ON THE AUTOMORPHISM-INVARIANCE OF FINITELY GENERATED IDEALS AND FORMAL MATRIX RINGS

LE VAN THUYET AND TRUONG CONG QUYNH

Abstract. In this paper, we study rings having the property that every finitely generated right ideal is automorphism-invariant. Such rings are called right fa-rings. It is shown that a right fa-ring with finite Goldie dimension is a direct sum of a semisimple artinian ring and a basic semiperfect ring. From this, we obtain that if R is a right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator and the right it's socle is essential in R_R, R is also indecomposable (as ring), not simple with non-trivial idempotents then R is QF. In this case, QF-rings are the same as $q-$, $fq-$, $a-$, fa-rings. We also obtain a result of the automorphism-invariance of formal matrix rings.

1. Introduction

Johnson and Wong [11] proved that a module M is invariant under any endomorphism of its injective envelope if and only if any homomorphism from a submodule of M to M can be extended to an endomorphism of M. A module satisfying one of these equivalent conditions is called a *quasi-injective* module. Clearly any injective module is quasi-injective. A module M which is invariant under automorphisms of it's injective envelope has been called an *automorphism-invariant* module. The class of these modules were investigated by many authors, e.g., [1], [2], [6], [8], [14], [18], [20]. The generalizations of quasi-injectivity were considered. Many results were obtained for a right *q-ring* (i.e., every right ideal is quasi-injective) [9], [7], for a right *a-ring* (i.e., every right ideal is automorphism-invariant) [12], for a right *fq-ring* (i.e., every finitely generated right ideal is quasi-injective), for a right *fa-ring* (i.e., every finitely generated right ideal is automorphism-invariant) [17]. In this paper, we continue consider the structure

[0]Corresponding author: Truong Cong Quynh

2010 *Mathematics Subject Classification.* 16D80; 16D40, 16W20.

Key words and phrases. automorphism-invariant module, fa-ring, finite Goldie dimension, formal matrix ring.

of a fa-ring with some addition conditions, for example, the finite Goldie dimension of the ring R, or R is semiperfect,.... Besides, we also consider the automorphism-invariance of formal matrix rings.

Throughout this article all rings are associative rings with identity and all modules are right unital unless stated otherwise. For a submodule N of M, we use $N \leq M$ ($N < M$, resp.) to mean that N is a submodule of M (proper submodule, resp.), and we write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule of M and N is a direct summand of M, respectively. We denote by $Soc(M)$ and $E(M)$, the socle and the injective envelope of M, respectively. The Jacobson radical of a ring R is denoted by $J(R)$ or J. A ring R is called *semiperfect* in case $R/J(R)$ is semisimple artinian and idempotents lift modulo $J(R)$. It is equivalent to every finitely generated right (left) R-module has a projective cover. A module is called *uniform* if the intersection of any two nonzero submodules is nonzero. A ring R is called *I-finite* if it contains no infinite orthogonal family of idempotents. A ring R is said to have *finite right Goldie dimension* if R does not contain an infinite direct sum of nonzero right ideals. A ring R is called right *pseudo-Frobenius* (briefly, right PF) if R is right self-injective, semiperfect and $\mathrm{Soc}(R_R) \leq^e R_R$. A ring R is *local* if R has a unique maximal left (right) ideal. We call an idempotent $e \in R$ *local* if $eRe \cong End_R(eR)$ is a local ring. For any term not defined here the reader is referred to [3], [5], [13] and [19].

Our paper will be structured as follows: In Section 1, we will give concepts, some known results that are used or cited throughout in this paper. Section 2 deals with rings whose every finitely generated ideal is automorphism-invariant. We have a right fa-ring with finite Goldie dimension is a direct sum of a semisimple artinian ring and a basic semiperfect ring. Next, we consider the right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator and the right it's socle is essential in R_R. We obtain some properties of the kind of these rings. From these, we have that for this ring and moreover it is also indecomposable (as ring), not simple with non-trivial idempotents then it is QF. In this case, QF-rings are the same as $q-$, $fq-$, $a-$, fa-rings. Section 3 discusses about the invariance of formal matrix rings. Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ and (X, Y, f, g) be a right K-module, $\tilde{f}$ and $\tilde{g}$ be isomorphisms. Then (X, Y, f, g) is an automorphism-invariant right K-module if and only if X is an automorphism-invariant right R-module and Y is an automorphism-invariant right S-module.

2. On fa-Rings with finite Goldie dimension

Recall that a ring R is a right fa-ring (fq-ring, resp.) if every finitely generated right ideal of R is automorphism-invariant (quasi-injective, resp.).

Remark 1. Applying [12, Lemma 2.1] we deduce the following result:
Let R be commutative ring. then R is a fa-ring if and only if it is an automorphism-invariant ring.

Example 2. It is clear that a-rings are fa-rings. And we have the example of a-rings but not self-injective. For example, consider the ring R consisting of all eventually constant sequences of elements from $\mathbb{F}_2$. Clearly, R is a commutative a-ring. But R is not self-injective. Thus, fa-rings are not fq-rings.

Example 3. The ring of linear transformations $R := End(V_D)$ of a vector space V infinite-dimensional over a division ring D. It follows that R is not a right a-ring. Because V is not finite dimensional. But R is a right fa-ring, since every finitely generated ideal is a direct summand of R and R is right self-injective.

Let R be a semiperfect ring. Then, there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1+e_2+\cdots+e_m$. We may assume that $\{e_iR/e_iJ(R)|\ 1 \leq i \leq n\}$ is a complete set of representatives of the isomorphism classes of the simple right R-modules. In this case, $\{e_1, e_2, \ldots, e_n\}$ is called the set of *basic idempotents* for R, and if $e = e_1 + e_2 + \cdots + e_n$, the ring eRe is called the *basic ring* of R. Note that $eR \cong fR$ if and only if $eR/eJ(R) \cong fR/fJ(R)$ for idempotents e and f of R by Jacobson's Lemma (see [16, Lemma B.12]). The ring R is itself called a *basic semiperfect* ring if $m = n$, that is, if $1 = e_1 + e_2 + \cdots + e_n$, where the e_i are a basic set of local idempotents.

