Delay-Dependent Positivity and Stability Analysis of Discrete-Time Systems with Delay

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Dedicated to Professor Vu Ngoc Phat, the Institute of Mathematics, Vietnam Academy of Science and Technology, on the occasion of his 70th birthday.

***Abstract*—In this paper, delay-dependent positivity and stability conditions, which are crucially different from existing delay-independent ones, are derived for discrete-time systems with time-varying delay. By utilizing a special property called non-oscillatory behavior of solutions of scalar difference equations with delays, the proposed conditions are formulated in terms of linear programming settings. The efficiency of the obtained results is illustrated by a numerical example with simulations.**

***Index Terms*—Positive systems, delay-dependent positivity, stability analysis.**

I. INTRODUCTION

VARIOUS aspects in the systems and control theory have been developed for discrete-time systems of the form

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau(k)) + Bd(k),\ k\in\mathbb{N}_0 \\ x(k) &= \phi(k),\ k\in\mathbb{Z}[-\tau_{\max},0] \end{aligned} \tag{1}$$

with positivity constraints on the system state, input and output vectors. Such systems with positivity constraints are typically referred to positive systems [1] or nonnegative dynamical systems [2]. A popular positivity concept, which has been extensively studied for linear systems, can be intuitively explained through invariant properties of input-output operator [3]–[5]. Specifically, system (1) is said to be (internally) positive if state trajectory $x(k)\succeq 0$ for any initial condition $\phi(k)\succeq 0$ and input $d(k)\succeq 0$ [6]. In this meaning, it is well-known that, system (1) is positive if and only if A, A_d and B are nonnegative matrices [7]–[9] and, subject to positivity, system (1) is globally asymptotically stable if and only if the matrix $A+A_d$ is Schur stable (see, [1], [3] or recent works [9], [10] for more details). The aforementioned conditions are referred to delay-independent positivity approach (i.e. positivity and stability conditions are not involved the magnitude of delays). The theory of positive linear systems based on delay-independent positivity has been extensively studied with a very rich literature (see, e.g., [5], [8]–[13] and the references therein). However, in practice, the positivity of the input vector $d(k)$ may not be predefined independently with that of initial sequence $\phi(k)$, for example, when $d(k)$ can be finite-time reference [14]. In other words, $d(k)$ may be dependent of $\phi(k)$ and the positivity of $d(k)$, for all $k\geq 0$, is not always available. To that case, the delay-independent criterion fails to apply. This urges to alternative results referred to delay-dependent positivity. Motivating examples of employing delay-dependent positivity conditions can be found in framer (a type of interval open-loop estimators) [15] or interval observer designs for delayed measurement systems [16]. For example, the design of interval observers requires that, besides stability conditions, some restrictions on the positivity of estimation error dynamics have to be imposed in order to envelop the system state trajectories. Thus, it is relevant to develop the theory of delay-dependent positivity. For another supportive reason, let us introduce a simple counterexample. Consider the following scalar equation

$$\begin{aligned} x(k+1) &= ax(k) - bx(k-1) + \omega(k),\ k\geq 0 \\ x(0) &= x_0,\ x(-1) = x_{-1} \end{aligned} \tag{2}$$

where $a,b\in\mathbb{R}$, $a>0$, $b>0$ and

$$\omega(k) = \begin{cases} \frac{a^2}{4}x_{-1} \text{ if } k=0 \\ 0 \text{ elsewhere.} \end{cases}$$

Clearly, equation (2) is neither positive nor stable by the delay-independent criterion. Assume that $0<b<\frac{a^2}{4}$. Then, $\delta_0 = a^2-4b>0$ and $\sqrt{\delta_0}<a$. By a recurrent process, we obtain

$$\begin{aligned} x(k) &= \frac{1}{\sqrt{\delta_0}}\left[\left(\frac{a+\sqrt{\delta_0}}{2}\right)^{k+1} - \left(\frac{a-\sqrt{\delta_0}}{2}\right)^{k+1}\right]x_0 \\ &\quad + \frac{\sqrt{\delta_0}}{4}\left[\left(\frac{a+\sqrt{\delta_0}}{2}\right)^{k} - \left(\frac{a-\sqrt{\delta_0}}{2}\right)^{k}\right]x_{-1}. \end{aligned} \tag{3}$$

Thus, $x(k)\geq 0$ for all $k\geq 0$ if $x_0\geq 0$ and $x_{-1}\geq 0$. In addition, if $a+\frac{\delta_0}{4}<1$, then it follows from (3) that $x(k)\to 0$ as $k\to\infty$ for any x_0, x_{-1}. This suggests a complementary delay-dependent positivity and stability criterion. For linear continuous-time systems, a few results concerning delay-dependent positivity and applications to stability analysis and controller/observer design have been reported, e.g., in [14], [16], [17]. However, the earlier mentioned works could not deal with delay-dependent positivity of systems as given in (1). Note also that the existing methods for continuous-time systems are not applicable to system (1).

