NEW CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS

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Abstract. In this paper, we firstly provide several new characterizations of quasi-Frobenius rings by using some generalized injectivity of rings with certain chain conditions. Namely, we prove among other results, that: (1) A ring R is quasi-Frobenius if and only if R is right C_{11}, right minfull with ACC on right annihilators; (2) A ring R is quasi-Frobenius if and only if R is two-sided min-CS with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$; (3) A ring R is quasi-Frobenius if and only if R is right Johns left C_{11}; (4) A ring R is quasi-Frobenius if and only if R is quasi-dual two-sided C_{11} with ACC on right annihilators. Moroever, we give more characterizations of quasi-Frobenius rings. For example, it is shown that a ring R is quasi-Frobenius if and only if R is a left P-injective left IN-ring with right RMC and $Z(R_R) = Z(_RR)$. Also, we prove that if R is a right duo, right QF-3^+ left quasi-duo ring satisfying ACC on right annihilators, then R is quasi-Frobenius. In this paper, several known results on quasi-Frobenius rings are reproved as corollaries.

1. Introduction

Throughout this paper, all rings R are associative with identity and all modules are unitary right R-module. The notations $N \leq_e M$ and $N \leq^{\oplus} M$ mean that N is an essential submodule and a direct summand, respectively. Let M be an R-module. Recall that the *singular submodule* $Z(M)$ of M is defined by

$$Z(M) = \{m \in M \mid mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

The *Goldie torsion submodule* $Z_2(M)$ of M (also known as the *second singular submodule* of M) is defined to be the submodule of M which contains $Z(M)$ such that $Z(M/Z(M)) = Z_2(M)/Z(M)$. The module M is called *singular* if $Z(M) = M$ and is called *nonsingular* if $Z(M) = 0$ (equivalently, $Z_2(M) = 0$). Recall that $M/Z_2(M)$ is a nonsingular module. For a ring R, we denote by $J(R)$ the Jacobson radical of R. If X is a subset of a ring R, the right (left) annihilator in R is denoted by $r(X)$ $(l(X))$.

The notion of self-injective rings is generalized by many authors. In [12], let R be a ring, then

• R is called right P-injective (resp., 2-injective) ring if every R-homomorphism from a principal (resp., 2-generated) right ideal of R extends to an endomorphims of R.

• R is said to be right mininjective if every R-homomorphism from a minimal right ideal of R extends to an endomorphims of R.

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• R is called right simple-injective if every R-linear map with simple image from a right ideal to R extends to R.

• R is called right dual if $rl(T) = T$ for all right ideals T of R.

• R is called right minfull if it is semiperfect right minijective and $Soc(eR) \neq 0$ for each local idempotent e of R.

• R is called right min-CS if every minimal right ideal is essential in a direct summand.

• R is said to be left IN ring if $r(T \cap T') = r(T) + r(T')$ for all left ideals T and T' of R .

A ring R is called right GP-injective if for each $0 \neq a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n$ ([2]).

Recall that a module M is said to be a C_{11}-module if every submodule of M has a complement which is a direct summand ([21]). A ring R is called a right C_{11}-ring if R_R is a C_{11}-module. Clearly, every CS-module satisfies the C_{11}-condition. However, the converse is not true in general (see [21, p. 1814]).

A submodule N of a module M is said to be an automorphism-invariant submodule if $f(N) \subseteq N$ for every automorphism f of M. A module is called automorphism-invariant if it is an automorphism-invariant of its injective hull ([14]). A ring R is called right automorphism-invariant if R_R is automorphism-invariant.

A module M is said to be satisfy the restricted minimum condition (briefly, RMC) if for every essential submodule N of M, M/N is an artinian module. A ring R is said to be have right RMC if R satisfies the RMC as a right R-module.

Recall that a ring R is quasi-Frobenius if R is two-sided artinian and two-sided self-injective. Quasi-Frobenius rings play an important role in the theory, and many interesting characterizations can be found in ([12]).

In Section 2, we provide several new characterizations of quasi-Frobenius rings by using some generalized injectivity of rings satisying certain chain conditions. We first prove that a right C_{11}, right minfull ring satisfying ACC on right annihilators is quasi-Frobenius. We prove that a two-sided min-CS ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$ is quasi-Frobenius. It is also shown that a left AGP-injective two-side min-CS ring satisfying ACC on left annihilators is quasi-Frobenius We prove that a right Johns left C_{11}-ring is quasi-Frobenius. Note that in this section, some known results on quasi-Frobenius are obtained as corollaries.

In section 3, quasi-Frobenius rings are characterized via two-side C_{11}-rings. We prove that a ring is quasi-Frobenius if and only if it is quasi-dual two-side C_{11} with ACC on right annihilators. Moroever, it is shown that a right artinian two-side C_{11}-ring R in which $Soc(R_R) = Soc(_RR)$ is quasi-Frobenius.