Lemma 4. *If R is a right automorphism-invariant I-finite ring, then R is a semiperfect ring.*

The following result is the main result of this section.

Theorem 5. *Let R be a right fa-ring with finite Goldie dimension. Then R is a direct sum of a semisimple artinian ring and a basic semiperfect ring.*

Proof. By Lemma 4, R is a semiperfect ring, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1 + e_2 + \cdots + e_m$. Suppose that $e_iR \ncong e_jR$ for all $i \neq j$ with $i, j \in \{1, 2, \ldots, m\}$. Then, we are done. Assume that e_i, for some $i \in \{1, 2, \ldots, m\}$, is a local idempotent of R such that there are direct summands isomorphic to e_iR in each decomposition of R_R as a direct sum of indecomposable modules. Thus, there exists an idempotent e' of R such that $e_iR \cap e'R = 0$ and $e_iR \cong e'R$. It follows, from [17, Lemma 4.2], that e_iR is a semisimple module. On the other and, we have that e_iR is an idecomposable module and obtain that e_iR is simple. Let eR be the direct sum of all copies of e_iR in the decomposition of $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$. Note that eR is a direct summand of R. We can assume that e is an idempotent of R. Then, we have a decomposition $R = eR \oplus (1-e)R$. Next, we show that eR and $(1-e)R$ are ideals of R. In order to show this, it is necessary to prove that $eR(1-e) = 0$ and $(1-e)Re = 0$.

Suppose $(1-e)Re \neq 0$. Take $(1-e)te \neq 0$ for some $t \in R$. Then there are primitive idempotents e_j and e_k such that $e_jR \cong e_iR, e_kR \ncong e_iR$ with $j, k \in \{1, 2, \dots, m\}$, $e_j \in eR, e_k \in (1-e)R$ and $e_kte_j \neq 0$. We consider the following map $\alpha : e_jR \to e_kR$ defined by $\alpha(e_jr) = e_kte_jr$ for all $r \in R$. One can check that α is a nonzero homomorphism. Note that e_jR is simple. Thus, α is a monomorphism. On the other hand, we have a direct sum $e_jR \oplus e_kR$. Since R is a right fa-ring, $e_jR \oplus e_kR$ is an automorphism-invariant module, and so e_jR is e_kR-injective by [14, Theorem 5]. From this, it immediately follows that α splits. We have that e_kR is simple and obtain that $e_jR \cong e_kR$, a contradiction. We deduce that $(1-e)Re = 0$, and so eR is an ideal of R.

Similarly to the above proof, suppose that $eR(1-e) \neq 0$. Call $eu(1-e) \neq 0$ for some $u \in R$. Then there are primitive idempotents e_p and e_q of R such that $e_pR \cong e_iR, e_qR \ncong e_iR$ with $p, q \in \{1, 2, \dots, m\}$, $e_p \in eR, e_q \in (1-e)R$ and $e_pue_q \neq 0$. We consider the following map $\beta : e_qR \to e_pR$ defined by $\beta(e_qr) = e_pue_qr$ for all $r \in R$. Then, β is a nonzero epimorphism by the simplicity of e_pR. Since e_pR is projective, β splits. One can check that $e_qR \cong e_pR$. This is a contradiction, and so $eR(1-e) = 0$. We deduce that $(1-e)R$ is an ideal of R.

Thus, eR is a semisimple artinian ring and $(1-e)R$ is a basic semiperfect ring. □

Next, we give some properties of minimal right and left ideals of R. Moreover, the self-injectivity of R is considered.

Lemma 6. *Let R be a right automorphism-invariant ring and $Soc(R_R) \leq^e R_R$ such that every minimal right ideal is a right annihilator.*

(1) *If xR is a minimal right ideal of R, then $l_Rr_R(x) = Rx$ and Rx is a minimal left ideal of R.*
(2) *If Ry is a minimal left ideal of R then yR is a minimal right ideal of R and $l_Rr_R(Ry) = Ry$. In particular, $Soc(R_R) = Soc(_RR)$ is denoted by S.*

(3) *$Soc(eR)$ and $Soc(Re)$ are simple for all local idempotents $e \in R$.*
(4) *If R is I-finite then R is a right PF-ring.*

Proof. (1) Assume that xR is a minimal right ideal of R. It is easy to see that $Rx \leq l_R r_R(x)$. For the converse, let $t \in l_R r_R(x)$ be a nonzero element. Then, we have $r_R(x) \leq r_R(t)$, and so $r_R(x) = r_R(t)$ by the maximality of $r_R(x)$. It follows that $Rx = Rt$ by [18, Lemma 1]. Then, $t \in Rx$ and so $l_R r_R(x) \leq Rx$ or $l_R r_R(x) = Rx$. On the other hand, for any nonzero element y in Rx, we have $r_R(x) \leq r_R(y)$, and so $r_R(x) = r_R(y)$ by the maximality of $r_R(x)$. It shows that $Rx = Ry$ is a minimal left ideal. We deduce that Rx is a minimal left ideal of R.

(2) Suppose that Ry is a minimal left ideal of R. Since $Soc(R_R) \leq^e R_R$, yR contains a minimal right ideal mR of R. Thus, $l_R(y) = l_R(m)$. It follows that $y \in r_R l_R(y) = r_R l_R(m) = mR \leq yR$ by our assumption, and so $yR = mR$. Thus, yR is a minimal right ideal of R. The rest is followed by (1).

