$ defined in [33, Sect. 4]. Here, we denote by $\mathfrak{X}_{\mathrm{sing}}$ the closed formal subscheme of $\mathfrak{X}$ on which the structural morphism $\mathfrak{X} \to \mathrm{Spf}R$ is non-smooth. By abuse of notation, the symbol $\widetilde{\mathbb{L}}$ will stand for both $\widetilde{\mathbb{L}}_{\mathfrak{X}_s}$ and $\widetilde{\mathbb{L}}_{\mathfrak{Y}_s}$.

Theorem 3.12. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be quasi-compact flat stft formal R-schemes endowed with good G-actions, purely of the same relative dimension. Assume $\mathfrak{X}$ is generically smooth and $\mathfrak{Y}$ is smooth over R. Let $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ be a G-equivariant morphism of formal R-schemes such that $\mathfrak{h}_\eta$ is étale and $\mathfrak{Y}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$. Then, for any naively exponentially G-integrable function α on $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$, so is $\alpha \circ \mathrm{Gr}(\mathfrak{h}) + \mathrm{ord}_\varpi(\det \mathrm{Jac}_\mathfrak{h})$ on $\mathrm{Gr}(\mathfrak{Y})$, and moreover in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$,*

$$\int_{\mathrm{Gr}(\mathfrak{X}) \backslash \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})} \widetilde{\mathbb{L}}^{-\alpha} d\mu_{\mathfrak{X}}^G = (\mathfrak{h}_s)! \int_{\mathrm{Gr}(\mathfrak{Y})} \widetilde{\mathbb{L}}^{-\alpha \circ \mathrm{Gr}(\mathfrak{h}) - \mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})} d\mu_{\mathfrak{Y}}^G.$$

Proof. Since $\mathfrak{Y}$ is smooth and $\mathfrak{h}_\eta$ is étale, it is shown in the proof of [9, Ch. 5, Prop. 3.1.4] that the fibers of $\mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})$ are constructible and $\mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})$ takes only finitely many values in $\mathbb{N}$. One can also show that all the fibers of $\mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})$ are stable cylinders. Let y be a point of $\mathrm{Gr}(\mathfrak{Y})$ with residue field k' and valuation ring $R' = R \widehat{\otimes}_k k'$, which corresponds to a morphism of formal R-scheme $\gamma \colon \mathrm{Spf}R' \to \mathfrak{Y}$. Then, for every g in G, the point $g \cdot y$ of $\mathrm{Gr}(\mathfrak{Y})$ corresponds to the morphism $\gamma' := g(k') \cdot \gamma = \theta \circ (g(k') \times \gamma)$ from $\mathrm{Spf}R'$ to $\mathfrak{Y}$, where $g(k')$ is the morphism $\mathrm{Spec}k' \to G$. Since $g(k')$ is in $G(k')$, $\gamma^* \mathrm{Jac}_\mathfrak{h}$ and $\gamma'^* \mathrm{Jac}_\mathfrak{h}$ have a same order in ϖ, thus

$$\mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h}(y) = \mathrm{ord}_\varpi \gamma^* \mathrm{Jac}_\mathfrak{h} = \mathrm{ord}_\varpi \gamma'^* \mathrm{Jac}_\mathfrak{h} = \mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h}(g \cdot y).$$

This means that all the fibers of $\mathrm{ord}_\varpi(\mathrm{Jac}_\mathfrak{h})$ are G-invariant. Because $\alpha^{-1}(n)$ is G-invariant for every $n \in \mathbb{N}$ (note that $\alpha^{-1}(n)$ is nonempty for finitely many $n \in \mathbb{N}$) and the fact that the G-equivariant $\mathfrak{h}$ induces a G-equivariant $\mathrm{Gr}(\mathfrak{h})$ (cf. Prop. 3.7), we deduce that $(\alpha \circ \mathrm{Gr}(\mathfrak{h}))^{-1}(n)$ is G-invariant for every $n \in \mathbb{N}$. Hence, the function

$$\beta := \alpha \circ \mathrm{Gr}(\mathfrak{h}) + \mathrm{ord}_\varpi(\det \mathrm{Jac}_\mathfrak{h})$$

is a naively exponentially G-integrable function on $\mathrm{Gr}(\mathfrak{Y})$.

Now, for any $n \in \mathbb{N}$, consider the decomposition

$$\beta^{-1}(n) = \bigsqcup_{e \in \mathbb{N}} \mathcal{A}_{n,e},$$

where

$$\mathcal{A}_{n,e} = (\alpha \circ \mathrm{Gr}(\mathfrak{h}))^{-1}(n-e) \cap (\mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h})^{-1}(e)$$

which is G-invariant for every $n, e \in \mathbb{N}$. Because $\mathrm{Gr}(\mathfrak{h})$ is G-equivariant (cf. Prop. 3.7), the semi-algebraic subset

$$\mathrm{Gr}(\mathfrak{h})(\mathcal{A}_{n,e}) = \alpha^{-1}(n-e) \cap \mathrm{Gr}(\mathfrak{h})((\mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h})^{-1}(e))$$

is G-invariant. For $m \geq 2e$, consider the G-equivariant morphism h in the commutative diagram

$$\begin{array}{ccc} \pi_{\mathfrak{Y},m}(\mathcal{A}_{n,e}) & \xrightarrow{\ h\ } & \pi_{\mathfrak{X},m}(\mathrm{Gr}(\mathfrak{h})(\mathcal{A}_{n,e})) \\ \downarrow & & \downarrow \\ \mathfrak{Y}_s & \xrightarrow{\ \mathfrak{h}_s\ } & \mathfrak{X}_s \end{array}$$

which is a restriction of the G-equivariant morphism $\mathrm{Gr}_m(h_m)$ (see Prop. 3.7). We deduce from [37, Lem. 2.28] (the corrected form of [38, Lemme 7.1.3]) that the fiber of h over any point x of $\pi_{\mathfrak{X},m}(\mathrm{Gr}(\mathfrak{h})(\mathcal{A}_{n,e}))$ is isomorphic to $\mathbb{A}^e_{\kappa(x)}$. By Theorem 2.2, when viewed as an $\mathfrak{X}_s$-morphism, h is G-equivariant piecewise trivial fibration with fiber $\mathbb{A}^e_{\mathfrak{X}_s}$, thus in $\widetilde{\mathcal{M}}^G_{\mathfrak{X}_s}$,

$$(\mathfrak{h}_s)_! \, \langle \pi_{\mathfrak{Y},m}(\mathcal{A}_{n,e}) \rangle = \langle \pi_{\mathfrak{X},m}(\mathrm{Gr}(\mathfrak{h})(\mathcal{A}_{n,e})) \rangle \widetilde{\mathbb{L}}^e.$$

Denote by d the relative dimension of $\mathfrak{X}$ and $\mathfrak{Y}$. Then we have

$$\begin{aligned}
(\mathfrak{h}_s)_! \mu^G_{\mathfrak{Y}}(\beta^{-1}(n)) \widetilde{\mathbb{L}}^{-n} &= \sum_{e \in \mathbb{N}} (\mathfrak{h}_s)_! \, \langle \pi_{\mathfrak{Y},m}(\mathcal{A}_{n,e}) \rangle \widetilde{\mathbb{L}}^{-(m+1)d-n} \\
&= \sum_{e \in \mathbb{N}} \left\langle \pi_{\mathfrak{X},m}\left(\alpha^{-1}(n-e) \cap \mathrm{Gr}(\mathfrak{h})((\mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h})^{-1}(e)) \right) \right\rangle \widetilde{\mathbb{L}}^{-(m+1)d-n+e} \\
&= \sum_{e \in \mathbb{N}} \mu^G_{\mathfrak{X}} \left(\alpha^{-1}(n-e) \cap \mathrm{Gr}(\mathfrak{h})((\mathrm{ord}_\varpi \mathrm{Jac}_\mathfrak{h})^{-1}(e)) \right) \widetilde{\mathbb{L}}^{-(n-e)},
\end{aligned}$$

from which, under the fact that $\mathrm{Gr}(\mathfrak{h})$ is a bijection, due to [37, Lem. 2.25], we have

$$(\mathfrak{h}_s)_! \int_{\mathrm{Gr}(\mathfrak{Y})} \widetilde{\mathbb{L}}^{-\beta} d\mu^G_{\mathfrak{Y}} = \sum_{n' \in \mathbb{N}} \mu^G_{\mathfrak{X}}(\alpha^{-1}(n')) \widetilde{\mathbb{L}}^{-n'} = \int_{\mathrm{Gr}(\mathfrak{X}) \backslash \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})} \widetilde{\mathbb{L}}^{-\alpha} d\mu^G_{\mathfrak{X}}.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.5. **Motivic G-integral on stft formal schemes of gauge forms.** Let $\mathfrak{X}$ be a purely relatively d-dimensional flat stft generically smooth formal R-scheme, and let $\widetilde{\omega}$ be a differential form in $\Omega^d_{\mathfrak{X}/R}(\mathfrak{X})$. Let x be in $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$ which is defined over some field extension k' of k. Put $R' = R_{k'}$, and consider the morphism of formal R-schemes $\gamma \colon \mathrm{Spf}\,R' \to \mathfrak{X}$ corresponding to x. Since $(\gamma^*\Omega^d_{\mathfrak{X}/R})/(\text{torsion})$ is a free $\mathcal{O}_{R'}$-module of rank one, we have either $\gamma^*\widetilde{\omega} = 0$ or $\gamma^*\widetilde{\omega} = \alpha\varpi^n$ for some nonzero $\alpha \in \mathcal{O}_{R'}$ and $n \in \mathbb{N}$. Then we define

$$\mathrm{ord}_\varpi(\widetilde{\omega})(x) = \begin{cases} \infty & \text{if } \gamma^*\widetilde{\omega} = 0 \\ n & \text{if } \gamma^*\widetilde{\omega} = \alpha\varpi^n. \end{cases}$$

Consider the canonical isomorphism

$$(3.4) \qquad\qquad \Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \cong \Omega^d_{\mathfrak{X}_\eta/K}(\mathfrak{X}_\eta)$$

shown in [5, Proposition 1.5].

Definition 3.13. A *gauge* form ω on $\mathfrak{X}_\eta$ is a global section of the differential sheaf $\Omega^d_{\mathfrak{X}_\eta/K}$ such that it generates the sheaf at every point of $\mathfrak{X}_\eta$.

For any gauge form ω on $\mathfrak{X}_\eta$, there exist $\widetilde{\omega} \in \Omega^d_{\mathfrak{X}/R}(\mathfrak{X})$ and $n \in \mathbb{N}$ such that $\omega = \varpi^{-n}\widetilde{\omega}$, hence we put

$$\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega) := \mathrm{ord}_\varpi(\widetilde{\omega}) - n,$$

which defines a $\mathbb{Z}$-value function

$$\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega) \colon \mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}}) \to \mathbb{Z} \cup \{\infty\}.$$

This definition is independent of the choice of $\widetilde{\omega}$ thanks to [29, Sect. 4.1].

Lemma 3.14. *Let $\mathfrak{X}$ be a flat stft generically smooth formal R-scheme of pure relative dimension d. Assume that $\mathfrak{X}$ is endowed with a good G-action $\theta \colon G \times_k \mathfrak{X} \to \mathfrak{X}$. If ω is in $\Omega^d_{\mathfrak{X}_\eta/K}(\mathfrak{X}_\eta)$ is a gauge form, the function $\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega)$ is naively exponentially G-integrable.*

Proof. For the first part of the proof, we use arguments from [29, Thm.-Def. 4.1.2]. Assume that there is an open dense smooth formal subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ such that $\mathfrak{Y}_\eta$ is an open rigid subspace of $\mathfrak{X}_\eta$ and $\mathfrak{Y}(R^{sh}) \to \mathfrak{X}_\eta(K^{sh})$ is a bijection. The module $\Omega^d_{\mathfrak{Y}/R}$ is locally free of rank one over $\mathcal{O}_\mathfrak{Y}$ because of the smoothness of $\mathfrak{Y}$, so there is a open covering $\{\mathfrak{U}_i\}$ of $\mathfrak{Y}$ such that $\Omega^d_{\mathfrak{Y}/R}(\mathfrak{U}_i)$ is free of rank one. Hence, for every i, there is an f_i in $\mathcal{O}_\mathfrak{Y}(\mathfrak{U}_i)$ such that

$$\widetilde{\omega}\mathcal{O}_\mathfrak{Y}(\mathfrak{U}_i) \otimes (\Omega^d_{\mathfrak{Y}/R}(\mathfrak{Y}))^{-1} \cong (f_i)\mathcal{O}_\mathfrak{Y}(\mathfrak{U}_i),$$

here we use the expression $\omega = \varpi^{-n}\widetilde{\omega}$. It implies that the restriction of the function $\mathrm{ord}_\varpi(\widetilde{\omega})$ to $\mathfrak{U}_i$ is equal to $\mathrm{ord}_\varpi(f_i)$ which assigns $\mathrm{ord}_\varpi(f_i(u))$ to a point $u \in \mathrm{Gr}(\mathfrak{U}_i)$. Let f be the global section of $\mathcal{O}_\mathfrak{Y}$ such that $f = f_i$ on $\mathfrak{U}_i$. By glueing $\mathrm{ord}_\varpi(f_i)$'s altogether we get a function

$$\mathrm{ord}_\varpi(f) \colon \mathrm{Gr}(\mathfrak{Y}) = \mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}}) \to \mathbb{Z} \cup \{\infty\}$$

which is equal to $\mathrm{ord}_\varpi(\widetilde{\omega})$. Since ω is a gauge form, f induces an invertible function on $\mathfrak{X}_\eta$, hence by the Maximum Modulus Principle (see [5]), the function $\mathrm{ord}_\varpi(f)$ has only finitely many values, and so does $\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega)$.