In this paper, we consider the problem of positivity and stability analysis for discrete-time systems with delay. Based on the theory of non-oscillation solutions of functional difference

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equations, delay-dependent positivity and stability conditions are derived in the form of linear programming (LP) problems. The obtained results nicely complement the existing delay-independent criteria, which can be regarded as the critical case when the upper bound of delay tends to infinity.

The remaining of this paper is organized as follows. Section II introduces preliminary results on delay-independent conditions. Main results on delay-dependent positivity and stability of discrete-time systems with delay are presented in Section III and Section IV, respectively. An illustrative example with simulations is given in Section V. The paper ends with a conclusion drawn in Section VI and cited references.

Notation. $\overline{1,n}$ denotes the set $\{1,2,\dots,n\}$ for an $n \in \mathbb{N}$ and $\mathbb{Z}[a,b] = \{a, a+1, \dots, b\}$ for integers $a \le b$. Comparisons between vectors are understood componentwise, that is, for $x = (x_i) \in \mathbb{R}^n$ and $y = (y_i) \in \mathbb{R}^n$, $x \preceq y$ iff $x_i \le y_i$ and $x \prec y$ iff $x_i < y_i$ for all $i \in \overline{1,n}$. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$ and $|x| = (|x_i|) \in \mathbb{R}^n_+$. For a matrix $A \in \mathbb{R}^{m\times n}$, we write as $[A]_{ij}$ the entry at row ith and column jth of A. A matrix A is a nonnegative matrix, $A \succeq 0$, if $[A]_{ij} \ge 0$ for all i,j; A is positive, $A \succ 0$, if $[A]_{ij} > 0$ for all i,j and A is Metzler if its off-diagonal entries are nonnegative (i.e., $[A]_{ij} \ge 0$ for all $i \ne j$). $A \succeq B$ ($A \succ B$) means $A - B \succeq 0$ ($A - B \succ 0$). We also denote by $\mathscr{D}_A$ the diagonal matrix obtained by stacking the diagonal entries of A.

II. Delay-independent conditions

Consider the following discrete-time system with delay

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-\tau(k)),\ k \in \mathbb{N}_0 \\ x(k) &= \phi(k),\ k \in \mathbb{Z}[-\tau_{\max}, 0] \end{aligned} \tag{4}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, A, $A_d \in \mathbb{R}^{n\times n}$ are given matrices, $\tau(k)$ is a time-varying delay satisfying $0 \le \tau(k) \le \tau_{\max}$ and $\phi(k)$ is the initial condition. We recall there that system (4) is (internally) positive if for any $\phi(k) \succeq 0$, $k \in \mathbb{Z}[-\tau_{\max}, 0]$, it holds that $x(k) \succeq 0$ for all $k \ge 0$. The following result is widely used for design problems of positive systems [7].

Lemma 1: System (4) is positive if and only if $A \succeq 0$ and $A_d \succeq 0$. Moreover, the positive system (4) is globally asymptotically stable (GAS) if and only if $\rho(A_0 + A_1) < 1$, where $\rho(M) = \max\{|\lambda| : \lambda \in \mathbb{C}, \det(M - \lambda I_n) = 0\}$ denotes the spectral radius of a matrix $M \in \mathbb{R}^{n\times n}$.