(3) Take kR a minimal right ideal of eR. Then, Rk is a minimal left ideal of R. Therefore, $l_R(kR) \geq R(1-e)$ and $l_R(kR) = l_R(k) \geq J(R)$. It follows that $l_R(kR) = J(R) + R(1-e)$ because $J(R) + R(1-e)$ is the unique maximal left ideal containing $R(1-e)$. By our assumption we have

$$kR = r_R l_R(kR) = r_R[J(R)+R(1-e)] = r_R(J(R)) \cap eR = Soc(R_R) \cap eR = Soc(eR)$$

It shows that $Soc(eR)$ is a minimal right ideal of R.

Similarly, we also have $Soc(Re)$ is simple for all local idempotents $e \in R$.

(4) From the hypothesis, we have R is a semiperfect ring. We have a decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$. By (2), we have that e_iR is uniform for any $i \in \{1, 2, \ldots, m\}$, and so R is right self-injective by [14, Corollary 15]. We deduce that R is a right PF-ring. □

Fact 7. All endomorphism rings of indecomposable automorphism-invariant modules are local rings.

Lemma 8. *Let R be a right fa-ring with finite Goldie dimension, e be a primitive idempotent of R. Then the following conditions are hold:*

(1) *If $\alpha : eR \to R$ is a nonzero homomorphism with $eR \cap \alpha(eR) = 0$ then $\alpha(eR)$ is a simple module.*
(2) *If $(1-e)Re \neq 0$ then $eR(1-e) \neq 0$.*

Proof. (1) Note that eR is local. Then, $\alpha(eR)$ is indecomposable. Let U be an arbitrary essential submodule of $\alpha(eR)$, then $E(U) = E(\alpha(eR))$. Since R has finite Goldie dimension, there exists a finitely generated right ideal I with $I \leq^e U$. It follows that $I \leq^e U \leq^e \alpha(eR)$, and so $E(I) = E(U) = E(\alpha(eR))$. Since $I \oplus eR$

is a finitely generated right ideal of R, $I \oplus eR$ is automorphism-invariant. It follows that I is $eR-$injective. On the other hand, there exists a homomorphism $\bar{\alpha} : E(eR) \to E(\alpha(eR))$ such that $\bar{\alpha}|_{eR} = \alpha$. We have that $E(I) = E(\alpha(eR))$ and I is $eR-$injective and obtain that $\bar{\alpha}(eR) \leq I \leq U$. It shows that $\alpha(eR) \leq U$. We deduce that $\alpha(eR) = Soc(\alpha(eR))$, and so $\alpha(eR)$ is semisimple. We deduce that $\alpha(eR)$ is simple.

(2) Assume that $(1-e)Re \neq 0$. Note that R is automorphism-invariant, eR is $(1-e)R$-injective and $(1-e)R$ is eR-injective. Call $\alpha : eR \to (1-e)R$ a nonzero homomorphism. Now, we assume that $eR(1-e) = 0$. Then, $eRe = eR$ is a local ring with its unique maximal ideal $eJ(R)$. If $eJ(R) = 0$ then eR is simple right R-module and so $\alpha(eR) \cong eR$. It follows that $\alpha^{-1} : \alpha(eR) \to eR$ is extended to a homomorphism from $(1-e)R$ to eR. It means that $eR(1-e) \neq 0$. Now, if $eJ(R)$ is nonzero, then we get a nonzero element x in $eJ(R)$. We have that eRe is local and obtain that there exists an eRe-epimorphism $\beta : xeR \to eR/eJ(R)$. On the other hand, we have $eRe = eR$ and so β is an R-homomorphism. From (1) it immediately infers that $eR/eJ(R) \cong \alpha(eR) \leq (1-e)R$. Then, there exists a nonzero homomorphism $\gamma : eR/eJ(R) \to (1-e)R$. It follows that composition of β and γ is a nonzero homomorphism $\gamma \circ \beta : xeR \to (1-e)R$. Again, $(1-e)R$ is eR-injective we have that there is a nonzero homomorphism $\theta : eR \to (1-e)R$ such that θ is an extension of $\gamma \circ \beta$. Moreover, we have $xeR \leq eJ(R) = \mathrm{Ker}(\theta)$ (by (1)) which implies that $(\gamma \circ \beta)(xeR) = \theta(xeR) = 0$, a contradiction. Thus, $eR(1-e) \neq 0$. □

Proposition 9. *An indecomposable right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator. Then the following conditions are equivalent:*

(1) *R has essential right socle.*
(2) $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$.

Proof. (1) ⇒ (2) by Lemma 6.

(2) ⇒ (1). Assume that $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$. Since R is semiperfect, $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$ with a set of orthogonal local idempotents $\{e_1, e_2, \dots, e_m\}$ of R. Since R is an indecomposable ring, $e_iR(1-e_i) \neq 0$ or $(1-e_i)Re_i \neq 0$ for all $i \in \{1, 2, \dots, m\}$. Suppose that $(1-e_i)Re_i \neq 0$. Then by Lemma 8 we have $e_iR(1-e_i) \neq 0$. We deduce that $e_iR(1-e_i) \neq 0$ for all $i \in \{1, 2, \dots, m\}$. Take $\alpha_i : (1-e_i)R \to e_iR$ a nonzero homomorphism. Then by Lemma 4.2 in [17], $\mathrm{Im}(\alpha_i)$ is semisimple. It follows that $\mathrm{Soc}(e_iR) \neq 0$ for all $i \in \{1, 2, \dots, m\}$.

