HILBERT COEFFICIENTS OF GOOD I-FILTRATIONS OF MODULES

LE XUAN DUNG
Faculty of Natural Sciences, Hong Duc University
No 565 Quang Trung, Dong Ve, Thanh Hoa, Vietnam
E-mail: lexuandung@hdu.edu.vn

Abstract. Let M be a finitely generated module of dimention d over a Noetherian local ring $(A, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. Let be a pair of good I-filtrations $\mathbb{F}$ and $\mathbb{F}'$ of M. We show that the Hilbert coefficients $e_i(\mathbb{F})$ are bounded below and above in terms of i, $e_0(\mathbb{F}'), ..., e_i(\mathbb{F}')$, and reduction numbers of $\mathbb{F}$ and $\mathbb{F}'$, for all $i \geq 1$.

1. Introduction

Let A be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and M be a finitely generated A-module of dimention d. Let I be an ideal of A; an I-filtration $\mathbb{F}$ of M is a collection of submodules F_n such that

$$M = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

with the property that $IF_n \subseteq F_{n+1}$ for all $n \geq 0$. In the present work we consider only good I-filtrations of M: this means that $IF_n = F_{n+1}$ for all sufficiently large n.

The Hilbert-Samuel function $H_{\mathbb{F}}(n) = \ell(M/F_{n+1})$ agrees with the Hilbert-Samuel polynomial $P_{\mathbb{F}}(n)$ for $n \gg 0$ and we may write

$$P_{\mathbb{F}}(n) = e_0(\mathbb{F})\binom{n+d}{d} - e_1(\mathbb{F})\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(\mathbb{F}).$$

The numbers $e_0(\mathbb{F}), e_1(\mathbb{F}), ..., e_d(\mathbb{F})$ are called the Hilbert coefficients of $\mathbb{F}$.

The notation of Hilbert function is central in communication algebra and is becoming increasingly importan in algebraic geometry and in computational algebra. Let be a good I-filtration $\mathbb{F}$ of M, the Hilbert-Samuel function and the Hilbert-Samuel polynomial of $\mathbb{F}$ give a lot of information on M. Therefore, it is of interest to examine properties of the Hilbert coefficients of $\mathbb{F}$, see ([5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 20]). For further applications, we need to consider another filtration related to I of M. Given a pair of good I-filtrations $\mathbb{F}$ and $\mathbb{F}'$ of M, we want to compare $\mathbb{F}$ with $\mathbb{F}'$. Atiyah-Macdnald ([1, Propsition 11.4]) and Brun-Hezog ([2, Proposition 4.6.5]) showed that $e_0(\mathbb{F}) = e_0(\mathbb{F}')$. In some special cases, Rossi-Vall in [15] gave alower bounds and upper bounds on $e_1(\mathbb{F})$ in terms of $e_0(\mathbb{F}')$, $e_1(\mathbb{F}')$, and other invarians of M. How about the other coefficients? The main goal of this paper is to show that $|e_i(\mathbb{F})|$ are bounded by a function depeding only i, $e_0(\mathbb{F}'), ..., e_i(\mathbb{F}')$, and reduction numbers of $\mathbb{F}$ and $\mathbb{F}'$, for all $i \geq 1$ (see Theorem 3.3). These bounds

2010 Mathematics Subject Classification: Primary 13D40, 13A30
Key words and phrases: good filtration, associated graded module, Hilbert coefficients, Castelnuovo-Mumford regularity.

are far from being sharp, but they have some interest because very little is known about relationships between $e_0(\mathbb{F}), ..., e_d(\mathbb{F})$ and $e_0(\mathbb{F}'), ..., e_d(\mathbb{F}')$.

Our paper is outlined as follows. In the next section, we collect notations and terminology used in the paper and start with a few preliminary results on bounding the length of local homology modules (see Lemma 2.5 and Lemma 2.6). In Section 3, we give new bounds on the Castelnuovo-Mumford regularity $\operatorname{reg}(G(\mathbb{F}))$ of $\mathbb{F}$ (see Theorem 3.2) and show that the Hilbert coefficients $e_i(\mathbb{F})$ are bounded below and above in terms of i, $e_0(\mathbb{F}'), ..., e_i(\mathbb{F}')$, and reduction numbers of $\mathbb{F}$ and $\mathbb{F}'$, for all $i \geq 1$ (see Theorem 3.3).

2. Hilbert coefficients and local cohomomology modules

In this section, we recall notations and terminology used in the paper, and a number of auxiliary results. Generally, we will follow standard texts in this research area (cf. [3, 4, 15]).

Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$ such that $R_0/\mathfrak{m}_0$ is an infinite field. Let E be a finitely generated graded R-module of dimension d. We denote the Hilbert function $\ell_{R_0}(E_t)$ and the Hilbert polynomial of E by $h_E(t)$ and $p_E(t)$, respectively. Writing $p_E(t)$ in the form:

$$p_E(t) = \sum_{i=0}^{d-1} (-1)^i e_i(E) \binom{t+d-1-i}{d-1-i},$$

we call the numbers $e_i(E)$ *Hilbert coefficients* of E.

Let $H^i_{R^+}(E)$, for $i \geq 0$, denote the i-th local cohomology module of E with respect to R^+. The *Castelnuovo-Mumford regularity of E* is defined by

$$\operatorname{reg}(E) := \max\{i + j \mid H^i_{R^+}(E)_j \neq 0, 0 \leq i \leq d\}$$

and the *Castelnuovo-Mumford regularity of E at and above level* 1 is defined by

$$\operatorname{reg}^1(E) := \max\{i + j \mid H^i_{R^+}(E)_j \neq 0, 0 < i \leq d\}.$$

From [19, Theorem 2], Dung-Hoa in [6] derived an explicit bound for $\operatorname{reg}^1(E)$ in terms of $e_i(E)$, $0 \leq i \leq d-1$ and the maximal generating degree of E.

$$\Delta'(E) = \max\{\Delta(E),\ 0\}.$$

Lemma 2.1. ([6, Lemma 1.2]) Let E be a finitely generated graded R-module of dimension $d \geq 1$. Put

$$\xi_{d-1}(E) = \max\{e_0(E), |e_1(E)|, ..., |e_{d-1}(E)|\}.$$

Then we have

$$\operatorname{reg}^1(E) \leq (\xi_{d-1}(E) + \Delta'(E) + 1)^{d!} - 2.$$

Our method in proving the main result is to pass to the associated grade modules, so we shall recall this notation and some more definitions.

Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field $K := A/\mathfrak{m}$ and M a finitely generated A-module. (Although the assumption K being infinite is not essential, because we can tensor A with $K(t)$.) Given a proper ideal I. A chain of submodules

$$\mathbb{F}:\ M = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

is called an *I-filtration* of M if $IF_i \subseteq F_{i+1}$ for all i, and a *good I-filtration* if $IF_i = F_{i+1}$ for all sufficiently large i. A module M with a filtration is called a *filtered module* (see [3, III 2.1]). If N is a submodule of M, then the sequence $\{F_n + N/N\}$ is a good I-filtration of M/N and will be denoted by $\mathbb{F}/N$.

Throughout the paper we always assume that I is an $\mathfrak{m}$-primary ideal and $\mathbb{F}$ is a good I-filtration. The *associated graded module* to the filtration $\mathbb{F}$ is defined by

$$G(\mathbb{F}) = \bigoplus_{n\geq 0} F_n/F_{n+1}.$$

We also say that $G(\mathbb{F})$ is the associated ring of the filtered module M. This is a finitely generated graded module over the standard graded ring $G := G(I, A) := \oplus_{n\geq 0} I^n/I^{n+1}$ (see [3, Proposition III 3.3]). In particular, when $\mathbb{F}$ is the I-adic filtration $\{I^nM\}$, $G(\mathbb{F})$ is just the usual associated graded module $G(I, M)$.

We call $H_{\mathbb{F}}(n) = \ell(M/F_{n+1})$ the Hilbert-Samuel function of M w.r.t $\mathbb{F}$. This function agrees with a polynomial - called the Hilbert-Samuel polynomial and denoted by $P_{\mathbb{F}}(n)$ - for $n \gg 0$. If we write

$$P_{\mathbb{F}}(t) = \sum_{i=0}^{d} (-1)^i e_i(\mathbb{F}) \binom{t+d-i}{d-i},$$

then the integers $e_i(\mathbb{F})$ are called *Hilbert coefficients* of $\mathbb{F}$ (see [15, Section 1]). When $\mathbb{F} = \{I^nM\}$, $H_{\mathbb{F}}(n)$ and $P_{\mathbb{F}}(n)$ are usually denoted by $H_{I,M}(n)$ and $P_{I,M}(n)$, respectively, and $e_i(\mathbb{F}) = e_i(I, M)$. Note that $e_i(\mathbb{F}) = e_i(G(\mathbb{F}))$ for $0 \leq i \leq d-1$. Then

Lemma 2.2. ([1, Proposition 11.4] and [2, Proposition 4.6.5]) *Let $\mathbb{F}$ and $\mathbb{F}'$ be good I-filtrations of M. Then we have*

$$e_0(G(\mathbb{F})) = e_0(\mathbb{F}) = e_0(\mathbb{F}').$$

We call

$$r(\mathbb{F}) = \min\{r \geq 0 \mid F_{n+1} = IF_n \ \text{ for all } \ n \geq r\}$$

the reduction number of $\mathbb{F}$ (w.r.t. I). When $\mathbb{F} = \{I^nM\}$, $r(\mathbb{F}) = 0$.

Denote the filtration $\mathbb{F}/H^0_{\mathfrak{m}}(M) = \overline{\mathbb{F}}$. Let

$$h^0(M) = \ell(H^0_{\mathfrak{m}}(M)).$$

The relationship between $\operatorname{reg}(G(\mathbb{F}))$ and $\operatorname{reg}(G(\overline{\mathbb{F}}))$ is given by the following lemma.

Lemma 2.3. ([5, Lemma 1.9]) $\operatorname{reg}(G(\mathbb{F})) \leq \max\{\operatorname{reg}(G(\overline{\mathbb{F}}));\ r(\mathbb{F})\} + h^0(M)$.

From now on, we will often use the following notation:

$$\xi_s(\mathbb{F}) = \max\{e_0(\mathbb{F}), |e_1(\mathbb{F})|, ..., |e_s(\mathbb{F})|\},$$

where $0 \leq s \leq d$. We see that

$$\xi_0(\mathbb{F}) \leq \xi_1(\mathbb{F}) \leq ... \leq \xi_d(\mathbb{F}) = \xi(\mathbb{F}). \tag{1}$$

Using the [15, Proposition 1.2 and Proposition 2.3] we get

Lemma 2.4. *Let $x_1, ..., x_d$ be an $\mathbb{F}$-superficial sequence for I and $\overline{M} = M/H^0_m(M)$. Set $M_i = M/(x_1, ..., x_i)M$ and $\mathbb{F}_i = \mathbb{F}/(x_1, ..., x_i)M$, where $F_0 = M$, $\mathbb{F}_0 = \mathbb{F}$, $0 \leq i \leq d-1$. Then we have*

i) $\xi_j(\overline{\mathbb{F}}) = \xi_j(\mathbb{F})$ *for all $j \leq d-1$,*

ii) $\xi_j(\overline{\mathbb{F}}/x_1\overline{M}) = \xi_j(\mathbb{F})$ *for all* $j \le d-1$,
iii) $\xi_j(\mathbb{F}_i) = \xi_j(\mathbb{F})$ *for all* $j \le d-i-1$.

