Discretized sum-product type problems: Energy variants and Applications

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Abstract

In this paper we establish non-trivial estimates for the additive discretized energy of

$$\sum_{c\in C}|\{(a_1,a_2,b_1,b_2)\in A^2\times B^2:|(a_1+cb_1)-(a_2+cb_2)|\le\delta\}|_\delta,$$

that depend on the non-concentration conditions of the sets. Our proofs introduce a number of novel approaches which make use of a combination of methods from both continuous and discrete settings including a pivoting argument, which has been used in the finite field setting due to Murphy and Petridis, the recent Guth-Katz-Zahl's method for the discretized sum-product problem and a Dabrowski-Orponen-Villa point-tube incidence bound. As applications, we obtain a number of improvements on the size of the δ-covering of sets $A+cB$ and $C(A+A)$. Furthermore, for compact sets $A,B\subset\mathbb{R}$, we also prove new explicit upper bounds on the quantity $\dim_H\{c\in\mathbb{R}:\ \dim_H(A+cB)\le\alpha+\epsilon\}$. Our approach leads to considerably shorter proofs over the previous works due to Bourgain and Orponen.

Contents

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1 Introduction

Let $\delta, \sigma \in (0,1)$. A set $A \subset \mathbb{R}$ is called δ-discretized if it is a union of closed intervals of length δ. A δ-discretized set A is called a (δ,σ)-set if $|A| \approx \delta^{1-\sigma}$ and it satisfies the non-concentration condition $|A \cap I| \lesssim |I|^{\sigma}|A|$ for all intervals I. Roughly speaking, we can consider a (δ,σ)-set a discrete analogue of a set of Hausdorff dimension σ.

Bourgain [4] proved that any (δ,σ)-set cannot be approximately closed under both addition and multiplication. More precisely, if $A \subset [1,2]$ is a (δ,σ)-set, then there exists $\epsilon = \epsilon(\sigma) > 0$ such that

$$\max\{|A+A|, |A \cdot A|\} \gtrsim \delta^{-\epsilon}|A|. \tag{1}$$

Here by $|X|$, we mean the Lebesgue measure of X. This result settles a conjecture of Katz and Tao in [21] for $\sigma = 1/2$. The recent work of Guth, Katz, and Zahl [15] provides a new proof of (1) with explicit exponent ϵ, namely, the estimate (1) holds for any $0 < \epsilon < \frac{\sigma(1-\sigma)}{4(7+3\sigma)}$.

Such results have found many applications in the literature on various topics in geometric measure theory and related areas including Borel rings in real line, distance sets, orthogonal and radial projections, Besicovitch and Furstenberg sets, and spectral gaps. We refer the reader to [2, 3, 4, 5, 6, 7, 10, 11, 12, 16, 21, 22, 24, 34, 35, 36] and references therein for more details. A number of generalizations with applications in different settings can also be found in [1, 8, 17, 18, 19, 23, 37].

Following this trend, in this paper, we explore more deeper properties of discretized sum-product type problems with an emphasis on energy variants and applications. Our work is motivated by earlier results on the $A + cB$ problem due to Bourgain in [4] and Orponen in [30, 29, 31]. Let us first start with the following theorem of Bourgain in [4]. Recall that a set $A \subset \mathbb{R}$ is called δ-separated if every two elements in A have distance greater than δ.

Theorem 1.1 (**Bourgain**, 2010)**.** *Given $\alpha \in (0,1)$ and $\gamma, \eta > 0$, there exist $\epsilon_0, \epsilon > 0$ such that the following holds for all sufficiently small $\delta > 0$.*

Let ν be a probability measure on $[0,1]$ satisfying $\nu(B(x,r)) \le r^{\gamma}$ for all $x \in \mathbb{R}$ and $\delta < r \le \delta^{\epsilon_0}$. Let additionally $A \subset [0,1]$ be a δ-separated set with $|A| = \delta^{-\alpha}$, which also satisfies the non-concentration condition $|A \cap B(x,r)| \le r^{\eta}|A|$ for $x \in \mathbb{R}$ and $\delta \le r \le \delta^{\epsilon_0}$.

Then, there exists a point $c \in spt(\nu)$ such that

$$|A + cA|_{\delta} \ge \delta^{-\epsilon}|A|.$$

Here $|\cdot|_{\delta}$ refers to the δ-covering number of A, namely the size of the smallest covering of A by intervals of length δ.

In the above theorem, $c \in \mathrm{spt}(\nu)$. To see the ABC version, i.e. assuming C is a δ-separated set satisfying $|C \cap B(x,r)| \le r^{\gamma}|C|$ for all $x \in \mathbb{R}$ and $\delta \le r \le \delta^{\epsilon_0}$, we choose the uniformly distributed probability measure ν on the δ-neighbourhood of C such that $\nu(B(x,r)) \lesssim r^{\gamma}$, namely, $\nu(X) = \frac{|X \cap C(\delta)|}{|C|_{\delta}}$. The theorem above tells us that there exists $c \in \mathrm{spt}(\nu)$ such that

$$|A + cA|_{\delta} \ge \delta^{-\epsilon}|A|.$$

This implies that there exists $c \in C$ with $|A + cA|_{\delta} \ge \delta^{-\epsilon}|A|$. This can be explained as follows. Assume c_{ν} is such an element in $\mathrm{spt}(\nu)$ and $c \in C$ is an element such that $|c - c_{\nu}| \le \delta$. We observe that if $\mathcal{C}$ is a covering of $A + cA$ by δ-balls, then for each ball in $\mathcal{C}$, adding two translations to the left and to the right by δ, we would have a covering of $A + c_{\nu}A$. This gives the desired conclusion.

Orponen [30] recently obtained a stronger result that extends Bourgain's result for different sets.

Theorem 1.2 (**Orponen**, 2021)**.** *Let $0 < \beta \le \alpha < 1$ and $\eta > 0$. Then, for every $\gamma \in ((\alpha-\beta)/(1-\beta), 1]$, there exist $\epsilon, \epsilon_0, \delta_0 \in (0, \frac{1}{2}]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let*

$\delta \in 2^{-\mathbb{N}}$ *with* $\delta \in (0, \delta_0]$, *and let* $A, B \subset \delta\mathbb{Z} \cap [0,1]$ *satisfy the following hypotheses:*

(A) $|A| \leq \delta^{-\alpha}$.

(B) $|B| \geq \delta^{-\beta}$, *and* B *satisfies the following Frostman (non-concentration) condition:*

$$|B \cap B(x,r)| \leq r^{\eta}|B|, \quad \forall x \in \mathbb{R}, \ \delta \leq r \leq \delta^{\epsilon_0}.$$

Further, let ν *be a Borel probability measure with spt*$(\nu) \subset [\frac{1}{2}, 1]$, *satisfying the Frostman condition* $\nu(B(x,r)) \leq r^{\gamma}$ *for* $x \in \mathbb{R}$ *and* $\delta \leq r \leq \delta^{\epsilon_0}$. *Then, there exists a point* $c \in$ *spt*(ν) *such that*

$$|A + cB|_{\delta} \geq \delta^{-\epsilon}|A|.$$

Orponen also made the conjecture that the sharp lower bound for γ should be $\gamma > \alpha - \beta$.

Conjecture 1.3. *Let* $\alpha, \beta, \gamma \in (0,1)$ *with* $\alpha \geq \beta$ *and* $\gamma > \alpha - \beta$. *Assume that* $A, B \subset [0,1]$ *and* $C \subset [1/2, 1]$ *are* δ*-separated sets with cardinalities* $|A| \leq \delta^{-\alpha}$, $|B| = \delta^{-\beta}$, *and* $|C| = \delta^{-\gamma}$. *Assume moreover that* $|B \cap B(x,r)| \lesssim r^{\beta}|B|$ *and* $|C \cap B(x,r)| \lesssim r^{\gamma}|C|$ *for all* $x \in \mathbb{R}$ *and* $r > 0$. *Then, there exist* $\epsilon = \epsilon(\alpha, \beta, \gamma) > 0$ *and a point* $c \in C$ *such that* $|A + cB|_{\delta} \gtrsim_{\alpha,\beta,\gamma} \delta^{-\epsilon}|A|$.

This conjecture is made based on a number of examples in the discrete setting. Let A, B, C be finite sets in $\mathbb{R}$. It is well-known that one can use the Szemerédi-Trotter theorem [33] to show that if $|B||C| \gtrsim |A|$ then there exists $c \in C$ such that $|A + cB| \gtrsim |A|$. For reader's convenience, we reproduce the argument here. For any $c \in C$, let L_c be the set of lines of the form $x = r - cy$ with $r \in A + cB$. Let $L = \cup_{c \in C} L_c$. Then it is clear that $|L| \leq |C| \max_{c \in C} |A + cB|$. We observe that $I(A \times B, L) \geq |A||B||C|$. Thus, the Szemerédi-Trotter incidence theorem gives

$$|A||B||C| \lesssim |A|^{2/3}|B|^{2/3}|C|^{2/3} \max_{c \in C} |A + cB|^{2/3},$$

which gives

$$\max_{c \in C} |A + cB| \gtrsim |A|^{1/2}|B|^{1/2}|C|^{1/2}.$$

In other words, if one wishes to have $|A + cB| \gtrsim |A|^{1+\epsilon}$ for some $\epsilon > 0$, then the condition $|B||C| \gtrsim |A|^{1+2\epsilon}$ is needed. The following example, taken from [30], also tells us that this condition is sharp. For $n \in \mathbb{N}$, define

$$A_n = \left\{\frac{1}{n^{1/2}}, \frac{2}{n^{1/2}}, \ldots, 1\right\}, \ B_n = \left\{\frac{1}{n^{1/4}}, \frac{2}{n^{1/4}}, \ldots, 1\right\} = C_n.$$

We can check that for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|A_n + C_n B_n| \leq n^{\epsilon}|A|$. The same happens in the finite field setting. We refer the reader to [32] for more discussions.

If we assume A and B are Ahlfors-David regular sets in $[0,1]$, then Conjecture 1.3 is known to be true in [29]. In fact, in [29], Orponen proved a much stronger statement as follows. Let $A, B \subset \mathbb{R}$ be closed sets, where A is α-regular and B is β-regular. Then

$$\dim_H \left\{c \in \mathbb{R} \colon \dim_H(A + cB) < \alpha + \frac{\beta(1-\alpha)}{2-\alpha}\right\} = 0. \tag{2}$$

For general sets, in another paper [31], Orponen proved the following result.

Theorem 1.4 (**Orponen**, 2022)**.** *Let* $0 < \beta \leq \alpha < 1$ *and* $\sigma > 0$. *Then there exists* $\epsilon = \epsilon(\alpha, \beta, \sigma) > 0$ *such that if* $A, B \subset \mathbb{R}$ *are Borel sets with* $\dim_H(A) = \alpha$, $\dim_H(B) = \beta$, *then*

$$\dim_H\{c \in \mathbb{R} \colon \dim_H(A + cB) \leq \alpha + \epsilon\} \leq \frac{\alpha - \beta}{1 - \beta} + \sigma.$$

In particular,

$$\dim_H \{c \in \mathbb{R}\colon \dim_H(A + cB) = \alpha\} \le \frac{\alpha - \beta}{1 - \beta}.$$

To prove this theorem, the following upgraded version of Theorem 1.2 is crucial.

Theorem 1.5 (**Orponen**, 2022)**.** *Let $0 < \beta \le \alpha < 1$ and $\eta > 0$. Then, for every $\gamma \in ((\alpha - \beta)/(1 - \beta), 1]$, there exist $\epsilon_0, \epsilon, \delta_0 \in (0, 1/2]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, and let $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ satisfy the following hypothesis:*

- $|A| \le \delta^{-\alpha}$
- $|B| \ge \delta^{-\beta}$*, and B satisfies the following Frostman condition*

$$|B \cap B(x, r)| \le r^{\eta}\ |B|, \quad \forall x \in \mathbb{R}, \delta \le r \le \delta_0^{\epsilon}.$$

Further, let ν be a Borel probability measure with $spt(\nu) \subset [0, 1]$, and satisfying the Frostman condition $\nu(B(x, r)) \le r^{\gamma}$ for all $x \in \mathbb{R}$ and $0 < r < \delta^{\epsilon_0}$. Then, there exists $c \in spt(\nu)$ such that the following holds: if $G \subset A \times B$ is any subset with $|G| \ge \delta^{\epsilon}|A||B|$, then

$$|\pi_c(G)|_\delta \ge \delta^{-\epsilon}|A|,$$

where $\pi_c(a, b) = a + cb$.

The main purpose of this paper is to study energy variants of these results. More precisely, let $\delta \in 2^{-\mathbb{N}}$, and $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose that

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \le \delta\}|_\delta = \frac{1}{K} \cdot |A|_\delta^{3/2}|B|_\delta^{3/2}|C|.$$

We aim to give a number of lower bounds on K depending on different non-concentration conditions of the sets A, B, and C. As applications, we obtain a number of improvements on the δ-covering problems and their Hausdorff dimensional versions.

Our first energy theorem is stated as follows.

Theorem 1.6 (First energy theorem)**.** *Let $\delta \in 2^{-\mathbb{N}}$, and $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose that*

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \le \delta\}|_\delta = \frac{1}{K} \cdot |A|_\delta^{3/2}|B|_\delta^{3/2}|C|.$$

Let $\alpha, \beta, \gamma, \eta \in (0, 1)$ with $\alpha \ge \beta$. Assume that $|A| = \delta^{-\alpha}$, $|B| = \delta^{-\beta}$, and $|C| = \delta^{-\gamma}$. There exist $\epsilon, \epsilon_0, \delta_0 \in (0, 1/2]$ such that the following holds for $\delta \in (0, \delta_0]$. If

$$|B \cap B(x, r)| \lesssim r^{\eta}|B|,\ \forall x \in \mathbb{R},\ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x, r)| \lesssim r^{\gamma}|C|,\ \forall x \in \mathbb{R},\ \delta < r < \delta^{\epsilon_0},$$

then K can be bounded from below by

$$K \gtrsim \min\{\delta^{-\epsilon/34}, \delta^{-\epsilon_0\beta/2}, \delta^{-\epsilon_0\gamma/8}\}.$$

We now list several applications of this energy theorem. The first one is a robust version of Theorem 1.1.

Theorem 1.7. *Let $\alpha, \gamma, \eta \in (0,1)$. There exist $\epsilon, \epsilon_0, \delta_0 \in (0,1/2]$ such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, $A \subset \delta\mathbb{Z} \cap [0,1]$ with $|A| = \delta^{-\alpha}$, and $C \subset \delta\mathbb{Z} \cap [1/2, 1]$. Assume that*

$$|A \cap B(x,r)| \lesssim r^{\eta}|A|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x,r)| \lesssim r^{\gamma}|C|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0}.$$

Then, there exists $c \in C$ such that the following holds: If $G \subset A \times A$ is any subset with $|G| \geq \delta^{\epsilon}|A|^2$, then

$$|\pi_c(G)|_{\delta} \geq \delta^{-\epsilon}|A|,$$

where $\pi_c(G) = \{a + cb \colon (a,b) \in G\}$.

We note that Theorem 1.7 was first proved in Bourgain's paper [4]. The higher dimensional version was also studied by He in [18]. Compared to their approach, our proof of Theorem 1.7 is much simpler, which is one of the novelties of this paper. As a consequence, overall, a short and self-contained proof of the following theorem, originally from [4], is derived.

Theorem 1.8. *Let $A \subset \mathbb{R}$ be a compact set with $\dim_H(A) = \alpha$ with $0 < \alpha < 1$, and $\sigma > 0$. Then there exists $\zeta = \zeta(\alpha, \sigma)$ such that*

$$\dim_H \{c \in \mathbb{R} \colon \dim_H(A + cA) \leq \alpha + \zeta\} \leq \sigma.$$

In particular,

$$\dim_H \{c \in \mathbb{R} \colon \dim_H(A + cA) = \alpha\} = 0.$$

Another application of Theorem 1.6 we have to mention here is the following discretized sum-product type estimate on the set $C(A + A)$.

Theorem 1.9. *Given $\alpha \in (0,1)$ and $\gamma, \eta > 0$, there exist $\epsilon_0, \epsilon > 0$ such that the following holds for all sufficiently small $\delta > 0$. Let $C \subset [1/2, 1]$ be a δ-separated set satisfying*

$$|C \cap B(x,r)| \lesssim r^{\gamma}|C|$$

for all $\delta \leq r \leq \delta^{\epsilon_0}$ and $x \in \mathbb{R}$. Let additionally $A \subset [0,1]$ be a δ-separated set with $|A| = \delta^{-\alpha}$, which also satisfies the non-concentration condition $|A \cap B(x,r)| \leq r^{\eta}|A|$ for $x \in \mathbb{R}$ and $\delta \leq r \leq \delta^{\epsilon_0}$.

Then, we have

$$|C(A + A)|_{\delta} \geq \delta^{-\epsilon}|A|.$$

In the next two energy theorems, we focus on finding explicit lower bounds of K. This task is important and crucial to prove versions of the dimensional estimate (2) for general sets A and B.

Our second energy theorem reads as follows.