For any $i \in \{1, 2, \dots, m\}$, take kR a minimal right ideal of e_iR. Then, Rk is a minimal left ideal of R. Therefore, $l_R(kR) \geq R(1-e_i)$ and $l_R(kR) = l_R(k) \geq$

$J(R)$. It follows that $l_R(kR) = J(R) + R(1 - e_i)$ because $J(R) + R(1 - e_i)$ is the unique maximal left ideal containing $R(1 - e_i)$. By our assumption we have

$$kR = r_R l_R(kR) = r_R[J(R)+R(1-e_i)] = r_R(J(R)) \cap e_i R = Soc(R_R) \cap e_i R = Soc(e_i R)$$

It shows that $Soc(e_i R)$ is a minimal right ideal of R for all $i \in \{1, 2, \ldots, m\}$. It follows that $Soc(e_i R)$ is essential in $e_i R$. Thus, $Soc(R)$ is essential in R_R. □

In this section, we assume that R **is a right** fa**-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator and** $\mathrm{Soc}(R_R)$ **is essential in** R_R. Moreover, R is semiperfect, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ of R such that $1 = e_1 + e_2 + \cdots + e_m$. Call $\{e_1, e_2, \ldots, e_n\}$ a set of basic idempotents for R with $n \leq m$.

Lemma 10. *If e and f are two orthogonal idempotents of R then $eRf \subseteq Soc(R_R)$.*

Proof. Suppose that e and f are two orthogonal idempotents of R. Then, $eR \cap fR = 0$. If $eRf = 0$, we are done. Otherwise, let exf be a nonzero arbitrary element of eRf. We consider a nonzero homomorphism $\alpha : fR \to eR$ defined by $\alpha(fr) = exfr$ for all $r \in R$. By [17, Lemma 4.2], we have that $\mathrm{Im}(\alpha) = exfR$ is semisimple. It follows that $exf \in Soc(R_R)$. We deduce that $eRf \subseteq Soc(R_R)$. □

Let R be a semiperfect ring with basic idempotents $\{e_1, e_2, \ldots, e_n\}$. A permutation σ of $\{1, 2, \ldots, n\}$ is called a *Nakayama permutation* for R if $Soc(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$ and $Soc(e_i R) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$ for each $i = \{1, 2, \ldots, n\}$. A ring R is called *quasi-Frobenius* (brief, QF) if R is one-sided artinian one-sided self-injective, see [16]. It is well-known that every QF-ring has a Nakayama permutation.

Lemma 11. *Let R be an indecomposable ring with non-trivial idempotents. Then, R has a Nakayama permutation σ of $\{1, 2, \ldots, n\}$. In particular, $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$ if R is not a simple ring.*

Proof. By the hypothesis, R is indecomposable and so R is either semisimple artinian or basic semiperfect by Theorem 5. If R is a semisimple artinian ring then R has a Nakayama permutation. Now, we assume that R is not a simple ring. It follows that R is a basic semiperfect ring.

For any $i \in \{1, 2, \ldots, n\}$, from the simplicity of $Soc(e_i R)$, it infers that there exists $\sigma(i) \in \{1, 2, \ldots, n\}$ such that $Soc(e_i R) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$. This map σ is a permutation of $\{1, 2, \ldots, n\}$ because $\sigma(i) = \sigma(j)$ implies that $Soc(e_i R) \cong Soc(e_j R)$. By the injectivity of $e_i R$ and $e_j R$, we infer that $e_i R \cong e_j R$, and so

$i = j$ (because the e_i are basic). Let $\alpha : e_{\sigma(i)}R/e_{\sigma(i)}J(R) \to Soc(e_iR)$ be an isomorphism and $s_i = \alpha(e_{\sigma(i)} + e_{\sigma(i)}J(R))$. It follows that $s_iR = Soc(e_iR)$ is a minimal right ideal of R. One can check that $J(R) + R(1 - e_i) \le l_R(s_i)$. But $R/[J(R)+R(1-e_i)] \cong Re_i/J(R)e_i$ is simple, and so $l_R(s_i) = J(R)+R(1-e_i)$. It follows that $Rs_i \cong Re_i/J(R)e_i$. Now observe that $s_i = s_ie_{\sigma(i)} \in Soc(_RR)e_{\sigma(i)} = Soc(Re_{\sigma(i)})$. We have, from Lemma 6, that $Soc(Re_{\sigma(i)})$ is simple and obtain that $Soc(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$. Thus, R has a Nakayama permutation σ of $\{1, 2, \dots, n\}$.

Next, we suppose that $\sigma(i) = i$ for some $i \in \{1, 2, \dots, n\}$ or $Soc(e_iR) \cong e_iR/e_iJ(R)$. Assume that $e_iR(1 - e_i) \neq 0$. Since R is a basic semiperfect ring, there would exist $j \in \{1, 2, \dots, n\}$ with $j \neq i$ such that $e_iRe_j \neq 0$. Then, there exists a nonzero homomorphism $\beta : e_jR \to e_iR$. By [12, Lemma 4.1] and e_iR is uniform, we infer that $\mathrm{Im}(\beta)$ is simple. It follows that $\mathrm{Im}(\beta) = Soc(e_iR)$ and $\mathrm{Ker}(\beta)$ is maximal in e_jR. Then, $\mathrm{Ker}(\beta) = e_jJ(R)$ which implies that $e_jR/e_jJ(R) \cong Soc(e_iR) \cong e_iR/e_iJ(R)$. From this, it immediately infers that $e_iR \cong e_jR$, a contradiction. It is shown that $e_iR(1 - e_i) = 0$. Similarly, we have $(1 - e_i)Re_i = 0$. In fact, if $(1 - e_i)Re_i \neq 0$, then $e_kRe_i \neq 0$ for some $k \in \{1, 2, \dots, n\}$ with $k \neq i$. By the above similar proof, we infer that $Soc(e_iR) \cong e_iR/e_iJ(R) \cong Soc(e_kR)$. By the injectivivity of e_iR and e_kR, we have $e_iR \cong e_kR$ which is impossible. It is shown that e_i is central, a contradiction. We deduce that $\sigma(i) \neq i$ for all $i = 1, 2, \dots, n$.