Remark 1: The stability condition given in Lemma 1 is satisfied if and only if [7] there exists a vector $\mu \in \mathbb{R}^n$, $\mu \succ 0$, such that

$$\mu^\top (A_0 + A_1) - \mu^\top \prec 0. \tag{5}$$

It can be seen that the delay $\tau(k)$ does not have any impact on the feasibility of condition (5). By this, condition (5) is typically mentioned as a *delay-independent* stability condition, which has been widely used in the literature concerning analysis and synthesis of discrete-time positive linear systems. In addition, condition (5) can also guarantee the exponential stability of system (4), that is, there exist scalars $0 < \alpha < 1$, $\beta > 0$ such that any solution $x(k)$ of (4) satisfies

$$\|x(k)\|_1 \le \beta \|\phi\|_{\mathbb{Z}[-\tau_{\max},0]} \alpha^k,\ k \ge 0$$

where $\|\phi\|_{\mathbb{Z}[-\tau_{\max},0]} = \sup_{k\in\mathbb{Z}[-\tau_{\max},0]} \|\phi(k)\|_1$. The exponential factor α can be determined as $\alpha = \max_{1\le i\le n} \alpha_i$ and, for each $i \in \overline{1,n}$, $\alpha_i \in (0,1)$ is the unique solution of the equation [18]

$$\frac{1}{\mu_i}\sum_{j=1}^n [A_0]_{ij}\mu_j + \frac{1}{\mu_i}\sum_{j=1}^n [A_1]_{ij}\mu_j \alpha_i^{-\tau_{\max}} = \alpha_i.$$

This shows that the magnitude of delay affects the convergent rate of the system states.

III. Delay-dependent positivity

A. Preliminary results: The case of scalar equations

Consider the following scalar difference equation

$$x(k+1) - x(k) + \sum_{j=0}^m a_j x(k - \tau_j(k)) = f(k) \tag{6}$$

where $a_j \in \mathbb{R}$, $\tau_j(.) : \mathbb{N}_0 \to \mathbb{N}_0$, $\tau_j(k) \le q_j$, and $f : \mathbb{N}_0 \to \mathbb{R}$. With initial condition $x(l) = 0$ for $l < 0$, $x(0) \in \mathbb{R}$, by the constant variation formula, the corresponding solution of (6) can be represented as

$$x(k) = X(k,0)x(0) + \sum_{l=0}^{k-1} X(k, l+1) f(l) \tag{7}$$

where, for a fixed $l \in \mathbb{N}_0$, $X(k,l)$ denotes the fundamental solution of the corresponding homogeneous equation of (6) (i.e. when f is set to be zero) defined by

$$\begin{aligned} &X(k+1,l) - X(k,l) + \sum_{j=0}^m a_j X(k - \tau_j(k), l) = 0,\ k \ge l \\ &X(i,l) = 0,\ i < l,\ X(l,l) = 1. \end{aligned} \tag{8}$$

Note that the representation according to constant variation formula (7) is also applicable for state-dependent inhomogeneous terms of the form $f(k) = f(k, x(k))$. On the other hand, in regard to (7), if the fundamental solution $X(k,l)$ is positive then any solution of (6) with nonnegative initial condition and input $f(k)$ is positive. Thus, the positivity of $X(k,l)$ is essential for the positivity of (6). Let $L : x \mapsto Lx$ be the linear difference operator defined by (6), that is,

$$(Lx)(k) = x(k+1) - x(k) + \sum_{j=0}^m a_j x(k - \tau_j(k)).$$

The following lemma will be used in our next derivation. See [19, Lemma 2] for more details.

Lemma 2: Assume that $a_j \ge 0$, $j = 0, 1, \dots, m$, and there exists a function $\Phi : \mathbb{N}_0 \to (0, \infty)$ such that $(L\Phi)(k) \le 0$, $k \in \mathbb{N}_0$. Then, it holds that $X(k,l) \ge \frac{\Phi(k)}{\Phi(l)}$ for all $k \ge l \ge 0$.

Typically, we could not derive explicit expression of the fundamental solution $X(k,l)$ due to time-varying delay. Thus, one needs to compare $X(k,l)$ with certain type of exponential functions $\Phi(k)$. On the basis of deduction, and as revealed in Lemma 2, let us consider the following characteristic equation

$$\lambda - 1 + \sum_{j=0}^m a_j \lambda^{-q_j} = 0. \tag{9}$$

Lemma 3: Assume that $a_j \geq 0$, $j = 0, 1, \ldots, m$, and Eq. (9) has a positive real root λ_*. Then, the fundamental solution $X(k,l)$ of (6) is positive.