Theorem 1.10 (Second energy theorem)**.** *Let $\delta \in 2^{-\mathbb{N}}$, and $A, B \subset \delta\mathbb{Z} \cap [0,1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose that*

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta\}|_{\delta} = \frac{1}{K} \cdot |A|_{\delta}^{3/2}|B|_{\delta}^{3/2}|C|.$$

Let $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta > 1$. Assume $|A| = \delta^{-\alpha}, |B| = \delta^{-\beta}$, and $|C| = \delta^{-\gamma}$, and

$$|A \cap B(x,r)| \leq Mr^{\alpha}|A|, \quad |B \cap B(x,r)| \leq Mr^{\beta}|B|, \ \forall \ \delta \leq r \leq 1, \ x \in \mathbb{R},$$

for some $M > 1$, then K can be bounded from below by

$$K \gtrsim \delta^{\frac{\alpha-3\beta-4\gamma+2\gamma(\alpha+\beta)-\alpha^2+\beta^2+2}{2(3-\alpha-\beta)}}.$$

Note that, in the statement of Theorem 1.10, taking $\alpha = \beta > 1/2$ and $\gamma > 1/2$, then we have the bound

$$K \gtrsim \delta^{\frac{-\gamma(4-4\alpha)+2-2\alpha}{2(3-2\alpha)}}.$$

Compared to the first energy theorem, while we require $\delta \le r \le 1$ instead of $\delta \le r \le \delta^{\epsilon_0}$ for some $0 < \epsilon_0 < 1$, no non-concentration condition on C is needed.

We now discuss some consequences of this new energy theorem. As above, we directly obtain δ-covering results for the $A + cB$ and $C(A + A)$ problems.

Theorem 1.11. *Let $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [0, 1]$ with $|A| \le \delta^{-\alpha}$, $|B| \ge \delta^{-\beta}$, and $|C| = \delta^{-\gamma}$. Suppose that $\alpha + \beta > 1$,*

$$|A \cap B(x, r)| \lesssim r^{\alpha}|A|,$$

and

$$|B \cap B(x, r)| \lesssim r^{\beta}|B|,$$

for all $\delta \le r \le 1$ and $x \in \mathbb{R}$. Then there exists $c \in C$ such that

$$|A + cB|_{\delta} \gtrsim \delta^{\frac{-6\beta-4\gamma-2\alpha^2+4\alpha+2\beta^2+2+2\gamma(\alpha+\beta)}{2(3-\alpha-\beta)}}|A|.$$

Theorem 1.12. *Let $A \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose $|A| = \delta^{-\alpha}$ and $|C| = \delta^{-\gamma}$, with $\alpha, \gamma \in (1/2, 1)$. Assuming*

$$|A \cap B(x, r)| \le Mr^{\alpha}|A|, \ \forall\ \delta \le r \le 1,\ x \in \mathbb{R},$$

for some $M > 1$, and sufficiently small $\delta > 0$. Then there exists $\varepsilon > 0$ such that

$$|C(A + A)|_{\delta} \ge \delta^{-\varepsilon}|A|,$$

where

$$\varepsilon = \frac{2\alpha + 4\gamma - 4\alpha\gamma - 2}{6 - 4\alpha}.$$

We now move to the next theorem, which presents an explicit energy variant of Theorem 1.2.

Theorem 1.13 (Third energy theorem)**.** *Let $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose that*

$$\sum_{c\in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \le \delta\}|_{\delta} = \frac{1}{K} \cdot |A|_{\delta}^{3/2}|B|_{\delta}^{3/2}|C|.$$

Let $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha \ge \beta$. Assume in addition that $|A| \le \delta^{-\alpha}$, $|B| = \delta^{-\beta}$, $|C| = \delta^{-\gamma}$ with $\delta \in (0, \delta_0]$, and

$$|B \cap B(x, r)| \lesssim r^{\eta}|B|,\ \forall x \in \mathbb{R},\ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x, r)| \lesssim r^{\gamma}|C|,\ \forall x \in \mathbb{R},\ \delta < r < \delta^{\epsilon_0},$$

for $\epsilon_0, \delta_0 \in (0, 1/2)$ depending on $\alpha, \beta, \gamma, \eta$. Then, for all $\epsilon > 0$, K can be bounded from below by

$$K \gtrsim |\log \delta|^{O(1)} \delta^{O(\epsilon)} \min\{M_0, M_1, M_2, M_3, M_4\},$$

where

$$M_0 = \delta^{-\frac{\gamma-\beta}{4}}, \quad (18)$$

$$M_1 = \delta^{-\frac{\gamma+2\beta+\epsilon_0\eta}{4}}, \quad (22)$$

$$M_2 = \max\left\{\delta^{-\frac{3\gamma-8\beta+1-\epsilon_0(3-\gamma)}{16m_2+36m_1}}, \delta^{-\frac{4\gamma-9\beta+1-\epsilon_0(3-\eta)}{16m_2+36m_1}}\right\}, \quad (28, 29)$$

$$M_3 = \max\left\{\delta^{-\frac{4\gamma-10\beta-\epsilon_0(3-\gamma)}{16m_2+44m_1}}, \delta^{-\frac{5\gamma-11\beta-\epsilon_0(2-\eta)}{16m_2+44m_1}}, \delta^{-\frac{5\gamma-12\beta+\eta-\epsilon_0(2-\gamma)}{16m_2+44m_1}}, \delta^{-\frac{6\gamma-13\beta+\eta-\epsilon_0(1-\eta)}{16m_2+44m_1}}\right\}, \quad (30, 31, 32, 33)$$

$$M_4 = \max\left\{\delta^{-\frac{6\gamma-14\beta-\epsilon_0(3-\gamma)}{20m_2+54m_1}}, \delta^{-\frac{7\gamma-16\beta+\eta-\epsilon_0(2-\gamma)}{20m_2+54m_1}}, \delta^{-\frac{7\gamma-15\beta-\epsilon_0(2-\eta)}{20m_2+54m_1}}, \delta^{-\frac{8\gamma-17\beta+\eta-\epsilon_0(1-\eta)}{20m_2+54m_1}}\right\}, \quad (38, 39, 40, 41)$$

for positive constants $m_1, m_2 \geq 1$ (given by the Balog-Szemerédi-Gower theorem 2.2 below).

Compared to the first two energy results, this theorem has more applications. In particular, the next two applications are explicit versions of Theorem 1.5.

Theorem 1.14. *Let $\alpha, \beta, \eta \in (0, 1)$, $\beta \leq \alpha \leq (21\beta+1)/22$. Then, for every $\gamma \in ((78\alpha-66\beta)/6, 1]$, there exist $\epsilon_0, \delta_0 \in (0, 1/2]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, and let $A, B \subset [0, 1]$ be δ-separated sets satisfying*

- $|A| \leq \delta^{-\alpha}$
- $|B| \geq \delta^{-\beta}$, *and B satisfies the following Frostman condition*

$$|B \cap B(x, r)| \leq r^{\eta}\ |B|, \quad \forall x \in \mathbb{R}, \delta \leq r \leq \delta^{\epsilon_0}.$$

Further, let C be a δ-separated set in $[1/2, 1]$ with $|C \cap B(x, r)| \lesssim r^{\gamma}\ |C|$ for all $x \in \mathbb{R}$ and $0 < r < \delta^{\epsilon_0}$. Then, there exists $c \in C$ such that the following holds for any ϵ satisfying

$$0 < \epsilon < \min\left\{\frac{4\gamma - 74\alpha + 65\beta + 1}{444}, \frac{6\gamma - 78\alpha + 66\beta}{468}\right\}.$$

If $G \subset A \times B$ is any subset with $|G| \geq \delta^{\epsilon}|A||B|$, then

$$|\pi_c(G)|_{\delta} \geq \delta^{-\epsilon}|A|.$$

Theorem 1.15. *Let $\alpha, \beta, \eta \in (0, 1)$, $\beta \leq \alpha$ and $\alpha > (21\beta + 1)/22$. Then, for every $\gamma \in ((74\alpha - 65\beta - 1)/4, 1]$, there exist $\epsilon_0, \delta_0 \in (0, 1/2]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, and let $A, B \subset [0, 1]$ be δ-separated sets satisfying the following hypotheses:*

- $|A| \leq \delta^{-\alpha}$
- $|B| \geq \delta^{-\beta}$, *and B satisfies the following Frostman condition*

$$|B \cap B(x, r)| \leq r^{\eta}\ |B|, \quad \forall x \in \mathbb{R}, \delta \leq r \leq \delta^{\epsilon_0}.$$

Further, let C be a δ-separated set in $[1/2, 1]$ with $|C \cap B(x, r)| \lesssim r^{\gamma}\ |C|$ for all $x \in \mathbb{R}$ and $0 < r < \delta^{\epsilon_0}$. Then, there exists $c \in C$ such that the following holds for any ϵ satisfying

$$0 < \epsilon < \min\left\{\frac{4\gamma - 74\alpha + 65\beta + 1}{444}, \frac{6\gamma - 78\alpha + 66\beta}{468}\right\}.$$

If $G \subset A \times B$ *is any subset with* $|G| \geq \delta^{\epsilon}|A||B|$*, then*

$$|\pi_c(G)|_\delta \geq \delta^{-\epsilon}|A|.$$

On the one hand, while Theorem 1.14 and Theorem 1.15 offer explicit exponents for ϵ, the ranges for γ is worse than that of Theorem 1.5 when $(\alpha-\beta) \to 0$. There is one point we should emphasise here, that if we want $\gamma \to 0$ as $(\alpha-\beta) \to 0$ then we would need $K \geq \delta^{-\epsilon}$, for some $\epsilon > 0$, whenever $\gamma > 0$. Unfortunately, the statement of the third energy theorem says that γ is bounded from below by a function in β. A quick explanation for this matter will be provided in the section "sketch of main ideas". On the other hand, since Theorem 1.5 was proved by using a number of reductions to Theorem 1.2, which is long and sophisticated, one might think that the same framework holds for the two theorems above. This is true, but the expilicit value of ϵ would be much smaller compared to those presented above.

With these two theorems in hand, we now can adapt an argument from [31] to prove the Hausdorff dimensional theorems for the $A + cB$ problem.

Theorem 1.16. *Let* $0 < \beta \leq \alpha < 1$ *with* $22\alpha \leq 21\beta + 1$*. If* $A, B \subset \mathbb{R}$ *are compact sets with* $\dim_H(A) = \alpha$, $\dim_H(B) = \beta$*, then, for any*

$$\sigma > \frac{39(21\beta + 1 - 22\alpha)}{699},$$

we have

$$\dim_H \{c \in \mathbb{R}\colon \dim_H(A + cB) < \alpha + x\} \leq \frac{78\alpha - 66\beta}{6} + \sigma,$$

for any x *smaller than*

$$\frac{1}{2} \cdot \min\left\{\frac{2\sigma - 22\alpha + 21\beta + 1}{518}, \frac{\sigma}{182}\right\}.$$

Theorem 1.17. *Let* $0 < \beta \leq \alpha < 1$ *with* $22\alpha > 21\beta + 1$*. If* $A, B \subset \mathbb{R}$ *are compact sets with* $\dim_H(A) = \alpha$, $\dim_H(B) = \beta$*, then, for any*

$$\sigma > \frac{74(33\alpha - \frac{63}{2}\beta - \frac{3}{2})}{870},$$

we have

$$\dim_H \{c \in \mathbb{R}\colon \dim_H(A + cB) < \alpha + x\} \leq \frac{74\alpha - 65\beta - 1}{4} + \sigma,$$

for any x *smaller than*

$$\frac{1}{2} \cdot \min\left\{\frac{3\sigma + 33\alpha - \frac{63\beta}{2} - \frac{3}{2}}{546}, \frac{\sigma}{259}\right\}.$$

The last application is an explicit version of Theorem 1.9 on the problem $C(A + A)$.

Theorem 1.18. *Let* $A \subset \delta\mathbb{Z} \cap [0, 1]$ *and* $C \subset [1/2, 1]$ *be* δ*-separated. Suppose* $|A| = \delta^{-\alpha}$, $|C| = \delta^{-\gamma}$ *with* $\gamma \in (2\alpha, 1)$, $\delta \in (0, \delta_0]$*, and*

$$|A \cap B(x, r)| \lesssim r^{\eta}|A|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x, r)| \lesssim r^{\gamma}|C|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

for $\epsilon_0, \delta_0 \in (0, 1/2)$ *depending on* α, γ, η*. Then for any*

$$0 < \varepsilon < \min\left\{\frac{4\gamma - 9\alpha + 1}{148}, \frac{6\gamma - 12\alpha}{156}\right\},$$

we have

$$|C(A+A)|_\delta \geq \delta^{-\varepsilon}|A|.$$

Sketch of main ideas

In this section, we briefly discuss methods/techniques we use in this paper. We recall that the main purpose is to find lower bounds of K in the following identity:

$$\sum_{c\in C}|\{(a_1,a_2,b_1,b_2)\in A^2\times B^2 : |(a_1+cb_1)-(a_2+cb_2)|\leq\delta\}|_\delta = \frac{1}{K}\cdot|A|_\delta^{3/2}|B|_\delta^{3/2}|C|.$$

The first step, which is needed for all the three energy theorems, is to prove the existence of subsets $B'\subset B$, $C'\subset C$, points $c^*\in C$, and $\frac{1}{K}\leq\rho<1$ such that

$$|c^*B'+cB'|_\delta \lesssim (\rho K)^{4m_2+6m_1}|B'|, \quad \forall c\in C',$$

$$|B'\pm B'|_\delta \lesssim (\rho K)^{2m_2+2m_1}|B'|,$$

$$|B'|\gtrsim \frac{|B|}{(\rho K)^{m_1}}, |C'|\gtrsim \frac{|C|\rho|\log\delta|^{-1}}{(\rho K)^{4m_1}}.$$

To prove the first energy theorem, we need to check that B' and C' satisfy the non-concentration condition of Theorem 1.1. As a consequence, Theorem 1.1 gives us a lower bound for $|B'+cB'|_\delta$ for all $c\in(c^*)^{-1}C$. Combining with the upper bound, we get the desired lower bound for K.

To prove the second energy theorem, we introduce an approach of using the recent point-tube incidence bound due to Dabrowski, Orponen, and Villa in [9]. More precisely, we observe that the number of tuples $(a_1,a_2,b_1,b_2,c)\in A^2\times B^2\times C$ such that

$$|(a_1+cb_1)-(a_2+cb_2)|\leq\delta$$

is $|A|^{3/2}|B|^{3/2}|C|K^{-1}$. For a fixed a_2, the expression $|(a_1-a_2)-c(b_2-b_1)|\leq\delta$ infers that the point (b_2,a_1-a_2) belongs to the δ-neighborhood of the line defined by $y=c(x-b_1)$. So the energy estimate is reduced to a point-tube incidence bound.

To prove the third energy theorem, we use Lemma 2.5 in the next section to give a lower bound for the set $D=(B'-b_1)\cap(b_2-b_3)C'$, for some $b_1,b_2,b_3\in B'$. More precisely, we have

$$|D|\gtrsim\frac{|C||\log\delta|^{-1}}{K^{4m_1}|B|}.$$

Note that we will need $|C|\geq K^{4m_1}|B|$ to guarantee that D is non-empty. The proof then proceeds by establishing upper and lower bounds on sum sets of the form $|d_1\widetilde{D}+d_2\widetilde{D}+\cdots+d_2\widetilde{D}|_\delta$, for elements $d_i\in C'-C'$ and the set $\widetilde{D}$ is an appropriately large subset of D satisfying small sum set condition. To prove an upper bound, we rely on the δ-covering variant of Plünnecke's inequality, Lemma 7.1, and the estimates of Theorem 3.1, making use of the fact that $D\subset B-b_1$. To obtain a lower bound, we follow the Guth-Katz-Zahl approach in [15] with suitable changes along the way clarifying and quantifying many of the steps. One of the biggest challenges in this approach is to optimize all parameters, roughly speaking, based on the definition of D, in the proof, if we have to bound the d-covering of D, for some $d\geq\delta$, then there are two ways one can proceed: either using the non-concentration condition on C or the non-concentration condition on B. Using the condition from only one set might imply an empty range. Overall, at the end, we have at least 64 ranges for K. This requires much work to figure out the best range for our purposes. It is very natural to ask if the argument presented in this paper can be improved to solve Conjecture 1.3 completely. At least to us, it is not possible when working with the set D. This can be seen clearly from the fact that we need $|C|\geq K^{4m_1}|B|$ to guarantee $D\neq\emptyset$. So the range $\gamma>\alpha-\beta$

is not sufficient for this purpose. The reader should keep in mind that the lower bounds of K only depend on the sets B and C, and are independent of the size or structural properties of A. Therefore, this approach might be useful for other geometric and sum-product type questions.

2 Basic lemmas from Additive Combinatorics

The first lemma is obtained by using the Cauchy-Schwarz inequality and a dyadic pigeon-hole argument.

Lemma 2.1 (Lemma 19, [26]). *For a finite set T and a collection $\{T_s : s \in S\}$ of subsets of T, i.e. $T_s \subset T$. Then*

$$\left(\sum_{s\in S} |T_s|\right)^2 \le |T| \sum_{s,s'\in S} |T_s \cap T_{s'}|.$$

Further, if there exists $\delta > 0$ such that

$$\sum_{s\in S} |T_s| \ge \delta |S||T|,$$

then there exists a subset $P \subseteq S \times S$ such that

1. *$|T_s \cap T_{s'}| \ge \delta^2 |T|/2$ for all pairs (s, s') in P.*
2. *$|P| \ge \delta^2 |S|^2/2$.*

The next result is known as the Balog-Szemerédi-Gowers theorem.