□

Lemma 12. *Let R be an indecomposable ring not simple with non-trivial idempotents. Then, e_iRe_i is a division ring for any $i \in \{1, 2, \dots, n\}$.*

Proof. By the hypothesis, R is a basic semiperfect ring and $1 = e_1 + e_2 + \dots + e_n$. For any $i \in \{1, 2, \dots, n\}$, there exists $j \neq i$ with $j \in \{1, 2, \dots, n\}$ such that $e_iRe_j \neq 0$ by Lemma 11. Suppose that $e_iR(1 - e_i) = 0$. Then, $e_iR(\sum_{k\neq i}^{n} e_k) = 0$ which implies that $e_iRe_j = 0$, a contradiction. Thus, $e_iR(1 - e_i) \neq 0$. Next, we show that $e_iJ(R)e_i = 0$. We have $e_iR(1 - e_i) \subset \mathrm{Soc}(eR)$ by Lemma 10, and so $e_iR(1 - e_i) = \mathrm{Soc}(e_iR)(1 - e_i)$. Now, we show that $e_iJ(R)e_i$ is a submodule of e_iR. Since R is right automorphism-invariant, $J(R) = \{a \in R : r_R(a) \le^e R_R\}$ by [8, Proposition 1] and so $J(R)\,\mathrm{Soc}(e_iR) = 0$. Now $(e_iJ(R)e_i)\,\mathrm{Soc}(e_iR) = e_iJ(R)\,\mathrm{Soc}(e_iR) = 0$ which implies $(e_iJ(R)e_i)(e_iR(1 - e_i)) = 0$. On the other hand, we have

$$e_iJ(R)e_iR = e_iJ(R)e_i(Re_i + R(1 - e_i)) = e_iJ(R)e_iRe_i \subset e_iJ(R)e_i.$$

Hence $e_iJ(R)e_i$ is an R-submodule of e_iR. Since $\mathrm{Soc}(e_iR)$ is simple, we have $e_iJ(R)e_i \cap \mathrm{Soc}(e_iR) = 0$ or $\mathrm{Soc}(e_iR) \leq e_iJ(R)e_i$. Suppose $\mathrm{Soc}(e_iR) \leq e_iJ(R)e_i$. Then $e_iR(1-e_i) = \mathrm{Soc}(e_iR)(1-e_i) \leq e_iJ(R)e_i(1-e_i) = 0$, a contradiction. It follows that $e_iJ(R)e_i \cap \mathrm{Soc}(e_iR) = 0$. Thus $e_iJ(R)e_i = 0$ because $\mathrm{Soc}(e_iR)$ is essential in e_iR. Note that $e_iRe_i \cong \mathrm{End}(e_iR)$ is a local ring. We deduce that e_iRe_i is a division ring.

□

Theorem 13. *If R is an indecomposable (as ring) ring not simple with non-trivial idempotents, then R is a QF-ring.*

Proof. By Lemma 6 and the hypothesis, R is a basic semiperfect right self-injective ring and $Soc(R_R)$ is an artinian right R-module. We have a decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$. Then

$$R = \sum_{i=1}^{n} e_iRe_i + \sum_{i\neq j}^{n} e_iRe_j$$

Note that $e_iRe_j \subseteq Soc(R_R)$ for all $i \neq j$ by Lemma 10. We consider the following mapping

$$\phi : R/Soc(R_R) \to \bigoplus_{i=1}^{n} e_iRe_i$$

via $\phi(\sum_{i=1}^{n} e_ir_ie_i) + Soc(R_R) = \sum_{i=1}^{n} e_ir_ie_i$ We show that ϕ is an isomorphism. If $\sum_{i=1}^{n} e_ir_ie_i \in S$, then $e_ir_ie_i \in e_iSe_i$ for all $i = 1, 2, \ldots, n$. Since $e_iJ(R)$ is the unique maximal submodule of e_iR, $e_iSoc(R_R) \leq e_iJ(R)$, and so $e_ir_ie_i \in e_iJ(R)e_i$. Note that $e_iJ(R)e_i = 0$ by Lemma 12. It shows that ϕ is a mapping. One can check that ϕ is a ring homomorphism. Moreover, ϕ is a bijection, and so ϕ is a ring isomorphism. It shows that $R/Soc(R_R)$ is a semisimple artinian ring. We deduce that R is a right artinian ring, and so R is QF. □

Corollary 14. *Let R be an indecomposable (as ring) ring not simple with non-trivial idempotents. Then, the following conditions are equivalent:*

(1) *R is a right q-ring.*
(2) *R is a right fq-ring.*
(3) *R is a right a-ring.*
(4) *R is a right fa-ring.*
(5) *$eRf \subseteq Soc(R_R)$ for each pair e, f of orthogonal idempotents of R.*
(6) *R is an QF-ring.*

Proof. (1) $\Rightarrow$ (2), (3); (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) by Lemma 10.

(5) $\Rightarrow$ (6). By Theorem 13, R is a basic semiperfect QF-ring.

(6) $\Rightarrow$ (1). Since R is QF, it follows that R_R is injective cogenerator. Thus, R is a right q-ring by [7, Theorem 2.9]. □

3. The automorphism-invariance of formal matrix rings

Let R and S be two rings and M be a $R-S$-bimodule and N be a $S-R$-bimodule. Take the set of matrices

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \;\middle|\; r \in R, s \in S,\ m \in M, n \in N \right\}$$

Assume that there exist an R-homomorphism $\varphi : M \otimes_S N \to R$ and an S-homomorphism $\psi : N \otimes_R M \to S$ such that

$$\varphi(m \otimes n)m' = m\psi(n \otimes m'),\ \psi(n \otimes m)n' = n\varphi(m \otimes n')$$

for all $m, m' \in M$ and $n, n' \in N$. For convenience in using notations, we can write $\varphi(m \otimes n) := mn$, $\psi(n \otimes m) := nm$ and $MN := \varphi(M \otimes_S N)$, $NM := \psi(N \otimes_R M)$.