Section 4 is devoted to automorphism-invariant rings and their generalizations. In this section, it is shwon among others results that every left automorphism-invariant ring R with ACC on right annihilators in which $Soc(_RR)$ is an essential right ideal is quasi-Frobenius. We prove also that every two-side pseudo-c^*-injective two-side C_{11}-ring with ACC on right annihilators is quasi-Frobenius.

In section 5, we provide more characterizations of quasi-Frobenius rings. Firstly, we prove that a left perfect right simple-injective ring such that for every injective right R-module M, $Z_2(M)$ is projective, is quasi-Frobenius. Also, it is shown that a two-sided minfull left (or right) pseudo-coherent ring R for which $J(R)$ is left or right T-nilpotent is quasi-Frobenius. Moroever, we prove that a left P-injective left

IN-ring with right RMC is quasi-Frobenius if and only if $Z(_RR) = Z(R_R)$. This result extends Theorem 13((1) $\Leftrightarrow$ (2)) in [10] and Proposition 18.6 in [5]. As a direct consequence of the last result, it is shown that a two-sided P-injective left IN-ring with right RMC is quasi-Frobenius. Finally, we show that if R is a right duo, right QF-3^+ left quasi-duo ring satisfying ACC on right annihilators, then R is quasi-Frobenius.

2. Quasi-Frobenius rings via the mimimal ideals

It is obvious that a quasi-Frobenius ring is right minfull with ACC on right annihiltors. However Examples 2.5 and 6.41(1) in [12] show that the converse is not true in general. In the next theorem, we provide some conditions which force a right minfull ring with ACC on right annihiltors to be quasi-Frobenius. We first prove the following lemma.

Lemma 2.1. *Let R be a right C_{11} right minifull ring. Then $Soc(eR)$ is a minimal right ideal for every local idempotent e of R and R is right finitely cogenerated.*

Proof. Since R is right minfull, R_R satisfies the C_2-condition by [12, Lemma 1.46 and Theorem 3.12]. Now, let e be a local idempotent of R. As R_R is a C_{11}-module, then by [21, Theorem 4.3], eR is also a C_{11}-module. Hence, since eR is indecomposable, it follows from [21, Proposition 2.3(iii)] that eR is uniform. Note that $Soc(eR) \neq 0$. Therefore, $Soc(eR)$ is a minimal right ideal. On the other hand, since R is semiperfect, there exits a decomposition $R_R = e_1R \oplus e_2R \oplus .... \oplus e_nR$ where each e_i is a local idempotent. Therefore, by what we shown above, $Soc(e_iR)$ is a minimal right ideal and $Soc(e_iR) \leq_e e_iR$. From this, we deduce that $Soc(R_R)$ is a finitely generated right ideal and $Soc(R_R) \leq_e R_R$. Therefore, R is right finitely cogenerated. □

Theorem 2.2. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right minfull with ACC on right annihilators and every complement right ideal is a right annihilator;*
3. *R is right C_{11} right minfull with ACC on right annihilators;*
4. *R is right C_{11} right minfull with right RMC.*

Proof. $(1) \Rightarrow (2), (4)$ are clear.

$(2) \Rightarrow (3)$ Being right minfull, R is left Kasch by [12, Theorem 3.12]. But every complement right ideal is a right annihilator. Then R is a right C_{11}-ring by [24, Theorem 10].

$(3) \Rightarrow (1)$ By Lemma 2.1, R is right finitely cogenerated. In addition, since R is right mininjective, $Soc(R_R) \subseteq Soc(_RR)$ by [12, Theorem 2.21]. Consequently, $Soc(_RR) \leq_e R_R$, and so $J(R) \subseteq Z(R)$. But R is semiperfect. Then $J(R) = Z(R)$ Note that R has ACC on right annihilators. Therefore, in view of [12, Lemma 3.29], $J(R)$ is nilpotent, from which it follows that R is semiprimary. Hence, by Lemma 2.1 and [22, Corollary 7], $Soc(Re)$ is a minimal left ideal for every local idempotent e of R. In addition, since R is right minfull, we infer from [12, Theorem 3.12] that R is right Kasch. So, using [12, Theorem 3.7(3)(a)], we deduce that $Soc(R_R) = Soc(_RR)$. Now, we claim that R is left mininjective. To see this fact, let e be a local idempotent of R. By Lemma 2.1, $Soc(eR)$ is a minimal right ideal. Therefore, being semiperfect,

R is left mininjective by [12, Theorem 3.2(1)]. Finally, since R is a right mininjective ring with ACC on right annihilators in which $Soc(R_R) \leq_e R_R$, R is quasi-Frobenius by [12, Theorem 3.31].