Theorem 2.2 (Theorem 6.10, [38]). *Let $k \ge 1$ be a parameter. Let A and B be bounded subsets of $\mathbb{R}^n$. If*

$$|\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + b_1) - (a_2 + b_2)| \le \delta\}| \gtrsim \frac{1}{K} \cdot |A|_\delta^{3/2} |B|_\delta^{3/2},$$

then there exist $A' \subset A$, $B' \subset B$, and constants $m_1, m_2 \ge 1$ such that

$$|A'|_\delta \gtrsim K^{-m_1} |A|_\delta, |B'|_\delta \gtrsim K^{-m_1} |B|_\delta,$$

and

$$|A' + B'|_\delta \lesssim K^{m_2} |A|_\delta^{1/2} |B|_\delta^{1/2}.$$

We remark that one can take $m_1 = 1$ and $m_2 = 7$ as in [39]. For a set X, we call a set $X^{'}$ a *δ-refinement* of X if $X^{'} \subset X$, and $|X^{'}|_\delta \ge |X|_\delta/2$. We recall the following δ-covering version of Plünnecke's inequality.

Lemma 2.3 (Corollary 3.4, [15]). *Let $X, Y_1, \ldots, Y_k$ be subsets of $\mathbb{R}$. Suppose that $|X + Y_i|_\delta \le K_i |X|_\delta$ for each $i = 1, \ldots, k$. Then there is a δ-refinement $X^{'}$ of X so that*

$$|X^{'} + Y_1 + Y_2 + \ldots + Y_k|_\delta \lesssim \left(\Pi_{i=1}^k K_i\right) |X^{'}|_\delta.$$

In particular,

$$|Y_1 + Y_2 + \ldots + Y_k|_\delta \lesssim \left(\Pi_{i=1}^k K_i\right) |X|_\delta.$$

As mentioned and proved in [15], we often need to replace the δ-covering of a set with a larger scale. The lemma below also plays an important role for this purpose.

Lemma 2.4 (Lemma 2.1, [15]). *Let $X \subset [1, 2]$ be a δ-separated subset and suppose that $|X| = \delta^{-\sigma}$ for some $0 < \sigma < 1$. Suppose that*

$$|X + X|_\delta \le K|X|.$$

Then for every $\epsilon > 0$, there is a subset $X' \subset X$ with $|X'| \gtrsim \delta^{\epsilon}|X|$, such that

$$|X' + X'|_t \lesssim \delta^{-10\epsilon} K |X'|_t,$$

for all $\delta < t < 1$, with the implicit constants depending on σ and ϵ.

We recall a simple, but useful result proved in Theorem C of Bourgain's paper [3].

Lemma 2.5. *Let X, Y be finite subsets of an arbitrary ring and let $M = \max_{y \in Y} |X + yX|$. Then there exist elements $x_1, x_2, x_3 \in X$ such that*

$$|(X - x_1) \cap (x_2 - x_3)Y| \gtrsim \frac{|Y||X|}{M}.$$

We also need Ruzsa's triangle inequality for finite sets and its δ-covering variant from [15].

Lemma 2.6 (Ruzsa triangle inequality)**.** *Let G be an Abelian group and let $X, Y, Z \subset G$ be finite subsets. Then*

$$|X - Z| \leq \frac{|X - Y| \cdot |Y - Z|}{|Y|},$$

and

$$|X + Z| \leq \frac{|X + Y| \cdot |Y + Z|}{|Y|}.$$

Lemma 2.7 (Proposition 3.5, [15])**.** *Let X, Y, Z be subsets of $\mathbb{R}$. Then*

$$|X - Z|_\delta \lesssim \frac{|X - Y|_\delta \cdot |Y - Z|_\delta}{|Y|_\delta}, \tag{3}$$

and

$$|X + Z|_\delta \lesssim \frac{|X + Y|_\delta \cdot |Y + Z|_\delta}{|Y|_\delta}. \tag{4}$$

3 A structural theorem: good subsets of B and C

The main theorem in this section is the following. Roughly speaking, it says that if we have the energy equality

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta\}|_\delta = \frac{1}{K} \cdot |A|_\delta^{3/2} |B|_\delta^{3/2} |C|,$$

then one can find subsets $B' \subset B$ and $C' \subset C$ such that the sets $B' + B'$ and $B' + cB'$ have small δ-covering for all $c \in C'$. This theorem can be viewed as the discretized version of [26, Proposition 4] due to Murphy and Petridis in the finite field setting.

Theorem 3.1. *Let $\delta \in 2^{-\mathbb{N}}$, and $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [1/2, 1]$ be δ-separated. Suppose that*

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta\}|_\delta = \frac{1}{K} \cdot |A|_\delta^{3/2} |B|_\delta^{3/2} |C|.$$

There exist subsets $B' \subset B$, $C' \subset C$, points $c^* \in C$, and $\frac{1}{K} \leq \rho < 1$ such that

$$|c^* B' + cB'|_\delta \lesssim (\rho K)^{4m_2 + 6m_1} |B'|, \quad \forall c \in C',$$

$$|B' \pm B'|_\delta \lesssim (\rho K)^{2m_2 + 2m_1} |B'|,$$

$$|B'| \gtrsim \frac{|B|}{(\rho K)^{m_1}}, |C'| \gtrsim \frac{|C|\rho|\log \delta|^{-1}}{(\rho K)^{4m_1}}.$$

Proof. For each $c \in C$, set

$$f_{\delta(A,cB)} := |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta\}|_\delta.$$

We are given

$$\sum_{c \in C} f_\delta(A, cB) \gtrsim \frac{(|A|_\delta |B|_\delta)^{3/2}|C|}{K}. \tag{5}$$

Observe that

$$f_\delta(A, cB) \leq |A|^2|B|, |A||B|^2,$$

so that we have $f_\delta(A, cB) \leq |A|^{3/2}|B|^{3/2}$. We now look at the equation (5). Without loss of generality, we can assume that $f_\delta(A, cB) \gtrsim |A|^{3/2}|B|^{3/2}/K$ for all $c \in C$. By dyadic pigeonholing, we can find an integer $N \in \mathbb{N}$ and a set $C_1 \subset C$ such that for all $c \in C_1$, one has

$$f_\delta(A, cB) \sim 2^N \frac{|A|^{3/2}|B|^{3/2}}{K} = \frac{|A|^{3/2}|B|^{3/2}}{\rho K}, \quad \text{where } \rho = 2^{-N}.$$

Notice that $1/K \leq \rho \leq 1$ and the fact that $2^N \lesssim K$ gives $N \lesssim \log(K)$. Thus,

$$|C_1| \gtrsim |C|\rho(\log K)^{-1}.$$

If $K > \delta^{-1}$, there is nothing to prove. So we may assume that $K \leq \delta^{-1}$ which gives $|C_1| \gtrsim |C|\rho|\log \delta|^{-1}$.

For each $c \in C_1$, by Theorem 2.2, we can find subsets $A^c \subset A$ and $B^c \subset B$ such that

$$|A^c| \gtrsim \frac{|A|}{(\rho K)^{m_1}}, \quad |B^c| \gtrsim \frac{|B|}{(\rho K)^{m_1}}, \quad |A^c + cB^c|_\delta \lesssim (\rho K)^{m_2}(|A|_\delta |B|_\delta)^{1/2} \tag{6}$$

for some positive constants $m_2 > m_1$. It is clear that we have $|A^c \times B^c| \gtrsim \frac{|A||B|}{(\rho K)^{2m_1}}$, for all $c \in C_1$. Then Lemma 2.1 ensures that there exists a subset $P \subset C_1 \times C_1$ such that

$$|(A^c \times B^c) \cap (A^{c'} \times B^{c'})| \gtrsim \frac{|A||B|}{(\rho K)^{4m_1}},$$

for all pairs $(c, c') \in P$ and $|P| \gtrsim \frac{|C_1|^2}{(\rho K)^{4m_1}}$. This also yields that there exists $c^* \in C_1$ and $C' \subset C_1$ such that $|C'| \gtrsim \frac{|C|\rho|\log \delta|^{-1}}{(\rho K)^{4m_1}}$ and

$$|(A' \times B') \cap (A^c \times B^c)| \gtrsim \frac{|A||B|}{(\rho K)^{4m_1}}, \tag{7}$$

for all $c \in C'$, where we write A' for A^{c^*} and B' for B^{c^*}.

For $c \in C_1$, applying the triangle inequality for δ-covering (4) with $X = Z = cB^{(c)}$, $Y = A^{(c)}$, we have

$$|B^{(c)} + B^{(c)}|_\delta \sim |cB^{(c)} + cB^{(c)}|_\delta \leq \frac{|A^{(c)} + cB^{(c)}|_\delta^2}{|A^{(c)}|} \lesssim (\rho K)^{2m_2} \frac{|A||B|}{|A^{(c)}|} \lesssim (\rho K)^{2m_2 + m_1}|B|. \tag{8}$$

Similarly, for $c \in C_1$, using the triangle inequality (3) with $X = Z = cB^{(c)}$, $Y = -A^{(c)}$, one obtains

$$|B^{(c)} - B^{(c)}|_\delta \sim |cB^{(c)} - cB^{(c)}|_\delta \leq \frac{|A^{(c)} + cB^{(c)}|_\delta^2}{|A^{(c)}|} \lesssim (\rho K)^{2m_2} \frac{|A||B|}{|A^{(c)}|} \lesssim (\rho K)^{2m_2+m_1}|B|.$$

Similarly, for all $c \in C_1$, one can check that

$$|A^{(c)} \pm A^{(c)}|_\delta \lesssim \frac{|A^{(c)} + cB^{(c)}|_\delta^2}{|B^{(c)}|} \lesssim (\rho K)^{2m_2+m_1}|A|. \tag{9}$$

In particular, these estimates imply

$$|A' \pm A'|_\delta \lesssim (\rho K)^{2m_2+m_1}|A|, \quad |B' \pm B'|_\delta \lesssim (\rho K)^{2m_2+m_1}|B|. \tag{10}$$

Now we turn to estimating $|c^*B' \pm cB'|_\delta$. Again, applying triangle inequalities for δ-covering from Lemma 2.7 with suitable sets, one has

$$\begin{aligned}
|c^*B' \pm cB'|_\delta &\leq \frac{|c^*B' + c(B^c \cap B')|_\delta |cB' + c(B^c \cap B')|_\delta}{|(B^c \cap B')|_\delta} \\
&\lesssim \frac{|c^*B' + cB^c|_\delta |B' + B'|_\delta}{|(B^c \cap B')|_\delta} \\
&\lesssim \frac{|A' + c^*B'|_\delta |A^c + cB^c|_\delta |B' + B'|_\delta}{|(A^c \cap A')|_\delta |(B^c \cap B')|_\delta} \\
&\lesssim (\rho K)^{4m_2+5m_1}|B|,
\end{aligned}$$

where the last inequality follows from (7). This gives that

$$|B' \pm (c^*)^{-1}cB'|_\delta \lesssim (\rho K)^{4m_2+5m_1}|B|, \ \forall\ c \in C', \tag{11}$$

and $|B' \pm B'|_\delta \lesssim (\rho K)^{2m_2+m_1}|B|$. □

4 Proof of Theorem 1.6

Using Theorem 3.1, we know that there exist subsets $B' \subset B$, $C' \subset C$, an element $c^* \in C$, and $\frac{1}{K} \leq \rho < 1$ such that

$$|c^*B' + cB'|_\delta \lesssim (\rho K)^{4m_2+6m_1}|B'|, \quad \forall c \in C',$$

$$|B' \pm B'|_\delta \lesssim (\rho K)^{2m_2+2m_1}|B'|,$$

$$|B'| \gtrsim \frac{|B|}{(\rho K)^{m_1}}, |C'| \gtrsim \frac{|C|\rho|\log\delta|^{-1}}{(\rho K)^{4m_1}}.$$

To find a lower bound for K, we apply Theorem 1.1 with $A := B'$ and $C := C'$.

We apply Theorem 1.1 with parameters $\beta/2$ and $\gamma/2$, namely, there exist $\epsilon, \epsilon_0 > 0$ such that the following holds. If

$$|B' \cap B(x,r)| \lesssim r^{\beta/2}|B'|,$$

and

$$|C' \cap B(x,r)| \lesssim r^{\gamma/2}|C'|,$$

then there exists $c \in C'$ such that $|B' + cB'| \gtrsim \delta^{-\epsilon}|B'|$. Next, we check the non-concentration conditions of B' and C'. We first have

$$|B' \cap B(x,r)| \leq r^\beta |B| \leq r^\beta K|B'|.$$

If $K > \delta^{-\epsilon_0\beta/2}$ or $K > \delta^{-\epsilon_0\gamma/8}|\log\delta|^{-1/4}$, then we are done. Thus, we can assume that $K \ll \delta^{-\epsilon_0\beta/2}$ and $K \ll \delta^{-\epsilon_0\gamma/8}|\log\delta|^{-1/5}$. This gives that

$$|B' \cap B(x,r)| \ll r^{\beta}K|B'| \le r^{\beta/2}|B'|,$$

for $\delta \le r \le \delta^{\epsilon_0}$.

Similarly,

$$|C' \cap B(x,r)| \ll r^{\gamma}|C| \le r^{\gamma}|C'|K^4|\log\delta| \ll r^{\gamma/2}|C'|,$$

for $\delta \le r \le \delta^{\epsilon_0}$. Note that we used the fact that $1/K \le \rho \le 1$.

Thus, as above, there exists $c \in C'$ such that

$$|B' + cB'|_{\delta} \gtrsim \delta^{-\epsilon}|B'|.$$

Notice that we may replace the set C' above by $(c^*)^{-1}C'$. On the other hand, we know from Theorem 3.1 (recall we may take $m_2 = 7$ and $m_1 = 1$) that

$$|c^*B' + cB'|_{\delta} \lesssim K^{34}|B'|.$$

This means that $K^{34} \gtrsim \delta^{-\epsilon}$, which gives $K \gtrsim \delta^{-\epsilon/34}$. This completes the proof.

Remark 4.1. *In the above proof, if we apply Theorem 1.2 to the set $A^c + cB^c$ from (6), then a better lower bound for K might be obtained, but we would need the condition that $\gamma > (\alpha-\beta)/(1-\beta)$.*

5 Proof of Theorem 1.7 and Theorem 1.8

Let us recall the statement of Theorem 1.7.

Theorem 5.1. *Let $\alpha, \gamma, \eta \in (0,1)$. There exist $\epsilon, \epsilon_0, \delta_0 \in (0, 1/2]$ such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, $A \subset \delta\mathbb{Z} \cap [0,1]$ with $|A| = \delta^{-\alpha}$, and $C \subset \delta\mathbb{Z} \cap [1/2, 1]$. Assume that*

$$|A \cap B(x,r)| \lesssim r^{\eta}|A|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x,r)| \lesssim r^{\gamma}|C|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0}.$$

Then, there exists $c \in C$ such that the following holds: If $G \subset A \times A$ is any subset with $|G| \ge \delta^{\epsilon}|A|^2$, then

$$|\pi_c(G)|_{\delta} \ge \delta^{-\epsilon}|A|,$$

where $\pi_c(G) = \{a + cb \colon (a,b) \in G\}$.

Proof. Let X be the set of $c \in C$ such that the conclusion of the theorem fails, i.e. for each $c \in X$, there exists $G_c \subset A \times A$ with $|G_c| \ge \delta^{\epsilon}|A|^2$ and $|\pi_c(G)|_{\delta} < \delta^{-\epsilon}|A|$. We want to show that $|X| < |C|$.

For each $c \in X$, by the Cauchy-Schwarz inequality, the number of tuples $(x,y,z,w) \in G_c^2$ such that $|(x+cy) - (z+cw)| \le \delta$ is at least $\frac{|G_c|^2}{\delta^{-\epsilon}|A|}$, which equals $\delta^{3\epsilon}|A|^3$. Summing over all $c \in X$ and using Theorem 1.6, one has

$$|X|\delta^{3\epsilon}|A|^3 \le \frac{|C||A|^3}{K}.$$

This infers

$$|X| \le \delta^{-3\epsilon}\frac{|C|}{K}.$$

Thus, as long as $K\delta^{3\epsilon} > 1$, we are done.

We now apply Theorem 1.6 with parameters α, η, γ to obtain $\overline{\epsilon}$, $\overline{\epsilon_0}$, and the conclusion that

$$K \gtrsim \min\left\{\delta^{-\overline{\epsilon}/34}, \delta^{-\overline{\epsilon_0}\alpha/2}, \delta^{-\overline{\epsilon_0}\gamma/8}\right\}.$$

Choose $\epsilon_0 = \overline{\epsilon_0}$ and

$$3\epsilon < \min\left\{\frac{\overline{\epsilon}}{34}, \frac{\overline{\epsilon_0}\alpha}{2}, \frac{\overline{\epsilon_0}\gamma}{8}\right\},$$

then the theorem follows. $\square$

We are now ready to prove Theorem 1.8. We learn this argument from Orponen in [31].