Proof. i) By [15, Proposition 2.3], $e_i(\mathbb{F}) = e_i(\overline{\mathbb{F}})$, for all $0 \le i \le d-1$. Hence $\xi_j(\overline{\mathbb{F}}) = \xi_j(\mathbb{F})$ for all $j \le d-1$.
ii) We have $\operatorname{depth}(\overline{M}) > 0$, by [15, Proposition 1.2],

$$e_i(\overline{\mathbb{F}}/x_1\overline{M}) = e_i(\overline{\mathbb{F}}), \text{ for all } 0 \le i \le d-1.$$

Therefor

$$\xi_j(\overline{\mathbb{F}}/x_1\overline{M}) = \xi_j(\overline{\mathbb{F}}), \text{ for all } 0 \le j \le d-1.$$

By i), we get $\xi_j(\overline{\mathbb{F}}/x_1\overline{M}) = \xi_j(\mathbb{F})$ for all $j \le d-1$.
iii) By [15, Proposition 1.2], $\dim(M_{i-1}) = d-i+1$ and

$$e_k(\mathbb{F}_i) = e_k(\mathbb{F}_{i-1}/x_iM_{i-1}) = e_k(\mathbb{F}_{i-1}), \text{ for all } 0 \le k \le d-i-1.$$

Hence $e_k(\mathbb{F}_i) = e_k(\mathbb{F})$ for all $0 \le k \le d-i-1, 0 \le i \le d-1$. Therefor $\xi_j(\mathbb{F}_i) = \xi_j(\mathbb{F})$ for all $j \le d-i-1$. □

We can improve the bounds in [6, Lemma 1.10 and Lemma 1.11]. In the following results, we can replace $\operatorname{reg}(G(\mathbb{F}))$ by the Hilbert coefficents of $\mathbb{F}$.

Lemma 2.5. *Let* $\mathbb{F}$ *a good* I*-filtration of* M *and* $x_1, x_2, ..., x_d$ *be an* $\mathbb{F}$*-superficial sequence for* I*. Set* $M_i = M/(x_1, ..., x_i)M$ *and* $\mathbb{F}_i = \mathbb{F}/(x_1, ..., x_i)M$ *where* $F_0 = M$ *and* $\mathbb{F}_0 = \mathbb{F}$*. Then we have*

$$h^0(M_i) \le \sum_{k=0}^{i} \xi_{d-i+k}(\mathbb{F})(\xi_{d-i-1+k}(\mathbb{F}) + r(\mathbb{F}) + 1)^{(d-i+k).(d-i+k)!},$$

for all $0 \le i \le d-1$.

Proof. i) By [5, Lemma 1.8] and Lemma 2.1, we have

$$\operatorname{reg}(G(\overline{\mathbb{F}_i})) = \operatorname{reg}^1(G(\overline{\mathbb{F}_i})) \le (\xi_{d-i-1}(\overline{\mathbb{F}_i}) + r(\overline{\mathbb{F}_i}) + 1)^{(d-i)!} - 2.$$

From Lemma 2.4 i) and iii) we get $\xi_{d-i-1}(\overline{\mathbb{F}_i}) = \xi_{d-i-1}(\mathbb{F}_i) = \xi_{d-i-1}(\mathbb{F})$ and $r(\overline{\mathbb{F}_i}) \le r(\mathbb{F})$, therefore

$$\operatorname{reg}(G(\overline{\mathbb{F}_i})) \le (\xi_{d-i-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{(d-i)!} - 2 =: m_i.$$

For $i = 0$, by Lemma [6, Lemma 1.6], we have

$$\begin{aligned} h^0(F_0) = h^0(M) \le P_{\mathbb{F}}(m_0) &\le \xi_d(\mathbb{F}) \sum_{j=0}^{d} \binom{d+m_0-j}{d-j} \\ &= \xi_d(\mathbb{F}) \binom{m_0+d+1}{d} \le \xi_d(\mathbb{F})(m_0+2)^d = \xi_d(\mathbb{F})(\xi_{d-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{d.d!}. \end{aligned}$$

For $0 < i \le d-1$, by [15, Proposition 1.2], we have $e_j(\mathbb{F}_i) = e_j(\mathbb{F}_{i-1})$ for all $0 \le j \le d-i-1$. Similarly, as in the proof of [6, Lemma 1.10] and Lemma 2.4 iii) we have

$$|e_{d-i}(\mathbb{F}_i)| \le \xi_{d-i}(\mathbb{F}_{i-1}) + h^0(M_{i-1}) \le \xi_{d-i}(\mathbb{F}) + h^0(M_{i-1}).$$

It implies that

$$\begin{aligned}
h^0(M_i) &\le \xi_{d-i}(\mathbb{F})\binom{m_i+d-i+1}{d-i} - \xi_{d-i}(\mathbb{F}) + |e_{d-i}(\mathbb{F}_i)| \\
&\le \xi_{d-i}(\mathbb{F})(m_i+2)^{d-i} + h^0(M_{i-1}) \\
&\le \xi_{d-i}(\mathbb{F})(\xi_{d-i-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{(d-i)(d-i)!} + \\
&+ \textstyle\sum_{k=0}^{i-1} \xi_{d-i+1+k}(\mathbb{F})(\xi_{d-i+k}(\mathbb{F}) + r(\mathbb{F}) + 1)^{(d-i+1+k).(d-i+1+k)!} \quad \text{(by induction hypothesis)} \\
&= \textstyle\sum_{k=0}^{i} \xi_{d-i+k}(\mathbb{F})(\xi_{d-i+k-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{(d-i+k).(d-i+k)!}.
\end{aligned}$$

□

Lemma 2.6. *Set $B = \ell(M/(x_1, x_2, ..., x_d)M)$, where $x_1, x_2, ..., x_d$ be an $\mathbb{F}$-superficial sequence for I and put $\xi_{-1} = 0$. We have*

$$B \le \sum_{k=0}^{d} \xi_k(\mathbb{F})(\xi_{k-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{k.k!}.$$

Proof. Take the proof of the [6, Lemma 1.11]. We have

$$B \le e_0(\mathbb{F}) + h^0(M_{d-1}). \tag{2}$$

By Lemma 2.5, $h^0(M_{d-1}) \le \sum_{k=0}^{d-1} \xi_{1+k}(\mathbb{F})(\xi_k(\mathbb{F}) + r(\mathbb{F}) + 1)^{(1+k).(1+k)!}$. From this estimation we immediately get

$$\begin{aligned}
B &\le e_0(\mathbb{F}) + \sum_{k=0}^{d-1} \xi_{1+k}(\mathbb{F})(\xi_k(\mathbb{F}) + r(\mathbb{F}) + 1)^{(1+k)(1+k)!} \\
&= \xi_0(\mathbb{F}) + \sum_{k=1}^{d} \xi_k(\mathbb{F})(\xi_{k-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{k.k!} \\
&= \sum_{k=0}^{d} \xi_k(\mathbb{F})(\xi_{k-1}(\mathbb{F}) + r(\mathbb{F}) + 1)^{k.k!}.
\end{aligned}$$

□

3. Main results

Throughout this section, $\mathbb{F}$ and $\mathbb{F}'$ will be a pair of good I-filtrations of a finitely generated module M over a local ring $(A, \mathfrak{m})$, where I is an $\mathfrak{m}$-primary ideal. The aim of this section is to show that the Hilbert coefficients $e_i(\mathbb{F})$ are bounded below and above in terms of $e_0(\mathbb{F}'), ..., e_i(\mathbb{F}')$, i, $r(\mathbb{F})$, and $r(\mathbb{F}')$, for all $i \ge 1$.