(4) ⇒ (1) By Lemma 2.1, R is right finitely cogenerated. Thus, by hypothesis, $R/Soc(R_R)$ is right noetherian, and so R has ACC on right annihilators. Therefore, R is quasi-Frobenius by (3). □

Corollary 2.3. *The following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right minfull right C_{11}-ring and $Z(R_R)$ is a noetherian right R-module.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) Assume that R has the stated condition. Then by Lemma 2.1, $Soc(R_R)$ is a finitely generated right ideal and essential in R_R. So, using [12, Lemma 6.43], we deduce that $R/Z(R_R)$ is right noetherian. Note that $Z(R_R)$ is a noetherian right R-module. Hence, R is right noetherian, which implies that R has ACC on right annihilators. Therefore, according to Theorem 2.2(2), R is quasi-Frobenius. □

Recall a ring R is called right (left) QF-2 if R is a direct sum of unform right (left) ideals.

Corollary 2.4 ([18, Theorem 4.4])**.** *If R is a QF-2 ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$, then R is quasi-Frobenius.*

Proof. By [18, Lemma 4.3], R is semiperfect and $Soc(Re) \neq 0$ for every local idempotent $e \in R$. Since R is left QF-2, Re is uniform, from which it follows that $Soc(Re)$ is simple. In addition, since $Soc(_RR) \leq_e R_R$, $Soc(R_R) \subseteq Soc(_RR)$. So, R is right mininjective by [12, Proposition 3.5] and consequently, R is right minfull. Note that R is a right C_{11}-ring (being right QF-2) by [21, Theorem 2.4]. Therefore, the result follows from Theorem 2.2(2). □

Corollary 2.5. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right C_{11} right GP-injective with ACC on right annihilators;*
3. *R is a right artinian right mininjective right CS-ring;*
4. *R is a right artinian right mininjective right C_{11}-ring.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) follows from [2, Theorem 3.7] and Theorem 2.2(2).

(1) ⇔ (3) ⇔ (4) follows from Theorem 2.2(2). □

Corollary 2.6. *The following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right C_{11}, left minannihilator and right artinian.*

Theorem 2.7. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is two-sided min-CS with ACC on right annihilators in which $Soc(_RR)$ is essential in R_R.*
3. *R is left AGP-injective two-sided min-CS with ACC on left annihilators;*

Proof. (1) $\Rightarrow$ (2), (3) are clear.

(2) $\Rightarrow$ (1) Since R has ACC on right annihilators and $Soc(_RR) \leq_e R_R$, R is semiprimary by [18, Lemma 4.3]. Thus, R is left Kasch by [12, Lemma 4.2]. As R is left min-CS, then it follows from [12, Lemma 4.5] that $Soc(Re)$ is simple for all local idempotent $e \in R$. On the other hand, the fact that $Soc(_RR) \leq_e R_R$ implies that $Soc(R_R) \subseteq Soc(_RR)$. Hence, being semiperfect, R is right mininjective by [12, Proposition 3.5], from which it follows that R is right minfull. Thus, using [12, Theorem 3.12], R is right Kasch. Since R is semiperfect right min-CS, we infer from [12, Lemma 4.5] that $Soc(eR)$ is simple for all local idempotent $e \in R$ for. But we have already seen that $Soc(Re)$ is simple for all local idempotent $e \in R$. Then, since R is right Kasch, it follows from [12, Theorem 3.7(3)] that $Soc(R_R) = Soc(_RR)$. So, by [12, Proposition 3.5] again, R is left mininjective. Finally, being a two-sided mininjective ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$, R is quasi-Frobenius by [12, Theorem 3.31].

(3) $\Rightarrow$ (1) Being left AGP-injective with ACC on left annihilators, R is semiprimary by [25, Corollary 1.6]. On the other hand, since R is left AGP-injective, $J(_RR) = Z(_RR)$ by [25, Lemma 1.3], and so $Soc(_RR) \subseteq Soc(R_R)$. This implies that $Soc(R_R) \leq_e {}_RR$. Therefore, according to (2) $\Rightarrow$ (1), R is quasi-Frobenius. □

A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M.

Corollary 2.8 ([18, Theorem 4.7])**.** *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right ef-extending with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$.*

Proposition 2.9. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right noetherian left AGP-injective two-sided ef-extending ring.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Since R is right noetherian, R contains no infinite orthogonal sets of idempotents. Hence, $_RR = Re_i \oplus ... \oplus Re_n$, where each Re_i is indecomposable. As, $_RR$ is an ef-extending module, each Re_i is uniform. Thus, $_RR$ has finite uniform dimension. So, using [14, Corollary 1.2], we deduce that R is semilocal. On the other hand, being right noetherian left AGP-injective, $J(R)$ is nilpotent by [14, Theorem 2.1]. Therefore, R is semiprimary, from which it follows that R is right artinian. So, R has ACC on left annihilators. Therefore, the claim follows from Theorem 2.7(3). □