Proof of Theorem 1.8. Using Frostman's lemma ([25, Theorem 8.8]), we can find probability measure μ_A supported on A such that

$$\mu_A(B(x,r)) \lesssim r^\alpha, \forall x \in \mathbb{R}, r > 0.$$

Our goal now is to show that

$$\dim_H(E) \le \sigma, \tag{12}$$

where $E := \{c \in \mathbb{R}\colon \dim_H(A + cA) \le \alpha + \zeta\}$ for some small ζ.

Without loss of generality, we may just assume $c \in [\frac{1}{2}, 1]$. Since if the result holds for any $A \subset \mathbb{R}$, and $c \in [1/2, 1]$, then we write $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} C_n \cup \{0\} \bigcup_{n\in\mathbb{Z}}(-C_n)$ where $C_n = [2^n, 2^{n+1}]$. Our proof below works equally well for each C_n and $-C_n$ with all the constants uniformly in n since we only need that the interval does not contain 0 to avoid degeneracy. When $c = 0$, the result holds trivially. In other words we have $\dim_H E \le \sigma$ when we restrict $c \in C_n$ for every n and this gives the result.

We will prove the result by contradiction. In other words, suppose the inequality (12) does not hold, i.e.

$$\dim_H(E) > \sigma,$$

then we will derive a contradiction when ζ is small enough. We first find a probability measure ν supported on E such that $\nu(B(x,r)) \lesssim r^\gamma$ for all $x \in \mathbb{R}$ and $r > 0$, where $\gamma > \sigma$. We note that in our argument we identify the measure ν with the maximal δ-separated subset $C \subset \mathrm{spt}(\nu)$. Notice that $|C| \sim \delta^{-\gamma}$ since $\nu(E) = 1$, and $|C \cap B(x,r)| \lesssim r^\gamma\, |C|$ for all $x \in \mathbb{R}$ and $r > 0$.

Fix an element $c \in E$ so that $\dim_H(A + cA) < \alpha + \zeta$. It follows that for a given number $\delta_0 := 2^{-j_0} \in 2^{-\mathbb{N}}$, there exists a family $\mathcal{I}_c$, a countable number of disjoint dyadic intervals of length $\ell(I) \le \delta_0$, which covers $A + cA$ such that

$$\sum_{I\in\mathcal{I}_c} \ell(I)^{\alpha+\zeta} \le 1. \tag{13}$$

Consider the sets $\mathcal{T}_c := \{\pi_c^{-1}(I)\}_{I\in\mathcal{I}_c}$ which cover $A \times A$, so that

$$\int_E \sum_{T\in\mathcal{T}_c} (\mu_A \times \mu_A)(T)\; d\nu(c) = 1.$$

Let $\mathcal{I}_c^j := \{I \in \mathcal{I}_c : \ell(I) = 2^{-j}\}$ for $j \ge j_0$ and write $\mathcal{T}_c^j := \{\pi_c^{-1}(I)\}_{I\in\mathcal{I}_c^j}$. By the pigeonhole principle, there exists $j^* \ge j_0$ such that

$$\int_E \sum_{T\in\mathcal{T}_c^{j^*}} (\mu_A \times \mu_A)(T)\; d\nu(c) \gtrsim j^{*-2}.$$

Let us also denote $\delta := 2^{-j^*}$ for this fixed index j^*. By the estimates above, we can find a subset $E'_\delta \subset E$ of measure $\nu(E'_\delta) \gtrsim j^{*-2} = \log(1/\delta)^{-2}$ such that for each $c \in E'_\delta$, the sets $T \in \mathcal{T}_c^{j^*}$ cover a subset $G_c \subset \operatorname{spt}(\mu_A \times \mu_A)$ of measure $(\mu_A \times \mu_A)(G_c) \gtrsim \log(1/\delta)^{-2}$. Moreover, note that we have

$$|\pi_c(G_c)|_\delta \le \delta^{-\alpha-\zeta}, \quad c \in E'_\delta, \tag{14}$$

by (13). For the rest of the proof, we use $f \lessapprox g$ to denote $f \lesssim \log(1/\delta)^C g$ for some absolute constant $C > 0$, which may only depend on the Frostman constant of A. In particular, $j^{*-2} \gtrapprox 1$.

Now we need to construct A_δ from A so that we can apply Theorem 1.7 to get a lower bound. The process of constructing these sets is to perform some averaging arguments so that we can pigeonhole to extract the sets. Now for $z \in \mathbb{R}$, let $I_\delta(z) \in \mathcal{D}_\delta$ be the unique dyadic interval of length δ with $z \in I_\delta(z)$. Fix $c \in C$ and given a dyadic number $\rho \in 2^{-\mathbb{N}}$, let $A(\rho) := \{z \in A : \rho \le \mu_A(I_\delta(z)) < 2\rho\}$ and write A as

$$A = \bigcup_{\rho \in 2^{-\mathbb{N}}} A(\rho).$$

Since $\mu_A(I_\delta(z)) \lesssim \delta^\alpha$, we see that $A(\rho) \neq \emptyset$ implying $\rho \lesssim \delta^\alpha$. We also note that $A(\rho)$ can be expressed as the intersection of A with certain dyadic intervals $\mathcal{A}(\rho) \subset \mathcal{D}_\delta$.

Let $\mu_A(\rho)$ be the restriction of μ_A to the intervals $\mathcal{A}(\rho)$. Then

$$\sum_{\rho_1}\sum_{\rho_2} \int_{E'_\delta} (\mu_A(\rho_1) \times \mu_A(\rho_2))(G_c) \approx 1,$$

so it follows from the pigeonhole principle that

$$\int_{E'_\delta} (\mu_A(\rho_u) \times \mu_A(\rho_v))(G_c) \approx 1$$

for some fixed choices $\rho_u \lesssim \delta^\alpha$ and $\rho_v \lesssim \delta^\alpha$. By the pigeonhole principle again, we can find a subset $E_\delta \subset E'_\delta$ so that $(\mu_A(\rho_u) \times \mu_A(\rho_v))(G_c) \approx 1$ for all $c \in E_\delta$. We now define

$$\bar{\mu}_u := \mu_A(\rho_u), \quad \bar{\mu}_v := \mu_A(\rho_v),$$

so $\|\bar{\mu}_u\| \approx 1$. The measure $\bar{\mu}_u$ is supported on the closure of the intervals in $\mathcal{A}(\rho_u)$. Let

$$A_\delta^u := (\delta \cdot \mathbb{Z}) \cap (\cup \mathcal{A}(\rho_u)).$$

Note that

$$\rho_u \cdot |A_\delta^u| \sim \|\bar{\mu}_u\| \approx 1 \implies \rho_u \approx |A_\delta^u|^{-1}.$$

Since $\rho_u \lesssim \delta^\alpha$, we have

$$|A_\delta^u| \approx \rho_u^{-1} \gtrapprox \delta^{-\alpha}. \tag{15}$$

In fact one also has $|A_\delta^u| \le \delta^{-\alpha-\zeta}$ if $\delta > 0$ is sufficiently small. To see this, fix an arbitrary $c \in E_\delta$. Since $(\bar{\mu}_u \times \bar{\mu}_v)(G_c) \approx 1$, there exists $b \in \operatorname{spt}(\bar{\mu}_v)$ such that

$$\bar{\mu}_u(G_c(b)) \approx 1, \quad \text{where} \quad G_c(b) = \{z \in \operatorname{spt}(\bar{\mu}_v) : (z, b) \in G_c\}.$$

Let $\mathcal{H}_c(b) := \{I \in \mathcal{A}(\rho_u) : G_c(b) \cap I \neq \emptyset\}$, we have that $\bar{\mu}_u(I) \sim \rho_u$ for all $I \in \mathcal{H}_c(b)$, and $\bar{\mu}_u(\cup \mathcal{H}_c(b)) \ge \bar{\mu}_u(G_c(b)) \approx 1$. Moreover, we observe that $|G_c(b)|_\delta \lesssim |\pi_c(G_c)|_\delta$, since $\pi_c(G_c) \supset G_c(b) + bc$. Therefore, we obtain

$$|A_\delta^u| \approx \rho_u^{-1} \lessapprox \rho_u^{-1} \cdot \bar{\mu}_u(\cup \mathcal{H}_c(b)) \lesssim |G_c(b)|_\delta \lesssim |\pi_c(G_c)|_\delta \le \delta^{-\alpha-\zeta}.$$

Next, we also need to claim the non-concentration condition for A_δ^u. For $r \geq \delta$, we note that every point $z' \in A_\delta^u \cap B(z,r)$ is contained in an interval $I_{z'}(\delta) \in \mathcal{A}(\rho_u)$ with $\mu_A(I_{z'}(\delta)) \geq \rho_u$. Since $I_{z'}(\delta) \subset B(z,2r)$, we thus have

$$|A_\delta^u \cap B(z,r)| \leq \rho_u^{-1} \cdot \mu_A(B(z,2r)) \lesssim \rho_u^{-1} \cdot (2r)^\alpha \lessapprox r^\alpha |A_\delta^u|.$$

We now choose a subset $A'_\delta \subset A_\delta$ with $|A'_\delta| \sim \delta^{-\alpha}$. This is possible since $|A_\delta^u| \gtrapprox \delta^{-\alpha}$ means that $|A_\delta^u| \gtrsim \delta^{-\alpha}(\log(1/\delta))^C$ for some positive constant C.

We now apply Theorem 1.7 for parameters α and $\eta = \alpha/2$ to obtain ϵ, $\epsilon_0, \delta_0 \in (0,1/2]$.

We now claim that

$$|A'_\delta \cap B(z,r)| \lesssim r^{\alpha/2}|A'_\delta|.$$

To see this, first we have

$$|A'_\delta \cap B(z,r)| \leq \rho_u^{-1} \cdot \mu_A(B(z,2r)) \lesssim \rho_u^{-1} \cdot (2r)^\alpha \lessapprox r^\alpha |A_\delta^u| \lesssim r^\alpha \delta^{-\zeta} |A'_\delta|.$$

For $\delta \leq r \leq \delta^{\epsilon_0}$, and sufficiently small δ and ζ, we have

$$|A'_\delta \cap B(z,r)| \lesssim r^{\alpha/2}|A'_\delta|.$$

Theorem 1.7 also tells us that $|\pi_c(G_c)|_\delta \geq \delta^{-\epsilon}|A'_\delta|$ for some $\epsilon > 0$. Therefore, if $\zeta > 0$ in the definition of E has been chosen small enough, then it contradicts the upper bound (14). $\square$

6 Proof of Theorem 1.10 and Theorem 1.11

To prove Theorem 1.10, we make use of the following point-tube incidence bound due to Dabrowski, Orponen, and Villa in [9].

Theorem 6.1. *Let $0 < n < d$ and $M \geq 1$. Let $\mathcal{V} \subset \mathcal{A}(d,n)$ be a δ-separated set of n-planes, and let $P \subset B(1) \subset \mathbb{R}^d$ be a δ-separated (δ, t, M)-set with $t > d-n$, i.e.*

$$|P \cap B(x,r)|_\delta \leq M r^t |P|_\delta, \ \forall \ r > 0, \ x \in \mathbb{R}^d.$$

For $r > 0$, define $\mathcal{I}_r(P, \mathcal{V}) = \{(p,V) \in P \times \mathcal{V} : p \in V(r)\}$. Then we have

$$|\mathcal{I}_{M\delta}(P,\mathcal{V})| \lesssim_{M,d,\varepsilon,t} |P| \cdot |\mathcal{V}|^{n/(d+n-t)} \cdot \delta^{n(t+1-d)(d-n)/(d+n-t)}.$$

Here, the Grassmannian $\mathcal{A}(d,n)$ is equipped with the metric $d_{\mathcal{A}}$ defined as follows. For $V, W \in \mathcal{A}(d,n)$, let V_0, W_0, and $a \in V_0^\perp, b \in W_0^\perp$ be unique subspaces and vectors such that

$$V = V_0 + a, \ W = W_0 + b.$$

The distance between V and W is defined by

$$d_{\mathcal{A}}(V,W) = ||\pi_{V_0} - \pi_{W_0}||_{op} + |a-b|,$$

where $|| \cdot ||_{op}$ is the operator norm.

In the plane, a direct computation shows that the distance between two lines $y = ax + b$ and $y = cx + d$ is

$$\left|\frac{(a,-1)}{|(a,-1)|} - \frac{(c,-1)}{|(c,-1)|}\right| + \left|\frac{b}{|(a,-1)|} - \frac{d}{|(c,-1)|}\right|.$$

Proof of Theorem 1.10. The first part is identical with Theorem 1.6. For the second part, since

A, B, C are δ-separated, the number of tuples $(a_1, a_2, b_1, b_2, c) \in A^2 \times B^2 \times C$ such that

$$|(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta$$

is $|A|^{3/2}|B|^{3/2}|C|K^{-1}$.

For a fixed a_2, the expression $|(a_1 - a_2) - c(b_2 - b_1)| \leq \delta$ infers that the point $(b_2, a_1 - a_2)$ belongs to the δ-neighborhood of the line defined by $y = c(x - b_1)$. Let L be the set of such lines.

Since C and B are δ-separated, one can directly check using the above metric to get the set L is $c_0\delta$ separated for some absolute constant $c_0 > 0$.

Set $P = B \times (A - a_2)$. Since B and A are δ-separated and

$$|B \cap B(x, r)| \leq Mr^{\beta}|B|,$$

and

$$|A \cap B(x, r)| \leq Mr^{\alpha}|A|,$$

for all $\delta \leq r \leq 1$. Therefore we have

$$|P \cap B(x, r)| \leq M^2 r^{\alpha+\beta}|P|.$$

If $\alpha + \beta > 1$, then we can apply Theorem 6.1 to obtain

$$|I_{M\delta}(P, L) \lesssim |P||L|^{\frac{1}{3-\alpha-\beta}}\delta^{\frac{\alpha+\beta-1}{3-\alpha-\beta}} = |A||B|(|B||C|)^{\frac{1}{3-\alpha-\beta}}\delta^{\frac{\alpha+\beta-1}{3-\alpha-\beta}}.$$

Summing over all $a_2 \in A$, we obtain

$$|A|^{3/2}|B|^{3/2}|C|K^{-1} \lesssim |A|^2|B|(|B||C|)^{\frac{1}{3-\alpha-\beta}}\delta^{\frac{\alpha+\beta-1}{3-\alpha-\beta}}.$$

This gives

$$K \gtrsim \delta^{\frac{\alpha-3\beta-4\gamma+2\gamma(\alpha+\beta)-\alpha^2+\beta^2+2}{2(3-\alpha-\beta)}},$$

concluding the proof. $\square$

Before proving Theorem 1.11, we first recall its statement. Its proof follows directly from Theorem 1.10 with the Cauchy-Schwarz inequality.

Theorem 6.2. *Let $A, B \subset \delta\mathbb{Z} \cap [0, 1]$ and $C \subset [0, 1]$ with $|A| \leq \delta^{-\alpha}$, $|B| \geq \delta^{-\beta}$, and $|C| = \delta^{-\gamma}$. Suppose that $\alpha + \beta > 1$,*

$$|A \cap B(x, r)| \lesssim r^{\alpha}|A|, \quad \textit{and} \quad |B \cap B(x, r)| \lesssim r^{\beta}|B|,$$

for all $\delta \leq r \leq 1$ and $x \in \mathbb{R}$. Then there exists $c \in C$ such that

$$|A + cB|_{\delta} \gtrsim \delta^{\frac{-6\beta-4\gamma-2\alpha^2+4\alpha+2\beta^2+2+2\gamma(\alpha+\beta)}{2(3-\alpha-\beta)}}|A|.$$

Proof. We recall from Theorem 1.10 that

$$\sum_{c \in C} |\{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : |(a_1 + cb_1) - (a_2 + cb_2)| \leq \delta\}|_{\delta} = \frac{1}{K} \cdot |A|_{\delta}^{3/2}|B|_{\delta}^{3/2}|C|,$$

where

$$K \gtrsim \delta^{\frac{\alpha-3\beta-4\gamma+2\gamma(\alpha+\beta)-\alpha^2+\beta^2+2}{2(3-\alpha-\beta)}}.$$

Thus, there exists $c \in C$ such that

$$|A+cB|_\delta \gtrsim \frac{K|B|^{1/2}}{|A|^{1/2}}|A| \gtrsim \delta^{\frac{-6\beta-4\gamma-2\alpha^2+4\alpha+2\beta^2+2+2\gamma(\alpha+\beta)}{2(3-\alpha-\beta)}}|A|.$$

This completes the proof of the theorem. $\square$

7 Proof of Theorem 1.13

Using Theorem 3.1, we know that there exist subsets $B' \subset B$, $C' \subset C$, an element $c^* \in C$, and $\frac{1}{K} \le \rho < 1$ such that

$$|c^*B' + cB'|_\delta \lesssim (\rho K)^{4m_2+6m_1}|B'|, \quad \forall c \in C',$$
$$|B' \pm B'|_\delta \lesssim (\rho K)^{2m_2+2m_1}|B'|,$$
$$|B'| \gtrsim \frac{|B|}{(\rho K)^{m_1}}, |C'| \gtrsim \frac{|C|\rho|\log\delta|^{-1}}{(\rho K)^{4m_1}}.$$

Notice that it is not possible to expect that the set $c^*B' + cB'$ is δ-separated for each $c \in C'$. This means that for each c, the size of $c^*B' + cB'$ is not the same as its δ-covering. So to proceed further, we start noting the following fact that

$$|B' + cB'| \le |B'|^2, \qquad \text{for all } c \in C. \tag{16}$$

We remark here that it is not possible to expect $|B' + cB'| \le |B'|^{2-\epsilon}$ for any $\epsilon > 0$ when $|C'|$ is much larger than $|B'|$. Indeed, if N is the number of tuples $(c, b_1, b_2, b_3, b_4) \in C' \times B' \times B' \times B' \times B'$ such that $b_1 + cb_2 = b_4 + cb_3$, by the Cauchy-Schwarz inequality, one has N is at least $|C'||B'|^{2+\epsilon}$. We observe that the equation $b_1 + cb_2 = b_4 + cb_3$ is equivalent to $b_1 - b_4 = c(b_2 - b_3)$. For a fixed b_4, the above identity gives an incidence between the line defined by $y = c(x - b_3) + b_4$ and the point $(b_2, b_1) \in B' \times B'$. So by the Szemerédi-Trotter theorem [33] and taking the sum over all $b_4 \in B'$, N is at most $|B'|^3|C'|^{2/3} + |B'|^2|C'|$, which is smaller than $|B'|^{2+\epsilon}|C'|$ whenever $|C'| \ge |B'|^{3-3\epsilon}$. With the fact (16) in hand, we can apply Lemma 2.5 to find $b_1, b_2, b_3 \in B'$ satisfying

$$\left|(B' - b_1) \cap (b_2 - b_3)C'\right| \ge \frac{|C'|}{|B'|} \gtrsim \frac{|C||\log\delta|^{-1}}{K^{4m_1}|B|}. \tag{17}$$

Define

$$D := (B' - b_1) \cap (b_2 - b_3)C'.$$

If $K^{4m_1} \ge \frac{|C|}{|B||\log\delta|}$, i.e.

$$K \ge \delta^{-\frac{\gamma-\beta}{4}}|\log\delta|^{1/4}, \tag{18}$$

then we are done. Otherwise, we can assume that $|C| > K^{4m_1}|B||\log\delta|$. This condition implies that D is non-empty.