Proof: Let λ_* be a positive real root of (9). Then, we have

$$1 - \lambda_* = \sum_{j=0}^{m} a_j \lambda_*^{-q_j} \geq 0.$$

Thus, $0 < \lambda_* \leq 1$. With the function $\Phi(k) = \lambda_*^k$, we have

$$\begin{aligned}(L\Phi)(k) &= \lambda_*^{k+1} - \lambda_*^k + \sum_{j=0}^{m} a_j \lambda_*^{k-\tau_j(k)} \\ &\leq \lambda_*^k \Big(\lambda_* - 1 + \sum_{j=0}^{m} a_j \lambda_*^{-q_j}\Big) = 0.\end{aligned}$$

By Lemma 2, the fundamental solution $X(k,l)$ of (6) satisfies $X(k,l) \geq \lambda_*^{k-l} > 0$ for all $k \geq l \geq 0$. ■

Next, the result of Lemma 3 will be applied to underline the positivity of solutions of the following equation

$$x(k+1) = ax(k) - bx(k-\tau(k)) + f(k),\ k \in \mathbb{N}_0 \quad (10)$$

where $a, b \in \mathbb{R}$ are given scalars, $0 \leq \tau(k) \leq q \in \mathbb{N}$, and $f : \mathbb{N}_0 \to \mathbb{R}$. With the initial condition

$$x(l) = 0,\ l < 0, \quad x(0) \geq 0, \quad (11)$$

if $a \geq 0$, $b \leq 0$ and $f(k) \geq 0$, then $x(k) \geq 0$ for all $k \geq 0$ according to delay-independent positivity. We assume that $b \geq 0$. Eq. (10) can be written as (6) with $a_0 = 1 - a$ and $a_1 = b$. By Lemma 3, the fundamental solution of (10) is positive if the following conditions are fulfilled

$$1 - a \geq 0, \quad b \geq 0 \quad (12a)$$
$$\exists \lambda > 0 :\ \lambda - a + b\lambda^{-q} = 0. \quad (12b)$$

To ensure the positivity of the fundamental solution of Eq. (10), it is essential to determine whether a positive real root of (12b) exists. This condition is satisfied if and only if there exists a positive real root of the polynomial $P(\lambda) = \lambda^{q+1} - a\lambda^q + b$, $\lambda \in \mathbb{R}$. Since $P(\lambda)$ attains its minimum at $\lambda_0 = \frac{qa}{q+1}$,

$$\min_{\lambda \in (0,\infty)} P(\lambda) = P(\lambda_0) = b - \frac{1}{q+1} \cdot \frac{a^{q+1}}{\left(1 + \frac{1}{q}\right)^q}$$

condition (12b) is satisfied if and only if it holds that

$$(q+1)b \leq \frac{a^{q+1}}{\left(1 + \frac{1}{q}\right)^q}. \quad (13)$$

Clearly, condition (13) is still valid for $b \leq 0$ and $a \in (0,1]$. Thus, for $a \in [0,1]$ and $b \in \mathbb{R}$ subject to condition (13), any solution of equation (10) with initial condition (11) is positive.

Next, we extend the initial condition (11) to the following general one

$$x(l) = \varphi(l),\ l \in \mathbb{Z}[-q, 0] \quad (14)$$

where $\varphi : \mathbb{Z}[-q,0] \to \mathbb{R}$ is a function that specifies an initial sequence of state $x(k)$ of (10).

Proposition 1: Let $a \in [0,1]$ and $b \in \mathbb{R}$ be given such that the inequality (13) is satisfied. If $\varphi(0) \geq 0$, $f(k) \geq 0$ for all $k \in \mathbb{N}_0$ and

$$f(k) \geq b\varphi(k - \tau(k)),\ \forall k \in \{k \in \mathbb{Z}[0,q] : k - \tau(k) < 0\}$$

then the corresponding solution of the problem defined by (10) and (14) satisfies $x(k) \geq 0$ for all $k \in \mathbb{N}_0$.