Theorem 2.10. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is left C_{11} right cogenerator with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) As R has ACC on right annihilators, then it has enough idempotents. So we can write $R = Re_1 \oplus Re_2 \oplus ... \oplus Re_n$, where $\{e_i\}_{i=1}^n$ is an orthogonal set of

primitive idempotents. Since R is right cogenerator, R is right Kasch. Thus, R is a left C_2-ring, and so $_RR$ is a C_3-module. Then, since $_RR$ is a C_{11}-module, it follows from [21, Proposition 2.3 (iii) and Theorem 4.3] that each Re_i is uniform. Consequently, $_RR$ has finite uniform dimension. As $_RR$ is a C_2-module, then R is semiperfect by [12, Lemma 4.26]. In particular, R has a finite number of isomorphism classes of simple right and (left) R-modules. Since R is right cogenerator, R is right self-injective by [12, Theorem 1.56]. Therefore, in view of [5, Proposition 18.9], R is quasi-Frobenius. □

Theorem 2.11. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a right noetherian left P-injective lef C_{11}-ring.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) Since R is right noetherian, R contains no infinite orthogonal sets of idempotents. So, we can write $_RR = Re_1 \oplus ... \oplus Re_n$, where each Re_i is a primitive orthogonal idempotent. Note that $_RR$ is a C_3-module. Then, since $_RR$ is a C_{11}-module, it follows from [21, Proposition 2.3(iii) and Theorem 4.3] that each Re_i is uniform. Consequently, $_RR$ has finite uniform dimension. Thus, using [25, Corollary 1.2], we deduce that R is semilocal. On the other hand, since R is right noetherian left AGP-injective, $J(R)$ is nilpotent by [25, Theorem 2.1]. This implies that R is semiprimary, and so R is right artinian. Hence, R has ACC on left annihilators. Note that R is left mininjective. Then, R is left minfull. Therefore, being left C_{11}, R is quasi-Frobenius by Theorem 2.2 (2). □

Corollary 2.12. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a right Johns left C_{11}-ring.*

Corollary 2.13. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a strongly right Johns left C_{11}-ring.*

3. Quasi-Frobenius rings via two-sided C_{11}-rings

Following [25], a ring R is called right (left) quasi-dual if every right (left) ideal is a direct summand of a right (left) annihilator.

Theorem 3.1. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is quasi-dual two-sided C_{11} with ACC on right annihilators;*
(3) *R is a two-sided C_{11}-ring with ACC on right annihilators in which $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (3) Since R is quasi-dual, $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R by [25, Corollary 3.3].

(3) ⇒ (1) Since R has ACC on right annihilators and $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R, we infer from [18, Lemma 4.3] that R is semiprimary. Thus, using [12, Lemma 4.2], we deduce that R is right Kasch. Hence,

by [12, Lemma 1.46], ${}_RR$ satisfies the C_2-condition. Now, we claim that R is right mininjective. To see this, let e be a local idempotent of R. Then $Soc(Re) \neq 0$. Since ${}_RR$ is a C_{11}-module satisfying the C_2-condition, it follows from [21, Proposition 2.3 (iii) and Theorem 4.3] that Re is a uniform module. Consequently, $Soc(Re)$ is simple. But $Soc(R_R) \subseteq Soc({}_RR)$. Then, R is right mininjective by [12, Proposition 3.5]. Therefore, by Theorem 2.2(2), R is quasi-Frobenius. □

Corollary 3.2. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right artinian two-sided C_{11} and $Soc(R_R) = Soc({}_RR)$.*

Corollary 3.3. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is two-sided C_{11} two-sided AGP-injective with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [18, Theorem 3.4] and its proof, R is semiprimary and $Soc(R_R) = Soc({}_RR)$. Therefore, by Theorem 3.1(3), R is quasi-Frobenius. □

The next example shows that the condition "$Soc(R_R) = Soc({}_RR)$" in the hypothesis of Corollary 3.2 is necessary.

Example 3.4 ([18, Remark 4.8(i)])**.** *Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. R is a two-sided artinian two-sided CS ring which is not quasi-Frobenius. However, $Soc(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ and $Soc({}_RR) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $Soc(R_R) \nleq Soc({}_RR)$ and $Soc({}_RR) \nleq Soc(R_R)$.*

4. Automorphism-invariant rings and their generalizations

Lemma 4.1. *If R is a left automorphism-invariant ring and containing no infinite orthogonal sets of idempotents, then R is semiperfect.*

Proof. Assume that R is a left automorphism-invariant ring and R contains no infinite orthogonal sets of idempotents. Let e be a primitive idempotent of R. Then, Re is an indecomposable autmorphism-invariant left R-module. It follows that $End(Re)$ is a local ring, and so e is a local idempotent of R. Thus, R is semiperfect. □

Proposition 4.2. *If R is left automorphism-invariant and has ACC on right annihilators with $Soc({}_RR)$ an essential right ideal, then R is a quasi-Frobenius ring*

Proof. Assume that R is left automorphism-invariant and has ACC on right annihilators with $Soc({}_RR)$ an essential right ideal. Then, R is semiperfect by Lemma 4.1. Moreover, $J(R)$ is nilpotent by [9, Corollary 1.5]. It follows that R is semiprimary and so R is left self-injective. This shows that R is quasi-Frobenius. □