Let $\kappa = \epsilon_0 \in (0, 1/2)$ be parameters that will be specified later. To proceed further, we need the following two lemmas.

Lemma 7.1. *Let $d_1 = c_1 - c_2$ and $d_2 = c_3 - c_4$ with $c_i \in C'$. Assume that $|c_1 - c_2| \le |c_3 - c_4|$ and $|c_3 - c_4| > \delta^\kappa = \delta^{\epsilon_0}$. Then, for any positive integer $k \ge 1$,*

(i) With the non-concentration on C, one has

$$|d_1D + \underbrace{d_2D + \cdots + d_2D}_{k \text{ terms}}|_\delta \lesssim \frac{|B|^{4k+4}K^{(8m_2+18m_1)(k+1)}|\log\delta|^{2k+2}}{|b_2 - b_3||C|^{2k+1}} \cdot \max\{|d_1|, |d_2|\}\delta^{\epsilon_0(\gamma-1)}.$$

(ii) With the non-concentration on B, one has

$$|d_1D + \underbrace{d_2D + \cdots + d_2D}_{k \text{ terms}}|_\delta \lesssim \frac{|B|^{4k+5}K^{(8m_2+18m_1)(k+1)}|\log\delta|^{2k+2}}{|C|^{2k+2}} \cdot \max\{|d_1|, |d_2|\}\delta^{\epsilon_0(\eta-1)}.$$

Proof. First, observe that for k bounded, the set $d_1D + \underbrace{d_2D + \cdots + d_2D}_{k \text{ terms}}$ is contained in an interval of length d, with $d \sim \max\{|d_1|, |d_2|\}$.

Let $c^* \in C$ as in Theorem 1.6, and let D' be a δ-refinement of D. Then we have

$$\begin{aligned} |d_1D + d_2D + \cdots + d_2D|_\delta &\lesssim |c^*D'|_d^{-1}|c^*D' + d_1D + d_2D + \cdots + d_2D|_\delta \\ &\leq \frac{|c^*D' + c_1D - c_2D + c_3D - c_4D + \cdots + c_3D - c_4D|_\delta}{|c^*D'|_d}. \end{aligned} \tag{19}$$

Thus, the task is now to estimate two terms $|c^*D'|_d$ and $|c^*D' + c_1D - c_2D + c_3D - c_4D + \cdots + c_3D - c_4D|_\delta$.

We will handle the second term first. By Theorem 1.6 and the fact that $D' \subset B' - b_1$, one has

$$|c^*D' + cD|_\delta \leq |c^*B' + cB'|_\delta \leq (\rho K)^{4m_2+5m_1}|B|.$$

Then we use Lemma 2.3 to bound the second term, namely, it is at most

$$\frac{K^{(4m_2+5m_1)(2k+2)}|B|^{2k+2}}{|c^*D'|_\delta^{2k+1}} \lesssim \frac{|B|^{4k+3}|\log\delta|^{2k+1}K^{(4m_2+9m_1)(2k+1)+4m_2+5m_1}}{|C|^{2k+1}}, \tag{20}$$

where the latter follows by $|c^*D|_\delta \sim |D|_\delta \gtrsim |D| \gtrsim \frac{|C||\log\delta|^{-1}}{|B|K^{4m_1}}$.

For the first term, there are two ways to bound it, which will give two different bounds. One way is to use the non-concentration condition on C, and the other way is to use the non-concentration condition on B.

(i) First approach: Non-concentration on C.

Recall that $D' \subset (b_2 - b_3)C'$ is a δ-refinement of D and $|b_2 - b_3| \geq \delta$ by the δ-separated property of B. In particular, this yields that

$$|D'|_d \sim \left|(b_2 - b_3)^{-1}D\right|_{d/(|b_2-b_3|)},$$

where $(b_2 - b_3)^{-1}D \subset C'$.

Let I be an arbitrary interval of length $d/|b_2 - b_3|$. From assumption, we know that $\frac{d}{|b_2-b_3|} \geq d > \delta^\kappa = \delta^{\epsilon_0}$. Thus, the non-concentration condition on the set C implies that

$$|C \cap I| \lesssim \frac{d}{|b_2 - b_3|}|C|\delta^{\epsilon_0(\gamma-1)} \sim d|b_2 - b_3|^{-1}|C|\delta^{\epsilon_0(\gamma-1)}.$$

Since c^* is greater than $1/2$, using the above estimate and (17), we will have an estimate for $|c^*D'|_d$, namely

$$|c^*D'|_d \gtrsim \frac{|D|}{d|b_2 - b_3|^{-1}|C|\delta^{\epsilon_0(\gamma-1)}} \gtrsim \frac{|\log\delta|^{-1}|b_2 - b_3|}{d\delta^{\epsilon_0(\gamma-1)}|B|K^{4m_1}}.$$

Substituting this and (20) into (19), one has the upper bound

$$|d_1D + d_2D + \cdots + d_2D|_\delta \lesssim \frac{|B|^{4(k+1)}K^{(8m_2+18m_1)(k+1)}|\log\delta|^{2k+2}}{|b_2-b_3||C|^{2k+1}} \cdot \max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\gamma-1)}.$$

(ii) Second approach: Non-concentration on B

From hypothesis, we know that $|c_3-c_4| \geq \delta^{\epsilon_0}$. By a similar argument, using the non-concentration assumption on B instead, we have

$$|c^*D'|_d \gtrsim \frac{|D|}{d\delta^{\epsilon_0(\eta-1)}|B|} \gtrsim \frac{|C||\log\delta|^{-1}}{|B|^2K^{4m_1}d\delta^{\epsilon_0(\eta-1)}}.$$

Putting this into (19), we obtain the bound

$$\frac{|B|^{4k+5}K^{(8m_2+18m_1)(k+1)}|\log\delta|^{2k+2}}{|C|^{2k+2}} \cdot \max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\eta-1)}.$$

We complete the proof of the lemma. □

We remark that the above lemma is most effective when $k=1$. For $k>1$, we have the following refinement.

Lemma 7.2. *Let $d_1 = c_1-c_2$ and $d_2 = c_3-c_4$ with $c_i \in C'$. Assume that $|c_1-c_2| \leq |c_3-c_4|$ and $|c_3-c_4| > \delta^\kappa = \delta^{\epsilon_0}$. Then, for a bounded positive integer $k \geq 1$ and for any $\epsilon > 0$, there exists $\widetilde{D} \subset D$ with $|\widetilde{D}| \geq \delta^\epsilon|D|$, such that the following holds.*

(i) With the non-concentration on C, one has

$$|d_1\widetilde{D} + \underbrace{d_2\widetilde{D} + \cdots + d_2\widetilde{D}}_{k \text{ terms}}|_\delta$$
$$\lesssim \delta^{-O(\epsilon)}\frac{|B|^{2k+6}K^{(2m_2+5m_1)k+14m_2+31m_1}|\log\delta|^{k+3}}{|C|^{k+2}|b_2-b_3|} \cdot \max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\gamma-1)}.$$

(ii) With the non-concentration on B, one has

$$|d_1\widetilde{D} + \underbrace{d_2\widetilde{D} + \cdots + d_2\widetilde{D}}_{k \text{ terms}}|_\delta$$
$$\lesssim \delta^{-O(\epsilon)}\frac{|B|^{2k+7}K^{(2m_2+5m_1)k+14m_2+31m_1}|\log\delta|^{k+3}}{|C|^{k+3}} \cdot \max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\eta-1)}.$$

Proof. As in the previous proof, we identify a δ-refinement D' of D, which will be needed when applying Lemma 2.3 later. Then we apply Lemma 2.4 to find a subset $\widetilde{D} \subset D'$ such that $|\widetilde{D}| \geq \delta^{O(\epsilon)}|D|$ for any given $\epsilon > 0$, so that

$$|\widetilde{D} + \widetilde{D}|_{\delta/d_2} \lesssim \delta^{-O(\epsilon)}\frac{|D+D|_\delta}{|D|}|d_2\widetilde{D}|_\delta.$$

On the other hand, using Theorem 1.6 and (17), we have

$$\frac{|D+D|_\delta}{|D|} \lesssim \frac{K^{2m_2+m_1}|B|}{|D|} = \frac{K^{2m_2+5m_1}|B|^2|\log\delta|}{|C|}.$$

Thus, when considering the sum of k terms, applying Lemma 2.3, we have

$$|\underbrace{d_2\widetilde{D}+\cdots+d_2\widetilde{D}}_{k\text{ terms}}|_\delta \lesssim \delta^{-O(\epsilon)}\frac{K^{(2m_2+5m_1)(k-1)}|B|^{2(k-1)}|\log\delta|^{k-1}}{|C|^{k-1}}\cdot|d_2\widetilde{D}|_\delta.$$

Next, applying (20) from the proof of the previous lemma with $k=1$, one obtains

$$|c^*\widetilde{D}+d_1D+d_2D|_\delta \lesssim \frac{|B|^7K^{16m_2+32m_1}|\log\delta|^3}{|C|^3}.$$

Thus,

$$\begin{aligned}
&|c^*\widetilde{D}+d_1\widetilde{D}+\underbrace{d_2\widetilde{D}+\cdots+d_2\widetilde{D}}_{k\text{ terms}}|_\delta\\
&\lesssim \frac{|d_2\widetilde{D}+d_2\widetilde{D}+\cdots+d_2\widetilde{D}|_\delta}{|d_2\widetilde{D}|_\delta}\cdot\frac{|c^*\widetilde{D}+d_1\widetilde{D}+d_2\widetilde{D}|_\delta}{|d_2\widetilde{D}|_\delta}\cdot|d_2\widetilde{D}|_\delta\\
&\lesssim \delta^{-O(\epsilon)}\frac{|B|^7K^{16m_2+32m_1}|\log\delta|^3}{|C|^3}\cdot\frac{K^{(2m_2+5m_1)(k-1)}|B|^{2(k-1)}|\log\delta|^{k-1}}{|C|^{k-1}}.
\end{aligned}$$

This gives the bound

$$\delta^{-O(\epsilon)}\frac{|B|^{2k+5}K^{(2m_2+5m_1)k+14m_2+27m_1}|\log\delta|^{k+2}}{|C|^{k+2}}. \tag{21}$$

Now based on the same argument as in the previous proof, we use different non-concentration conditions to bound the following estimate

$$|d_1\widetilde{D}+\underbrace{d_2\widetilde{D}+\cdots+d_2\widetilde{D}}_{k\text{ terms}}|_\delta \lesssim |c^*\widetilde{D}|_\delta^{-1}|c^*\widetilde{D}+d_1\widetilde{D}+\underbrace{d_2\widetilde{D}+\cdots+d_2\widetilde{D}}_{k\text{ terms}}|_\delta.$$

(i) Non-concentration on C:

Using the non-concentration condition on C, one has the first term is bounded below by

$$|c^*\widetilde{D}|_d \gtrsim \frac{|D|}{d|b_2-b_3|^{-1}|C|\delta^{\epsilon_0(\gamma-1)}} \gtrsim \frac{|\log\delta|^{-1}|b_2-b_3|}{|B|K^{4m_1}d\delta^{\epsilon_0(\gamma-1)}}.$$

Combining with the estimate for the second term (21), we get

$$\delta^{-O(\epsilon)}\frac{|B|^{2k+6}K^{(2m_2+5m_1)k+14m_2+31m_1}|\log\delta|^{k+3}}{|C|^{k+2}|b_2-b_3|}\cdot\max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\gamma-1)}.$$

(ii) Non-concentration on B:

In this case, the only difference is that

$$|c^*\widetilde{D}|_d \gtrsim \frac{|D|}{d\delta^{\epsilon_0(\eta-1)}|B|} \gtrsim \frac{|C||\log\delta|^{-1}}{|B|^2K^{4m_1}d\delta^{\epsilon_0(\eta-1)}}.$$

This follows by the assumption on B. Therefore, we obtain the bound

$$\delta^{-O(\epsilon)}\frac{|B|^{2k+7}K^{(2m_2+5m_1)k+14m_2+31m_1}|\log\delta|^{k+3}}{|C|^{k+3}}\cdot\max\{|d_1|,|d_2|\}\delta^{\epsilon_0(\eta-1)}.$$

This completes the proof of the lemma. ☐

We now consider the set

$$R := R(\widetilde{D}) = \left\{ \frac{d_1 - d_2}{d_3 - d_4} : d_i \in \widetilde{D}, |d_3 - d_4| > \delta^{\kappa} \right\}.$$

Since $R(\widetilde{D}) \subseteq R(C^{'})$ and $C^{'} \subset [1/2, 1]$, we have that $R \subset [-\delta^{-\kappa}/2, \delta^{-\kappa}/2]$.

Lemma 7.3. *Suppose $\widetilde{D}$ is non-empty. We have either R is non-empty or*

$$K \gtrsim |\log \delta|^{O(1)} \delta^{-\frac{\gamma - 2\beta + \kappa\eta}{4}}. \tag{22}$$

Proof. To prove that R is non-empty, it is enough to show that there are at least two elements $d_3, d_4 \in \widetilde{D}$ such that $|d_3 - d_4| > \delta^{\kappa} = \delta^{\epsilon_0}$. Since the set $\widetilde{D}$ is assumed to be non-empty, there exists $d_3 \in \widetilde{D}$. If $|\widetilde{D}| \gtrsim \delta^{\epsilon_0 \eta}|B|$, then the existence of d_4 follows from the non-concentration of B. Otherwise, from the lower bound on the size of $\widetilde{D}$ in (17), we have

$$K^{4m_1} \gtrsim \frac{|C| |\log \delta|^{-1}}{|B|^2 \delta^{\epsilon_0 \eta}}.$$

From this, we will get the lower bound for K and complete the proof of the lemma. □

Note that we also have $0, 1 \in R$. Choose a positive integer m so that $2^{-m} \sim \delta^{1-2\kappa}|b_2 - b_3|$. Define $s = 2^{-m}$.

The following result was proved in [15], and it turns out to be useful for our proofs.

Lemma 7.4. *At least one of the following two things must happen.*

(A): There exists a point $r \in R \cap [0, 1]$ with

$$\max \left\{ \operatorname{dist}(r/2,\ R), \quad \operatorname{dist}\left(\frac{r+1}{2},\ R\right) \right\} \geq s.$$

(B): $|R \cap [0, 1]|_s \gtrsim s^{-1}$.

As in [15], we refer to the case (A) as the "gap" case, and (B) as the "dense" case. We proceed to obtain lower bounds for K corresponding to each case (A) and (B).

7.1 Dense case

Suppose we are in the dense case (B), that is $|R \cap [0, 1]|_s \gtrsim s^{-1}$. By pigeonholing, we can select points $c_1, c_2, c_3, c_4 \in C^{'}$ with $|c_3 - c_4||b_2 - b_3| > \delta^{\kappa}$ and $|c_1 - c_2| \leq |c_3 - c_4|$ so that

$$\left| \left\{ (d_1, ..., d_4) \in \widetilde{D}^4 \colon \left| \frac{d_1 - d_2}{d_3 - d_4} - \frac{c_1 - c_2}{c_3 - c_4} \right| < \delta^{1-2\kappa}|b_2 - b_3|,\ |d_3 - d_4| > \delta^{\kappa} \right\} \right| \lesssim |\widetilde{D}|^4 \delta^{1-2\kappa}|b_2 - b_3|. \tag{23}$$

Here and below, we write $d_1' - d_2' = (b_2 - b_3)(c_1 - c_2)$ and $d_3' - d_4' = (b_2 - b_3)(c_3 - c_4)$.