Also by the proof of [29, Thm.-Def. 4.1.2], the fibers of the function $\mathrm{ord}_\varpi(\widetilde{\omega}) = \mathrm{ord}_\varpi(f)$ are stable cylinders, thus we only need to show that these fibers are G-invariant. Let x be a point of $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$, which we assume to have residue field k'. Put $R' = R_{k'}$. Let $\gamma \colon \mathrm{Spf}\,R' \to \mathfrak{X}$ be the morphism corresponding to x. For every g in G, we denote by $g' := g(k')$ the corresponding element in $G(k')$. Then the point $g \cdot x$ of $\mathrm{Gr}(\mathfrak{X})$ corresponds to

the morphism $\gamma' := g' \cdot \gamma = \theta \circ (g' \times \gamma)$ from $\mathrm{Spf}\, R'$ to $\mathfrak{X}$. Since $g' \in G(k')$, we can prove that $g \cdot x$ is also in $\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})$ and that $\gamma^* \widetilde{\omega}$, $\gamma'^* \widetilde{\omega}$ have the same order in ϖ. Hence two points x and $g \cdot x$ belong to a same fiber of $\mathrm{ord}_\varpi(\widetilde{\omega})$, i.e. the function $\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega)$ is naively exponentially G-integrable. $\square$

The following definition is supported by Lemma 3.14 and [33, Sec. 6].

Definition 3.15. Let $\mathfrak{X}$ be a purely relatively d-dimensional flat stft generically smooth formal R-scheme endowed with a good G-action. For any gauge form ω on $\mathfrak{X}_\eta$, the integral

$$\int_{\mathfrak{X}} |\omega| := \int_{\mathrm{Gr}(\mathfrak{X}) \setminus \mathrm{Gr}(\mathfrak{X}_{\mathrm{sing}})} \widetilde{\mathbb{L}}^{-\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega)} d\mu_{\mathfrak{X}}^G \ \in \widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$$

is called the *motivic G-integral of ω on $\mathfrak{X}$*, and the integral

$$\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{X}_s} \int_{\mathfrak{X}} |\omega| \ \in \widetilde{\mathcal{M}}_k^G$$

is called the *motivic G-integral of ω on $\mathfrak{X}_\eta$*.

Remark 3.16. The proof of Lemma 3.14 still works for any differential form $\omega \in \Omega_{\mathfrak{X}_\eta/K}^d(\mathfrak{X}_\eta)$, which is not necessarily a gauge form, provided that $\mathrm{ord}_{\varpi,\mathfrak{X}}(\omega)$ has only finitely many values. Then the integral $\int_{\mathfrak{X}} |\omega|$ (which takes value in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$) in Definition 3.15 is also well defined for this kind of differential form ω.

Lemma 3.17. *Let $\mathfrak{X}$ be a flat smooth stft formal R-scheme of pure relative dimension d, which is endowed with a good G-action. Let ω be a gauge form on $\mathfrak{X}_\eta$. Suppose that $\mathfrak{X}$ has connected components $\mathfrak{X}_i$, $i \in I$ (where I is a finite set). Then, for every $i \in I$, the function $\mathrm{ord}_{\varpi,\mathfrak{X}_i}(\omega) := \mathrm{ord}_{\varpi,\mathfrak{X}_i}(\omega|_{(\mathfrak{X}_i)_\eta})$ is constant on $\mathrm{Gr}(\mathfrak{X}_i)$. If $(\mathfrak{X}_i)_s$ is G-invariant for every $i \in I$, then*

$$\int_{\mathfrak{X}} |\omega| = \widetilde{\mathbb{L}}^{-d} \sum_{i \in I} \langle (\mathfrak{X}_i)_s \hookrightarrow \mathfrak{X}_s \rangle \, \widetilde{\mathbb{L}}^{-\mathrm{ord}_{\varpi,\mathfrak{X}_i}(\omega)}$$

in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$. Denote by $\mathrm{ord}_C(\omega)$ the constant $\mathrm{ord}_{\varpi,\mathfrak{X}_i}(\omega)$ if C is a connected component of $(\mathfrak{X}_i)_s$. Let $\mathcal{C}(\mathfrak{X}_s)$ be a family of constructible subsets of $\mathfrak{X}_s$ such that

- *$\mathfrak{X}_s = \bigcup_{C \in \mathcal{C}(\mathfrak{X}_s)} C$; $C \cap C' = \emptyset$ if $C, C' \in \mathcal{C}(\mathfrak{X}_s)$ and $C \neq C'$;*
- *every $C \in \mathcal{C}(\mathfrak{X}_s)$ is a union of connected components $C_1, \ldots, C_r$ of $\mathfrak{X}_s$ such that $\mathrm{ord}_{C_1}(\omega) = \cdots = \mathrm{ord}_{C_r}(\omega)$ (denote this number by $\mathrm{ord}_C(\omega)$) and C is G-invariant.*

Then the following holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G$:

$$\int_{\mathfrak{X}} |\omega| = \widetilde{\mathbb{L}}^{-d} \sum_{C \in \mathcal{C}(\mathfrak{X}_s)} \langle C \hookrightarrow \mathfrak{X}_s \rangle \, \widetilde{\mathbb{L}}^{-\mathrm{ord}_C(\omega)}.$$

Proof. The first statement is the same as in [33, Lem. 6.4.] (see also [29, Prop. 4.3.1]) using the hypothesis that ω is a gauge form on $\mathfrak{X}_\eta$. The rest completely follows from the definition of motivic G-integral of a gauge form (cf. Def. 3.10, 3.15). $\square$

4. EQUIVARIANT MOTIVIC INTEGRATION ON SPECIAL FORMAL SCHEMES

4.1. Special formal schemes. Let $R\{x_1, \ldots, x_m\}[[y_1, \ldots, y_{m'}]]$ be the mixed formal power series R-algebra which is the $R\{x_1, \ldots, x_m\}$-algebra of formal power series in $y_1, \ldots, y_{m'}$. In fact, one can prove that

$$R\{x_1, \ldots, x_m\}[[y_1, \ldots, y_{m'}]] = R[[y_1, \ldots, y_{m'}]]\{x_1, \ldots, x_m\}.$$

A topological R-algebra A is called *special* if A is a Noetherian adic ring and the R-algebra A/J is finitely generated for some ideal of definition J of A. By [3], we have an equivalent definition, which states that a topological R-algebra A is special if and only if A is topologically R-isomorphic to a quotient the R-algebra $R\{x_1, \ldots, x_m\}[[y_1, \ldots, y_{m'}]]$ for some $m, m' \in \mathbb{N}^*$.

Definition 4.1. A *special* formal scheme is a separated Noetherian adic formal scheme $\mathfrak{X}$ that is a finite union of open affine formal schemes of the form $\mathrm{Spf}\,A$ with A a Noetherian special R-algebra. A morphism $\mathfrak{Y} \to \mathfrak{X}$ of special formal R-schemes is said to be *locally of finite type* if locally it is isomorphic to a morphism of the form $\mathrm{Spf}\,B \to \mathrm{Spf}\,A$, where $A \to B$ is a morphism of finite type of Noetherian special R-algebras. A morphism $\mathfrak{Y} \to \mathfrak{X}$ of special formal R-schemes is said to be *(locally) adic* if locally it is isomorphic to a morphism of the form $\mathrm{Spf}\,B \to \mathrm{Spf}\,A$, where $A \to B$ is an adic morphism, i.e. there is an ideal of definition J of A such that the topology on B is J-adic.

The category of special formal R-schemes admits fiber products, and it contains the category of formal R-schemes topologically of finite type as a full subcategory. If $\mathfrak{X}$ is a special formal R-scheme, any formal completion of $\mathfrak{X}$ is also a special formal R-scheme.

Remark 4.2. Let $\mathcal{A}$ be the category of special formal R-schemes whose morphisms are adic morphisms. Then the correspondence
$$\mathfrak{X} \mapsto \mathfrak{X}_0$$
from $\mathcal{A}$ to the category of k-schemes is functorial. Furthermore, the natural closed immersion $\mathfrak{X}_0 \to \mathfrak{X}$ is a homeomorphism.

In the category of stft formal R-schemes every morphism is automatically an adic morphism, but in general this is not true in the category of special formal R-schemes. In the rest of this article, we shall always consider *adic* morphisms between special formal R-schemes $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$, because it allows to induce k-morphisms at the reduction level $\mathfrak{f}_0\colon \mathfrak{Y}_0 \to \mathfrak{X}_0$. For short, from now on, saying morphisms of special formal R-schemes we means adic morphisms.

As explained in [7, 0.2.6] and [32, Sect. 2.1], one first considers the affine case $\mathfrak{X} = \mathrm{Spf}\,A$, where A is a special R-algebra. Denote by J the largest ideal of definition of A and consider for each $n \in \mathbb{N}^*$ the subalgebra $A\left[\varpi^{-1}J^n\right]$ of $A \otimes_R K$ generated by A and $\varpi^{-1}J^n$. Let B_n be the J-adic completion of $A[\varpi^{-1}J^n]$. Then we have the affinoid K-algebra $C_n := B_n \otimes_R K$. The inclusion $J^{n+1} \subseteq J^n$ gives rise naturally to a morphism of affinoid K-algebras $C_{n+1} \to C_n$, which in its turn induces an open embedding of affinoid K-spaces $\mathrm{Spm}(C_n) \to \mathrm{Spm}(C_{n+1})$. Then the *generic fiber* $\mathfrak{X}_\eta$ of $\mathfrak{X}$ is defined to be
$$\mathfrak{X}_\eta = \varinjlim_n \mathrm{Spm}(C_n) = \bigcup_{n \in \mathbb{N}^*} \mathrm{Spm}(C_n).$$
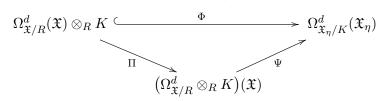
For the general case, the *generic fiber* of a special formal R-scheme $\mathfrak{X}$ can be obtained by a glueing procedure. More precisely, by the construction in [7], one covers $\mathfrak{X}$ by open affine formal R-subscheme $\mathfrak{X}_i$, $i \in I$, and glues $\mathfrak{X}_{i\eta}$ into the generic fiber $\mathfrak{X}_\eta$ of $\mathfrak{X}$ due to the method introduced in [7, Prop. 0.2.3].

In general, the generic fiber $\mathfrak{X}_\eta$ is a rigid K-variety, which is separated but not necessarily quasi-compact (cf. [7]). The correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor from the category of special formal R-schemes to the category of separated rigid K-varieties.

Note that the reduction $\mathfrak{X}_0$ of any Noetherian ϖ-adic formal scheme is already mentioned in Section 3.1. The *specialization map* $\mathrm{sp}\colon \mathrm{Spf}\,A_\eta \to \mathrm{Spf}\,A$ for the case of affine sepcial formal R-schemes is defined as follows. Let x be in $\mathrm{Spf}\,A_\eta$, and let $I \subseteq A \otimes_R K$ be the maximal ideal in $A \otimes_R K$ corresponding to x. Put $I' = I \cap A \subseteq A$. Then, by construction, $\mathrm{sp}(x)$ is the

unique maximal ideal of A containing ϖ and I'. If Z is a locally closed subscheme of $(\mathrm{Spf}\,A)_0$, $\mathrm{sp}^{-1}(Z)$ is an open rigid K-subvariety of $(\mathrm{Spf}\,A)_\eta$, which is canonically isomorphic to the generic fiber of the formal completion of $\mathrm{Spf}\,A$ along Z (cf. [15, Section 7.1]). In general, the construction of the specialization map $\mathrm{sp}\colon \mathfrak{X}_\eta \to \mathfrak{X}$ can be generalized to any special formal R-scheme $\mathfrak{X}$ using a glueing procedure (see [15]).