Proof: Let $\hat{x} : \mathbb{Z} \cap [-q, \infty) \to \mathbb{R}$ be the function defined as

$$\hat{x}(k) = \begin{cases} x(k) - \varphi(k) \text{ for } k < 0 \\ x(k) \text{ for } k \in \mathbb{N}_0. \end{cases}$$

Then, equation (10) with initial condition (14) can be recast into the following equation

$$\begin{aligned}\hat{x}(k+1) &= a\hat{x}(k) - b\hat{x}(k-\tau(k)) + \hat{f}(k),\ k \in \mathbb{N}_0 \\ \hat{x}(l) &= 0,\ l < 0, \quad \hat{x}(0) = \varphi(0)\end{aligned} \quad (15)$$

where

$$\hat{f}(k) = \begin{cases} f(k) - b\varphi(k - \tau(k)) \text{ if } k - \tau(k) < 0 \\ f(k) \text{ otherwise.} \end{cases}$$

Since $\hat{f}(k) \geq 0$ for all $k \in \mathbb{N}_0$ and condition (13) is satisfied, by Lemma 3, $\hat{x}(k) \geq 0$, and hence $x(k) \geq 0$, for $k \geq 0$. ■

B. Delay-dependent positivity

Proposition 1 provides a so-called delay-dependent criterion for the positivity of scalar equation (10). This result can be extended to n-dimensional systems given by

$$\begin{aligned}x(k+1) &= Ax(k) + A_d x(k - \tau(k)) + w(k) \\ x(k) &= \phi(k), \quad k \in \mathbb{Z}[-\tau_{\max}, 0]\end{aligned} \quad (16)$$

where $w(k) \in \mathbb{R}^n$ is a time-dependent parameter to enforce the positivity, which will be determined.

Proposition 2: Asume that

$$A \succeq 0,\ A_d \text{ is Metzler},\ \mathscr{D}_A \preceq I_n \quad (17a)$$

$$\mathscr{D}_{A_d} + \frac{1}{1+\tau_{\max}} \cdot \frac{\mathscr{D}_A^{1+\tau_{\max}}}{\left(1 + \frac{1}{\tau_{\max}}\right)^{\tau_{\max}}} \succeq 0. \quad (17b)$$

If $w(k) \succeq 0$ for all $k \geq 0$, $\phi(0) \succeq 0$ and

$$w(k) \succeq -A_d \phi(k - \tau(k)) \text{ for } k - \tau(k) < 0 \quad (18)$$

then the solution of (16) satisfies $x(k) \succeq 0$ for all $k \in \mathbb{N}_0$.

Proof: Inspired by the proof of Proposition 1, we consider the following n-dimensional system

$$\begin{aligned}\tilde{x}(k+1) &= A\tilde{x}(k) + A_d\tilde{x}(k - \tau(k)) + \tilde{w}(k),\ k \in \mathbb{N}_0 \\ \tilde{x}(l) &= 0,\ l < 0, \quad \tilde{x}(0) = \phi(0)\end{aligned} \quad (19)$$

where

$$\tilde{w}(k) = \begin{cases} w(k) + A_d\phi(k - \tau(k)) \text{ if } k - \tau(k) < 0 \\ w(k) \text{ elsewhere.} \end{cases}$$

Clearly, it follows from (16) and (19) that $x(k) = \tilde{x}(k)$ for all $k \in \mathbb{N}_0$. On the other hand, for each $i \in \overline{1,n}$, by (19), we have

$$\begin{aligned}\tilde{x}_i(k+1) &= [A]_{ii}\tilde{x}_i(k) + [A_d]_{ii}\tilde{x}_i(k-\tau(k)) + \tilde{w}_i(k)\\ &\quad + \sum_{j=1, j\neq i}^{n} ([A]_{ij}\tilde{x}_j(k) + [A_d]_{ij}\tilde{x}_j(k-\tau(k)))\\ \tilde{x}_i(l) &= 0,\ l<0, \quad \tilde{x}_i(0) = \phi_i(0).\end{aligned} \tag{20}$$

By virtue of the induction method, assume that $\tilde{x}(k) \succeq 0$ for $0 \le k \le p$. Then, take into account condition (18), we have

$$\begin{aligned}f_i(k,\tilde{x}_i(k)) = &\sum_{j=1,j\neq i}^{n} ([A]_{ij}\tilde{x}_j(k) + [A_d]_{ij}\tilde{x}_j(k-\tau(k)))\\ &+ \tilde{w}_i(k) \ge 0, \quad 0 \le k \le p.\end{aligned}$$