In order to prove the main result of this paper, we need bound on the Castelnouvo-Mumford regularity $\operatorname{reg}(G(\mathbb{F}))$ of $\mathbb{F}$ in terms of d, $e_0(\mathbb{F}'), ..., e_d(\mathbb{F}')$, $r(\mathbb{F})$, and $r(\mathbb{F}')$.

Lemma 3.1. ([5, Proof of Theorem 1.5]) *Let $\dim M = d \ge 2$, x be an $\mathbb{F}$-superficial sequence for I. We have*

$$\operatorname{reg}^1(G(\overline{\mathbb{F}})/x^*G(\overline{\mathbb{F}})) = \operatorname{reg}^1(G(\overline{\mathbb{F}}/x\overline{M})).$$

Theorem 3.2. *Let $\mathbb{F}$ and $\mathbb{F}'$ be are good I-filtrations of M with $\dim(M) = d \ge 1$*

$$\mathbb{F}: \ M = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

$$\mathbb{F}': \ M = F'_0 \supseteq F'_1 \supseteq F'_2 \supseteq \cdots \supseteq F'_n \supseteq \cdots$$

Then

i) $\operatorname{reg}(G(\mathbb{F})) \le (\xi(\mathbb{F}') + r(\mathbb{F}') + 1)(\xi(\mathbb{F}') + r(\mathbb{F}) + 1) - 2$ *if* $d = 1$,

ii) $\operatorname{reg}(G(\mathbb{F})) \le (\xi(\mathbb{F}') + r(\mathbb{F}') + 1)^6(\xi(\mathbb{F}') + r(\mathbb{F}) + 1) - 3$ *if* $d = 2$,

iii) $\operatorname{reg}(G(\mathbb{F})) \leq (\xi(\mathbb{F}')+r(\mathbb{F}')+1)^{(d-1)(d+1)!-d}(\xi(\mathbb{F}')+r(\mathbb{F})+1)^{(d-1)!}-d$ *if* $d\geq 3$.

Proof. Let $\xi := \xi(\mathbb{F}')$, $r := r(\mathbb{F})$ and $r' := r(\mathbb{F}')$. We distinguish two cases
If $d=1$, then $\overline{M}$ is a Cohen-Macaulay module. By [5, Lemma 1.8], [10, Lemma 2.2], Lemma 2.2, $r(\overline{\mathbb{F}}) \leq r$ and (1)

$$\operatorname{reg}(G(\overline{\mathbb{F}})) \leq e_0(G(\overline{\mathbb{F}}))+r(\overline{\mathbb{F}})-1 \leq e_0(\mathbb{F}')+r-1 \leq \xi+r-1.$$

Hence, by Lemma 2.3 and applying Lemma 2.5 to $\mathbb{F}'$, we then obtain

$$\begin{aligned}\operatorname{reg}(G(\mathbb{F})) &\leq \max\{\operatorname{reg}(G(\overline{\mathbb{F}}));\ r\}+h^0(M)\\ &\leq \xi+r-1+\xi(\xi+r'+1)\\ &\leq \xi+r-1+\xi(\xi+r')+(\xi+r')\\ &= (\xi+r)+(\xi+1)(\xi+r')-1\\ &\leq (\xi+r+1)+(\xi+r+1)(\xi+r')-2\\ &\leq (\xi+r'+1)(\xi+r+1)-2.\end{aligned}$$

If $d\geq 2$, let $x_1, x_2, ..., x_d$ be an $\mathbb{F}$-superficial sequence and $\mathbb{F}'$-superficial sequence for I. Put $\overline{\mathbb{F}} = \mathbb{F}/H^0_m(M)$ and $\overline{\mathbb{F}'} = \mathbb{F}'/H^0_m(M)$. We have $\overline{\mathbb{F}}/x_1\overline{M}$ and $\overline{\mathbb{F}'}/x_1\overline{M}$ be are good I-filtrations of $\overline{M}/x_1\overline{M}$. Let $m \geq \max\{\operatorname{reg}(G(\overline{\mathbb{F}}/x_1\overline{M})), r\}$, by Lemma 3.1, we have

$$\operatorname{reg}^1(G(\overline{\mathbb{F}})/x_1^*G(\overline{\mathbb{F}})) = \operatorname{reg}^1(G(\overline{\mathbb{F}}/x_1\overline{M})) \leq m.$$

Hence, by [10, Theorem 2.7],

$$\operatorname{reg}^1(G(\overline{\mathbb{F}})) \leq m + P_{G(\overline{\mathbb{F}})}(m).$$

Since [5, Lemma 1.6] and [5, Lemma 1.7 (i)]

$$\begin{aligned}P_{G(\overline{\mathbb{F}})}(m) &\leq H_{I,\overline{M}/x_1\overline{M}}(m)\\ &\leq \tbinom{m+d-1}{d-1}\ell\left(\left(\overline{M}/x_1\overline{M}\right)/(x_2,...,x_n)\left(\overline{M}/x_1\overline{M}\right)\right) \leq B\tbinom{m+d-1}{d-1}.\end{aligned}$$

Therefor, by Lemma 2.3, we get

$$\operatorname{reg}(G(\mathbb{F})) \leq m + h^0(M) + B\binom{m+d-1}{d-1}. \tag{3}$$