Let $\mathfrak{X}$ be a special formal scheme of pure relative dimension d. Thanks to [15, Sect. 7] and [32, Sect. 2.1], we have a natural injective map $\Phi\colon \Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \to \Omega^d_{\mathfrak{X}_\eta/K}(\mathfrak{X}_\eta)$, which factors uniquely through the sheafification map $\Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K \to \big(\Omega^d_{\mathfrak{X}/R} \otimes_R K\big)(\mathfrak{X})$, namely,

$$
\begin{array}{ccc}
\Omega^d_{\mathfrak{X}/R}(\mathfrak{X}) \otimes_R K & \xrightarrow{\quad\Phi\quad} & \Omega^d_{\mathfrak{X}_\eta/K}(\mathfrak{X}_\eta) \\
& \overset{\Pi}{\searrow} \qquad \overset{\Psi}{\nearrow} & \\
& \big(\Omega^d_{\mathfrak{X}/R} \otimes_R K\big)(\mathfrak{X}) &
\end{array}
$$

If $\mathfrak{X}$ is a stft formal R-scheme, Φ is an isomorphism (see Eq. (3.4)); also, if $\mathfrak{X}$ is an affine special formal R-scheme, the homomorphism Π is an isomorphism. In the general case, we always have $\mathrm{Im}(\Phi) = \mathrm{Im}(\Psi)$. We now recall Definition 2.11 in [32].

Definition 4.3. A gauge form ω on $\mathfrak{X}_\eta$ which lies in $\mathrm{Im}(\Phi)$ is called $\mathfrak{X}$-*bounded*.

Clearly, if $\mathfrak{X}$ is a stft formal R-scheme, every gauge form on $\mathfrak{X}_\eta$ is an $\mathfrak{X}$-bounded gauge form; but this does not hold in general. The $\mathfrak{X}$-boundedness is very important for a gauge form ω on $\mathfrak{X}_\eta$ because it allows to get a well-defined order of ω with respect to a connected component of $\mathfrak{X}_0$.

Let G be a finite group k-scheme. In Definition 3.5 we defined good G-action on any formal R-scheme. To be more useful in studying special formal R-schemes we need a nicer action of G as follows.

Definition 4.4. Let $\mathfrak{X}$ be a special formal R-scheme. An *adic G-action* on $\mathfrak{X}$ is a G-action $\theta\colon G \times_k \mathfrak{X} \to \mathfrak{X}$ such that regarding as a morphism over $\mathrm{Spf}\,R$, θ is adic.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be special formal R-schemes each of which is endowed with an adic G-action. An (adic) morphism of special formal schemes $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ is called *adic G-equivariant* if it is compatible with the adic actions $G \times_k \mathfrak{X} \to \mathfrak{X}$ and $G \times_k \mathfrak{Y} \to \mathfrak{Y}$.

Clearly, if $\theta\colon G \times_k \mathfrak{X} \to \mathfrak{X}$ is an adic G-action on $\mathfrak{X}$, it induces naturally a G-action on $\mathfrak{X}_0$ which is the k-morphism $\theta_0\colon G \times_k \mathfrak{X}_0 \to \mathfrak{X}_0$.

4.2. **Motivic G-integral on special formal schemes.** Let $\mathfrak{X}$ be a Noetherian adic formal R-scheme, let $\mathcal{J}$ be the largest ideal of definition of $\mathfrak{X}$, and let $\mathcal{I}$ be a coherent ideal sheaf on $\mathfrak{X}$. The *formal blowup of $\mathfrak{X}$ with center $\mathcal{I}$* is the morphism of formal schemes

$$
(4.1) \qquad\qquad \pi\colon \varinjlim_{n \in \mathbb{N}^*} \mathrm{Proj}\Big(\bigoplus_{m \ge 0} \mathcal{I}^m \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n)\Big) \to \mathfrak{X}.
$$

The formal blowup of $\mathfrak{X}$ with center $\mathcal{I}$ is an adic formal R-scheme on which the ideal generated by $\mathcal{I}$ is invertible. The blowup has the universality, it also commutes with flat base change, with the completion of $\mathfrak{X}$ along a closed k-subscheme of $\mathfrak{X}_s$ (cf. [32, Proposition 2.16]). If $\mathfrak{X}$ is a special formal R-scheme and $\mathcal{I}$ is open with respect to the ϖ-adic topology, then the blowup π is called *admissible*. By [32, Corollary 2.17], if $\mathfrak{Y} \to \mathfrak{X}$ is an admissible blowup, then $\mathfrak{Y}$ is a special formal R-scheme, and if, in addition, $\mathfrak{X}$ is R-flat, so is $\mathfrak{Y}$. Furthermore, the induced morphism of rigid K-varieties $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is an isomorphism due to [32, Proposition 2.19].

Definition 4.5. Let $\mathfrak{X}$ be a flat special formal R-scheme, and $\mathcal{I}$ a coherent ideal sheaf on $\mathfrak{X}$ which contains ϖ. Let $\pi \colon \mathfrak{Y} \to \mathfrak{X}$ be the admissible blowup with center $\mathcal{I}$. If $\mathfrak{U}$ is the open formal subscheme of $\mathfrak{Y}$ where $\mathcal{I}\mathcal{O}_{\mathfrak{Y}}$ is generated by ϖ, the restriction $\pi \colon \mathfrak{U} \to \mathfrak{X}$ is called *the dilatation of $\mathfrak{X}$ with center $\mathcal{I}$.*

It is easy to prove that the dilatation of a special formal R-scheme $\mathfrak{X}$ always exists and that it is a flat special formal R-scheme. Like admissible blowups, dilatations have the universality (cf. [32, Proposition 2.22]), they commute with the formal completion along closed subschemes (cf. [32, Propositions 2.21, 2.23]). In the sequel, we shall use the following important properties, see more in [32, Proposition 2.22].

Proposition 4.6 (Nicaise [32])**.** *Let $\mathfrak{X}$ be a flat special formal R-scheme, and let $\mathfrak{U} \to \mathfrak{X}$ be the dilatation of $\mathfrak{X}$ with center $\mathcal{I}$ containing ϖ. If $\mathfrak{X}' \to \mathfrak{X}$ is a morphism of flat special formal R-schemes such that the induced morphism $\mathfrak{X}'_s \to \mathfrak{X}_s$ factors through the closed formal subscheme of $\mathfrak{X}_s$ defined by $\mathcal{I}$, then there exists a unique morphism of formal R-schemes $\mathfrak{X}' \to \mathfrak{U}$ that makes the diagram*

$$
\begin{array}{ccc}
 & & \mathfrak{U} \\
 & \nearrow & \downarrow \\
\mathfrak{X}' & \longrightarrow & \mathfrak{X}.
\end{array}
$$

commute. Furthermore, if $\mathcal{I}$ is open, then $\mathfrak{U}$ is a stft formal R-scheme.

Let G be a finite group k-scheme. Due to Definition 3.4, a G-action on $\mathfrak{X}$ is a G-action on $\mathfrak{X}$ when viewed as a formal k-scheme such that the structural morphism is G-equivariant.

Proposition 4.7. *Let $\mathfrak{X}$ be a flat special formal R-scheme endowed with a good adic G-action θ. Let $\pi \colon \mathfrak{U} \to \mathfrak{X}$ be the dilatation of $\mathfrak{X}$ with center $\mathcal{I}$ containing ϖ, and let $\mathfrak{Z}$ be the closed formal subscheme of $\mathfrak{X}_s$ defined by $\mathcal{I}$. Assume that $\mathfrak{Z}$ is G-invariant. Then there exists a good adic G-action $\pi^*\theta$ on $\mathfrak{U}$ such that the dilatation π is G-equivariant. The formal scheme $\mathfrak{U}$ together with this action is called the G-**dilatation** of $\mathfrak{X}$*

Proof. Consider the following morphism of formal k-schemes

$$\pi' = \theta \circ (\mathrm{Id} \times \pi) \colon G \times_k \mathfrak{U} \to G \times_k \mathfrak{X} \to \mathfrak{X}.$$

Since π is the dilatation with center $\mathcal{I}$, and since $\mathfrak{Z}$ is G-invariant, it follows that the induced morphism

$$\pi'_s \colon G \times_k \mathfrak{U}_s \to G \times_k \mathfrak{X}_s \to \mathfrak{X}_s$$

factors through $\mathfrak{Z}$. Note that $\mathfrak{X}$ is a formal R-scheme and therefore π' can be regarded as a morphism of formal R-schemes. Applying Proposition 4.6 to π' we obtain a unique morphism $\pi^*\theta \colon G \times_k \mathfrak{U} \to \mathfrak{U}$ making the diagram

$$
\begin{array}{ccc}
G \times_k \mathfrak{U} & \xrightarrow{\ \pi^*\theta\ } & \mathfrak{U} \\
{\scriptstyle \mathrm{Id}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
G \times_k \mathfrak{X} & \xrightarrow{\ \theta\ } & \mathfrak{X}
\end{array}
$$

commutes. Similarly, applying again Proposition 4.6 one can show that $\pi^*\theta$ defines a good adic action of G on $\mathfrak{U}$, and hence the dilatation π is G-equivariant due to the commutative diagram. $\qquad\qquad \square$
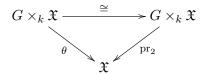
The following definition is a generalization of [32, Definition 4.1] to the G-equivariant context.

Definition 4.8. A *G-Néron smoothening* for a special formal R-scheme $\mathfrak{X}$ is a G-equivariant morphism of special formal R-schemes $\mathfrak{Y} \to \mathfrak{X}$, with $\mathfrak{Y}$ adic smooth over R, such that the induced morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is an open embedding satisfying $\mathfrak{Y}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$, and the induced morphism $\mathfrak{Y}_s \to \mathfrak{X}_s$ factors through $\mathfrak{X}_0$.

Proposition 4.9. *Let G be a smooth finite group k-scheme. Then every flat generically smooth special formal R-scheme $\mathfrak{X}$ endowed with a good adic G-action admits a G-Néron smoothening $\mathfrak{h}: \mathfrak{Y} \to \mathfrak{X}$.*

Proof. Let us first mention the following simple but useful observations:
  (i) Every adic action θ of G on $\mathfrak{X}$ is smooth, since G is smooth. This follows immediately from the following diagram

$$\begin{array}{ccc} G \times_k \mathfrak{X} & \xrightarrow{\;\cong\;} & G \times_k \mathfrak{X} \\ & {\scriptstyle \theta} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \mathfrak{X} & \end{array}$$

  where the upper arrow is defined as $(g, x) \mapsto (g, g^{-1}x)$.
  (ii) The R-smooth locus $Sm(\mathfrak{X})$ of $\mathfrak{X}$ is G-invariant. In fact, since the structural morphism $\mathfrak{f}: \mathfrak{X} \to \mathrm{Spf}\,R$ is G-equivariant, one has the following commutative diagram

$$\begin{array}{ccc} G \times_k \mathfrak{X} & \xrightarrow{\;\theta\;} & \mathfrak{X} \\ {\scriptstyle \mathrm{Id}\times\mathfrak{f}} \downarrow & \searrow {\scriptstyle \widetilde{\mathfrak{f}}} & \downarrow {\scriptstyle \mathfrak{f}} \\ G \times_k \mathrm{Spf}\,R & \xrightarrow{\;\rho\;} & \mathrm{Spf}\,R. \end{array}$$

By (i), the morphisms ρ is smooth, then so is the morphism $\widetilde{\mathfrak{f}}|_{G\times_k Sm(\mathfrak{X})}$. Combining with the smoothness of θ (by (i)) we can deduce that for all $g \in G$ and all $x \in Sm(\mathfrak{X})$, $\mathfrak{f}$ is smooth at gx ([39, Tag 02K5]). This means that $Sm(\mathfrak{X})$ is G-invariant.