Then, K is a ring with the addition and multiplication as follows:

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} + \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} r + r' & m + m' \\ n + n' & s + s' \end{pmatrix}$$

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} rr' + mn' & rm' + ms' \\ nr' + sn' & nm' + ss' \end{pmatrix}$$

The ring K is called a *formal matrix ring or generalized matrix rings* (see [13] or [15]). It is well-known that the category of right K-module Mod-K is equivalent to the category $\mathcal{A}(K)$ of objects (X, Y, f, g), where X is a right R-module, Y is a right S-module, $f : X \otimes_R M \to Y$ is an S-homomorphism and $g : Y \otimes_S N \to X$ is an R-homomorphism. The right K-module (X, Y, f, g) is the additive group $X \oplus Y$ with right K-action given by

$$(x\ y) \begin{pmatrix} r & m \\ n & s \end{pmatrix} = (xr + g(y \otimes n), f(x \otimes m) + ys)$$

such that the following diagrams are commutative

$$\begin{array}{ccccc}
X \otimes_R M \otimes_S N & \xrightarrow{f \otimes 1_N} & Y \otimes_S N & \xrightarrow{g} & X \\
\downarrow{\scriptstyle 1_X \otimes \varphi} & & & & \downarrow{\scriptstyle 1_X} \\
X \otimes_R R & & \xrightarrow{\mu} & & X
\end{array}$$

$$\begin{array}{ccccc}
Y \otimes_S N \otimes_R M & \xrightarrow{g \otimes 1_M} & X \otimes_R M & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle 1_Y \otimes \psi} & & & & \downarrow{\scriptstyle 1_Y} \\
Y \otimes_S S & & \xrightarrow{\nu} & & Y
\end{array}$$

where $\mu : X \otimes_R R \to X$ and $\nu : Y \otimes_S S \to Y$ are canonical isomorphisms.

Next, we consider homomorphisms of K-modules. Let (X_1, Y_1, f_1, g_1) and (X_2, Y_2, f_2, g_2) be right K-modules. A right K-homomorphism $\varphi : (X_1, Y_1, f_1, g_1) \to (X_2, Y_2, f_2, g_2)$ is a pair (φ_1, φ_2) where $\varphi_1 : X_1 \to X_2$ is an R-homomorphism and $\varphi_2 : Y_1 \to Y_2$ is an S-homomorphism such that the following diagrams are commutative

$$\begin{array}{ccc}
X_1 \otimes_R M & \xrightarrow{f_1} & Y_1 \\
\downarrow{\scriptstyle \varphi_1 \otimes 1_M} & & \downarrow{\scriptstyle \varphi_2} \\
X_2 \otimes_R M & \xrightarrow{f_2} & Y_2
\end{array}$$

$$\begin{array}{ccc}
Y_1 \otimes_S N & \xrightarrow{g_1} & X_1 \\
\downarrow{\scriptstyle \varphi_2 \otimes 1_N} & & \downarrow{\scriptstyle \varphi_1} \\
Y_2 \otimes_S N & \xrightarrow{g_2} & X_2
\end{array}$$

Note that a K-homomorphism $\varphi = (\varphi_1, \varphi_2) : (X_1, Y_1, f_1, g_1) \to (X_2, Y_2, f_2, g_2)$ is a monomorphism (epimorphism, resp.) if and only if φ_1 and φ_2 are monomorphisms (epimorphisms, resp.).

A submodule of a right K-module (X, Y, f, g) is a quadrupe $(X_0,\ Y_0, f_0, g_0)$, where $X_0 \leq X_R$, $Y_0 \leq Y_S$ such that the following diagrams are commutative.

$$\begin{array}{ccc} X_0 \otimes_R M & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle \iota_1 \otimes 1_M}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ X \otimes_R M & \xrightarrow{f} & Y \end{array}$$

$$\begin{array}{ccc} Y_0 \otimes_S N & \xrightarrow{g_0} & X_0 \\ {\scriptstyle \iota_2 \otimes 1_N}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ Y \otimes_S N & \xrightarrow{g} & X \end{array}$$

with $\iota_1 : X_0 \to X$, $\iota_2 : Y_0 \to Y$ the inclusion maps. This is equivalent $X_0M \subseteq Y_0$ and $Y_0N \subseteq X_0$.

Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ and X be a right R-module. Denote by $H(X) = \mathrm{Hom}_R(N, X)$. We consider the following homomorphisms

$$\begin{aligned} u_X : X \otimes_R M &\longrightarrow \mathrm{Hom}_R(N, X) \\ x \otimes m &\longmapsto u(x \otimes m) : N \to X \\ &\qquad\qquad n \mapsto u(x \otimes m)(n) = x(mn) \end{aligned}$$

and

$$\begin{aligned} v_X : \mathrm{Hom}_R(N, X) \otimes_S N &\longrightarrow X \\ \alpha \otimes n &\longmapsto \alpha(n) \end{aligned}$$

One can check that $(X, H(X), u_X, v_X)$ is a right K-module. Similarly, we also have that $(H(Y), Y, v_Y, u_Y)$ is a right K-module for all right S-module Y with $H(Y) = \mathrm{Hom}_S(M, Y)$ and $v_Y : H(Y) \otimes_R M \to Y$ and $u_Y : Y \otimes_S N \to H(Y)$.