Observe that $|c_3 - c_4| > \delta^{\kappa} = \delta^{\epsilon_0}$. Applying Lemma 7.2 with $k = 1$, the following quantity

$$\left|(d_1' - d_2')\widetilde{D} + (d_3' - d_4')\widetilde{D}\right|_{\delta|b_2 - b_3|} = \left|(c_1 - c_2)\widetilde{D} + (c_3 - c_4)\widetilde{D}\right|_{\delta}$$

is bounded from above by

$$\delta^{-O(\epsilon)} \frac{|B|^8 K^{16m_2 + 36m_1} |\log \delta|^4}{|C|^3 |b_2 - b_3|} \cdot |c_3 - c_4| \delta^{\epsilon_0(\gamma - 1)} \tag{24}$$

if we use the non-concentration condition on C, and by

$$\delta^{-O(\epsilon)}\frac{|B|^9K^{16m_2+36m_1}|\log\delta|^4}{|C|^4}\cdot|c_3-c_4|\delta^{\epsilon_0(\eta-1)}, \tag{25}$$

if we use the non-concentration condition on B instead.

We will now establish a lower bound for $\left|(d_1'-d_2')\widetilde{D}+(d_3'-d_4')\widetilde{D}\right|_{\delta|b_2-b_3|}$. Define $Q\subset\widetilde{D}^4$ to be the set of quadruples obeying

$$(d_3'-d_4')d_1+(d_1'-d_2')d_4=(d_3'-d_4')d_2+(d_1'-d_2')d_3+O(\delta|b_2-b_3|). \tag{26}$$

By the Cauchy-Schwarz inequality, one has $\left|(d_1'-d_2')\widetilde{D}+(d_3'-d_4')\widetilde{D}\right|_\delta\gtrsim|\widetilde{D}|^4/|Q|$. Thus our goal is now to find an upper bound for $|Q|$. Note that equation (26) can be written as

$$d_1+\frac{c_1-c_2}{c_3-c_4}d_4=d_2+\frac{c_1-c_2}{c_3-c_4}d_3+O(\delta|d_3'-d_4'|^{-1}|b_2-b_3|),$$

which implies that

$$\left|\frac{d_1-d_2}{d_3-d_4}-\frac{c_1-c_2}{c_3-c_4}\right|\lesssim\delta|d_3'-d_4'|^{-1}|d_3-d_4|^{-1}|b_2-b_3|. \tag{27}$$

At this step, we consider two separate cases:

(i): At least $\frac{|Q|}{2}$ quadruples $(d_1,...,d_4)\in Q$ satisfy $|d_3-d_4|>\delta^\kappa$. For each such quadruple, from inequality (27) above, one has

$$\left|\frac{d_1-d_2}{d_3-d_4}-\frac{c_1-c_2}{c_3-c_4}\right|\lesssim\delta^{1-2\kappa}|b_2-b_3|.$$

Comparing with (23), we see that the number of such quadruples is $\lesssim|\widetilde{D}|^4\delta^{1-2\kappa}|b_2-b_3|$. Thus if at least half the quadruples $(d_1,...,d_4)\in Q$ satisfy $|d_3-d_4|>\delta^\kappa$, then

$$\left|(c_1-c_2)\widetilde{D}+(c_3-c_4)\widetilde{D}\right|_\delta\gtrsim\frac{|\widetilde{D}|^4}{|Q|}\gtrsim\delta^{2\kappa-1}|b_2-b_3|^{-1}.$$

Combining with the upper bound in (24), we have

$$\delta^{-O(\epsilon)}\frac{|B|^8K^{16m_2+36m_1}|\log\delta|^4}{|b_2-b_3||C|^3}\cdot|c_3-c_4|\delta^{\epsilon_0(\gamma-1)}\gtrsim\frac{\delta^{2\kappa-1}}{|b_2-b_3|}.$$

This infers that

$$K^{16m_2+36m_1}\gtrsim|\log\delta|^{O(1)}\delta^{O(\epsilon)}\frac{|C|^3}{\delta^{1-2\kappa}|c_3-c_4|\delta^{\epsilon_0(\gamma-1)}|B|^8}.$$

In other words, one obtains a lower bound for K, namely

$$K\gtrsim|\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{3\gamma-8\beta+1-\kappa(3-\gamma)}{16m_2+36m_1}}. \tag{28}$$

Here we use the fact that $1>|c_3-c_4|>\delta^\kappa$. Similarly, if we use the bound (25) in place of (24), we have

$$K\gtrsim|\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{4\gamma-9\beta+1-\kappa(3-\eta)}{16m_2+36m_1}}. \tag{29}$$

(ii) More than half of the quadruples $(d_1,...,d_4)\in Q$ satisfy $|d_3-d_4|\le\delta^\kappa$. We will make use of non-concentration assumptions on C and B again to estimate the upper bound for the number of these quadruples. As a result, we will obtain the corresponding lower bounds for K.

(a) Non-concentration on C: We begin by choosing elements $d_1, d_4 \in \widetilde{D}$. Since $\widetilde{D} \subset (b_2 - b_3)C$, according to the Frostman condition on C and the requirement that $|d_3 - d_4| \leq \delta^\kappa$, the number of admissible d_3 is at most

$$|\widetilde{D} \cap B(d_4, \delta^\kappa)| \leq \frac{|C|\delta^\kappa}{|b_2 - b_3|}\delta^{\epsilon_0(\gamma-1)}.$$

Here we used the fact that $\frac{\delta^\kappa}{|b_2-b_3|} \geq \delta^\kappa = \delta^{\epsilon_0}$.

Next, observe that from (27), d_2 must lie in an interval of length at most $\delta|c_3 - c_4|^{-1}$. Notice that since $\epsilon_0 + \kappa < 1$, we must have

$$\frac{\delta}{|c_3 - c_4||b_2 - b_3|} = \frac{\delta}{|d_3' - d_4'|} < \delta^{1-\kappa} < \delta^{\epsilon_0}.$$

The Frostman condition on C yields that the number of admissible d_2 is bounded by

$$\delta^\gamma |C||c_3 - c_4|^{-\gamma}|b_2 - b_3|^{-\gamma} = |d_3' - d_4'|^{-\gamma} \lesssim \delta^{-\kappa\gamma}.$$

Thus the set of quadruples of this type has size at most $\frac{|\widetilde{D}|^2|C|\delta^\kappa\delta^{\epsilon_0(\gamma-1)}}{|b_2-b_3|\delta^{\kappa\gamma}}$. From (17), one has

$$|(c_1 - c_2)\widetilde{D} + (c_3 - c_4)\widetilde{D}|_\delta \gtrsim \frac{|\widetilde{D}|^4}{|Q|} \gtrsim \frac{|\widetilde{D}|^2|b_2 - b_3|}{|C|\delta^{\kappa+\epsilon_0(\gamma-1)}\delta^{-\kappa\gamma}} \gtrsim \delta^{O(\epsilon)}\frac{|C||\log\delta|^{-2}|b_2 - b_3|}{K^{8m_1}|B|^2}.$$

Altogether, the lower and the upper bounds from (24) imply that

$$K^{16m_2+44m_1} \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\frac{|C|^4}{|B|^{10}}\frac{|b_2 - b_3|^2}{|c_3 - c_4|\delta^{\epsilon_0(\gamma-1)}}.$$

Using the fact that $|b_2 - b_3||c_3 - c_4| > \delta^\kappa$, one gets

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{4\gamma-10\beta-\kappa(3-\gamma)}{16m_2+44m_1}}. \tag{30}$$

On the other hand, if we use (25) instead of (24), we have

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{5\gamma-11\beta-\kappa(2-\eta)}{16m_2+44m_1}}. \tag{31}$$

(b) Non-concentration on B:
Similarly, for given elements $d_4 \in \widetilde{D}$, the Frostman condition on B implies that the number of admissible d_3 is at most $|B|\delta^{\kappa\eta}$.
Next, observe that from inequality (27), we must have d_2 lie in an interval of length at most $\delta|c_3 - c_4|^{-1}$, for any fixed $d_1 \in \widetilde{D}$. Because $\epsilon_0 + \kappa < 1$, one has $\delta|c_3 - c_4|^{-1} < \delta^{1-\kappa} < \delta^{\epsilon_0}$. Hence the number of admissible d_2 is at most $\delta^\eta|B||c_3 - c_4|^{-\eta}$, where we make use of the Frostman condition on B.
Altogether, the set of quadruples of this type has size at most $\frac{|\widetilde{D}|^2|B|^2\delta^{\kappa\eta+\eta}}{|c_3-c_4|^\eta}$, which implies

$$|(c_1 - c_2)\widetilde{D} + (c_3 - c_4)\widetilde{D}|_\delta \gtrsim \frac{|\widetilde{D}|^4}{|Q|} \gtrsim \frac{|\widetilde{D}|^2|c_3 - c_4|^\eta}{|B|^2\delta^{\kappa\eta+\eta}} \gtrsim \delta^{O(\epsilon)}\frac{|C|^2|\log\delta|^{-2}}{|B|^4K^{8m_1}\delta^\eta}.$$

Then combining this estimate with the upper bound (24), one has

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{5\gamma-12\beta+\eta-\kappa(2-\gamma)}{16m_2+44m_1}}. \tag{32}$$

Using inequality (25) instead, one has

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{6\gamma-13\beta+\eta-\kappa(1-\eta)}{16m_2+44m_1}}. \tag{33}$$

7.2 Gap case

In this section, we will suppose that we are in the gap case (A). This means that there exists $r \in R \cap [0,1]$ so that either

(A.1) $r/2$ is at least s-separated from R or

(A.2) $\frac{r+1}{2}$ is at least s-separated from R, where $s \sim \delta^{1-2\kappa}|b_2 - b_3|$.

Notice that it follows from the definition of $\widetilde{D}$ that $r = \frac{c_1-c_2}{c_3-c_4}$ for some $c_1, c_2, c_3, c_4 \in C'$ with $|c_3 - c_4| \geq |c_1 - c_2|$ and $|c_3 - c_4| \geq \delta^{\kappa} = \delta^{\epsilon_0}$.

In Case (A.1), we write $r/2 = e_1/e_2$ with $e_1 = x_1$ and $e_2 = x_2 + x_2$, where $x_1, x_2 \in C' - C'$.

In Case (A.2), we write $\frac{r+1}{2}$ as e_1/e_2 with $e_1 = x_1 + x_2,\ e_2 = x_2 + x_2$, where $x_1, x_2 \in C' - C'$.

The first task in this section is to find a lower bound on $|e_1\widetilde{D} + e_2\widetilde{D}|_{\delta}$. One needs to keep in mind that $|e_2| \sim |c_3 - c_4|$.

Define $Q \subset \widetilde{D}^4$ to be the set of quadruples obeying

$$e_2 d_1 + e_1 d_4 = e_2 d_2 + e_1 d_3 + O(\delta). \tag{34}$$

As in the dense case, we only need to find an upper bound for $|Q|$. Since by the Cauchy-Schwarz inequality, one has $|e_1\widetilde{D} + e_2\widetilde{D}|_{\delta} \gtrsim \frac{|\widetilde{D}|}{|Q|}$.

Dividing equation (34) by e_2 gives

$$\left|\frac{d_1-d_2}{d_3-d_4} - \frac{e_1}{e_2}\right| \lesssim \delta|e_2|^{-1}|d_3-d_4|^{-1}. \tag{35}$$

Assume that there exists a quadruple $(d_1, ..., d_4) \in Q$ such that $|d_3 - d_4| \geq \delta^{\kappa}$. In other words, we have $\frac{d_1-d_2}{d_3-d_4} \in R$. Then using the fact that $|b_2 - b_3||e_2| \gtrsim \delta^{\kappa}$, equation (35) implies

$$\left|\frac{d_1-d_2}{d_3-d_4} - \frac{e_1}{e_2}\right| \lesssim \delta^{1-2\kappa}|b_2-b_3|.$$

Since we are in the gap case, $r = e_1/e_2$ is at least $s \sim \delta^{1-2\kappa}|b_2 - b_3|$ separated from R, which is a contradiction. It turns out that every quadruple in Q satisfies $|d_3 - d_4| \leq \delta^{\kappa}$.

As in the dense case, by using the non-concentration assumptions on C and B, respectively, we obtain the following bounds.

(i) Non-concentration on C:

$$|e_1\widetilde{D} + e_2\widetilde{D}|_{\delta} \gtrsim \delta^{O(\epsilon)}\frac{|C||\log\delta|^{-2}|b_2-b_3|}{K^{8m_1}|B|^2}. \tag{36}$$

(ii) Non-concentration on B:

$$|e_1\widetilde{D} + e_2\widetilde{D}|_{\delta} \gtrsim \delta^{O(\epsilon)}\frac{|C|^2|\log\delta|^{-2}}{|B|^4K^{8m_1}|B|^2\delta^{\eta}}. \tag{37}$$

In the next step, we apply Lemma 7.2 to get upper bounds on $|e_1\widetilde{D}+e_2\widetilde{D}|_\delta$. Then all possibilities for bounds of K will be examined.

(a) First upper bound

Recall that $|e_1\widetilde{D}+e_2\widetilde{D}|_\delta \lesssim |x_1\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}|_\delta$. Applying Lemma 7.2 (i) with $k=3$, one has

$$|x_1\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}|_\delta \lesssim \delta^{-O(\epsilon)} \cdot \frac{|B|^{12}K^{20m_2+46m_1}|\log\delta|^6|e_2|\delta^{\epsilon_0(\gamma-1)}}{|b_2-b_3||C|^5}.$$

Then one can combine with the lower bound (36) to get

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{6\gamma-14\beta-\kappa(3-\gamma)}{20m_2+54m_1}}. \tag{38}$$

Similarly, replacing (36) with the lower bound (37), one obtains

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{7\gamma-16\beta+\eta-\kappa(2-\gamma)}{20m_2+54m_1}}. \tag{39}$$

(b) Second upper bound

Applying Lemma 7.2(ii) with $k=3$, one has

$$|x_1\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}+x_2\widetilde{D}|_\delta \lesssim \delta^{-O(\epsilon)} \cdot \frac{|B|^{13}K^{20m_2+46m_1}|\log\delta|^6|e_2|\delta^{\kappa(\eta-1)}}{|C|^6}.$$

Incorporating with the lower bounds (36) and (37), we have

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{7\gamma-15\beta-\kappa(2-\eta)}{20m_2+54m_1}}, \tag{40}$$

and

$$K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{8\gamma-17\beta+\eta-\kappa(1-\eta)}{20m_2+54m_1}}. \tag{41}$$

7.3 Concluding the proof

Let us summarize the lower bounds for K here:

- (28) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{3\gamma-8\beta+1-\kappa(3-\gamma)}{16m_2+36m_1}}$,
- (29) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{4\gamma-9\beta+1-\kappa(3-\eta)}{16m_2+36m_1}}$,
- (30) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{4\gamma-10\beta-\kappa(3-\gamma)}{16m_2+44m_1}}$,
- (31) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{5\gamma-11\beta-\kappa(2-\eta)}{16m_2+44m_1}}$,
- (32) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{5\gamma-12\beta+\eta-\kappa(2-\gamma)}{16m_2+44m_1}}$,
- (33) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{6\gamma-13\beta+\eta-\kappa(1-\eta)}{16m_2+44m_1}}$,
- (38) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{6\gamma-14\beta-\kappa(3-\gamma)}{20m_2+54m_1}}$,
- (39) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{7\gamma-16\beta+\eta-\kappa(2-\gamma)}{20m_2+54m_1}}$,
- (40) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{7\gamma-15\beta-\kappa(2-\eta)}{20m_2+54m_1}}$,
- (41) $K \gtrsim |\log\delta|^{O(1)}\delta^{O(\epsilon)}\delta^{-\frac{8\gamma-17\beta+\eta-\kappa(1-\eta)}{20m_2+54m_1}}$.

where κ can be chosen arbitrarily in $(0, 1/2)$.

Now we put $M_0 = \delta^{-\frac{\gamma-\beta}{4}}$ and $M_1 = \delta^{-\frac{\gamma+2\beta+\kappa\eta}{4}}$, which are given by (18) and (22), respectively. Then define

$$\begin{aligned} M_2 &= \max\big\{(28), (29)\big\}, \\ M_3 &= \max\big\{(30), (31), (32), (33)\big\}, \\ M_4 &= \max\big\{(38), (39), (40), (41)\big\}. \end{aligned}$$

Following the proof, one can see that K can be bounded from below by

$$K \gtrsim |\log \delta|^{O(1)} \delta^{O(\epsilon)} \min\{M_0, M_1, M_2, M_3, M_4\}.$$

We complete the proof of Theorem 1.13.