Let $\pi: \mathfrak{U} \to \mathfrak{X}$ be the G-dilatation of $\mathfrak{X}$ with center $\mathfrak{X}_0$. Then $\mathfrak{U}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$ by the universal property of the dilatation (see also [32, Lemma 4.3]). Applying [4, 3.4/2] (see also, [6, Thm. 3.1]) for $E := \mathfrak{U}(R^{sh})$ we obtain a morphism $\mathfrak{U}' \to \mathfrak{U}$ which consists of a finite sequence of (E-permissible) blowing-ups with centers contained in the non-smooth parts of the corresponding formal R-schemes, such that the R-smooth locus $Sm(\mathfrak{U}')$ is a Néron smoothening of $\mathfrak{U}$. Let us consider the finite sequence of (E-permissible) blowing-ups $\mathfrak{h}: \mathfrak{U}' \to \mathfrak{U}$ constructed in [4, 3.4/2]. We first prove that the centers of these blowing-ups are G-invariants. By induction, it suffices to prove that the center Z in $\mathfrak{U}_s$ of the first blowing-up $\mathfrak{h}_1: \mathfrak{U}_1 \to \mathfrak{U}$ is G-equivariant, since then, by the universal property of blowing-up, $\mathfrak{U}_1$ admits an adic action of G such that $\mathfrak{h}_1$ is G-equivariant (see, Proposition 4.7). The center $Z = Y_\ell$ of the first blowup $\mathfrak{h}_1$ is defined in [4, 3.4/2] as follows. Let $(\cdot)_k$ be the map defined as

$$(\cdot)_k: \mathfrak{U}(R^{sh}) \to \mathfrak{U}(\overline{k}) \hookrightarrow \mathfrak{U} \times_R k = \mathfrak{U}_s,$$

where the first map is induced by the specilization $R^{sh} \to \overline{k}$. Note that, the adic action of G on $\mathfrak{U}$ induces actions of $G(R^{sh})$ on $\mathfrak{U}(R^{sh})$ and G on $\mathfrak{U}_s$ respectively. Therefore, the map $(\cdot)_k$ is equivariant, i.e. the following diagram commutative

$$\begin{array}{ccc} G(R^{sh}) \times \mathfrak{U}(R^{sh}) & \longrightarrow & \mathfrak{U}(R^{sh}) \\ {\scriptstyle (\cdot)_k} \downarrow & & \downarrow {\scriptstyle (\cdot)_k} \\ G \times_k \mathfrak{U}_s & \longrightarrow & \mathfrak{U}_s. \end{array}$$

Set $F^1 := E$ and $Y_1 := \overline{F_k^1}$ the Zariski closure of $F_k^1 := (\cdot)_k(F^1)$ in $\mathfrak{U}_0$. Let U_1 be the largest open subscheme of Y_1 which is smooth over k and where $\Omega^1_{\mathfrak{X}/R}|_{U^1}$ is locally free, and define

$$E^1 := \left\{a \in F^1 | a_k \in U_1\right\} = (\cdot)_k^{-1}(U_1).$$

Proceeding in the same way with $F^2 := F^1 \setminus E^1$, and so on, we obtain

(a) a decreasing sequence $F^1 \supseteq F^2 \supseteq \ldots$ in E,
(b) subsets $E^1, E^2, \ldots$such that E decomposes into a disjoint union

$$E = E^1 \sqcup \cdots \sqcup E^i \sqcup F^{i+1} \ \forall i \geq 1,$$

(c) dense open subschemes $U_i \subseteq Y_i := \overline{F_k^i}$ such that $E_k^i \subseteq U_i$ and, moreover, $Y_{i+1} \subseteq Y_i \setminus U_i$; in particular, $\dim Y_{i+1} < \dim Y_i$ if $Y_i \neq \emptyset$.

We see that $Y_{i+1} = \emptyset$ for all $i \geq \dim \mathfrak{U}_0$, denote by ℓ the smallest i with this property, i.e. $Y_\ell \neq \emptyset$ and $Y_{\ell+1} = \emptyset$. Then, by construction, $Z = Y_\ell$ is the center of the first blowing-up of the sequence $\mathfrak{h}: \mathfrak{U}' \to \mathfrak{U}$. We will prove the G-invariance of $Z = Y_\ell$. It is obviously that $Y_1 = \mathfrak{U}_0$ is G-invariant. By our first observation, $U_1 = Sm(Y_1) \cap \iota^{-1}(Sm(\mathfrak{U}))$ is also G-invariant, where $\iota: \mathfrak{U}_s \hookrightarrow \mathfrak{U}$ is the natural inclusion which is also G-equivariant. Since the map $(\cdot)_k$ is equivariant, the set E^1, and so F^2 are $G(R^{sh})$-invariant and F_k^2 is G-invariant. Note that, the Zariski closure Y_2 of F_k^2 in $\mathfrak{U}_s$ consists of points $\operatorname{Spec} A \to \mathfrak{U}_s$ whose specialzations $\operatorname{Spec} \overline{k} \to \operatorname{Spec} A \to \mathfrak{U}_s$ belong to F_k^2. Moreover, every (specialization) morphism $\varphi: A \to \overline{k}$ of k-algebras gives rise to a commutative diagram

$$
\begin{array}{ccc}
G(A) \times \mathfrak{U}_s(A) & \xrightarrow{\theta} & \mathfrak{U}_s(A) \\
\varphi \downarrow & & \downarrow \varphi \\
G(\overline{k}) \times \mathfrak{U}_s(\overline{k}) & \xrightarrow[\theta]{} & \mathfrak{U}_s(\overline{k})
\end{array}
$$

It yields that the Zariski closure Y_2 of F_k^2 is G-invariant. By induction, we can conclude that Y_ℓ is G-invariant. This proves our first assertion that the centers of the blowing-ups occuring in $\mathfrak{h}: \mathfrak{U}' \to \mathfrak{U}$ are G-invariants. This implies, by the universal property of the blowing-up, there exists an adic action of G on $\mathfrak{U}'$ such that $\mathfrak{h}$ is G-equivariant. Again, by our first observation $\mathfrak{Y} = Sm(\mathfrak{U}')$ is also G-invariant, hence $\mathfrak{h}: \mathfrak{Y} \to \mathfrak{U}$ is a G-Néron smoothening of $\mathfrak{U}$ and $\mathfrak{X}$. $\quad\square$

Proposition 4.10. *Let G be a smooth finite group k-scheme. Let $\mathfrak{X}$ be a flat, generically smooth stft formal R-scheme endowed with a good G-action, and let U be a G-invariant closed subscheme of $\mathfrak{X}_0$. Let $\pi: \mathfrak{U} \to \mathfrak{X}$ be the G-dilatation of $\mathfrak{X}$ with center U. If ω is a gauge form on $\mathfrak{X}_\eta$, then the identity*

$$\int_{\mathfrak{U}} |\omega| = \pi_s^* \int_{\mathfrak{X}} |\pi_\eta^* \omega|$$

holds in $\widetilde{\mathcal{M}}_{\mathfrak{U}_s}^G$.

Proof. Let $\mathfrak{h}: \mathfrak{Y}' \to \mathfrak{X}$ be the blowup of $\mathfrak{X}$ with center U. Using the same argument as in the proof of Proposition 4.7, we can construct an action of G on $\mathfrak{Y}'$ extending the action on $\mathfrak{U}$ such that $\mathfrak{h}$ is G-equivariant. Then the proof works on the same line as in [32, Prop. 4.5] by using the G-Néron smoothening $\mathfrak{g}: \mathfrak{Z}' \to \mathfrak{Y}'$ as constructed in Proposition 4.9. $\quad\square$

Remark from Proposition 4.6 that if $\pi: \mathfrak{U} \to \mathfrak{X}$ is the dilatation of a flat special formal R-scheme $\mathfrak{X}$ with center $\mathfrak{X}_0$, then $\mathfrak{U}$ is a flat stft formal R-scheme, and for any gauge form ω on $\mathfrak{X}_\eta$, the differential form $\pi_\eta^* \omega$ is also a gauge form on $\mathfrak{U}_\eta$.

Definition 4.11. Let G be a finite group k-scheme. Let $\mathfrak{X}$ be a flat generically smooth special formal R-scheme endowed with a good adic G-action, and let $\pi\colon \mathfrak{U} \to \mathfrak{X}$ be the G-dilatation of $\mathfrak{X}$ with center $\mathfrak{X}_0$. For any gauge form ω on $\mathfrak{X}_\eta$, we define

$$\int_{\mathfrak{X}} |\omega| := \pi_{s!} \int_{\mathfrak{U}} |\pi_\eta^* \omega| \quad \text{in} \quad \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$$

and call it the *motivic G-integral* of ω on $\mathfrak{X}$. The integral $\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{U}_\eta} |\omega|$ in $\widetilde{\mathcal{M}}_k^G$ is called the *motivic G-integral* of ω on $\mathfrak{X}_\eta$.

If $\mathfrak{X}$ is a generically smooth special formal R-scheme endowed with a good adic G-action, we denote by $\mathfrak{X}^{\text{flat}}$ its maximal flat closed subscheme (obtained by killing ϖ-torsion), and define the *motivic G-integral* of a gauge form ω on $\mathfrak{X}$ to be

$$\int_{\mathfrak{X}} |\omega| := \int_{\mathfrak{X}^{\text{flat}}} |\omega| \quad \text{in} \quad \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G.$$

In this case, the integral $\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{X}_\eta^{\text{flat}}} |\omega|$ in $\widetilde{\mathcal{M}}_k^G$ is called the *motivic G-integral* of ω on $\mathfrak{X}_\eta$.

Remark 4.12. By Remark 3.16, the integral $\int_{\mathfrak{X}} |\omega|$ can be defined with value in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$ for any differential form of maximal degree ω on $\mathfrak{X}_\eta$ provided $\operatorname{ord}_{\varpi,\mathfrak{U}}(\pi_\eta^* \omega)$ has only finitely many values in $\mathbb{Z}$.

Let us remark that in the stft case, by Proposition 4.10, this integral is obtained from the integral defined in Definition 3.15 by the base change

$$\widetilde{\mathcal{M}}_{\mathfrak{X}_s}^G \to \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G.$$

Therefore, we shall use the same notation for these two integrals but mention the ring of integral values.

Proposition 4.13. *Let $\mathfrak{X}$ be a generically smooth special formal R-scheme endowed with a good adic action of G and let $\phi\colon \mathfrak{Y} \to \mathfrak{X}$ be a G-Néron smoothening. Then, for any gauge form ω on $\mathfrak{X}_\eta$, we have*

$$\int_{\mathfrak{X}} |\omega| = \phi_{s!} \int_{\mathfrak{Y}} |\phi_\eta^* \omega| \quad \text{in} \quad \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G.$$

Proof. Let $\pi\colon \mathfrak{U} \to \mathfrak{X}$ be the G-dilatation of $\mathfrak{X}$ with center $\mathfrak{X}_0$. Then therer exists a unique morphism of formal R-schemes $\psi\colon \mathfrak{Y} \to \mathfrak{U}$ by the universal property (cf. Proposition 4.7) of π such that $\phi = \pi \circ \psi$. Combining Proposition 3.12, Definition 3.15 and [29, Lemma 4.1.1] we obtain the identity

$$\int_{\mathfrak{U}} |\omega| = \phi_{s!} \int_{\mathfrak{Y}} |\omega|,$$

which holds in $\widetilde{\mathcal{M}}_{\mathfrak{U}_s}^G$. Hence,

$$\int_{\mathfrak{X}} |\omega| = \pi_{s!} \int_{\mathfrak{U}} |\omega| = \pi_{s!} \left(\psi_{s!} \int_{\mathfrak{Y}} |\omega| \right) = \phi_{s!} \int_{\mathfrak{Y}} |\omega|,$$

which holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$. $\qquad\qquad\square$

Theorem 4.14 (Special G-equivariant change of variables formula)**.** *Let G be a smooth finite group k-scheme. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be generically smooth special formal R-schemes endowed with*

good adic actions of G, and let $\mathfrak{h}\colon \mathfrak{Y} \to \mathfrak{X}$ be an adic G-equivariant morphism of formal R-schemes such that the induced morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is an open embedding and $\mathfrak{Y}_\eta(K^{sh}) = \mathfrak{X}_\eta(K^{sh})$. If ω is a gauge form on $\mathfrak{X}_\eta$, then

$$\int_{\mathfrak{X}} |\omega| = \mathfrak{h}_{0!} \int_{\mathfrak{Y}} |\mathfrak{h}_\eta^* \omega| \quad in \ \ \widetilde{\mathcal{M}}^G_{\mathfrak{X}_0}.$$