In addition to this, since $0 \le [A]_{ii} \le 1$ and, according to (17),

$$-(1+\tau_{\max})[A_d]_{ii} \le \frac{[A]_{ii}^{1+\tau_{\max}}}{\left(1+\frac{1}{\tau_{\max}}\right)^{\tau_{\max}}},$$

the fundamental solution of the homogeneous equation of (20) (without the term $f_i(k,\tilde{x}_i(k))$) is positive. This, together with (7), yields

$$\begin{aligned}\tilde{x}_i(p+1) &= X_i(p+1,0)\phi_i(0)\\ &\quad + \sum_{k=0}^{p} X_i(p+1,k+1)f_i(k,\tilde{x}_i(k)) \ge 0,\ i \in \overline{1,n}.\end{aligned}$$

Thus, $\tilde{x}(p+1) \succeq 0$. By the reduction method, we can conclude that $\tilde{x}(p) \succeq 0$ for all $p \ge 0$. The proof is completed. ■

Remark 2: Proposition 2 gives a delay-dependent criterion for the positivity of solutions of (16). Since nonnegativity of the matrix A_d and initial sequence is not imposed, the obtained result is clearly different from the delay-independent one given in Lemma 1. Moreover, it is noticing that when $A_d \succeq 0$, condition (17b) is ignored and condition (18) is obviously satisfied for nonnegative initial condition $\phi(k)$. Thus, the result of Proposition 2 encompasses the delay-independent positivity as a particular case.

We now define a transformation $\phi \mapsto w_\phi$ from a sequence $\phi : \mathbb{Z}[-\tau_{\max}, 0] \to \mathbb{R}^n$ to a sequence $w_\phi : \mathbb{N}_0 \to \mathbb{R}^n$ by

$$w_\phi(k) = \begin{cases} \frac{1}{1+\tau_{\max}} \cdot \frac{\mathscr{D}_A^{1+\tau_{\max}}}{\left(1+\frac{1}{\tau_{\max}}\right)^{\tau_{\max}}}\phi(k-\tau(k) \text{ if } k-\tau(k) \le 0\\ 0 \text{ otherwise.}\end{cases} \tag{21}$$

According to (17b) and (21), if $\phi(l) \succeq 0$ for $l \in \mathbb{Z}[-\tau_{\max}, 0]$ then $w_\phi(k) + A_d\phi(k-\tau(k)) \succeq 0$ for $k \in \mathbb{Z}[0, \tau_{\max}]$ such that $k - \tau(k) \le 0$. Thus, condition (18) is fulfilled. By this fact, the following corollary can be obtained from Proposition 2.

Corollary 1: Under the assumptions of Proposition 2, if $\phi(k) \succeq 0$, $k \in \mathbb{Z}[-\tau_{\max}, 0]$, the solution of the system

$$\begin{aligned}x(k+1) &= Ax(k) + A_d x(k-\tau(k)) + w_\phi(k),\ k \in \mathbb{N}_0\\ x(k) &= \phi(k),\ k \in \mathbb{Z}[-\tau_{\max}, 0]\end{aligned} \tag{22}$$

satisfies $x(k) \succeq 0$ for all $k \in \mathbb{N}_0$.

IV. Stability analysis under delay-dependent positivity

In this section, we derive delay-dependent stability conditions for discrete-time positive linear systems in the form of (4). Note at first that if $x(k)$ is a solution of (22) then $z(k) = x(k+k_0)$ is a solution of (4) with initial condition $\phi_z(l) = x(l+k_0)$, where $k_0 = 1+\tau_{\max}$. In addition, for any solutions $x(k)$, $z(k)$ of (22) and (4), $e(k) = x(k) - z(k)$ is also a solution of (22). This shows that the asymptotic stability of (22) implies that of (4) and vice versa. For convenience, we state this fact in the following lemma.

Lemma 4: The discrete-time linear system (4) is GAS if and only if system (22) is GAS.

Under the assumptions of Proposition 2, any solution $x(k,\phi)$ of system (22) satisfies $|x(k,\phi)| \preceq x(k,|\phi|)$ for all $k \in \mathbb{N}_0$. By this observation, from (22) we have the following result.