Proposition 4.3. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right automorphism-invariant right C_{11} with ACC on left annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Since R has ACC on left annihilators, it has enough idempotents. So, we can write $R_R = e_iR \oplus ... \oplus e_nR$ where each e_iR is a primitive orthogonal idempotent. Being autmomorphism-invariant, R_R is a C_3-modue by [14, page 26]. Thus, since R_R is a C_{11}-module, each e_iR is uniform by [21, Proposition 2.3 (iii)] and Theorem 4.3]. Therefore, according to the proof of ((5) $\Rightarrow$ (1) of [14, Theorem 2], R is right self-injective. Thus, using [12, Proposition 18.9], we deduce that R is quasi-Frobenius. □

Corollary 4.4. *A left noetherian right automorphism-invariant C_{11}-ring is quasi-Frobenius.*

Recall from [13] that a module N is said to be pseudo M-c^*-injective if for any submodule A of M which is isomorphic to a closed submodule of M, every monomorphism from A to N can be extended to a homomorphism from M to N. A module M is called pseudo-c^*-injective if M is pseudo M-c^*-injective. A ring is called right pseudo-c^*-injective if R_R is pseudo-c^*-injective.

Proposition 4.5. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left 2-injective with ACC on right annihilators and $Soc(_RR) \leq_e R_R$.*
(3) *R is left 2-injective right AGP-injective with ACC on right annihilators;*
(4) *R is left 2-injective right pseudo-c^*-injective with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2), (3), (4) are clear.

(2) $\Rightarrow$ (1) Since R has ACC on right annihilators and $Soc(_RR) \leq_e R_R$, R is semiprimary by [18, Lemma 4.3]. Then by [12, Theorem 5.31], R is left Kasch. Consequently, R is right P-injective by [12, Lemma 5.21]. Therefore, by [12, Theorem 3.31], R is quasi-Frobenius.

(3) $\Rightarrow$ (2) Since R is right AGP-injective with ACC on right annihilators, R is semiprimary, by [25, Corollary 1.6]. Moreover, $J(R) = Z(R_R)$ by [25, Lemma 1.3], and so $Soc(R_R) \subseteq Soc(_RR)$. Hence, $Soc(_RR) \leq_e R_R$.

(4) $\Rightarrow$ (2) Since R is right pseudo-c^*-injective with ACC on right annihilators, it follows from [13, Corollary 3.6] that R is semirpimary. Hence, by [12, Theorem 5.31], $Soc(_RR) \leq_e R_R$. □

A ring R is strongly right Johns if $M_n(R)$ is right Johns for all $n \geq 1$. By [12, Lemma 8.10], if $M_2(R)$ is right Johns, then so is R. We have the following result:

Corollary 4.6. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is strongly right Johns right pseudo-c^*-injective;*
(3) *R is strongly right Johns and $Soc(_RR) \leq_e R_R$;*
(4) *$M_2(R)$ is right Johns right pseudo-c^*-injective;*
(5) *$M_2(R)$ is right Johns and $Soc(_RR) \leq_e R_R$.*

Theorem 4.7. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is two-sided pseudo-c^*-injective, two-sided C_{11} and has ACC on right annihilators.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) Since R is right pseudo-c^*-injective and has ACC on right annihilators, by [13, Corollary 3.6], R is semiprimary. Hence, we can write $R_R = e_iR \oplus ... \oplus e_nR$ where each e_iR is a primtive orthogonal idempotent. Being right pseudo-c^*-injective, R_R is a C_3-modue by [13, Theorem 3.1]. Thus, since R_R is a C_{11}-module, each e_iR is uniform by [21, Proposition 2.3 (iii) and Theorem 4.3]. Therefore, according to [13, Theorem 3.4], R is right continuous. Similary, since R is left C_{11}, we can easily show that R is left continuous. Now, being two-sided continuous with ACC on right annihilators, R is quasi-Frobenius by [18, Corollary 4.11]. □

5. More characterizations

In the next result, we a provide a necessary and sufficient condition for a left perfect right simple-injective ring to be quasi-Frobenius.

Theorem 5.1. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect right simple-injective and for every projective right R-module M, $Z_2(M)$ is injective;*
(3) *R is left perfect right simple-injective and for every injective right R-module M, $Z_2(M)$ is projective;*
(4) *R is left perfect right simple-injective and $Z(R_R)$ is a noetherian right R-module.*

Proof. (1) ⇒ (2), (3), (4) are clear.

(2) ⇒ (1) By [12, Theorem 2.21], $Soc(R_R) \subseteq Soc(_RR)$, from which it follows that $Soc(_RR) \leq_e R_R$. Using [12, Lemma 4.2], we deduce that R is left Kasch and $rl(T)$ is essential in a direct summand of R for all right ideals T of R. Also, R is right Kasch by [12, Theorem 3.12]. Therefore, according to [12, Proposition 6.14], $rl(T) = T$ for all right ideals T of R. Hence, $J(R) \leq Z_2(R_R)$ by [8, Lemma 2]. Let M be any projective R-module. Then, by [7, p. 48 Exercise 22], $M = Z_2(M) \oplus M'$ for some injective R-module. Therefore, by hypothesis, R is quasi-Frobenius.