If $d=2$. Let $m = (\xi+r'+1)(\xi+r+1)-2$. Since (i) of the theorem, $r(\overline{\mathbb{F}'}/x_1\overline{\mathbb{F}'}) \leq r'$, $r(\overline{\mathbb{F}}/x_1\overline{\mathbb{F}}) \leq r$ and by Lemma 2.4 ii), we get

$$\begin{aligned}\operatorname{reg}(G(\overline{\mathbb{F}}/x_1\overline{M})) &\leq (\xi_1(\overline{\mathbb{F}'}/x_1\overline{M})+r(\overline{\mathbb{F}'}/x_1\overline{M})+1)(\xi_1(\overline{\mathbb{F}'}/x_1\overline{M})+r(\overline{\mathbb{F}}/x_1\overline{M})+1)-2.\\ &= (\xi_1(\mathbb{F}')+r(\overline{\mathbb{F}'}/x_1\overline{M})+1)(\xi_1(\mathbb{F}')+r(\overline{\mathbb{F}}/x_1\overline{M})+1)-2.\\ &\leq (\xi+r'+1)(\xi+r+1)-2 = m.\end{aligned}$$

Hence, $\max\{\operatorname{reg}(G(\overline{\mathbb{F}}/x_1\overline{M})), r\} \leq m$. From (1), (3), and applying Lemma 2.5, Lemma 2.6 to $\mathbb{F}'$, we get

$$\begin{aligned}\operatorname{reg}(G(\mathbb{F})) &\leq m+h_0(M)+B(m+1)\\ &\leq (\xi+r'+1)(\xi+r+1)-1+\xi(\xi+r'+1)^4+\\ &+[\xi+\xi(\xi+r'+1)+\xi(\xi+r'+1)^4](\xi+r'+1)(\xi+r+1)\\ &\leq (\xi+r'+1)(\xi+r+1)+\xi(\xi+r'+1)^3(\xi+r'+1)(\xi+r+1)+\\ &+[\xi+\xi(\xi+r'+1)^2+\xi(\xi+r'+1)^4](\xi+r'+1)(\xi+r+1)-3\\ &\leq [1+\xi+\xi(\xi+r'+1)^2+\xi(\xi+r'+1)^3+\xi(\xi+r'+1)^4](\xi+r'+1)(\xi+r+1)-3\\ &\leq (\xi+r'+1)^5(\xi+r'+1)(\xi+r+1)-3\\ &= (\xi+r'+1)^6(\xi+r+1)-3.\end{aligned}$$

If $d\geq 3$. The case $m=0$ is trivial. Then for all $m>0$.

By the induction hypothessis, $r(\overline{\mathbb{F}}/x_1\overline{M}) \le r$, $r(\overline{\mathbb{F}'}/x_1\overline{M}) \le r'$ and by Lemma 2.4 ii), we have

$$\begin{aligned}
\operatorname{reg}(G(\overline{\mathbb{F}}/x\overline{M})) &\le (\xi_{d-1}(\overline{\mathbb{F}'}/x_1\overline{M}) + r(\overline{\mathbb{F}'}/x_1\overline{M}) + 1)^{(d-2)d!-d}(\xi_{d-1}(\overline{\mathbb{F}'}/x_1\overline{M}) + r(\overline{\mathbb{F}}/x_1\overline{M}) + 1)^{(d-2)!} - d + 1 \\
&= (\xi_{d-1}(\mathbb{F}') + r(\overline{\mathbb{F}'}/x_1\overline{M}) + 1)^{(d-2)d!-d}(\xi_{d-1}(\mathbb{F}') + r(\overline{\mathbb{F}}/x_1\overline{M}) + 1)^{(d-2)!} - d + 1 \\
&\le (\xi + r' + 1)^{(d-2)d!-d}(\xi + r + 1)^{(d-2)!} - d + 1.
\end{aligned}$$

We can take

$$m = (\xi + r' + 1)^{(d-2)d!-d}(\xi + r + 1)^{(d-2)!} - d + 1.$$

From (3) and applying Lemma 2.5, Lemma 2.6 to $\mathbb{F}'$, we get

$$\begin{aligned}
\operatorname{reg}(G(\mathbb{F})) &\le m + \xi_d(\mathbb{F}')(\xi_{d-1}(\mathbb{F}') + r' + 1)^{d.d!} + \sum_{k=0}^{d} \xi_k(\mathbb{F}')(\xi_{k-1}(\mathbb{F}') + r' + 1)^{k.k!}\binom{m+d-1}{d-1} \\
&< \sum_{k=0}^{d} \xi_k(\mathbb{F}')(\xi_{k-1}(\mathbb{F}') + r' + 1)^{k.k!}(m+1)^{d-1} - 1 \\
&\le (\xi_d(\mathbb{F}') + r' + 1)^{d.d!+1}(m+1)^{d-1} - 1 \\
&\le (\xi_d(\mathbb{F}') + r' + 1)^{d.d!+1}\left[(\xi + r' + 1)^{(d-2)d!-d}(\xi + r + 1)^{(d-2)!} - d\right]^{d-1} - 1 \\
&\le (\xi + r' + 1)^{(d-1)(d+1)!-d}(\xi + r + 1)^{(d-1)!} - d.
\end{aligned}$$

□

Now we are going to prove the main result of this paper.

Theorem 3.3. *Let $\mathbb{F}$ and $\mathbb{F}'$ be good I-filtrations of M with* $\dim(M) = d \ge 1$

$$\mathbb{F}: \; M = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

$$\mathbb{F}': \; M = F'_0 \supseteq F'_1 \supseteq F'_2 \supseteq \cdots \supseteq F'_n \supseteq \cdots$$

Then

i) $|e_1(\mathbb{F})| \le \xi_1(\mathbb{F}')(\xi_1(\mathbb{F}') + r(\mathbb{F}') + 1)^2(\xi_1(\mathbb{F}') + r(\mathbb{F}) + 1)$;

ii) $|e_2(\mathbb{F})| \le \xi_2(\mathbb{F}')(\xi_2(\mathbb{F}') + r(\mathbb{F}') + 1)^{17}(\xi_2(\mathbb{F}') + r(\mathbb{F}) + 1)^2$;

iii) $|e_i(\mathbb{F})| \le \xi_i(\mathbb{F}')(\xi_i(\mathbb{F}') + r(\mathbb{F}') + 1)^{(i^2-1)i!-i+1}(\xi_i(\mathbb{F}') + r(\mathbb{F}) + 1)^{i!}$ *if* $i \ge 3$.