Proof. Let $\phi\colon \mathfrak{Z} \to \mathfrak{Y}$ be a G-Néron smoothening of $\mathfrak{Y}$. Then $\mathfrak{h} \circ \phi\colon \mathfrak{Z} \to \mathfrak{X}$ is a G-Néron smoothening of $\mathfrak{X}$. It follows from Proposition 4.13 that

$$\int_{\mathfrak{Y}} |\omega| = \phi_{s!} \int_{\mathfrak{Z}} |\omega| \quad in \ \ \widetilde{\mathcal{M}}^G_{\mathfrak{Y}_0}$$

and

$$\int_{\mathfrak{X}} |\omega| = (\mathfrak{h} \circ \phi)_{s!} \int_{\mathfrak{Z}} |\omega| \ in \ \widetilde{\mathcal{M}}^G_{\mathfrak{X}_0}.$$

Since $\mathfrak{h}$ is adic, $\mathfrak{h}_0\colon \mathfrak{Y}_0 \to \mathfrak{X}_0$ is the restriction of $\mathfrak{h}_s$ on $\mathfrak{Y}_0$. Therefore,

$$\int_{\mathfrak{X}} |\omega| = \mathfrak{h}_{0!} \int_{\mathfrak{Y}} |\omega|,$$

which holds in $\widetilde{\mathcal{M}}^G_{\mathfrak{X}_0}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 4.15. *Let G be a smooth finite group k-scheme. Let $\mathfrak{X}$ be a generically smooth special formal R-scheme endowed with a good adic action of G, and let U be a G-invariant locally closed subscheme of $\mathfrak{X}_0$. Denote by $\mathfrak{U}$ the formal completion of $\mathfrak{X}$ along U. Then for every gauge form ω on $\mathfrak{X}_\eta$, the integral $\int_{\mathfrak{U}} |\omega|$ is the image of $\int_{\mathfrak{X}} |\omega|$ under the base change*

$$\widetilde{\mathcal{M}}^G_{\mathfrak{X}_0} \to \widetilde{\mathcal{M}}^G_U.$$

Proof. We prove only for the case that U is a closed subscheme of $\mathfrak{X}_0$ since the proof for the case that U is open is similar (and simpler). Let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ and $\pi'\colon \mathfrak{U}' \to \mathfrak{U}$ be the G-dilatations of $\mathfrak{X}$ with center $\mathfrak{X}_0$ and of $\mathfrak{U}$ with center U, respectively. By [32, Prop. 2.23], there exists a morphism $\mathfrak{i}'\colon \mathfrak{U}' \to \mathfrak{X}'$ such that the following diagram

$$\begin{array}{ccc} \mathfrak{U}' & \xrightarrow{\ \pi'\ } & \mathfrak{U} \\ {\scriptstyle \mathfrak{i}'} \downarrow & & \downarrow {\scriptstyle \mathfrak{i}} \\ \mathfrak{X}' & \xrightarrow{\ \pi\ } & \mathfrak{X} \end{array}$$

commutes, and the morphism $\mathfrak{i}'\colon \mathfrak{U}' \to \mathfrak{X}'$ is the dilatation of $\mathfrak{X}'$ with center $\mathfrak{X}'_s \times_{\mathfrak{X}_0} U$. Moreover, it follows from [32, Prop. 2.21], $\mathfrak{U}'$ is actually the formal completion of $\mathfrak{X}'$ along $\mathfrak{X}'_s \times_{\mathfrak{X}_0} U$, i.e. the following induced diagram is Cartesian

$$\begin{array}{ccc} \mathfrak{U}'_s & \xrightarrow{\ \pi'_s\ } & \mathfrak{U}_s \\ {\scriptstyle \mathfrak{i}'_s} \downarrow & & \downarrow {\scriptstyle \mathfrak{i}_s} \\ \mathfrak{X}'_s & \xrightarrow{\ \pi_s\ } & \mathfrak{X}_0 \end{array}$$

Since π_s and π'_s factors through $\mathfrak{X}_0$ and U respectively, the diagram

$$\begin{array}{ccc} \mathfrak{U}'_s & \xrightarrow{\ \pi'_s\ } & U \\ {\scriptstyle \mathfrak{i}'_s} \downarrow & & \downarrow {\scriptstyle \mathfrak{i}_0} \\ \mathfrak{X}'_s & \xrightarrow{\ \pi_s\ } & \mathfrak{X}_0 \end{array}$$

is also Cartesian, therefore $i_0^* \circ \pi_{s!} = \pi_{s!}' \circ i_s'^*$. We can conclude that

$$\int_{\mathfrak{U}} |\omega| = \pi_{s!}' \int_{\mathfrak{U}'} |\omega| = \pi_{s!}' \left(i_s'^* \int_{\mathfrak{X}'} |\omega| \right)$$

$$= (i_0^* \circ \pi_{s!}) \int_{\mathfrak{X}'} |\omega|$$

$$= i_0^* \int_{\mathfrak{X}} |\omega|,$$

which hold in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$. Here, the second equality is due to Proposition 4.10. $\hfill\square$

The following corollary is an immediate consequence of Proposition 4.15.

Corollary 4.16 (Additivity of motivic integrals). *Let G be a smooth finite group k-scheme. Let $\mathfrak{X}$ be a generically smooth special formal R-scheme endowed with a good adic action of G and let ω be a gauge form on $\mathfrak{X}_\eta$.*

(i) *If $\{U_i, i \in I\}$ is a finite stratification of $\mathfrak{X}_0$ into G-invariant locally closed subsets, and $\mathfrak{U}_i$ is the formal completion of $\mathfrak{X}$ along U_i, then*

$$\int_{\mathfrak{X}} |\omega| = \sum_{i \in I} \int_{\mathfrak{U}_i} |\omega| \quad in \ \ \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G.$$

(ii) *If $\{\mathfrak{U}_i, i \in I\}$ is a finite covering of G-invariant open subsets of $\mathfrak{X}$, then*

$$\int_{\mathfrak{X}} |\omega| = \sum_{I' \subseteq I} (-1)^{|I'|-1} \int_{\mathfrak{U}_{I'}} |\omega| \quad in \ \ \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G,$$

where $\mathfrak{U}_{I'} = \cap_{i \in I'} \mathfrak{U}_i$.

Here the pushforward morphisms $\widetilde{\mathcal{M}}_{U_{I'}}^G \to \widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$ are applied to the RHS in both statements.

Proposition 4.17. *Let $\mathfrak{X}$ be a smooth special formal R-scheme of pure relative dimension d endowed with a good adic G-action. Suppose that ω is an $\mathfrak{X}$-bounded gauge form on $\mathfrak{X}_\eta$. Let $\mathcal{C}(\mathfrak{X}_0)$ be a family of constructible subsets of $\mathfrak{X}_0$ such that*

- $\mathfrak{X}_0 = \bigcup_{C \in \mathcal{C}(\mathfrak{X}_0)} C$; $C \cap C' = \emptyset$ *if $C, C' \in \mathcal{C}(\mathfrak{X}_0)$ and $C \neq C'$;*
- *every $C \in \mathcal{C}(\mathfrak{X}_0)$ is a union of connected components $C_1, \ldots, C_r$ of $\mathfrak{X}_0$ such that $\mathrm{ord}_{C_1}(\omega) = \cdots = \mathrm{ord}_{C_r}(\omega)$ (denote this number by $\mathrm{ord}_C(\omega)$) and C is G-invariant.*

Then the identity

$$\int_{\mathfrak{X}} |\omega| = \widetilde{\mathbb{L}}^{-d} \sum_{C \in \mathcal{C}(\mathfrak{X}_0)} \langle C \hookrightarrow \mathfrak{X}_0 \rangle \, \widetilde{\mathbb{L}}^{-\mathrm{ord}_C(\omega)}$$

holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$.

Proof. The proof of this proposition is an adic G-action analogue of that of [32, Prop. 5.14]. By the definition of integral, we can assume that $\mathfrak{X}$ is flat. By Corollary 4.16 and [32, Cor. 5.12], we can also assume that $\mathfrak{X}_0$ and connected. Let $\pi \colon \mathfrak{U} \to \mathfrak{X}$ be a G-ditalation with center $\mathfrak{X}_0$. Then

$$\int_{\mathfrak{X}} |\omega| = (\pi_0)_! \int_{\mathfrak{U}} |\pi_\eta^* \omega|.$$

Since $\mathfrak{U}$ is a flat smooth stft formal R-scheme of pure relative dimension d, we deduce from the proof of [32, Prop. 4.15] that $\langle \mathfrak{U}_0 \to \mathfrak{X}_0 \rangle = \langle \mathfrak{X}_0 \to \mathfrak{X}_0 \rangle \widetilde{\mathbb{L}}^{\mathrm{ord}_\varpi \mathrm{Jac}_\pi}$. In particular, $\mathfrak{U}_0 = \mathfrak{U}_s$ is

connected, thus by Lemma 3.17,

$$\int_{\mathfrak{U}} |\pi_\eta^* \omega| = \widetilde{\mathbb{L}}^{-d} \langle \mathfrak{U}_0 \to \mathfrak{U}_0 \rangle \widetilde{\mathbb{L}}^{-\mathrm{ord}_{\mathfrak{U}_0}(\pi_\eta^* \omega)}$$

which holds in $\widetilde{\mathcal{M}}_{\mathfrak{U}_0}^G$. Remark that the equality on orders in [32, Lem. 5.13] has a small confusion, as one can see in [32, Lem. 5.5], and here we can correct it as follows

$$\mathrm{ord}_{\mathfrak{U}_0}(\pi_\eta^* \omega) = \mathrm{ord}_{\mathfrak{X}_0}(\omega) + \mathrm{ord}_\varpi \mathrm{Jac}_\pi.$$

Using the previously obtained equalities in the proof we have

$$\int_{\mathfrak{X}} |\omega| = \widetilde{\mathbb{L}}^{-d} \langle \mathfrak{X}_0 \to \mathfrak{X}_0 \rangle \widetilde{\mathbb{L}}^{-\mathrm{ord}_{\mathfrak{X}_0}(\omega)},$$

which holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^G$. The proposition is now proved. $\qquad\square$

4.3. **Monodromic volume Poincaré series and motivic volumes.** A special formal R-scheme $\mathfrak{X}$ is called *regular* if $\mathcal{O}_{\mathfrak{X},x}$ is regular for every $x \in \mathfrak{X}$. By [32, Def. 2.33], a closed formal subscheme $\mathfrak{E}$ of a purely relatively d-dimensional special formal R-scheme $\mathfrak{X}$ is called a *strict normal crossings divisor* if, for every x in $\mathfrak{X}$, there exists a regular system of local parameters $(x_0, \ldots, x_d)$ in $\mathcal{O}_{\mathfrak{X},x}$ such that the ideal defining $\mathfrak{E}$ at x is locally generated by $\prod_{i=0}^d x_i^{N_i}$ for some $N_i \in \mathbb{N}$, $0 \le i \le d$, and such that the irreducible components of $\mathfrak{E}$ are regular (see [32, Sect. 2.4] for definition of irreducibility). If $\mathfrak{E}'$ is an irreducible component of $\mathfrak{E}$ which is defined locally at x by the ideal $(x_i^{N_i})$, it is a fact that N_i is constant when x varies on $\mathfrak{E}'$, which is called *the multiplicity of* $\mathfrak{E}'$ and denoted by $N(\mathfrak{E}')$. Then we have $\mathfrak{E} = \sum_{i=1}^r N(\mathfrak{E}_i)\mathfrak{E}_i$, where $\mathfrak{E}_i$'s are irreducible components of $\mathfrak{E}$. The divisor $\mathfrak{E}$ is called a *tame strict normal crossings divisor* if $N(\mathfrak{E}_i)$ is prime to the characteristic exponent of k for every i. Any special formal R-scheme $\mathfrak{X}$ is said to *have tame strict normal crossings* if $\mathfrak{X}$ is regular and $\mathfrak{X}_s$ is a tame strict normal crossings divisor.

Definition 4.18. Let $\mathfrak{X}$ be a flat generically smooth special formal R-scheme. A *resolution of singularities* of $\mathfrak{X}$ is a proper morphism of flat special formal R-schemes $\mathfrak{h}\colon \mathfrak{Y} \to \mathfrak{X}$ such that $\mathfrak{h}_\eta$ is an isomorphism and $\mathfrak{Y}$ is regular with $\mathfrak{Y}_s$ being a strict normal crossings divisor. The resolution of singularities $\mathfrak{h}$ is said to be *tame* if $\mathfrak{Y}_s$ is a tame strict normal crossings divisor.