(3) ⇒ (1) Let M be an injective R-module. Thus, by the proof of (2) ⇒ (1), $M = Z_2(M) \oplus M'$ for some projective R-module. By hypothesis, R is quasi-Frobenius.

(4) ⇒ (1) As shown in the proof of (2) ⇒ (1), R is left Kasch and $rl(T) = T$ for all right ideals T of R. Thus, by [12, Proposition 5.20], $Soc(_RR) \leq_e R_R$. It follows from [12, Corollary 5.53] that R is right finitely cogenerated. Using [12, Lemma 6.43], we deduce that $R/Z(R_R)$ is right noetherian. Note that $Z(R_R)$ is a noetherian right R-module. Hence, we infer from [12, Lemma 8.6] that right artinian. Finally, R is quasi-Frobenius by [12, Theorem 3.31]. □

Corollary 5.2. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect right self-injective and for every projective right R-module M, $Z_2(M)$ is injective;*
(3) *R is left perfect right self-injective and for every injective right R-module M, $Z_2(M)$ is projective.*

Recall that a ring R is said to be left pseudo-coherent if the left annihilator of every finite subset of R is finitely generated.

Theorem 5.3. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is two-sided minfull left (or right) pseudo-coherent and $J(R)$ is left (or right) T-nilpotent.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [12, Corollary 5.53], $Soc(_RR)$ is a finitely generated right ideal. Note that R is left pseudo-coherent. Thus, $J(R)$ is finitely generated as a left ideal. Since $J(R)$ is left T-nilpotent, we infer from [12, Lemma 5.64] that R is right perfect. Therefore, according to [12, Lemma 6.50], R has ACC on left annihilators. On the other hand, $Soc(R_R) = Soc(_RR)$ is left finitely generated as a right R-module by [12, Corollary 5.53]. Hence, by [12, Lemma 3.30], R is right artinian and we conclude by [12, Theorem 3.31] that R is quasi-Frobenius. □

Corollary 5.4. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a dual left (or right) pseudo-coherent ring in which $J(R)$ is left (or right) T-nilpotent.*

Corollary 5.5. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect, two-sided mininjective and left (or right) pseudo-coherent.*

Theorem 5.6. *Let R be a right C_{11} right minifull ring such that $J^2(R) = r(A)$ for a finite subset A of R. Then $J(R)/J^2(R)$ is a finitely generated right R-module.*

Proof. Let $J^2(R) = r(a_1, ..., a_n)$. Define $\phi : R/J^2(R) \longrightarrow R^n_R$ via $\phi(a + J^2(R)) = r(a_1a, a_2a..., a_na)$ for $a \in R$. Then ϕ is a monomorphism. Hence, we may regard $J^2(R)/J(R)$ as a submodule of R^n_R. Also, we have $J(R)/J^2(R) = Soc(J(R)/J^2(R)) \subseteq Soc(R^n_R) = (Soc(R_R))^n$. On the other hand, $Soc(R_R)$ is finitely generated by Lemma 2.1. Therefore, as a direct summand of $(Soc(R_R))^n$, $J(R)/J^2(R)$ is a finitely generated right R-module. □

Corollary 5.7. *Let R be a left perfect right C_{11} right mininjective ring. If $J^2(R) = r(A)$ for a finite subset A of R, then R is quasi-Frobenius.*

Proof. Since R is left perfect right mininjective, it is right minfull. Thus, $J(R)/J^2(R)$ is a finitely generated right R-module by Theorem 5.6. Now, being left perfect, R is right artinian by [4, Lemma 2.9]. Thus, using Corollary 2.5(5), we deduce that R is quasi-Frobenius. □

The following theorem is motivated by Theorem 3.13 in [10]. First, we prove the following lemmas.

Lemma 5.8. *Let R be a left continuous ring right RMC. Then R is semiperfect.*

Proof. Assume that R is left continuous right RMC. Let $\overline{S}_1 = Soc(\overline{Q}_{\overline{Q}})$ where $\overline{Q} = R/J(R)$. By [8, Lemma 2], $\overline{Q}$ is a von Neumann regular left continuous ring. Consequently, $\overline{Q}/\overline{S}_1$ is von Neumann regular. In addition, since $\overline{Q}$ has right RMC, $\overline{Q}/\overline{S}_1$ has finite right uniform dimension by [5, Lemma 5.14]. It follows that $\overline{Q}/\overline{S}_1$ is semisimple. As $\overline{Q}$ is semiprime, then $\overline{S}_1 = Soc(_{\overline{Q}}\overline{Q})$. Thus, $\overline{Q}$ satisfies DCC on essential left ideals. Therefore, $\overline{Q}$ is an artinian ring by [5, Corollary 18.7(2)], and we conclude by [8, Lemma 2] that R is semiperfect. □

Lemma 5.9. *Let R be a left CS ring with right RMC such that every principal right ideal is right annihilator. Then $r(J(R))$ is a noetherian right R-module.*