Proof. i) By [5, (8)] we have

$$\ell(M/F_{m+1}) = \sum_{i=0}^{d}(-1)^i e_i(\mathbb{F})\binom{m+d-i}{d-i} \tag{4}$$

for any $m \ge \operatorname{reg}(G(\mathbb{F}))$. For short we write $\xi_i := \xi_i(\mathbb{F}'), r := r(\mathbb{F})$, and $r' := r(\mathbb{F}')$.

Assume that $d = 1$. Putting $m := (\xi_1 + r' + 1)(\xi_1 + r + 1) - 1$, by Theorem 3.2 i) and (4), we have

$$e_1(\mathbb{F}) = (m+1)e_0(\mathbb{F}) - \ell(M/F_{m+1}) \tag{5}$$

Since $F_n = I^{n-r}F_r$ for $n \ge r$ and $M_r \ne 0$

$$\begin{aligned}
\ell(M/F_{m+1}) &= \ell(M/F_{(m+1-r)-r}) \\
&\ge \ell(F_{r(\mathbb{F})}/IF_r) + \cdots + \ell(I^{m-r}F_r/I^{m-r+1}F_r) \ge m - r + 1.
\end{aligned}$$

By (5) and Lemma 2.2, this implies

$$\begin{aligned}
e_1 &\le (\xi_1 + r' + 1)(\xi_1 + r + 1)\xi_0 - [(\xi_1 + r' + 1)(\xi_1 + r + 1) - r + 1] \\
&\le \xi_1(\xi_1 + r' + 1)^2(\xi_1 + r + 1).
\end{aligned}$$

By [5, Lemma 1.7 i)], Lemma 2.2 and Lemma 2.6

$$\begin{aligned} -e_1(\mathbb{F}) &\le B(m+1) - (m+1)e_0(\mathbb{F}) = (B-\xi_0)(m+1) \\ &\le \xi_1(\xi_1+r')(\xi_1+r'+1)(\xi_1+r+1) = \xi_1(\xi_1+r'+1)^2(\xi_1+r+1). \end{aligned}$$

Hence

$$|e_1(\mathbb{F})| \le \xi_1(\xi_1+r'+1)^2(\xi_1+r+1).$$

Assume that $d \ge 2$. Let $x_1,...,x_d$ be $\mathbb{F}$-superficial sequence and $\mathbb{F}'$-superficial sequence for M and I. Put $\overline{\mathbb{F}} = \mathbb{F}/H^0_m(M)$ and $\overline{\mathbb{F}'} = \mathbb{F}'/H^0_m(M)$. We have $\mathbb{F}_i = \overline{\mathbb{F}}/(x_1,...,x_i)\overline{M}$, $\mathbb{F}'_i = \overline{\mathbb{F}'}/(x_1,...,x_i)\overline{M}$ be are good I-filtrations of $M_i = \overline{M}/(x_1,...,x_i)\overline{M}$, and $\dim M_i = i$. By [15, Proposition 1.2 and Prosition 2.3], we get

$$e_i(\mathbb{F}) = e_i(\mathbb{F}_{d-i}) \text{ for all } i \le d-1. \tag{6}$$

By Theorem 3.2, $\operatorname{reg}(G(\mathbb{F})) \le m$, (4), [5, Lemma 1.7 ii)] and [2, Corollary 4.7.11 a)], we have

$$\begin{aligned} |e_d(\mathbb{F})| &= \left|\ell(M/F_{m+1}) - e_0(\mathbb{F})\binom{m+2}{2} + ... + e_{d-1}(\mathbb{F})(m+1)\right| \\ &\le \max\left\{B\binom{m+d}{d}, e_0(\mathbb{F})\binom{m+d}{d}\right\} + \sum_{i=1}^{d-1}|e_i(\mathbb{F})|\binom{m+d-i}{d-i} \\ &\le B(m+d)^d + \sum_{i=1}^{d-1}|e_i(\mathbb{F})|\binom{m+d-i}{d-i} \end{aligned} \tag{7}$$

If $d=2$, by (6), $e_1(\mathbb{F}) = e_1(\mathbb{F}_1)$. Using the induction hypothessis, $r(\mathbb{F}_1) \le r$, $r(\mathbb{F}'_1) \le r'$ and by Lemma 2.4 ii), we have

$$\begin{aligned} |e_1(\mathbb{F})| &\le \xi_1(\mathbb{F}'_1)(\xi(\mathbb{F}'_1) + r(\mathbb{F}'_1)+1)^2(\xi(\mathbb{F}'_1) + r(\mathbb{F}_1)+1) \\ &= \xi_1(\xi_1+r'+1)^2(\xi_1+r+1). \end{aligned}$$

By Lemma 2.6, Theorem 3.2 ii) and putting $m = (\xi+r'+1)^6(\xi+r+1)-2$ into (7), we have

$$\begin{aligned} |e_2(\mathbb{F})| &\le B(m+2)^2 + |e_1(\mathbb{F})|(m+1) \\ &\le B\left[(\xi+r'+1)^6(\xi+r+1)\right]^2 + \xi(\xi+r'+1)^2(\xi+r+1)(\xi+r'+1)^6(\xi+r+1) \\ &\le [\xi+\xi(\xi+r'+1)^2+\xi(\xi+r'+1)^4]\left[(\xi+r'+1)^6(\xi+r+1)\right]^2 + \\ &+\xi(\xi+r'+1)^2(\xi+r+1)(\xi+r'+1)^6(\xi+r+1) \\ &\le \xi[1+(\xi+r'+1)^2+(\xi+r'+1)^4+1]\left[(\xi+r'+1)^6(\xi+r+1)\right]^2 \\ &\le (\xi+r'+1)^5(\xi+r'+1)^{12}(\xi+r+1)^2 \\ &\le (\xi+r'+1)^{17}(\xi+r+1)^2. \end{aligned}$$