Let (X, Y, f, g) be a right K-module. Then, we have the following R-homomorphism

$$\begin{aligned} \tilde{f} : X &\longrightarrow \mathrm{Hom}_S(M, Y) = H(Y) \\ x &\longmapsto \tilde{f}(x) : M \to Y \\ &\qquad\qquad m \mapsto \tilde{f}(x)(m) = f(x \otimes m) \end{aligned}$$

and S-homomorphism

$$\begin{aligned} \tilde{g} : Y &\longrightarrow \mathrm{Hom}_S(N, X) = H(X) \\ y &\longmapsto \tilde{g}(y) : N \to X \\ &\qquad\qquad n \mapsto \tilde{g}(y)(n) = g(y \otimes n) \end{aligned}$$

Theorem 15. *Let* $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *and* (X, Y, f, g) *be a right* K*-module. Assume that* $\tilde{f}$ *and* $\tilde{g}$ *are isomorphisms. Then the following conditions are equivalent:*

(1) (X, Y, f, g) *is an automorphism-invariant right* K*-module.*
(2) (a) X *is an automorphism-invariant right* R*-module.*
  (b) Y *is an automorphism-invariant right* S*-module.*

Proof. (2) $\Rightarrow$ (1). By Lemma 2.3 in [15], there exist isomorphisms $\tilde{\mu} : E(X) \to Hom_S(M, E(Y))$ and $\tilde{\eta} : E(Y) \to Hom_R(N, E(X)$ such that $(E(X), E(Y), \mu, \eta)$ is the injective envelope of (X, Y, f, g). Let $\varphi = (\varphi_1, \varphi_2)$ be an automorphism of $(E(X), E(Y), \mu, \eta)$ then φ_1 is an R-automorphism of $E(X)$ and φ_2 is an S-automorphism of $E(Y)$. Since X is an automorphism-invariant right R-module and Y is an automorphism-invariant right S-module, it follows that (X, Y, f, g) is an automorphism-invariant right K-module.

(1) $\Rightarrow$ (2) Assume that (X, Y, f, g) is an automorphism-invariant right K-module. We show that X is an automorphism-invariant right R-module. To prove this, firstly we show that $(X, Y, f, g) \cong (X, H(X), u_X, v_X)$. In fact that we consider the mapping $(1_X, \tilde{g}) : (X, Y, f, g) \to (X, H(X), u_X, v_X)$.

Since (X, Y, f, g) is a right K-module, $g \circ (f \otimes 1_N) = \mu \circ (1_X \otimes \varphi)$, where $\mu : X \otimes_R R \to X$ is the canonical isomorphism and $\varphi : M \otimes_S N \to R$ is the multipilication in K. Then, for all $x \in X$, $m \in M$ and $n \in M$, we have

$$(\tilde{g} \circ f)(x \otimes m)(n) = g(f(x \otimes m) \otimes n) = \mu(1_X \otimes \varphi)(x \otimes m \otimes n) = x(mn)$$

and

$$u_X(1_X \otimes 1_M)(x \otimes m)(n) = u_X(x \otimes m)(n) = x(mn)$$

It shows that $\tilde{g} \circ f = u_X \circ (1_X \otimes 1_M)$ and so the following diagram is commutative.

$$\begin{array}{ccc} X \otimes_R M & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle 1_X \otimes 1_M} & & \downarrow{\scriptstyle \tilde{g}} \\ X \otimes_R M & \xrightarrow{u_X} & H(X) \end{array}$$

On the other hand, for all $y \in Y$ and $n \in N$, we have

$$v_X(\tilde{g} \otimes 1_N)(y \otimes n) = v_X(\tilde{g}(y) \otimes n) = \tilde{g}(y)(n) = g(y \otimes n) = 1_X g(y \otimes n)$$

and so $1_X \circ g = v_X \circ (\tilde{g} \otimes 1_N)$. It means that the following diagram is commutative.

$$\begin{array}{ccc} Y \otimes_S N & \xrightarrow{g} & X \\ {\scriptstyle \tilde{g} \otimes 1_N}\downarrow & & \downarrow{\scriptstyle 1_X} \\ H(X) \otimes_S N & \xrightarrow{v_X} & X \end{array}$$

Thus, $(1_X, \tilde{g}) : (X, Y, f, g) \to (X, H(X), u_X, v_X)$ is a K-homomorphism. By our assumption, $\tilde{g}$ is an isomorphism, $(1_X, \tilde{g})$ is an isomorphism. Then, $(X, H(X), u_X, v_X)$ is an automorphism-invariant right K-module.

Now, we show that X is an automorphism-invariant right R-module. Let $\alpha : A \to X$ be an R-monomorphism. Then, we have that $(A, H(A), u_A, v_A)$ is a submodule of $(X, H(X), u_X, v_X)$. We consider the mapping $\beta : H(A) \to H(X)$ via by the relation $\beta(h)(n) = \alpha(v_A(h \otimes n))$. One can check that β is an S-homomorphism. For all $a \in A$, $m \in M$ and $n \in M$, we have

$$(\beta \circ u_A)(a \otimes m)(n) = \alpha(v_A(u_A(a \otimes m) \otimes n)) = \alpha(\mu(1_A \otimes \varphi)(a \otimes m \otimes n)) = \alpha(a)mn$$

and

$$u_X(\alpha \otimes 1_M)(a \otimes m)(n) = u_X(\alpha(a) \otimes m)(n) = \alpha(a)mn$$

It shows that $\beta \circ u_A = u_X \circ (\alpha \otimes 1_M)$ and so the following diagram is commutative.

$$\begin{array}{ccc} A \otimes_R M & \xrightarrow{u_A} & H(A) \\ \downarrow{\scriptstyle \alpha \otimes 1_M} & & \downarrow{\scriptstyle \beta} \\ X \otimes_R M & \xrightarrow{u_X} & H(X) \end{array}$$

On the other hand, for all $h \in H(A)$ and $n \in N$, we have

$$v_X(\beta \otimes 1_N)(h \otimes n) = v_X(\beta(h) \otimes n) = \beta(h)(n) = \alpha v_A(h \otimes n)$$

and so $\alpha \circ v_A = v_X \circ (\beta \otimes 1_N)$. It means that the following diagram is commutative.

$$\begin{array}{ccc} H(A) \otimes_S N & \xrightarrow{v_A} & A \\ \Big\downarrow{\scriptstyle \beta\otimes 1_N} & & \Big\downarrow{\scriptstyle \alpha} \\ H(X) \otimes_S N & \xrightarrow{v_X} & X \end{array}$$

Thus, $(\alpha, \beta) : (A, H(A), u_A, v_A) \to (X, H(X), u_X, v_X)$ is a K-monomorphism. Since $(X, H(X), u_X, v_X)$ is an automorphism-invariant right K-module, there exists an endomorphism (γ, θ) of $(X, H(X), u_X, v_X)$ such that (γ, θ) is an extension of (α, β). Thus, $\gamma : X \to X$ is an extension of α. We deduce that X is an automorphism-invariant right R-module.