Proof. Since every principal right ideal is right annihilator, R is a left C_2-ring by [12, Proposition 5.10]. Thus, by Lemma 5.8, R is semiperfect. Using [12, Theorem 5.52], we deduce that $r(J(R))$ is a noetherian right R-module, as required. □

Lemma 5.10. *Let R be a left CS ring with right RMC such that every principal right ideal is right annihilator. Then following conditions are equivalent:*

(1) *R is quasi-Frobenius;*
(2) *$Z(_RR) = Z(R_R)$.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) By Lemma 5.9, $r(J(R))$ is a noetherian right R-module. By hypothesis, $Z(_RR) = Z(R_R)$. Thus, as $Z(_RR) = J$ by [8, Lemma 2], then it follows that $Soc(R_R)$ is right finitely generated. Therefore, according to [5, Lemma 5.14], R has finite right uniform dimension. Using [10, Proposition 2.4(e)], we deduce that $Z(R_R)$ is right artinian. Hence, by hypothesis, R has ACC on left annihilators. Clearly, R is right minannihilator by [12, Lemma 5.1] (i.e every minimal right ideal of R is an annihilator). Therefore, R is quasi-Frobenius by [12, Theorem 4.22((1) ⇔ (2))]. □

Now, we are able to prove the following result which improve Theorem 3.13((1) ⇒ (2) in [10] and Proposition 18.6 in [5].

Theorem 5.11. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a left P-injective left IN-ring with right RMC and $J(R)$ is nil-ideal;*
(3) *R is a left P-injective left IN-ring with right RMC and $Z(R_R) = Z(_RR)$.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (3) Assume that R has the stated condition. By [10, Propositon 2.4(a)], $J(R)$ is nilpotent. It follows from [12, Proposition 5.10 and Theorem 6.32] and Lemma 5.8 that R is semiprimary. Since R is left P-injective, we infer from [12, Theorem 5.31] that $Z(R_R) = Z(_RR)$.

(3) ⇒ (1) As R is a left IN-ring, it is left CS by [12, Theorem 6.32]. It is clear that every principal right ideal is right annihilator (for, R is left P-injective). But by hypothesis, $Z(_RR) = Z(R_R)$. Therefore, according to Lemma 5.10, R is quasi-Frobenius. □

Corollary 5.12. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a two-sided P-injective left IN-ring with right RMC.*

Proposition 5.13. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a right P-injective right IN-ring with right RMC.*

Proof. (1) ⇒ (2) is clear.

(2) $\Rightarrow$ (1) By [12, Proposition 5.10 and Theorem 6.32], R is right continuous. Using [12, Proposition 18.14], we deduce that R is right artinian. Hence, R has ACC on right annihilators. Since R is left minannihilator, we infer from [12, Theorem 4.22((1) $\Leftrightarrow$ (2))] that R is quasi-Frobenius. □

Proposition 5.14. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left Kasch, every closed right ideal is a right annihilator and $Z_2(R_R)$ is an injective artinian right R-module.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [24, Theorem 10], R is semiperfect right continuous. Using [8, Lemma 2], we deduce that $J(R) \leq Z_2(R_R)$. Therefore, from the hypothesis, we can write $R = Z_2(R_R) \oplus K$, where K is a semisimple right ideal. It follows that R is quasi-Frobenius. □

Let (P) be a property of rings. A ring R is called completely P if each factor ring of R has the property (P).

Proposition 5.15. *A left perfect right completely simple-injective ring is quasi-Frobenius.*

Proof. Let $\overline{R}$ be a factor ring of R. By the proof (2) $\Rightarrow$ (1) of Theorem 5.1, $\overline{R}$ is right continuous and $rl(T) = T$ for all right ideals T of R. It follows that $\overline{R}$ has finite right uniform dimension. Hence, every cyclic right R-module is finitely cogenerated. Thus, R is right artinian by [12, Lemma 1.52]. But R is two-sided mininjective. Therefore, R is quasi-Frobenius by [12, Theorem 3.31]. □

Surjeet Singh and Yousef Al-Shaniafi (see [20, Theorem 1.10]) proved that: Let R be any commutative ring such that the injective envelope $E(R)$ of R is a projective R-module. Then $R = E(R)$, i.e., R is self-injective. From this, it is easy to see that for a commutative ring R satisfying ACC on annihilators such that the injective envelope $E(R)$ of R is a projective R-module then R is quasi-Frobenius. Now we will extend this result to the noncommutative case. A ring R is called *right duo* if every right ideal is an ideal.

For a subset X of a right R-module M over a ring R, we denote that $r_R(X)$ or $r(X)$ the right annihilator of X in R. Now let X and Y are two subset of a right R-module M, the subset $\{r \in R | Xr \subseteq Y\}$ of R is denoted by $[Y : X]$. Recall that if $Y \leq M_R$ then $[Y : X] \leq R_R$ and if $X \leq M_R$ and $Y \leq M_R$ then $[Y : X]$ is an ideal of R.