Theorem 1: Assume that

$$A \succeq 0, \quad A_d + \frac{1}{1+\tau_{\max}} \cdot \frac{\mathscr{D}_A^{1+\tau_{\max}}}{\left(1+\frac{1}{\tau_{\max}}\right)^{\tau_{\max}}} \succeq 0 \tag{23}$$

and there exists a vector $\eta \in \mathbb{R}^n$, $\eta \succ 0$, such that

$$\eta^\top (A + A_d - I_n) + \frac{1}{1+\tau_{\max}} \cdot \frac{\mathscr{D}_A^{1+\tau_{\max}}}{\left(1+\frac{1}{\tau_{\max}}\right)^{\tau_{\max}}}\eta^\top \prec 0. \tag{24}$$

Then, there exist scalars $\beta > 0$, $\delta \in (0,1)$ such that any solution $x(k,\phi)$ of system (22) satisfies

$$\|x(k,\phi)\|_1 \le \beta\|\phi\|_{\mathbb{Z}[-\tau_{\max},0]}\delta^k,\ k \in \mathbb{N}_0.$$

Consequently, $x(k,\phi) \to 0$ as $k \to \infty$ and system (4) is GAS.

Proof: It follows from conditions (23) and (24) that $A \succeq 0$, A_d is Metzler and $\mathscr{D}_A \prec I_n$. Thus, the derived conditions in (16) are fulfilled and by Proposition 1, system (22) is positive.

Let $x(k,\phi)$ be a solution of (22) with $\phi(k) \succeq 0$. By Corollary 1, $x(k,\phi) \succeq 0$ for all $k \in \mathbb{N}_0$. Consider the following scaled system

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + (A_d + J_A^\tau)\,\hat{x}(k-\tau(k)),\ k \ge 0\\ \hat{x}(k) &= \hat{\phi}(k),\ k \in \mathbb{Z}[-\tau_{\max}, 0]\end{aligned} \tag{25}$$

where $J_A^\tau = \frac{1}{1+\tau_{\max}} \cdot \frac{\mathscr{D}_A^{1+\tau_{\max}}}{\left(1+\frac{1}{\tau_{\max}}\right)^{\tau_{\max}}}$. By induction, it is found that if $\phi(k) \preceq \hat{\phi}(k)$, $k \in \mathbb{Z}[-\tau_{\max}, 0]$, then $x(k,\phi) \preceq \hat{x}(k,\hat{\phi})$ for all $k \ge 0$. As a consequence, we have

$$|x(k,\phi)| \preceq x(k,|\phi|) \preceq \hat{x}(k,|\phi|),\ k \in \mathbb{N}_0$$

for any solution $x(k,\phi)$ of (22).

On the other hand, system (25) is delay-independent positive by condition (17b). Thus, according to Lemma 1, system (25) is GAS if and only if the LP-based condition (24) is feasible for a positive vector $\eta \in \mathbb{R}^n$. Moreover, in that case, there exists scalars $\beta > 0$, $\delta \in (0,1)$ such that

$$\|\hat{x}(k,|\phi|)\|_1 \le \beta\|\phi\|_{\mathbb{Z}[-\tau_{\max},0]}\delta^k,\ k \in \mathbb{N}_0$$

which completes the proof as $\|x(k,\phi)\|_1 \le \|\hat{x}(k,|\phi|)\|_1$. ■

Remark 3: It should be pointed out that, regardless to the positivity, condition (24), in general, cannot guarantee the stability of system (22). Thus, Theorem 1 in this paper provides a delay-dependent stability criterion for discrete-time positive linear systems with delay. This result is crucially different from delay-independent results in the existing literature.

Remark 4: For any matrix A with $0 \preceq \mathscr{D}_A \preceq I_n$, we have

$$\limsup_{q\to\infty} \frac{1}{1+q} \cdot \frac{\mathscr{D}_A^{1+q}}{\left(1+\frac{1}{q}\right)^q} = 0.$$

Thus, when $q = \tau_{\max} \to \infty$, conditions (23) and (24) are reduced to delay-independent conditions given in Lemma 1. By this, Theorem 1 is an extension of delay-independent results.

Remark 5: It should be mentioned additionally that the delay-dependent positivity and stability conditions derived in this paper are sufficient, which may produce certain conservatism. How to prove the necessity of the derived stability conditions in Theorem 1 or how to formulate alternative necessary and sufficient stability conditions based on the delay-dependent positivity are interesting questions. This motivates further investigation.

Remark 6: In this paper, the problem of delay-dependent positivity and stability is studied for discrete time-delay linear systems. It can be noticed that the results presented in this paper are also utilizable to address various problems in the systems and control theory of discrete positive systems with delays [4], [10], [20], [21]. Potential results involving such problems prove to be meaningful contributions to enrich the literature in the area of positive systems theory.