Theorem 4.19 (Temkin [42]). *Suppose that the base field k has characteristic zero. Then any generically smooth flat special formal R-scheme $\mathfrak{X}$ admits a resolution of singularities.*

As shortly explained in [9, Thm. 6.3.3], this theorem can be proved by using [42, Thm. 1.1.13]. For the affine case, it was realized early in [32, Prop. 2.43] by means of a result in [41].

We now study the $\hat{\mu}$-equivariant setting of volume Poincaré series and motivic volume of special formal R-schemes. Note that their older version (without action) was performed early in [32, Sect. 7].

Notation 4.20. For $n \in \mathbb{N}^*$, we put $R(n) = R[\tau]/(\tau^n - \varpi)$ and $K(n) = K[\tau]/(\tau^n - \varpi)$. For any formal R-scheme $\mathfrak{X}$, we define its ramifications as follows: $\mathfrak{X}(n) = \mathfrak{X} \times_R R(n)$, $\mathfrak{X}_\eta(n) = \mathfrak{X}_\eta \times_K K(n)$. If ω is a gauge form on $\mathfrak{X}_\eta$, let $\omega(n)$ be its pullback via the natural morphism $\mathfrak{X}_\eta(n) \to \mathfrak{X}_\eta$, which is a gauge form on $\mathfrak{X}_\eta(n)$.

Studying action of $\mu_n = \mathrm{Spec}\,(k[\xi]/(\xi^n - 1))$ on $\mathfrak{X}$, the below lemma is straightforward.

Lemma 4.21. *Let $\mathfrak{X}$ be a formal R-scheme and n in $\mathbb{N}^*$. Then there is a natural good adic μ_n-action on both $\operatorname{Spf}R(n)$ and $\mathfrak{X}(n)$ which is induced from the ring homomorphism $R(n) \to k[\xi]/(\xi^n - 1) \otimes_k R(n)$ given by $\tau \mapsto \xi \otimes \tau$. Moreover, the structural morphism of the formal $\operatorname{Spf}R(n)$-scheme $\mathfrak{X}(n)$ is μ_n-equivariant.*

Remark that if $\mathfrak{X}$ is a generically smooth special formal R-scheme and $n \in \mathbb{N}^*$, then $\mathfrak{X}(n)$ is a generically smooth special formal $R(n)$-scheme.

Definition 4.22. Let $\mathfrak{X}$ be a generically smooth special formal R-scheme, and let ω be a gauge form on $\mathfrak{X}_\eta$. The formal power series

$$P(\mathfrak{X}, \omega; T) := \sum_{n \geq 1} \Big(\int_{\mathfrak{X}(n)} |\omega(n)| \Big) T^n \ \in \widetilde{\mathcal{M}}^{\hat{\mu}}_{\mathfrak{X}_0}[[T]]$$

is called the motivic volume Poincaré series of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a generically smooth flat special formal R-scheme of pure relative dimension d. Assume that $\mathfrak{X}$ admits a resolution of singularities $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ (this is always available if the characteristic of k is zero, cf. Theorem 4.19). Let $\mathfrak{E}_i$, $i \in S$, be the irreducible components of $(\mathfrak{Y}_s)_{\mathrm{red}}$. Let N_i be the multiplicity of $\mathfrak{E}_i$ in $\mathfrak{Y}_s$. Put

$$E_i := (\mathfrak{E}_i)_0$$

for $i \in S$ and

$$\mathfrak{E}_I := \bigcap_{i \in I} \mathfrak{E}_i, \quad E_I := \bigcap_{i \in I} E_i, \quad E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j$$

for any nonempty subset $I \subseteq S$. We can check that $\mathfrak{E}_I$ is regular and $E_I = (\mathfrak{E}_I)_0$. Let $\{\mathfrak{U}\}$ be a covering of $\mathfrak{Y}$ by affine open subschemes. If $\mathfrak{U}_0 \cap E_I^\circ \neq \emptyset$, the composition $\mathfrak{f} \circ \mathfrak{h} \colon \widehat{\mathfrak{U}_I} \to \operatorname{Spf}R$ is defined on the ring level by

$$\varpi \mapsto u \prod_{i \in I} y_i^{N_i},$$

where $\widehat{\mathfrak{U}_I}$ is the formal completion of $\mathfrak{U}$ along $\mathfrak{U}_0 \cap E_I^\circ$, u is a unit in $(y_i)_{i \in I}$ and y_i is a local coordinate defining E_i. Put $N_I := \gcd(N_i)_{i \in I}$. Then we can construct as in [20] an unramified Galois covering $\widetilde{E}_I^\circ \to E_I^\circ$ with Galois group μ_{N_I} which is given over $\mathfrak{U}_0 \cap E_I^\circ$ as follows

$$\left\{ (z, y) \in \mathbb{A}_k^1 \times (\mathfrak{U}_0 \cap E_I^\circ) \mid z^{N_I} = u(y)^{-1} \right\}.$$

Notice that $\widetilde{E}_I^\circ$ is endowed with a natural good adic μ_{N_I}-action over E_I° obtained by multiplying the z-coordinate with elements of μ_{N_I}. Denote by $\big[\widetilde{E}_I^\circ\big]$ (resp. $\big\langle\widetilde{E}_I^\circ\big\rangle$) the class of the μ_{N_I}-equivariant morphism

$$\widetilde{E}_I^\circ \to E_I^\circ \to \mathfrak{X}_0$$

in the ring $\mathcal{M}^{\mu_{N_I}}_{\mathfrak{X}_0}$ (resp. $\widetilde{\mathcal{M}}^{\mu_{N_I}}_{\mathfrak{X}_0}$).

Theorem 4.23. *Let $\mathfrak{X}$ be a generically smooth flat special formal R-scheme of pure relative dimension d. Suppose that we are given a tame resolution of singularities $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ with $\mathfrak{Y}_s = \sum_{i \in S} N_i \mathfrak{E}_i$ and an $\mathfrak{X}$-bounded gauge form ω on $\mathfrak{X}_\eta$ with order $\alpha_i := \operatorname{ord}_{\mathfrak{E}_i}(\mathfrak{h}_\eta^* \omega)$ for every $i \in S$. If $n \in \mathbb{N}^*$ is prime to the characteristic exponent of k, then the identity*

$$\int_{\mathfrak{X}(n)} |\omega(n)| = \widetilde{\mathbb{L}}^{-d} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I|-1} \big\langle \widetilde{E}_I^\circ \big\rangle \left(\sum_{\substack{k_i \geq 1, i \in I \\ \sum_{i \in I} k_i N_i = n}} \widetilde{\mathbb{L}}^{-\sum_{i \in I} k_i \alpha_i} \right).$$

holds in $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^{\mu_n}$.

In order to prove this theorem we need the following two lemmas, in which the first lemma is trivial to prove.

Lemma 4.24. *Let* $\mathfrak{h} \colon \mathfrak{Y} \to \mathfrak{X}$ *be a resolution of singularities of a generically smooth special formal R-schemes. Then* $\mathfrak{h}(n) \colon \mathfrak{Y}(n) \to \mathfrak{X}(n)$ *is an adic μ_n-equivariant morphism of formal $R(n)$-schemes with* $\mathfrak{h}(n)_\eta$ *an isomorphism.*

Let $\mathfrak{Y}$ be a regular special formal R-scheme whose special fiber is a tame strict normal crossings divisor $\mathfrak{Y}_s = \sum_{i \in S} N_i \mathfrak{E}_i$. Recall from [32, Def. 2.38] that a number $n \in \mathbb{N}^*$ is said to be $\mathfrak{Y}_s$-*linear* if there exists a nonempty subset $I \subseteq S$ of cardinal $|I| \geq 2$ such that $E_I^\circ \neq \emptyset$ and the linear equation $\sum_{i \in I} k_i N_i = n$ has solutions in $(\mathbb{N}^*)^I$.

Lemma 4.25. *Let* $\mathfrak{Y}$ *be a regular special formal R-scheme whose special fiber is a tame strict normal crossings divisor* $\mathfrak{Y}_s = \sum_{i \in S} N_i \mathfrak{E}_i$. *If n is prime to the characteristic exponent of k and not* $\mathfrak{Y}_s$-*linear, then*

$$\phi \colon Sm(\widetilde{\mathfrak{Y}(n)}) \to \mathfrak{Y}(n)$$

is an adic μ_n-equivariant morphism of formal $R(n)$-schemes such that ϕ_η is an open embedding and $Sm(\widetilde{\mathfrak{Y}(n)})_\eta(K(n)^{sh}) = \mathfrak{Y}(n)_\eta(K(n)^{sh})$. *Furthermore,*

$$Sm(\widetilde{\mathfrak{Y}(n)})_0 = \bigsqcup_{N_i | n} \left((\widetilde{\mathfrak{Y}(n)})_0 \times_{\mathfrak{Y}_0} E_i^\circ \right),$$

in which for i with $N_i | n$, $(\widetilde{\mathfrak{Y}(n)})_0 \times_{\mathfrak{Y}_0} E_i^\circ$ *is a μ_{N_i}-invariant constructible subset of* $Sm(\widetilde{\mathfrak{Y}(n)})_0$ *and μ_{N_i}-equivariant canonically isomorphic to* $\widetilde{E}_i^\circ$ *over* E_i°.

Proof. By [32, Thm. 5.1], ϕ is a morphism of special formal $R(n)$-scheme which is the restriction of the normalization $\widetilde{\mathfrak{Y}(n)} \to \mathfrak{Y}(n)$. The normalization is clearly an adic μ_n-equivariant because of its definition and by the natural adic μ-action defined in Lemma 4.21. Since μ_n is smooth, it follows from the second observation in the proof of Proposition 4.9 that $Sm(\widetilde{\mathfrak{Y}(n)})$ is μ_n-invariant in $\widetilde{\mathfrak{Y}(n)}$, thus ϕ is an adic μ_n-equivariant morphism of formal $R(n)$-schemes. The properties that ϕ_η is an open embedding, $Sm(\widetilde{\mathfrak{Y}(n)})_\eta(K(n)^{sh}) = \mathfrak{Y}(n)_\eta(K(n)^{sh})$ and the decomposition of $Sm(\widetilde{\mathfrak{Y}(n)})_0$ in the lemma are verified in the proof of [32, Thm. 5.1].

Let y be any point in $\mathfrak{E}_i^\circ$, and let $\mathfrak{U} = \mathrm{Spf}\, A$ be an affine open formal neighborhood of y in $\mathfrak{Y}$ (in fact, $\mathfrak{U}$ is the formal completion of $\mathfrak{Y}$ along an affine open neighborhood of y in $\mathfrak{Y}_0$). Then the formal R-scheme structure at y is given by $\varpi = u y_i^{N_i}$ with u a unit. Similarly as in the proof of [34, Lem. 4.4] we may write

$$\widetilde{\mathfrak{U}(n)} \cong \mathrm{Spf}(A \otimes_R R(n))\{T\}/(\varpi(n)^{n/N_i} T - y_i, u T^{N_i} - 1),$$

where $\varpi(n)$ denotes the uniformizing parameter of $R(n)$. Then we have

$$(\widetilde{\mathfrak{U}(n)})_0 \times_{\mathfrak{U}_0} E_i^\circ \cong \mathrm{Spec}\Big(A[T]/(y_i, u T^{N_i} - 1) \Big),$$

which is endowed with the natural good adic μ_{N_i}-action given by $T \mapsto \xi \otimes T$, induced from the μ_{N_i}-action on $\widetilde{\mathfrak{U}(n)}$, and μ_{N_i}-equivariant canonically isomorphic to the restriction of $\widetilde{E}_i^\circ$ over $\mathfrak{U}_0$. The conclusion comes from the glueing procedure. $\square$

Proof of Theorem 4.23. We first prove the theorem for the case where n is not $\mathfrak{Y}_s$-linear. Since the conclusion of Lemma 4.24 satisfies the hypothesis of Theorem 4.14, we deduce from the proposition that

$$\int_{\mathfrak{X}(n)} |\omega(n)| = (\mathfrak{h}(n)_0)_! \int_{\mathfrak{Y}(n)} |\mathfrak{h}(n)_\eta^* \omega(n)|.$$

Similarly, applying Theorem 4.14 once again, in the setting of Lemma 4.25, we get

$$\int_{\mathfrak{Y}(n)} |\mathfrak{h}(n)_\eta^* \omega(n)| = (\phi_0)_! \int_{Sm(\widetilde{\mathfrak{Y}(n)})} |\phi_\eta^* \mathfrak{h}(n)_\eta^* \omega(n)|.$$

By Lemma 4.25, all $D_i := (\widetilde{\mathfrak{Y}(n)})_0 \times_{\mathfrak{Y}_0} E_i^\circ$ for $N_i | n$ are μ_{N_i}-invariant constructible subsets of $Sm(\widetilde{\mathfrak{Y}(n)})_0$. Using [32, Prop. 7.11] and the notation in Proposition 4.17 we get

$$\mathrm{ord}_{D_i}(\phi_\eta^* \mathfrak{h}(n)_\eta^* \omega(n)) = (n/N_i) \cdot \mathrm{ord}_{\mathfrak{E}_i}(\mathfrak{h}_\eta^* \omega) = (n/N_i) \cdot \alpha_i$$

for all $i \in S$ with $N_i | n$, we deduce from Proposition 4.17 and, again, Lemma 4.25 that

$$\int_{Sm(\widetilde{\mathfrak{Y}(n)})} |\phi_\eta^* \mathfrak{h}(n)_\eta^* \omega(n)| = \widetilde{\mathbb{L}}^{-d} \sum_{N_i | n} \langle \widetilde{E}_i^\circ \rangle \widetilde{\mathbb{L}}^{-n\alpha_i/N_i}.$$

Therefore, the case where n is not $\mathfrak{Y}_s$-linear has been completely proved.