Let R be a right duo ring and P be a maximal ideal of R. Then it is easy to prove that $R \setminus P$ is multiplicatively closed and satisfies condition (S1): $\forall s \in R \setminus P$ and $r \in R$, there exist $t \in R \setminus P$ and $u \in R$ such that $su = rt$. Moreover, if R satisfies ACC on right annihilators then by [17, Proposition 1.5], $R \setminus P$ is a right denominator set. In this case, the ring $R(R \setminus P)^{-1}$ is called the *right localization with respect to* P and we write R_P and M_P instead of $R(R \setminus P)^{-1}$ and $M(R \setminus P)^{-1} = M \otimes_R R_P$, respectively. A ring R is called *right localizable* if for each maximal right ideal P of R, the right localization R_P exists. A ring R is said to be left quasi-duo if each of its maximal left ideals is an ideal of R. A ring R is called right QF-3^+ if the injective envelope $E(R_R)$ of R_R is a projective right R-module.

Theorem 5.16. *Let R be a right duo, right QF-3^+, left quasi-duo ring satisfying ACC on right annihilators. Then R is quasi-Frobenius.*

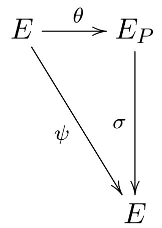
Proof. Now let P be a maximal ideal of R and $\theta : E \to E_P$ be the canonical map. Then the right localization R_P exists. Since E is projective, we have $E \oplus A = R^{(X)}$ with some A_R and index set X. We know that $E_P = E \otimes_R R_P$, so

$$\begin{aligned}(E \oplus A) \otimes_R R_P &= (E \otimes_R R_P) \oplus (A \otimes_R R_P) \\ &= R^{(X)} \otimes_R R_P \cong R_P^{(X)}\end{aligned}$$

Hence E_P is a projective right R_P-module.

Let $F = \{x \in E | [EP : x] \nsubseteq P\}$. With assumption $\theta(1) \in E_P P$ and by [19, Lemma 3.17], $[EP : 1] \nsubseteq P$. So $1 \in F$. Similarly, by [19, Lemma 3.17], $\theta(x) \in E_P P$ if and only if $[EP : x] \nsubseteq P$. So $F = \{x \in E | \theta(x) \in E_P P\}$. Because θ is an R-homomorphism, we can prove easily that F is a submodule of E.

Now we will prove that F is quasi-injective. Now since $E(F)$ is a direct summand of E, we can assume that we take any homomorphism $\psi : E \longrightarrow E$. There exists an R_P-homomorphism $\sigma : E_P \longrightarrow E$ such that $\sigma\theta = \psi$, i.e., the following diagram is commutative:



Now, let $t \in F$ then $t \in E$ and there exists $r \notin P$ such that $tr \in EP$. Moreover, $\theta(t) \in E_P P$. Hence there exists $p \in P, e_t \in E_p$ such that $\theta(t) = e_t p$. So $\psi(t) = (\sigma\theta)(tr) = \sigma(\theta(t))r = (\sigma\theta)(e_t p)r = (\sigma\theta)(e_t)pr \in EP$. It follows that $\psi(t) \in L$.

Since F is invariant under any homomorphism of E, F is quasi-injective. Now since $1 \in F$, there exist $r \in EP$ such that $r \notin P$. Let $e \in E$ then since $r \in (EP) \cap R$, $er \in E[(EP) \cap R] \leq EP$. So $e \in F$. Hence $E = F$. Hence $E_P \neq E_P P$. So there exists an $e \in E$ such that $\theta(e) \notin E_P P$. Since $E = L$, $e \in L$, so $[EP : e] \nsubseteq P$. Then there exists $v \notin P$ such that $ev \in EP$. Hence $\theta(e) \in EP$. Contradiction. Hence $\theta(1) \notin E_P P$. Since R_P is a local ring and E_P is a non-zero projective R_P-module, so it is free and then

$$E_P = \bigoplus_{i \in I} A_i, \quad A_i \cong R_P.$$

Now we prove that E/R is a flat right R-module. By [17, Exe. 39, p. 48] we need to prove that for every maximal left ideal P of R, $EP \neq E$. Note that P is an ideal and since $\theta(1) \notin E_P P$, $R \cap EP \leq P$. Assume that $EP = E$ then $x \in R \Rightarrow x \in E \Rightarrow x \in EP \Rightarrow x \in P$. So $R = P$. Contradiction. Since E is projective and by [12, Lemma 7.30], E is also finitely generated, so for some $n \in \mathbb{N}$, we obtain that $R^n \to E/R \to 0$ is exact and then by [17, Cor. 11.4, p.38], E/R is projective. Then $E = R$. And R is right self-injective. Then R is quasi-Frobenius.

Corollary 5.17. ([20, Theorem 1.10]) *Let R be any commutative, QF-3^+ ring satisfying ACC on annihilators. Then R is quasi-Frobenius.*

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