iv) Assume that $d \ge 3$. Putting Using the induction hypothessis, $r(\mathbb{F}_i) \le r$, $r(\mathbb{F}'_i) \le r'$, for all $1 \le i \le d-1$ and by Lemma 2.4 ii), we have

$$\begin{aligned} |e_1(\mathbb{F})| &= |e_1(\mathbb{F}_{d-1})| \le (\xi_1(\mathbb{F}'_{d-1}) + r(\mathbb{F}'_{d-1})+1)^2(\xi_1(\mathbb{F}'_{d-1}) + r(\mathbb{F}_{d-1})+1) \\ &\le (\xi_1+r'+1)^2(\xi_1+r+1). \quad (8) \\ |e_2(\mathbb{F})| &= |e_2(\mathbb{F}_{d-2})| \le \xi_2(\mathbb{F}'_{d-2})(\xi_2(\mathbb{F}'_{d-2}) + r(\mathbb{F}'_{d-2})+1)^{17}(\xi_2(\mathbb{F}'_{d-2}) + r(\mathbb{F}_{d-2})+1)^2 \\ &\le \xi_2(\xi_2+r'+1)^{17}(\xi_2+r+1)^2. \quad (9) \\ |e_i(\mathbb{F})| &= |e_i(\mathbb{F}_{d-i})| \le (\xi_i(\mathbb{F}'_{d-i}) + r(\mathbb{F}'_{d-i})+1)^{(i^2-1)i!-i+1}(\xi_i(\mathbb{F}'_{d-i}) + r(\mathbb{F}_{d-i})+1)^{i!} \\ &\le (\xi_i+r'+1)^{(i^2-1)i!-i+1}(\xi_i+r+1)^{i!} \text{ if } 3 \le i \le d-1. \quad (10) \end{aligned}$$

To prove the inequallity for $e_d(\mathbb{F})$, we set

$$m = (\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!} - d.$$

By (7), Theorem 3.2, $\operatorname{reg}(G(\mathbb{F})) \leq m$ and (4), we have

$$\begin{aligned} |e_d(\mathbb{F})| &\leq B(m+d)^d + |e_1(\mathbb{F})|(m+d-1)^{d-1} + \sum_{i=2}^{d-1} |e_i(\mathbb{F})|(m+d-i)^{d-i} \\ &\leq Bm^d + |e_1(\mathbb{F})|m^{d-1} + \sum_{i=2}^{d-1} |e_i(\mathbb{F})|m^{d-i} \\ &= \left(B + \frac{|e_1(\mathbb{F})|}{m} + \frac{|e_2(\mathbb{F})|}{m^2} + \sum_{i=3}^{d-1} \frac{|e_i(\mathbb{F})|}{m^i}\right) m^d \end{aligned} \tag{11}$$

By (8)-(10), we get

$$\begin{aligned} \frac{|e_1(\mathbb{F})|}{m} &\leq \frac{\xi_1(\mathbb{F}')(\xi_1(\mathbb{F}')+r'+1)^2(\xi_1(\mathbb{F}')+r+1)}{(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r(\mathbb{F})+1)^{(d-1)!}} \\ &\leq \frac{\xi(\xi+r'+1)^2(\xi+r+1)}{(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}} \leq \frac{\xi}{2}. \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{|e_2(\mathbb{F})|}{m^2} &\leq \frac{\xi_2(\mathbb{F}')(\xi_2(\mathbb{F}')+r'+1)^{17}(\xi_2(\mathbb{F}')+r+1)^2}{\left[(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}\right]^2} \\ &\leq \frac{\xi(\xi+r'+1)^{17}(\xi+r+1)^2}{\left[(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}\right]^2} < \frac{\xi}{2^2}. \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{|e_i(\mathbb{F})|}{m^i} &\leq \frac{\xi_i(\mathbb{F}')(\xi_i+r'+1)^{(i^2-1)i!-i+1}(\xi_i(\mathbb{F}')+r+1)^{i!}}{\left[(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}\right]^i} \\ &\leq \frac{\xi(\xi+r'+1)^{(i^2-1)i!-i+1}(\xi+r+1)^{i!}}{\left[(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}\right]^i} \leq \frac{\xi}{2^i} \end{aligned}$$

if $3 \leq i \leq d-1$. (14)

From (11)-(14) and Lemma 2.6, we obtain

$$\begin{aligned} |e_d(\mathbb{F})| &\leq \left[B + \xi\left(\frac{1}{2} + \dots + \frac{1}{2^{d-1}}\right)\right] m^d < (B+\xi)m^d \\ &\leq \xi\left[1 + (\xi+r'+1) + (\xi+r'+1)^{2.2!} + .. + (\xi+r'+1)^{d.d!} + 1\right] \times \\ &\times \left[(\xi+r'+1)^{(d-2)(d+1)!-d}(\xi+r+1)^{(d-1)!}\right]^d \\ &\leq \xi(\xi+r'+1)^{d.d!+1}(\xi+r'+1)^{d(d-2)(d+1)!-d^2}(\xi+r+1)^{d!} \\ &\leq (\xi+r'+1)^{d(d-2)(d+1)!-d^2+d.d!+1}(\xi+r+1)^{d!} \\ &\leq \xi(\xi+r'+1)^{(d^2-1)d!-d+1}(\xi+r+1)^{d!}. \end{aligned}$$

□

we immediately obtain the following consequence

Corollary 3.4. *Let $\mathbb{F}$ be a good I-filtration of M with $\dim(M) = d \geq 1$. Then*
i) $|e_1(\mathbb{F})| \leq \xi_1(I,M)(\xi_1(I,M)+1)^2(\xi_1(I,M)+r(\mathbb{F})+1)$;
ii) $|e_2(\mathbb{F})| \leq \xi_2(I,M)(\xi_2(I,M)+1)^{17}(\xi_2(I,M)+r(\mathbb{F})+1)^2$;
iii) $|e_i(\mathbb{F})| \leq \xi_i(I,M)(\xi_i(I,M)+1)^{(i^2-1)i!-i+1}(\xi_i(I,M)+r(\mathbb{F})+1)^{i!}$ *if* $i \geq 3$.