8 Proof of Theorems 1.14, 1.15, 1.16, and 1.17

By the Cauchy-Schwarz inequality and Theorem 1.13, it can be seen easily that there exists $c \in C$ such that

$$|A + cB|_\delta \gtrsim \frac{K|B|^{1/2}}{|A|^{1/2}}|A|.$$

Therefore, to show $|A + cB|_\delta \geq \delta^{-\epsilon'}|A|$ for some $\epsilon' > 0$, we need

$$\frac{K|B|^{1/2}}{|A|^{1/2}} \gtrsim \delta^{-\epsilon'}. \tag{42}$$

On the other hand, from Theorem 1.13, we have

$$K \gtrsim |\log \delta|^{O(1)} \delta^{O(\epsilon)} \min\{M_0, M_1, M_2, M_3, M_4\}.$$

We now compute the ranges of γ corresponding to the above cases such that (42) holds, namely,

- (18) $\gamma > 2\alpha - \beta$.
- (22) $\gamma > 2\alpha - 4\beta - \kappa\eta$.
- (28) $\gamma > \frac{74\alpha - 66\beta - 1 + 3\kappa}{3+\kappa}$.
- (29) $\gamma > \frac{74\alpha - 65\beta - 1 + \kappa(3-\eta)}{4}$.
- (30) $\gamma > \frac{78\alpha - 68\beta + 3\kappa}{4+\kappa}$.
- (31) $\gamma > \frac{78\alpha - 67\beta + \kappa(2-\eta)}{5}$.
- (32) $\gamma > \frac{78\alpha - 66\beta - \eta + 2\kappa}{5+\kappa}$.
- (33) $\gamma > \frac{78\alpha - 65\beta - \eta + \kappa(1-\eta)}{6}$.
- (38) $\gamma > \frac{97\alpha - 83\beta + 3\kappa}{6+\kappa}$.
- (39) $\gamma > \frac{97\alpha - 81\beta - \eta + 2\kappa}{7+\kappa}$.
- (40) $\gamma > \frac{97\alpha - 82\beta + \kappa(2-\eta)}{7}$.
- (41) $\gamma > \frac{97\alpha - 80\beta - \eta + \kappa(1-\eta)}{8}$.

Set $\eta = \beta$ and choose $\kappa = \kappa(\alpha, \beta)$ close to zero.

Among (30, 31, 32, 33), the widest range for γ is

$$\gamma > \frac{78\alpha - 66\beta}{6},$$

which comes from (33).

Among (38, 39, 40, 41), the widest range for γ is

$$\gamma > \frac{97\alpha - 81\beta}{8},$$

which comes from (41).

On the other hand, since $\alpha \geq \beta$, one has

$$\frac{78\alpha - 66\beta}{6} > \frac{97\alpha - 81\beta}{8}.$$

This means that we end up with the following two ranges:

1. The case (18, 22, 28, 33, 41):

$$\gamma > \max\left\{2\alpha - \beta, \frac{74\alpha - 66\beta - 1}{3}, \frac{78\alpha - 66\beta}{6}\right\}.$$

   Using the fact $\alpha \geq \beta$, we conclude that

   If $\alpha < \frac{33}{35}\beta + \frac{1}{35}$, then

$$\gamma > \frac{78\alpha - 66\beta}{6}.$$

   If $\alpha > \frac{33}{35}\beta + \frac{1}{35}$, then

$$\gamma > \frac{74\alpha - 66\beta - 1}{3}.$$

2. The case (18, 22, 29, 33, 41):

$$\gamma > \max\left\{2\alpha - \beta, \frac{74\alpha - 65\beta - 1}{4}, \frac{78\alpha - 66\beta}{6}\right\}.$$

   If $\alpha > \frac{63\beta}{66} + \frac{1}{22}$, then

$$\gamma > \frac{74\alpha - 65\beta - 1}{4}.$$

   If $\alpha < \frac{63\beta}{66} + \frac{1}{22}$, then

$$\gamma > \frac{78\alpha - 66\beta}{6}.$$

Comparing between the two cases, we infer that (18, 22, 29, 33, 41) gives the best range for γ. Namely,

If $\alpha \leq \frac{63}{66}\beta + \frac{1}{22}$, then the condition

$$\gamma > \frac{78\alpha - 66\beta}{6}$$

would be enough to have (42). Moreover, we also have

$$K \gtrsim \min\left\{\delta^{-\frac{4\gamma - 9\beta + 1}{148}},\ \delta^{-\frac{6\gamma - 12\beta}{156}}\right\}.$$

If $\alpha > \frac{63}{66}\beta + \frac{1}{22}$, then the condition

$$\gamma > \frac{74\alpha - 65\beta - 1}{4}$$

would be enough to have (42). Similarly, we also have

$$K \gtrsim \min\left\{\delta^{-\frac{4\gamma-9\beta+1}{148}},\ \delta^{-\frac{6\gamma-12\beta}{156}}\right\}.$$

With these computations, we are ready to prove Theorem 1.14 and Theorem 1.15. Let us first recall the statements.

Theorem 8.1 (Theorem 1.14)**.** *Let $\alpha, \beta, \eta \in (0,1)$, $\beta \le \alpha \le (21\beta+1)/22$. Then, for every $\gamma \in ((78\alpha - 66\beta)/6, 1]$, there exist $\epsilon_0, \delta_0 \in (0, 1/2]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, and let $A, B \subset [0,1]$ be δ-separated sets satisfying the following hypotheses:*

- $|A| \le \delta^{-\alpha}$
- $|B| \ge \delta^{-\beta}$*, and B satisfies the following Frostman condition*

$$|B \cap B(x,r)| \le r^{\eta}\ |B|, \quad \forall x \in \mathbb{R}, \delta \le r \le \delta^{\epsilon_0}.$$

Further, let $C \subset [1/2, 1]$ be a δ-separated set with $|C \cap B(x,r)| \lesssim r^{\gamma}\ |C|$ for all $x \in \mathbb{R}$ and $0 < r < \delta^{\epsilon_0}$. Then, there exists $c \in C$ such that the following holds for any ϵ satisfying

$$0 < \epsilon < \min\left\{\frac{4\gamma - 74\alpha + 65\beta + 1}{444},\ \frac{6\gamma - 78\alpha + 66\beta}{468}\right\}.$$

If $G \subset A \times B$ is any subset with $|G| \ge \delta^{\epsilon}|A||B|$, then

$$|\pi_c(G)|_\delta \ge \delta^{-\epsilon}|A|.$$

Theorem 8.2 (Theorem 1.15)**.** *Let $\alpha, \beta, \eta \in (0,1)$, $\beta \le \alpha$ and $\alpha > (21\beta+1)/22$. Then, for every $\gamma \in ((74\alpha - 65\beta - 1)/4, 1]$, there exist $\epsilon_0, \delta_0 \in (0, 1/2]$, depending only on $\alpha, \beta, \gamma, \eta$, such that the following holds. Let $\delta \in 2^{-\mathbb{N}}$ with $\delta \in (0, \delta_0]$, and let $A, B \subset [0,1]$ be δ-separated sets satisfying the following hypotheses:*

- $|A| \le \delta^{-\alpha}$
- $|B| \ge \delta^{-\beta}$*, and B satisfies the following Frostman condition*

$$|B \cap B(x,r)| \le r^{\eta}\ |B|, \quad \forall x \in \mathbb{R}, \delta \le r \le \delta^{\epsilon_0}.$$

Further, let C be a δ-separated set in $[1/2, 1]$ with $|C \cap B(x,r)| \lesssim r^{\gamma}\ |C|$ for all $x \in \mathbb{R}$ and $0 < r < \delta^{\epsilon_0}$. Then, there exists $c \in C$ such that the following holds for any ϵ satisfying

$$0 < \epsilon < \min\left\{\frac{4\gamma - 74\alpha + 65\beta + 1}{444},\ \frac{6\gamma - 78\alpha + 66\beta}{468}\right\}.$$

If $G \subset A \times B$ is any subset with $|G| \ge \delta^{\epsilon}|A||B|$, then

$$|\pi_c(G)|_\delta \ge \delta^{-\epsilon}|A|.$$

The proofs of these two theorems are the same, so we only present a proof for the first one. Again, the proof is short and follows directly from the energy estimates.

Proof of Theorem 1.14. Let X be the set of $c \in C$ such that the conclusion of the theorem fails, i.e. for each $c \in X$, there exists $G_c \subset A \times B$ with $|G_c| \geq \delta^\epsilon |A||B|$ and $|\pi_c(G)|_\delta < \delta^{-\epsilon}|A|$. We want to show that $|X| < |C|$.

For each $c \in X$, by the Cauchy-Schwarz inequality, the number of tuples $(x, y, z, w) \in G_c^2$ such that $|(x + cy) - (z + cw)| \leq \delta$ is at least $\frac{|G_c|^2}{\delta^{-\epsilon}|A|}$, which equals $\delta^{3\epsilon}|A||B|^2$. Summing over all $c \in X$ and using Theorem 1.13, one has

$$|X|\delta^{3\epsilon}|A||B|^2 \leq \frac{|C||A|^{3/2}|B|^{3/2}}{K}.$$

This infers

$$|X| \leq \delta^{-3\epsilon}\frac{|A|^{1/2}}{|B|^{1/2}K}|C|.$$

Using the computations above, we know that

$$\frac{K|B|^{1/2}}{|A|^{1/2}} \gtrsim \min\left\{\delta^{-\frac{4\gamma-74\alpha+65\beta+1}{148}},\ \delta^{-\frac{6\gamma-78\alpha+66\beta}{156}}\right\}.$$

Thus,

$$\delta^{-3\epsilon}\frac{|A|^{1/2}}{|B|^{1/2}K} \lesssim \delta < 1$$

as long as

$$\epsilon < \min\left\{\frac{4\gamma - 74\alpha + 65\beta + 1}{444},\ \frac{6\gamma - 78\alpha + 66\beta}{468}\right\}.$$

This completes the proof of the theorem. □

With these two theorems, we can run the same argument as we did in the proof of Theorem 1.8 to prove the Hausdorff dimensional versions, beginning with Theorem 1.16, which we restate below.

Theorem 8.3. *Let $0 < \beta \leq \alpha < 1$ with $22\alpha \leq 21\beta + 1$. If $A, B \subset \mathbb{R}$ are compact sets with $\dim_H(A) = \alpha$, $\dim_H(B) = \beta$, then, for any*

$$\sigma > \frac{39(21\beta + 1 - 22\alpha)}{699},$$

we have

$$\dim_H \{c \in \mathbb{R} \colon \dim_H(A + cB) < \alpha + x\} \leq \frac{78\alpha - 66\beta}{6} + \sigma,$$

for any x smaller than

$$\frac{1}{2}\cdot\min\left\{\frac{2\sigma - 22\alpha + 21\beta + 1}{518},\ \frac{\sigma}{182}\right\}.$$

Proof of Theorem 1.16. Without loss of generality, we may assume that A and B are compact sets in $[0, 1]$. Using Frostman's lemma ([25, Theorem 8.8]), we can find probability measures μ_A and μ_B supported on A and B, respectively, such that

$$\mu_A(B(x, r)) \lesssim r^\alpha,\ \mu_B(B(x, r)) \lesssim r^\beta.$$

We want to show that

$$\dim_H (E) \leq \frac{78\alpha - 66\beta}{6} + \sigma, \tag{43}$$

where $E := \{c \in [1/2, 1] \colon \dim_H(A + cB) < \alpha + x\}$, and x is a parameter to be chosen later.

This would imply the full conclusion for $c \in \mathbb{R}$. Since if the result holds for any $A, B \subset \mathbb{R}$, and

$c \in [1/2, 1]$, then we write $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} C_n \cup \{0\} \bigcup_{n\in\mathbb{Z}} (-C_n)$ where $C_n = [2^n, 2^{n+1}]$. Hence for each n, i.e. when $c \in C_n$, we know that $\dim_H(A+cB) = \dim_H(2^{-(n+1)}(A+cB)) = \dim_H(2^{-(n+1)}A+c'B) = \dim_H(A'+c'B)$, where $A' = 2^{-(n+1)}A$. But $\dim_H A' = \dim_H A = \alpha$ and $c' \in [1/2, 1]$. Hence it can be reduced to the assumption above.

Suppose the inequality (43) does not hold, i.e.

$$\dim_H(E) > \frac{78\alpha - 66\beta}{6} + \sigma,$$

then we can find a probability measure ν supported on E such that $\nu(B(z,r)) \lesssim r^\gamma$ for all $z \in \mathbb{R}$ and $r > 0$, where $\gamma \geq \frac{78\alpha-66\beta}{6} + \sigma$. We note that in our argument we identify the measure ν with the maximal δ-separated subset $C \subset \mathrm{spt}(\nu)$. Notice that $|C| \sim \delta^{-\gamma}$ since $\nu(E) = 1$, and $|C \cap B(z,r)| \lesssim r^\gamma |C|$ for all $z \in \mathbb{R}$ and $r > 0$.

Set

$$\overline{\gamma} := \frac{78\alpha - 66\beta}{6} + \frac{\sigma}{2}.$$

We now choose parameters $\overline{\alpha}, \overline{\beta}$ such that

$$\overline{\alpha} > \alpha,\ \overline{\beta} < \beta,\ \overline{\gamma} > \frac{78\overline{\alpha} - 66\overline{\beta}}{6}.$$

Indeed, we can set $\overline{\alpha} = \alpha + 2x$, $\overline{\beta} = \beta - y$ for some $x, y > 0$ satisfying $\frac{156x+66y}{6} < \sigma/2$.

Now, we apply Theorem 1.14 with parameters $\overline{\alpha}, \overline{\beta}, \overline{\gamma},\ \eta = \overline{\beta}$, we obtain $\overline{\epsilon}, \overline{\epsilon_0}, \overline{\delta_0}$. In particular, if $G \subset A \times B$ is any subset with $|G| \geq \delta^{\overline{\epsilon}}|A||B|$, then

$$|\pi_c(G)|_\delta \geq \delta^{-\overline{\epsilon}}|A|,$$

where

$$\overline{\epsilon} \geq \min\left\{\frac{4\overline{\gamma} - 74\overline{\alpha} + 65\overline{\beta} + 1 - \zeta}{444},\ \frac{6\overline{\gamma} - 78\overline{\alpha} + 66\overline{\beta} - \zeta}{468}\right\},$$

for any $\zeta > 0$.

To proceed further, we need to do some computations:

$$\frac{4\overline{\gamma} - 74\overline{\alpha} + 65\overline{\beta} + 1 - \zeta}{444} = \frac{2\sigma - 22\alpha + 21\beta - 148x - 65y + 1 - \zeta}{444},$$

and

$$\frac{6\overline{\gamma} - 78\overline{\alpha} + 66\overline{\beta} - \zeta}{468} = \frac{3\sigma - 156x - 66y - \zeta}{468}.$$

Choose

$$2x = \min\left\{\frac{2\sigma - 22\alpha + 21\beta - 65y + 1 - 2\zeta}{518},\ \frac{3\sigma - 66y - 2\zeta}{546}\right\} \in (0,1).$$

It is clear that $2x < \overline{\epsilon}$. We now need to check two conditions.

The first condition is that $156x + 66y < 3\sigma$. To guarantee this, one has to have

$$y < \frac{699\sigma - 39(21\beta + 1 - 22\alpha) + 78\zeta}{14559},\ \sigma > \frac{39(21\beta + 1 - 22\alpha) - 78\zeta}{699},$$

and

$$y < \frac{18\sigma + 2\zeta}{396}.$$

The second condition is that $x \in (0,1)$. This is clear when y and ζ are sufficiently close to zero.

With these facts in hand, the goal is to find a set G so that, on the one hand, we apply Theorem 1.14 to G as above to get a lower bound for $|\pi_c(G)|_\delta \geq \delta^{-\alpha-\overline{\epsilon}}$. On the other hand, we will also derive an upper bound such as $|\pi_c(G_c)|_\delta \leq \delta^{-\alpha-x}$. This immediately gives a contradiction since $x < \overline{\epsilon}$.

We now proceed to prove the result. Fix some parameters as above such that $\bar{\alpha} > \alpha$, $\bar{\beta} < \beta$, with $\eta = \bar{\beta}$ and $\bar{\gamma} < \gamma$ such that $\bar{\gamma}$ is still in the range associated with $\bar{\alpha}$ and $\bar{\beta}$. Before we apply Theorem 1.14, we would need to construct sets A_δ and B_δ from A, B so that these two sets satisfy the assumptions in Theorem 1.14. That is, the δ-separatedness and non-concentration conditions with these modified parameters.