Similarly, we also prove that Y is an automorphism-invariant right S-module. □

By [13, Lemma 3.8.1] and Theorem 15, we have the following result:

Corollary 16. *Let* $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *and* (X, Y, f, g) *be a right* K*-module. Assume that* $MN = R$ *and* $NM = S$*. Then the following conditions are equivalent:*

(1) (X, Y, f, g) *is an automorphism-invariant right* K*-module.*
(2) (a) X *is an automorphism-invariant right* R*-module.*
  (b) Y *is an automorphism-invariant right* S*-module.*

Corollary 17. *Let* e *be a non-zero idempotent of a ring* R*,* $K = \begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$ *and* (X, Y, f, g) *be a right* K*-module. Assume that* $\tilde{f}$ *and* $\tilde{g}$ *are isomorphisms. Then* (X, Y, f, g) *is an automorphism-invariant right* K*-module if and only if* X *is an automorphism-invariant right* R*-module and* Y *is an automorphism-invariant right* eRe*-module.*

If e is an idempotent of a ring R such that $ReR = R$ then $R \approx eRe$. So in this case, we have:

Corollary 18. *Let* e *be an idempotent of a ring* R *such that* $ReR = R$ *and* $K = \begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$*. Assume that* R *is a right fa-ring and* $\tilde{f}$*,* $\tilde{g}$ *are isomorphisms. Then* (eR, Re, f, g) *is an automorphism-invariant right* K*-module.*

Acknowledgment. The authors acknowledge the support/partial support of the Core Research Program of Hue University, Grant No. NCM.DHH.2020.15. Parts of this paper were written during a stay of the authors in the Vietnam Institute For Advanced Study in Mathematics (VIASM) and they would like to thank the members of VIASM for their hospitality, as well as to gratefully acknowledge the received support.

References

[1] A.N. Abyzov, T.C. Quynh and A. A. Tuganbaev, Modules that are Invariant with Respect to Automorphisms and Idempotent Endomorphisms of Their Hulls and Covers. J. Math. Sci. 256(2021), 235-277.

[2] A. Alahmadi, A. Facchini, N. K. Tung, Automorphism-invariant modules, Rend. Sem. Mat. Univ. Padova, 133 (2015), 241-259.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, (Springer-Verlag, 1992).

[4] S. E. Dickson and K. R. Fuller, Algebras for which every indecompoable right module is invariant in its injective envolope, Pacific J. Math., 31, 3 (1969), 655-658.

[5] N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer, *Extending modules*, Vol. 313 (Harlow: Longman 1994).

[6] N. Er, S. Singh, A. K. Srivastava, Rings and modules which are stable under automorphisms of their injective hulls, J. Algebra 379, (2013), 223–229.

[7] D. A. Hill, Semi-perfect q-rings. Math. Ann. 200 (1973), 113-121

[8] P. A. Guil Asensio, A. K. Srivastava, Automorphism-invariant modules satisfy the exchange property, J. Algebra 388 (2013), 101-106.

[9] S. K. Jain, S. Mohamed, S. Singh, Rings in which every right ideal is quasi-injective, Pacific J. Math. 31 (1969), 73-79.

[10] S. K. Jain, S. Singh, Quasi-injective and pseudo-injective modules, Canad. Math. Bull. 18 (1975), 359-366.

[11] R.E. Johnson and E.T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. 36(1961), 260-268.

[12] M. T. Koşan, T. C. Quynh and A. K. Srivastava, Rings with each right ideal automorphism-invariant, J. Pure Appl. Algebra 220(2016) 1525-1537.

[13] P. Krylov, A. Tuganbaev, Formal Matrices, Springer International Publishing, Switzerland, 2017

[14] T.K. Lee and Y. Zhou, Modules which are invariant under automorphisms of their injective hulls, J. Algebra Appl. 12 (2013) 1250159.

[15] M. Müller, Rings of quotients of generalized matrix rings. Commun. Algebra 15(1987), 1991-2015

[16] W. K. Nicholson, M. F. Yousif, Quasi-Frobenius Rings, Cambridge Univ. Press. (2003).

[17] T. C. Quynh, A. N. Abyzov, D. T. Trang, Rings all of whose finitely generated ideals are automorphism-invariant, J. Algebra Appl. (2022) 2250159.

[18] T. C. Quynh, M. T. Koşan, L. V. Thuyet, On automorphism-invariant rings with chain conditions. Vietnam J. Math. 48 (2020), 23-29.

[19] A. K. Srivastava, A. A. Tuganbaev and P. A. Guil Asensio, Invariance of Modules under Automorphisms of their Envelopes and Covers, Cambridge University Press, March 2021.

[20] A. A. Tuganbaev, Automorphism-Invariant Modules, Journal of Mathematical Sciences volume 206, (2015), pages 694–698.

Department of Mathematics, College of Education, Hue University, 34 Le Loi, Hue city, Viet Nam

Email address: lvthuyet@hueuni.edu.vn

Department of Mathematics, The University of Danang - University of Science and Education, 459 Ton Duc Thang, Danang city, Vietnam

Email address: tcquynh@ued.udn.vn