For the case where n is $\mathfrak{Y}_s$-linear, we can extend the computation in [34, Lem. 7.5] to the special formal scheme setting and to the μ_n-equivariant setting, which is natural, and use the same arguments in the proof of [32, Thm. 7.12] (see also the proof of [34, Thm. 7.6.]). $\square$

Corollary 4.26. *Suppose that the base field k has characteristic zero. Let $\mathfrak{X}$ be a generically smooth flat special formal R-scheme of relative dimension d. Let $\mathfrak{h}: \mathfrak{Y} \to \mathfrak{X}$ be a resolution of singularities with $\mathfrak{Y}_s = \sum_{i \in S} N_i \mathfrak{E}_i$. Suppose that ω is an $\mathfrak{X}$-bounded gauge form on $\mathfrak{X}_\eta$ with $\alpha_i := \mathrm{ord}_{\mathfrak{E}_i}(\mathfrak{h}_\eta^* \omega)$ for every $i \in S$. Then, in $\mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}[[T]]$,*

$$P(\mathfrak{X}, \omega; T) = \mathbb{L}^{-d} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ] \prod_{i \in I} \frac{\mathbb{L}^{-\alpha_i} T^{N_i}}{1 - \mathbb{L}^{-\alpha_i} T^{N_i}}.$$

Here, we remark that when k has characteristic zero, we have $\widetilde{\mathbb{L}} = \mathbb{L}$ and $\widetilde{\mathcal{M}}_{\mathfrak{X}_0}^{\hat{\mu}} = \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$. We deduce from Corollary 4.26 that the limit

$$\lim_{T \to \infty} P(\mathfrak{X}, \omega; T) = -\mathbb{L}^{-d} \sum_{\emptyset \neq I \subseteq S} (1 - \mathbb{L})^{|I|-1} [\widetilde{E}_I^\circ] \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$$

is independent of the choice of the $\mathfrak{X}$-bounded gauge form ω. However, it depends on $\widehat{K^s}$, the completion of the separable closure of K, because it depends on the choice of the uniformizing parameter (see [32, Rem. 7.40]).

Definition 4.27. Suppose that k has characteristic zero. Let $\mathfrak{X}$ be a generically smooth flat special formal R-scheme. The motivic quantity

$$\mathrm{MV}(\mathfrak{X}; \widehat{K^s}) := \sum_{\emptyset \neq I \subseteq S} (1 - \mathbb{L})^{|I|-1} [\widetilde{E}_I^\circ] \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$$

is called the *motivic volume* of $\mathfrak{X}$.

Proposition 4.28 (Additivity of MV)**.** *Suppose that k has characteristic zero. Let $\mathfrak{X}$ be a generically smooth special formal R-scheme. The following hold.*

(i) *If $\{U_i, i \in Q\}$ is a finite stratification of $\mathfrak{X}_0$ into locally closed subsets, and $\mathfrak{U}_i$ is the formal completion of $\mathfrak{X}$ along U_i, then*

$$\mathrm{MV}(\mathfrak{X}; \widehat{K^s}) = \sum_{i \in Q} \mathrm{MV}(\mathfrak{U}_i; \widehat{K^s}).$$

(ii) *If $\{\mathfrak{U}_i, i \in Q\}$ is a finite open covering of $\mathfrak{X}$, then by putting $\mathfrak{U}_I = \bigcap_{i \in I} \mathfrak{U}_i$, we have*

$$\mathrm{MV}(\mathfrak{X}; \widehat{K^s}) = \sum_{\emptyset \neq I \subseteq Q} (-1)^{|I|-1} \mathrm{MV}(\mathfrak{U}_I; \widehat{K^s}).$$

Proof. Since $\mathfrak{X}$ admits a resolution of singularities (cf. Theorem 4.19), we can identify $\mathfrak{X}$ with its resolution of singularities. It implies from [32, Prop.-Def. 7.38] that $\mathfrak{X}$ has a finite open covering $\{\mathfrak{V}_j\}_j$ such that each $\mathfrak{V}_j$ admits a $\mathfrak{V}_j$-bounded gauge form ω on $\mathfrak{V}_{j\eta}$. Thus we can apply Corollary 4.16 to the coefficients of the volume Poincaré series $P(\mathfrak{U}_i \cap \mathfrak{V}_j, \omega; T)$ and deduce the proposition. $\qquad \square$

4.4. **Motivic zeta functions and motivic nearby cycles of formal power series.**
Consider the mixed formal power series R-algebra $R\{x\}[[y]]$, with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_{m'})$. Let $d = m + m'$. Let f be in $k\{x\}[[y]]$ such that $f(x, 0)$ is non-constant, and let $\mathfrak{X}_f$ be the formal completion of $\mathrm{Spf}(k\{x\}[[y]])$ along (f). Then $\mathfrak{X}_f$ is a special formal R-scheme of pure relative dimension $d - 1$, with structural morphism defined by $\varpi \mapsto f$. The reduction of $\mathfrak{X}_f$ is the algebraic k-variety $(\mathfrak{X}_f)_0 = \mathrm{Spec}\, k[x]/(f(x, 0))$. The following lemma will be used many times in the rest of this section.

Lemma 4.29. *Let f be in $k\{x\}[[y]]$ such that $f(x, 0)$ is non-constant. Then, there is an isomorphism of special formal R-schemes $\mathfrak{X}_f \cong \mathrm{Spf}\,(R\{x\}[[y]]/(f - \varpi))$. Consequently, $\mathfrak{X}_f$ is a generically smooth special formal R-scheme of pure relative dimension $d - 1$.*

Proof. Consider the k-algebra homomorphism $\varphi \colon R\{x\}[[y]] \to k\widehat{\{x\}[[y]]}_{(f)}$ given by $\varphi(\varpi) = f$, $\varphi(x) = x$ and $\varphi(y) = y$. This induces a canonical isomorphism of k-algebras

$$\overline{\varphi} \colon R\{x\}[[y]]/(f - \varpi) \to k\widehat{\{x\}[[y]]}_{(f)},$$

which makes the following diagram

$$
\begin{array}{ccc}
R\{x\}[[y]]/(f - \varpi) & \xrightarrow{\ \overline{\varphi}\ } & k\widehat{\{x\}[[y]]}_{(f)} \\
& \underset{\varpi \mapsto [\varpi]}{\searrow} \quad \underset{\varpi \mapsto f}{\swarrow} & \\
& R &
\end{array}
$$

commute. The lemma is now proved. $\qquad \square$

By [1, Sect. 4], we can see that $\mathfrak{X}_f$ is a formal scheme of pseudo-finite type over k, the sheaf of continuous differential form $\Omega^i_{\mathfrak{X}_f/k}$ is coherent for any i, and that there exists a morphism of coherent $\mathcal{O}_{\mathfrak{X}_f}$-modules $d\varpi \wedge (.) \colon \Omega^{d-1}_{\mathfrak{X}_f/R} \to \Omega^d_{\mathfrak{X}_f/k}$ defined by taking the exterior product with the differential df. By [15, Sect. 7] and [32, Lem. 2.5], we have an exact functor $(\cdot)_{\mathrm{rig}}$ from the category of coherent modules over $\mathcal{O}_{\mathfrak{X}_f}$ to the category of coherent modules over $\mathcal{O}_{(\mathfrak{X}_f)_\eta}$. Taking this functor we get a morphism of coherent $\mathcal{O}_{(\mathfrak{X}_f)_\eta}$-modules

$$d\varpi \wedge (.) \colon \Omega^{d-1}_{(\mathfrak{X}_f)_\eta/K} \to (\Omega^d_{\mathfrak{X}_f/k})_{\mathrm{rig}}.$$

By [32, Prop. 7.19], the morphism $d\varpi \wedge (.)$ is an isomorphism. If ω is a global section of $\Omega^d_{\mathfrak{X}_f/k}$, we denote as in [32, Def. 7.21] by ω/df the inverse image of ω under $d\varpi \wedge (.)$. Now we fix a gauge form ω on $\mathfrak{X}_f$, such as $\omega = dx_1 \wedge \cdots \wedge dx_m \wedge dy_1 \wedge \cdots \wedge dy_{m'}$.

Let $\mathfrak{h}: \mathfrak{Y} \to \mathfrak{X}_f$ be a tame resolution of singularities of $\mathfrak{X}_f$. Then, by [32, Lem. 7.24], $(\mathfrak{h}^*\omega)/d\varpi = \mathfrak{h}^*_\eta(\omega/df)$ in $\Omega^{d-1}_{\mathfrak{Y}_\eta/K}(\mathfrak{Y}_\eta)$. Since $\mathfrak{h}$ is a tame resolution of singularities of $\mathfrak{X}_f$ and ω is a gauge form on $\mathfrak{X}_f$, we have $\mathfrak{h}^*\omega$ to be a gauge form on $\mathfrak{Y}$. Since $\mathfrak{Y}$ is a regular flat special formal R-scheme, it follows from [32, Cor. 7.23] that $(\mathfrak{h}^*\omega)/dt$ is a $\mathfrak{Y}$-bounded gauge form on $\mathfrak{Y}_\eta$. This together with Remark 4.12 guarantees that the integral $\int_{\mathfrak{X}_f(n)} |(\omega/df)(n)|$ makes sense as a motivic quantity in $\widetilde{\mathcal{M}}^{\mu_n}_{(\mathfrak{X}_f)_0}$ even though ω/df is possibly not a gauge form.

Assume that the data of $\mathfrak{Y}$ are given as in the setting before Theorem 4.23 and that $K_{\mathfrak{Y}/\mathfrak{X}_f} = \sum_{i \in S} (\nu_i - 1)\mathfrak{E}_i$. Using the same argument in the proof of [32, Lem. 7.30] we get $\mathrm{ord}_{\mathfrak{E}_i} \mathfrak{h}^*_\eta(\omega/df) = \nu_i - N_i$ for all $i \in S$. Note that these numbers do not depend on ω. Similarly as in the proof of Theorem 4.23 we have the following result.