Fix a point $c \in E$ so that $\dim_H(A + cB) < \alpha + x$ (recall that x can be made explicit as mentioned above). This gives that for a given number $\delta_0 := 2^{-j_0} \in 2^{-\mathbb{N}}$, there exists a family $\mathcal{I}_c$, a countable number of disjoint dyadic intervals of length $\ell(I) \leq \delta_0$, which covers $A + cB$ such that

$$\sum_{I \in \mathcal{I}_c} \ell(I)^{\alpha+x} \leq 1. \tag{44}$$

Consider the sets $\mathcal{T}_c := \{\pi_c^{-1}(I)\}_{I \in \mathcal{I}_c}$ which cover $A \times B$, so that

$$\int_E \sum_{T \in \mathcal{T}_c} (\mu_A \times \mu_B)(T)\, d\nu(c) = 1.$$

Let $\mathcal{I}_c^j := \{I \in \mathcal{I}_c : \ell(I) = 2^{-j}\}$ for $j \geq j_0$ and write $\mathcal{T}_c^j := \{\pi_c^{-1}(I)\}_{I \in \mathcal{I}_c^j}$. By the pigeonhole principle, there exists $j^* \geq j_0$ such that

$$\int_E \sum_{T \in \mathcal{T}_c^{j^*}} (\mu_A \times \mu_B)(T)\, d\nu(c) \gtrsim j^{*-2}.$$

Let us also denote $\delta := 2^{-j^*}$ for the index j^*. By the estimates above, we can find a subset $E'_\delta \subset E$ of measure $\nu(E'_\delta) \gtrsim j^{*-2} = \log(1/\delta)^{-2}$ such that for each $c \in E'_\delta$, the sets $T \in \mathcal{T}_c^{j^*}$ cover a subset $G_c \subset \operatorname{spt}(\mu_A \times \mu_B)$ of measure $(\mu_A \times \mu_B)(G_c) \gtrsim \log(1/\delta)^{-2}$. Moreover, note that we have

$$|\pi_c(G_c)|_\delta \leq \delta^{-\alpha-x}, \qquad c \in E'_\delta, \tag{45}$$

by (44). Recall that, we use $f \lessapprox g$ to denote $f \lesssim \log(1/\delta)^C g$ for some absolute constant $C > 0$, which may only depend on the Frostman constants of A, B . In particular, $j^{-2} = \log(1/\delta)^{-2} \gtrapprox 1$.

So now we need to construct two sets A_δ and B_δ from A, B so that we can apply Theorem 1.14 to them to get a lower bound for (45). The process of constructing these sets is to perform some averaging arguments to extract these sets through pigeonholing. Now for $z \in \mathbb{R}$, let $I_\delta(z) \in \mathcal{D}_\delta$ be the unique dyadic interval of length δ with $z \in I_\delta(z)$. Fix $c \in C$ and given a dyadic number $\rho \in 2^{-\mathbb{N}}$, let $A(\rho) := \{z \in A : \rho \leq \mu_A(I_\delta(z)) < 2\rho\}$ and write A as

$$A = \bigcup_{\rho \in 2^{-\mathbb{N}}} A(\rho).$$

The set $B(\rho) \subset B$ is defined in the same way. Since $\mu_A(I_\delta(z)) \lesssim \delta^\alpha$ and $\mu_B(I_\delta(z')) \lesssim \delta^\beta$, we see that $A(\rho) \neq \emptyset$ implies $\rho \lesssim \delta^\alpha$, and $B(\rho) \neq \emptyset$ implies $\rho \lesssim \delta^\beta$. We also note that $A(\rho)$ can be expressed as the intersection of A with certain dyadic intervals $\mathcal{A}(\rho) \subset \mathcal{D}_\delta$. The same is true for $B(\rho)$, for certain dyadic intervals $\mathcal{B}(\rho) \subset \mathcal{D}_\delta$.

Let $\mu_A(\rho)$ be the restriction of μ_A to the intervals $\mathcal{A}(\rho)$, and similarly let $\mu_B(\rho)$ be the restriction of μ_B to the intervals in $\mathcal{B}(\rho)$. Then

$$\sum_{\rho_1}\sum_{\rho_2}\int_{E'_\delta}(\mu_A(\rho_1)\times\mu_B(\rho_2))(G_c)\approx 1,$$

so it follows from the pigeonhole principle that

$$\int_{E'_\delta}(\mu_A(\rho_A)\times\mu_B(\rho_A))(G_c)\approx 1$$

for some fixed choices $\rho_A\lessapprox\delta^\alpha$ and $\rho_B\lessapprox\delta^\beta$. By the pigeonhole principle again, we can find a subset $E_\delta\subset E'_\delta$ so that $(\mu_A(\rho_A)\times\mu_B(\rho_B))(G_c)\approx 1$ for all $c\in E_\delta$. We now define

$$\bar\mu_A:=\mu_A(\rho_A)\quad\text{and}\quad\bar\mu_B:=\mu_B(\rho_B),$$

so $\|\bar\mu_A\|\approx 1\approx\|\bar\mu_B\|$. The measure $\bar\mu_A$ is supported on the closure of the intervals in $\mathcal{A}(\rho_A)$, and $\bar\mu_B$ is supported on the closure of the intervals in $\mathcal{B}(\rho_B)$. Let

$$A_\delta:=(\delta\cdot\mathbb{Z})\cap(\cup\mathcal{A}(\rho_A))\quad\text{and}\quad B_\delta:=(\delta\cdot\mathbb{Z})\cap(\cup\mathcal{B}(\rho_B)).$$

Note that

$$\rho_A\cdot|A_\delta|\sim\|\mu_A\|\approx 1\quad\Longrightarrow\quad\rho_A\approx|A_\delta|^{-1},$$

and similarly $\rho_B\approx|B_\delta|^{-1}$. Since $\rho_A\lessapprox\delta^\alpha$, we have

$$|A_\delta|\approx\rho_A^{-1}\gtrapprox\delta^{-\alpha}.$$

In fact one also has $|A_\delta|\le\delta^{-\bar\alpha}$ if $\delta>0$ is sufficiently small. To see this, fix an arbitrary $c\in E_\delta$. Since $(\bar\mu_A\times\bar\mu_B)(G_c)\approx 1$, there exists $b\in\operatorname{spt}(\bar\mu_B)$ such that

$$\bar\mu_A(G_c(b))\approx 1,\quad\text{where}\quad G_c(b)=\{z\in\operatorname{spt}(\bar\mu_A):(z,b)\in G_c\}.$$

Let $\mathcal{H}_c(b):=\{I\in\mathcal{A}(\rho_A):G_c(b)\cap I\neq\emptyset\}$, we have that $\bar\mu_A(I)\sim\rho_A$ for all $I\in\mathcal{H}_c(b)$, and $\bar\mu_A(\cup\mathcal{H}_c(b))\ge\bar\mu_A(G_c(b))\approx 1$. Moreover, we observe that $|G_c(b)|_\delta\lessapprox|\pi_c(G_c)|_\delta$, since $\pi_c(G_c)\supset G_c(b)+bc$. Therefore we obtain

$$|A_\delta|\approx\rho_A^{-1}\lessapprox\rho_A^{-1}\cdot\bar\mu_A(\cup\mathcal{H}_c(b))\lessapprox|G_c(b)|_\delta\lessapprox|\pi_c(G_c)|_\delta\le\delta^{-\alpha-x}.$$

For x chosen at the beginning of the proof,

$$x<\min\{\bar\epsilon,\bar\alpha-\alpha\}.\tag{46}$$

Hence it is clear to see that $\alpha+x<\bar\alpha$ by (46) which in turn gives $|A_\delta|\le\delta^{-\bar\alpha}$ if $\delta>0$ is sufficiently small.

Next, we also need to claim the non-concentration condition for B_δ. First we note that $\rho_B\lessapprox\delta^\beta$, which gives

$$|B_\delta|\approx\rho_B^{-1}\gtrapprox\delta^{-\beta}\ge\delta^{-\bar\beta}.\tag{47}$$

Moreover, for $r\ge\delta$, we note that every point $z'\in B_\delta\cap B(z,r)$ is contained in an interval $I_{z'}(\delta)\in\mathcal{B}(\rho_B)$ with $\mu_B(I_{z'}(\delta))\ge\rho_B$. Since $I_{z'}(\delta)\subset B(z,2r)$, we have

$$|B_\delta\cap B(z,r)|\le\rho_B^{-1}\cdot\mu_B(B(z,2r))\lessapprox\rho_B^{-1}\cdot(2r)^\beta\lessapprox r^\beta|B_\delta|.$$

Therefore with the constant $\bar\epsilon_0>0$ mentioned above, we also have $|B_\delta\cap B(z,r)|\le r^{\bar\beta}|B_\delta|$ for $\delta\le r\le\delta^{\bar\epsilon_0}$ as long as $\delta>0$ is sufficiently small. Combining all the estimates above, A_δ,B_δ are now constructed. Now we apply Theorem 1.14 to obtain $|\pi_c(G_c)|_\delta \gtrapprox \delta^{-\overline{\epsilon}-\alpha}$. This immediately contradicts (45), since x is chosen to be smaller than $\overline{\epsilon}$. $\square$

Next, we recall the statement for Theorem 1.17.

Theorem 8.4. *Let $0 < \beta \leq \alpha < 1$ with $22\alpha > 21\beta + 1$. If $A, B \subset \mathbb{R}$ are compact sets with $\dim_H(A) = \alpha$, $\dim_H(B) = \beta$, then, for any*

$$\sigma > \frac{74(33\alpha - \frac{63}{2}\beta - \frac{3}{2})}{870},$$

we have

$$\dim_H \{c \in \mathbb{R} \colon \dim_H(A + cB) < \alpha + x\} \leq \frac{74\alpha - 65\beta - 1}{4} + \sigma,$$

for any x smaller than

$$\frac{1}{2} \cdot \min \left\{ \frac{3\sigma + 33\alpha - \frac{63\beta}{2} - \frac{3}{2}}{546}, \ \frac{\sigma}{259} \right\}.$$

Proof of Theorem 1.17. The proof is the same as the previous proof except that we have to compute the parameters. We do the computations below.

Set

$$\overline{\gamma} := \frac{74\alpha - 65\beta - 1}{4} + \frac{\sigma}{2}.$$

We now choose parameters $\overline{\alpha}, \overline{\beta}$ such that

$$\overline{\alpha} > \alpha, \ \overline{\beta} < \beta, \ \overline{\gamma} > \frac{74\overline{\alpha} - 65\overline{\beta} - 1}{4}.$$

Indeed, we can set $\overline{\alpha} = \alpha + 2x$, $\overline{\beta} = \beta - y$ for some $x, y > 0$ satisfying $\frac{148x+65y}{4} < \sigma/2$.

Next, we apply Theorem 1.15 with parameters $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$, $\eta = \overline{\beta}$, to obtain $\overline{\epsilon}, \overline{\epsilon_0}, \overline{\delta_0}$. So taking $G \subset A \times B$ to be any subset with $|G| \geq \delta^{\overline{\epsilon}}|A||B|$, then

$$|\pi_c(G)|_\delta \geq \delta^{-\overline{\epsilon}}|A|,$$

where

$$\overline{\epsilon} \geq \min \left\{ \frac{4\overline{\gamma} - 74\overline{\alpha} + 65\overline{\beta} + 1 - \zeta}{444}, \ \frac{6\overline{\gamma} - 78\overline{\alpha} + 66\overline{\beta} - \zeta}{468} \right\},$$

for any $\zeta > 0$.

To proceed further, we need to do some computations:

$$\frac{4\overline{\gamma} - 74\overline{\alpha} + 65\overline{\beta} + 1 - \zeta}{444} = \frac{2\sigma - 148x - 65y - \zeta}{444},$$

and

$$\frac{6\overline{\gamma} - 78\overline{\alpha} + 66\overline{\beta} - \zeta}{468} = \frac{3\sigma + 33\alpha - \frac{63\beta}{2} - 156x - 66y - \frac{3}{2} - \zeta}{468}.$$

Choose

$$2x = \min \left\{ \frac{2\sigma - 65y - 2\zeta}{518}, \ \frac{3\sigma + 33\alpha - \frac{63\beta}{2} - 66y - \frac{3}{2} - 2\zeta}{546} \right\} \in (0, 1).$$

It is clear that $2x < \overline{\epsilon}$. We now need to check two conditions.

The first condition is that $148x + 65y < 2\sigma$. To guarantee this, one has to have

$$y < \frac{870\sigma - 74(33\alpha - \frac{63}{2}\beta - \frac{3}{2}) + 148\zeta}{30606}, \quad \sigma > \frac{74(33\alpha - \frac{63}{2}\beta - \frac{3}{2}) - 148\zeta}{870},$$

and

$$y < \frac{370\sigma + 148\zeta}{28860}.$$

The second condition is that $x \in (0,1)$. This is clear when y and ζ are small enough, i.e. close to zero.

$\square$

9 On the $C(A + A)$ problem

This section is devoted to prove results on the $C(A+A)$ problem mentioned in the introduction. Let us recall all statements here. (That is Theorems 1.9, 1.12 and 1.18 respectively).

Theorem 9.1. *Given $\alpha \in (0,1)$ and $\gamma, \eta > 0$, there exist $\epsilon_0, \epsilon > 0$ such that the following holds for all sufficiently small $\delta > 0$. Let $C \subset [1/2, 1]$ be a δ-separated set satisfying*

$$|C \cap B(x,r)| \lesssim r^{\gamma}|C|$$

for all $\delta \le r \le \delta^{\epsilon_0}$. Let additionally $A \subset [0,1]$ be a δ-separated set with $|A| = \delta^{-\alpha}$, which also satisfies the non-concentration condition $|A \cap B(x,r)| \le r^{\eta}|A|$ for $x \in \mathbb{R}$ and $\delta \le r \le \delta^{\epsilon_0}$.

Then, we have

$$|C(A+A)|_{\delta} \ge \delta^{-\epsilon}|A|.$$

Theorem 9.2. *Let $A \subset \delta\mathbb{Z} \cap [0,1]$ and $C \subset [1/2,1]$ be δ-separated. Suppose $|A| = \delta^{-\alpha}$ and $|C| = \delta^{-\gamma}$, with $\alpha, \gamma \in (1/2, 1)$. Assuming*

$$|A \cap B(x,r)| \le Mr^{\alpha}|A|, \ \forall \ \delta \le r \le 1, \ x \in \mathbb{R},$$

for some $M > 1$, for sufficiently small $\delta > 0$. Then there exists $\varepsilon > 0$ such that

$$|C(A+A)|_{\delta} \ge \delta^{-\varepsilon}|A|,$$

where

$$\varepsilon = \frac{2\alpha + 4\gamma - 4\alpha\gamma - 2}{6 - 4\alpha}.$$

Theorem 9.3. *Let $A \subset \delta\mathbb{Z} \cap [0,1]$ and $C \subset [1/2,1]$ be δ-separated. Suppose $|A| = \delta^{-\alpha}$, $|C| = \delta^{-\gamma}$ with $\gamma \in (2\alpha, 1)$, $\delta \in (0, \delta_0]$, and*

$$|A \cap B(x,r)| \lesssim r^{\eta}|A|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

and

$$|C \cap B(x,r)| \lesssim r^{\gamma}|C|, \ \forall x \in \mathbb{R}, \ \delta < r < \delta^{\epsilon_0},$$

for $\epsilon_0, \delta_0 \in (0, 1/2)$ depending on α, γ, η. Then for any

$$0 < \varepsilon < \min\left\{\frac{4\gamma - 9\alpha + 1}{148}, \frac{6\gamma - 12\alpha}{156}\right\},$$

we have

$$|C(A+A)|_{\delta} \ge \delta^{-\varepsilon}|A|.$$

All these theorems have the same proof as follows.

Let D be a maximal $\delta/2$-separated subset of $C(A+A)$, then

$$|D| \lesssim |C(A+A)|_{\delta/4} \sim |C(A+A)|_{\delta}.$$

Notice that for each $(a_1, a_2, c) \in A \times A \times C$, $c(a_1+a_2)$ is contained in a ball centered at some point $x \in D$, of radius at most $\delta/2$. Thus

$$\sum_{x \in D} |\{(a_1, a_2, c) \in A \times A \times C : |x - c(a_1+a_2)| \leq \delta/2\}|_{\delta} = |C||A|^2.$$

By the Cauchy-Schwarz inequality, we have

$$|A|^4|C|^2 \leq \left(\sum_{x \in D} |\{(a_1, a_2, c) \in A \times A \times C : |x - c(a_1+a_2)| \leq \delta/2\}|_{\delta}^2 \right) \cdot |D|.$$

This implies that $|C(A+A)|_{\delta} \gtrsim |D| \gtrsim \frac{|C|^2|A|^4}{N}$, where

$$N = \left|\{(a_1, a_2, a_3, a_4, c_1, c_2) \in A^4 \times C^2 : |c_1(a_1 - a_2) - c_2(a_3 - a_4)| \leq \delta\}\right|_{\delta}.$$

By the pigeonhole principle, there exists $c_1 \in C$ such that

$$|\{(a_1, a_2, a_3, a_4, c_2) \in A^4 \times C : |c_1(a_1 - a_2) - c_2(a_3 - a_4)| \leq \delta\}|_{\delta} \geq \frac{N}{|C|}.$$

Therefore

$$\begin{aligned} N &\leq |C| \cdot \left| \left\{ (a_1, a_2, a_3, a_4, c_2) \in A^4 \times C \colon |(a_1 - a_2) - \frac{c_2}{c_1}(a_3 - a_4)| \leq \frac{\delta}{c_1} \right\} \right|_{\delta} \\ &= |C| \cdot \left|\{(a_1, a_2, a_3, a_4, c) \in A^4 \times (c_1^{-1}C) \colon |(a_1 - a_2) - c(a_3 - a_4)| \leq 2\delta\}\right|_{\delta}. \end{aligned}$$

Now, we may conclude that $N \leq \frac{|A|^3|C|^2}{K}$, which implies $|C(A+A)|_{\delta} \geq |A|K$, where the lower bounds on K come from the three energy theorems (Theorems 1.6, 1.10 and 1.13) to get the desired conclusions.

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