V. An Illustrative Example

Consider system (4) with the following data

$$A = \begin{bmatrix} 0.6 & 0.12 & 0.05 & 0.16 \\ 0.05 & 0.6 & 0.07 & 0.05 \\ 0.15 & 0.08 & 0.45 & 0.1 \\ 0.11 & 0.09 & 0.15 & 0.45 \end{bmatrix}$$

$$A_d = \begin{bmatrix} -0.0011 & 0.05 & 0 & 0.1 \\ 0.05 & -0.0031 & 0.06 & 0.05 \\ 0.08 & 0.1 & 0.0009 & 0.11 \\ 0.05 & 0 & 0.07 & -0.0006 \end{bmatrix}.$$

To this system, the delay-independent criterion (Lemma 1) is no longer applicable as A_d is not nonnegative. Moreover, it can be verified that condition (23) is satisfied if and only if $\tau_{\max} \le 5$. For example, when $\tau_{\max} = 6$, by the notation defined in Theorem 1, we have

$$A_d + J_A^\tau = \begin{bmatrix} 0.0005 & 0.05 & 0 & 0.1 \\ 0.05 & -0.0015 & 0.06 & 0.05 \\ 0.08 & 0.1 & 0.0012 & 0.11 \\ 0.05 & 0 & 0.07 & -0.0003 \end{bmatrix} \not\succeq 0.$$

This demonstrates the dependence on the magnitude of delay of the derived conditions. In addition, for $\tau_{\max} \le 5$, according to (23),

$$w_\phi(k) + A_d\phi(k-\tau(k)) = (A_d + J_A^\tau)\,\phi(k-\tau(k)) \succeq 0$$

for $k - \tau(k) \le 0$, where $w_\phi(k)$ is defined in (21). Thus, condition (18) is satisfied. For $\tau_{\max} = 5$, we compute the LP-based matrix $\mathcal{M} = A + A_d - I_4 + J_A^\tau$ in (24) as

$$\mathcal{M} = \begin{bmatrix} -0.398 & 0.17 & 0.05 & 0.26 \\ 0.1 & -0.4 & 0.13 & 0.1 \\ 0.23 & 0.18 & -0.5485 & 0.21 \\ 0.16 & 0.09 & 0.22 & -0.55 \end{bmatrix}.$$

It can be verified by using the Matlab `linprog` Toolbox that the LP-based condition (24) is feasible with

$$\eta = [0.2 \quad 0.1568 \quad 0.1992 \quad 0.1643]^\top \succ 0.$$

Thus, by Theorem 1, system (4) is positive and GAS, which validates the obtained theoretical results. A simulation result with initial sequence $(1, 1.5, 0.5, 2)^\top$ and random delay $0 \le \tau(k) \le \tau_{\max} = 5$ is given in Fig. 1. It can be seen that the conducted trajectory is positive and converges to the origin in long time. This illustrates the effectiveness of the derived conditions.

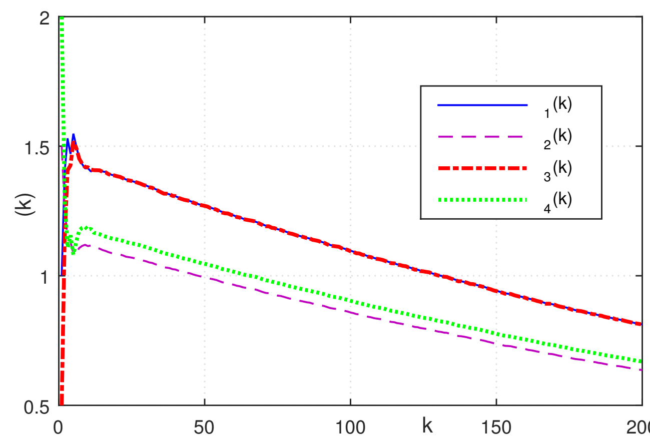

Fig. 1. Positive and convergent trajectories of system (4)

VI. Conclusion

In this paper, new delay-dependent positivity and stability conditions have been derived for discrete-time systems with a bounded time-varying delay based on non-oscillatory behavior of solutions of the corresponding diagonal scalar difference equations. It has been shown that the obtained results are essentially different from existing delay-independent criteria, which can be regarded as a complementary extension of delay-independent ones.

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