Proposition 4.30. *With the previous notation and hypotheses, if $n \in \mathbb{N}^*$ is prime to the characteristic exponent of k, then the identity*

$$\int_{\mathfrak{X}_f(n)} |(\omega/df)(n)| = \widetilde{\mathbb{L}}^{n+1-d} \sum_{\emptyset \neq I \subseteq S} (\mathbb{L}-1)^{|I|-1} \langle \widetilde{E}^\circ_I \rangle \left(\sum_{\substack{k_i \geq 1, i \in I \\ \sum_{i \in I} k_i N_i = n}} \widetilde{\mathbb{L}}^{\sum_{i \in I} k_i(N_i - \nu_i)} \right).$$

holds in $\widetilde{\mathcal{M}}^{\mu_n}_{(\mathfrak{X}_f)_0}$. If, in addition, k has characteristic zero, then

$$P(\mathfrak{X}_f, \omega/df; T) = \mathbb{L}^{-(d-1)} \frac{\mathbb{L}T}{1 - \mathbb{L}T} * \sum_{\emptyset \neq I \subseteq S} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}^\circ_I] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}},$$

where $*$ is the Hadamard product of formal series in $\mathcal{M}^{\hat{\mu}}_{(\mathfrak{X}_f)_0}[[T]]$ (cf. Section 2.3). Moreover,

$$\mathrm{MV}(\mathfrak{X}_f; \widehat{K^s}) = \sum_{\emptyset \neq I \subseteq S} (1 - \mathbb{L})^{|I|-1} [\widetilde{E}^\circ_I] \in \mathcal{M}^{\hat{\mu}}_{(\mathfrak{X}_f)_0}.$$

In preparation for Corollary 4.31 we consider the case where $d = m$ and f is a polynomial in $k[x]$, as well as let k be of characteristic zero, $R = k[[t]]$ and $K = k((t))$, with t as a traditional notaton in Denef-Loeser's setting [16, 17, 20] replacing ϖ. Assume that the zero locus X_0 of f is nonempty. Denote by $\mathcal{L}_n(\mathbb{A}^d_k)$ the n-jet scheme of $\mathbb{A}^d_k$ (cf. [16, 17]). The contact loci and motivic zeta function of f are defined as follows

$$\mathfrak{X}_n(f) = \{ \gamma \in \mathcal{L}_n(\mathbb{A}^d_k) \mid f(\gamma) = t^n \mod t^{n+1} \},$$
$$Z_f(T) = \sum_{n \geq 1} [\mathfrak{X}_n(f)] \mathbb{L}^{-nd} T^n \in \mathcal{M}^{\hat{\mu}}_{X_0}[[T]],$$

where the μ_n-action on the $\mathfrak{X}_n(f)$ is given by $\xi \cdot \gamma(t) = \gamma(\xi t)$. By [16], $Z_f(T)$ is rational, thus we can define $\mathcal{S}_f = -\lim_{T \to \infty} Z_f(T) \in \mathcal{M}^{\hat{\mu}}_{X_0}$ called the *motivic nearby cycles* of f. For any closed point $\mathbf{x} \in X_0$ we also obtain the local version $\mathfrak{X}_{n,\mathbf{x}}(f)$ and $Z_{f,\mathbf{x}}(T) \in \mathcal{M}^{\hat{\mu}}_k[[T]]$ (cf. [16]).

Corollary 4.31. *Let k be a field of characteristic zero, let f be a polynomial in $k[x_1, \ldots, x_d]$. With the above notation we have*

$$[\mathfrak{X}_n(f)] = \mathbb{L}^{(n+1)(d-1)} \int_{\mathfrak{X}_f(n)} |(\omega/df)(n)|,$$

$$[\mathfrak{X}_{n,\mathbf{x}}(f)] = \mathbb{L}^{(n+1)(d-1)} \int_{\widehat{(\mathfrak{X}_f)/\mathbf{x}}(n)} |(\omega/df)(n)|.$$

As a consequence, the following hold

$$Z_f(T) = \mathbb{L}^{d-1} \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} * P(\mathfrak{X}_f, \omega/df; T),$$

$$Z_{f,\mathbf{x}}(T) = \mathbb{L}^{d-1} \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} * P(\widehat{(\mathfrak{X}_f)/\mathbf{x}}, \omega/df; T),$$

as well as

$$\mathcal{S}_f = \mathrm{MV}(\mathfrak{X}_f; \widehat{K^s}) \in \mathcal{M}^{\hat{\mu}}_{(\mathfrak{X}_f)_0},$$

$$\mathcal{S}_{f,\mathbf{x}} = \mathrm{MV}(\widehat{(\mathfrak{X}_f)/\mathbf{x}}; \widehat{K^s}) \in \mathcal{M}^{\hat{\mu}}_k.$$

Proof. Let $h\colon Y \to \mathbb{A}^d_k$ be an embedded resolution of singularities of X_0 with strict normal crossing divisor $Y_s = \sum_{i \in S} N_i E_i$. By [34, Lem. 2.4], the induced morphism $\hat{h}\colon \widehat{Y} \to \mathfrak{X}_f$ is a resolution of singularities, where $\widehat{Y}$ is the formal completion of Y along $(f \circ h)$ and $(\widehat{Y})_s = Y_s$. Assume that $K_{Y/\mathbb{A}^d_k} = \sum_{i \in S}(\nu_i - 1)E_i$. By [34, Lem. 9.6], $\mathrm{ord}_{E_i} \hat{h}^*(\omega/df) = \nu_i - N_i$. By Remark 4.12, Theorem 4.23 and [20, Thm. 2.4] we complete the proof. $\square$

Definition 4.32. Let k be a field of characteristic zero. Let f be in $k\{x\}[[y]]$ such that $f(x, 0)$ is non-constant. Let $\mathbf{x}$ be a closed point in $(\mathfrak{X}_f)_0$. The *motive zeta function* $Z_f(T)$ of f and the *local motivic zeta function* $Z_{f,\mathbf{x}}(T)$ of f at $\mathbf{x}$ are defined as follows

$$Z_f(T) := \mathbb{L}^{d-1} P(\mathfrak{X}_f, \omega/df; T)$$

and

$$Z_{f,\mathbf{x}}(T) := \mathbb{L}^{d-1} P(\widehat{(\mathfrak{X}_f)/\mathbf{x}}, \omega/df; T).$$

The *motivic nearby cycles* $\mathcal{S}_f$ of f and the *motivic Milnor fiber* $\mathcal{S}_{f,\mathbf{x}}$ of f at $\mathbf{x}$ are defined as

$$\mathcal{S}_f := \mathbb{L}^{d-1} \mathrm{MV}(\mathfrak{X}_f; \widehat{K^s}) \in \mathcal{M}^{\hat{\mu}}_{(\mathfrak{X}_f)_0}$$

and

$$\mathcal{S}_{f,\mathbf{x}} := \mathbb{L}^{d-1} \mathrm{MV}(\widehat{(\mathfrak{X}_f)/\mathbf{x}}; \widehat{K^s}) \in \mathcal{M}^{\hat{\mu}}_k.$$

5. Two conjectures

5.1. A description of motivic Milnor fiber of formal power series.
Let k be a field of characteristic zero, and let f be a formal power series in $k[[x_1, \ldots, x_d]]$ such that $f(O) = 0$, where O is the origin of k^d. Let $R = k[[t]]$ and $K = k((t))$, with t replacing ϖ. By Lemma 4.29, we can consider the special formal R-scheme $\mathfrak{X}_f = \mathrm{Spf}(R[[x_1, \ldots, x_d]]/(f - t))$, whose relative dimension is $d - 1$ and whose reduction is $(\mathfrak{X}_f)_0 = \mathrm{Spec}\, k$. As in Definition 4.32, using a Gelfand-Leray form, we have the concept of motivic zeta function

$$Z_f(T) = \mathbb{L}^{d-1} P(\mathfrak{X}_f, \omega/df; T) \in \mathcal{M}^{\hat{\mu}}_k[[T]]$$

and that of motivic Milnor fiber

$$\mathcal{S}_f = \mathbb{L}^{d-1} \mathrm{MV}(\mathfrak{X}_f; \widehat{K^s}) \in \mathcal{M}^{\hat{\mu}}_k$$

of the formal power series f at O.

For $n \in \mathbb{N}^*$, let f_n denote the sum of all the degree k homogeneous parts of f over $1 \le k \le n$. Since f has no free coefficient, O is a zero of f_n. Consider the algebraic k-variety

$$\mathfrak{X}_{n,O}(f_n) = \{\gamma \in \mathcal{L}_n(\mathbb{A}_k^d) \mid f_n(\gamma) = t^n \mod t^{n+1}, \ \gamma(0) = O\},$$

which admits the good μ_n-action given by $\xi \cdot \gamma(t) = \gamma(\xi t)$. It is not simple to prove that the series

$$Z(T) = \sum_{n \ge 1} [\mathfrak{X}_{n,O}(f_n)] \mathbb{L}^{-nd} T^n$$

is rational because one can not find a common log resolution for all f_n's. As a solution, we recommend the following conjecture.

Conjecture 5.1. *Let f be a formal power series in $k[[x_1, \dots, x_d]]$ such that $f(O) = 0$. Put $\omega = dx_1 \wedge \cdots \wedge dx_d$. Then the identity*

$$[\mathfrak{X}_{n,O}(f_n)] = \mathbb{L}^{(n+1)(d-1)} \int_{\mathfrak{X}_f(n)} |(\omega/df)(n)|$$

holds in $\mathcal{M}_k^{\mu_n}$.

This conjecture can provide further applications to problems set up in the formal geometry context.

5.2. Motivic Milnor fiber of holomorphic function germs.

We consider the case $k = \mathbb{C}$. Let $f \in \mathbb{C}\{x_1, \dots, x_d\}$ be a complex analytic function vanishing at $O \in \mathbb{C}^d$. Using Denef-Loeser's theory of motivic integration [16, 17], it seems impossible to define *directly* $\mathfrak{X}_{n,O}(f)$ except f is a polynomial, but we can define $\mathfrak{X}_{n,O}(f)$ to be $\mathfrak{X}_{n,O}(f_n)$ with f_n understood as in Section 5.1. On the other hand, the rationality of the series $Z(T) = \sum_{n \ge 1} [\mathfrak{X}_{n,O}(f_n)] \mathbb{L}^{-nd} T^n$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[[T]]$ is a big problem because of the lack of existence of a common log resolution for all hypersurfaces defined by the vanishing of f_n. We even need to distinguish a resolution of singularities of an algebraic variety from that of an analytic complex manifold. Hence, if Conjecture 5.1 is not proved, we can not define the motivic Milnor fiber of a complex analytic function f as $-\lim_{T \to \infty} Z(T)$. In particular, it is important to require a complete explanation for the statement in Remark 1.17 of [21].

Denote by B_ϵ the closed ball of radius ϵ centered at O in $\mathbb{C}^d$ and by D_δ the closed disk of radius δ centered at 0 in $\mathbb{C}$. Put $D_\delta^\times = D_\delta \setminus \{0\}$. An important topological invariant of the germ (f, O) is the Milnor fibration (see [31] and [27])

$$f_{\epsilon,\delta} = f|_{B_\epsilon \cap f^{-1}(D_\delta^\times)} : B_\epsilon \cap f^{-1}(D_\delta^\times) \to D_\delta^\times$$

with $0 < \delta \ll \epsilon \ll 1$. Consider the topological Milnor fiber $F_{f,O} = f_{\epsilon,\delta}^{-1}(\delta)$ and the monodromy (cohomology) operator $M_{f,O} : H^*(F_{f,O}, \mathbb{Q}) \to H^*(F_{f,O}, \mathbb{Q})$. By [40], the groups $H^i(F_{f,O}, \mathbb{Q})$ carry a natural mixed Hodge structure, which is compatible with the semi-simplification of $M_{f,O}$. Denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structure endowed with an automorphism of finite order, and by $K_0(\mathrm{HS}^{\mathrm{mon}})$ the Grothendieck ring of $\mathrm{HS}^{\mathrm{mon}}$. One can define the Hodge characteristic $\chi_h(F_{f,O})$ of $F_{f,O}$ as in [19], that is,

$$\chi_h(F_{f,O}) := \sum_{i \ge 0} (-1)^i [H^i(F_{f,O}, \mathbb{Q})] \in K_0(\mathrm{HS}^{\mathrm{mon}})$$

taking monodromy $M_{f,O}$ into account. By abuse of notation, we also denote by χ_h the canonical homomorphism $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \to K_0(\mathrm{HS}^{\mathrm{mon}})$ induced by the assignment

$$X \mapsto \sum_{i \geq 0} (-1)^i \big[H_c^i(X, \mathbb{Q})\big],$$

where $H_c^i(X, \mathbb{Q})$ is the simplicial cohomology group with compact support, endowed with a natural mixed Hodge structure.

Similarly, we also use the same symbol χ_c for the topological Euler characteristic with compact support and for the Euler characteristic with compact support of complex constructible sets.

We now define *the motivic Milnor fiber* $\mathcal{S}_{f,O}$ of the complex analytic function germ (f, O) to be the motivic Milnor fiber of a Taylor expansion of f at O as in Definition 4.32.

Conjecture 5.2. *Let f be a complex analytic function in d variables which vanishes at O. Then the following statements hold.*

   (i) $\chi_c(\mathcal{S}_{f,O}) = \chi_c(F_{f,O})$.
   (ii) $\chi_h(\mathcal{S}_{f,O}) = \chi_h(F_{f,O})$ *in* $K_0(\mathrm{HS}^{\mathrm{mon}})$.

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