Proof. The reduction number of the I-adic filtration $\{I^nM\}$ is 0. Therefore, applying Theorem 3.3 to $\mathbb{F}' = \{I^nM\}$, we then obtain. □

Let $x_1, ..., x_d$ be an $\mathbb{F}$-superficial sequence for I and $Q := (x_1, ..., x_d)$. It is not difficult to prove that also the $\mathbb{F}$ is a good Q-filtration of M. Rossi-Valla in [15] gave the following fitration

$$\mathbb{E}: \ M = F_0 \supseteq F_1 \supseteq QF_1 \supseteq Q^2F_1 \supseteq \cdots \supseteq Q^nF_1 \supseteq \cdots .$$

This filtration is a good Q-filtration of M. As in consequence of the Theorem 3.3 we have a relationship between $\mathbb{E}$ and $\{Q^nM\}$ as follows:

Corollary 3.5. *Let $x_1, ..., x_d$ be an $\mathbb{F}$-superficial sequence for I and $Q := (x_1, ..., x_d)$. Then*

i) $|e_1(\mathbb{E})| \leq \xi_1(Q,M)(\xi_1(Q,M)+1)^2(\xi_1(Q,M)+2)$;

ii) $|e_2(\mathbb{E})| \leq \xi_2(Q,M)(\xi_2(Q,M)+1)^{17}(\xi_2(Q,M)+2)^2$;

iii) $|e_i(\mathbb{E})| \leq \xi_i(Q,M)(\xi_i(Q,M)+1)^{(i^2-1)i!-i+1}(\xi_i(Q,M)+2)^{i!}$ *if* $i \geq 3$.

Proof. The reduction number of the good Q-filtration $\mathbb{E}$ is 1 and the reduction number of Q-adic filtration $\{Q^nM\}$ is 0. Therefore, applying Theorem 3.3 to $\mathbb{F} = \mathbb{E}$ and $\mathbb{F}' = \{Q^nM\}$, we then obtain. □

Remark 3.6. Let p be an integer such that $IM \subseteq \mathfrak{m}^pM$. Rossi-Valla in [15, Proposition 2.10 and Proposition 2.11] gave a sharp upper bounds for $e_1(\mathbb{F})$ in terms of $e_0(Q,M)$, $e_1(Q,M)$, and p and a sharp lower bounds for $e_1(\mathbb{E})$ in terms of $e_0(Q,M)$, $e_1(Q,M)$ and other invarians of M, respectively. The bounds of Corollary 3.4 and Corollary 3.5 are far from being sharp, but they show that the Hilbert coefficients $e_i(\mathbb{F})$ and $e_i(\mathbb{E})$ are bounded below and above in terms of $e_0(Q,M), ..., e_i(Q,M)$, i, and $r(\mathbb{F})$ (only for $e_i(\mathbb{F})$), for all $i \geq 1$.

Acknowledgment: The paper was completed during the stay of the author at the Vietnam Institute for Advanced Study in Mathematics.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley, 1969).

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings* (Cambridge University, 1993).

[3] N. Bourbaki, *Algebèbre commutative* (Hermann, Paris, 1961 - 1965).

[4] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*. Cambridge Studies in Advanced Mathematics, 136 (Cambridge University, 2012).

[5] L. X. Dung and L. T. Hoa, Castelnuovo-Mumford regularity of associated graded modules and fiber cones of filtered modules, *Comm. Algebra* **40** (2012), 404 - 422.

[6] L. X. Dung and L. T. Hoa, Dependence of Hilbert coefficients, *Manuscripta math.* **149** (2016), 235 - 249 . Corrigendum, ArXiv 1706.08669.

[7] S. Goto and K. Ozeki, Uniform bounds for Hilbert coefficients of parameters, in "Commutative algebra and its connections to geometry", pp. 97118, *Contemp. Math.*, **555**, Amer. Math. Soc., Providence, RI, 2011.

[8] K. Hanumanthu and C. Huneke, Bounding the first Hilbert coefficient, *Proc. Amer. Math. Soc.* **140** (2012), 109-117

[9] D. Kirby and H. A. Mehran, A note on the coefficients of the Hilbert-Samuel polynomial for a Cohen-Macaulay module, *J. London Math. Soc.* (2) **25** (1982), 449 - 457.

[10] C. H. Linh, Upper bound for Castelnuovo-Mumford regularity of associated graded modules, *Comm. Algebra.* **33** (2005), 1817 - 1831.

[11] C. H. Linh, Castelnuovo-Mumford regularity and degree of nilpotency, *Math. Proc. Cambridge Philos. Soc.* **142** (2007), 429 - 437.

[12] M. Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, *Proc. Cambridge Philos. Soc.* **59** (1963), 269 - 275.

[13] N. G. Northcott, A note on the coefficients of the abstract Hilbert function, *J. London Math. Soc.* (1), **35** (1960), 209 - 214.

[14] C. P. L. Rhodes, The Hilbert-Samuel polynomial in a filtered module, *J. London Math. Soc.* (1) **3** (1971), 73 - 85.

[15] M. E. Rossi and G. Valla, *Hilbert functions of filtered modules* Lecture Notes of the Unione Matematica Italiana, **9**. (Springer, Heidelberg, 2010)

[16] M. E. Rossi, N. V. Trung and G. Valla, Castelnuovo-Mumford regularity and extended degree, *Trans. Amer. Math. Soc.* **355** (2003), 1773 - 1786.

[17] V. Srinivas and V. Trivedi, A finiteness theorem for the Hilbert functions of complete intersection local rings, *Math. Z.* **225** (1997), 543 - 558.

[18] V. Srinivas and V. Trivedi, On the Hilbert function of a Cohen-Macaulay local ring, *J. Algebraic Geom.* **6** (1997), 733 - 751.

[19] V. Trivedi, Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers, *Manuscripta Math.* **94** (1997), 485 - 499.

[20] V. Trivedi, Finiteness of Hilbert functions for generalized Cohen-Macaulay modules. *Comm. Algebra* **29** (2001), 805 - 813.