Event-triggered control for differential-algebraic systems with arbitrary index

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Abstract

We investigate ...

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Nomenclature

$\mathbb{R}^{m\times n}$	set of all $m\times n$ matrix
$\mathbb{C}_-$	$=\{\ \lambda\in\mathbb{C}\ \mid\ \mathrm{Re}\,\lambda<0\ \}$
$\|x\|$	the Euclidean norm of $x\in\mathbb{R}^n$

1. Introduction

(adding later).....

2. Preliminaries

Consider the linear systems

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Ex(t)&=Ax(t)+Bu(t),\quad t\geq 0,\\ x(0)&=x^0,\end{aligned}\tag{1}$$

where $E,\ A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control vector, $\operatorname{rank}E=r<n$.

Definition 2.1. A matrix pair (E,A), $E,A\in\mathbb{R}^{n\times n}$ is called *regular* if there exists $s\in\mathbb{C}$ such that $\det(sE-A)$ is different from zero. Otherwise, if $\det(sE-A)=0$ for all $s\in\mathbb{C}$, then we say that (E,A) is *singular*.

If (E,A) is regular, then a complex number λ is called a *(generalized finite) eigenvalue* of (E,A) if $\det(\lambda E-A)=0$. The set of all (finite) eigenvalues of (E,A) is called the *(finite) spectrum of the pencil* (E,A) and denoted by $\sigma(E,A)$. If E is singular and the pair is regular, then we say that (E,A) has the eigenvalue ∞.

Regular pairs (E,A) can be transformed to *Weierstraß-Kronecker canonical form*, see [2, 4], i.e., there exist nonsingular matrices W, $T\in\mathbb{R}^{n\times n}$ such that

$$E=W\begin{bmatrix}I_r & 0\\ 0 & N\end{bmatrix}T^{-1},\quad A=W\begin{bmatrix}J & 0\\ 0 & I_{n-r}\end{bmatrix}T^{-1},\tag{2}$$

where I_r, I_{n-r} are identity matrices of indicated size, $J\in\mathbb{R}^{r\times r}$, and $N\in\mathbb{R}^{(n-r)\times(n-r)}$ are matrices in Jordan canonical form and N is nilpotent. If E is invertible, then $r=n$, i.e., the second diagonal block does not occur.

Definition 2.2. Consider a regular pair (E,A) with $E,A\in\mathbb{R}^{n\times n}$ in Weierstraß-Kronecker form (2). If $r<n$ and N has nilpotency index $\nu\in\{1,2,\dots\}$, i.e., $N^\nu=0, N^i\neq 0$ for $i=1,2,\dots,\nu-1$, then ν is called the *index of the pair* (E,A) and we write $\operatorname{index}(E,A)=\nu$. If $r=n$ then the pair has index $\nu=0$.

We note that $\nu=\operatorname{index}(E,A)$ does not depend on the special transformation to canonical form. If $E\in\mathbb{R}^{n\times n}$ then the quantity $\nu=\operatorname{index}(E,I)$ is called index (of nilpotency) of E and is denoted by $\nu=\operatorname{index}(E)$.

Definition 2.3. The regular system (1) with $u(t)=0$ is said to be asymptotically stable if there exist positive numbers M,α such that

$$\|x(t)\|\leq Me^{-\alpha t}\|x^0\|,\quad t>0.$$

Definition 2.4. Let $E\in\mathbb{R}^{n\times n}$ have $\nu=\operatorname{index}E$. A matrix $X\in\mathbb{R}^{n\times n}$ satisfying

$$EX=XE,\tag{3a}$$

$$XEX=X,\tag{3b}$$

$$XE^{\nu+1}=E^\nu,\tag{3c}$$

is called a Drazin inverse of E.

Theorem 2.5. *Every $E\in\mathbb{R}^{n\times n}$ has one and only one Drazin inverse E^D. Moreover, if $E\in\mathbb{R}^{n\times n}$ is nonsingular then*

$$E^D=E^{-1},\tag{4}$$

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and for arbitrary nonsingular $T \in \mathbb{R}^{n\times n}$, we have

$$(T^{-1}ET)^D = T^{-1}E^DT. \tag{5}$$

Theorem 2.6. *Let $E \in \mathbb{R}^{n\times n}$ with $\nu = \operatorname{index} E$. There is one and only one decomposition*

$$E = \widetilde{C} + \widetilde{N} \tag{6}$$

with the properties

$$\widetilde{C}\widetilde{N} = \widetilde{N}\widetilde{C} = 0,\ \widetilde{N}^\nu = 0,\ \widetilde{N}^{\nu-1} \neq 0,\ \operatorname{index}\widetilde{C} \leq 1. \tag{7}$$

In particular, the following statements hold:

$$\widetilde{C}^D\widetilde{N} = 0, \qquad \widetilde{N}\widetilde{C}^D = 0, \tag{8a}$$

$$E^D = \widetilde{C}^D, \tag{8b}$$

$$\widetilde{C}\widetilde{C}^D\widetilde{C} = \widetilde{C}, \tag{8c}$$

$$\widetilde{C}^D\widetilde{C} = E^DE, \tag{8d}$$

$$\widetilde{C} = EE^DE, \quad \widetilde{N} = E(I - E^DE). \tag{8e}$$

Theorem 2.7. *Let $E, A \in \mathbb{R}^{n\times n}$ satisfy $AE = EA$. Then we have*

$$EA^D = A^DE,\ E^DA = AE^D,\ E^DA^D = A^DE^D. \tag{9}$$

Moreover, if

$$\ker E \cap \ker A = \{0\} \tag{10}$$

then we have

$$(I - E^DE)A^DA = I - E^DE. \tag{11}$$

Note that for the commuting matrices E and A, condition (10) is equivalent to the regularity of (E,A) and in formula (2) we can choose $T = W$.

In order to formulate an explicit solution representations of (1), the matrices E and A are required commuting. If they do not commute, we can obtain commuting matrices by multiplication with a scaling factor. Indeed, because of the regularity of matrix pencil (E,A), there exists $\lambda_0 \in \mathbb{C}$ with $\det(\lambda_0E - A) \neq 0$. Set

$$\widetilde{E} = (\lambda_0E - A)^{-1}E,$$
$$\widetilde{A} = (\lambda_0E - A)^{-1}A,$$
$$\widetilde{B} = (\lambda_0E - A)^{-1}B.$$

It is easy to check that $\widetilde{E}\widetilde{A} = \widetilde{A}\widetilde{E}$ and the system (1) is equivalent to

$$\begin{aligned}\widetilde{E}\dot{x}(t) &= \widetilde{A}x(t) + \widetilde{B}u(t), \quad t \geq 0,\\ x(0) &= x^0.\end{aligned} \tag{12}$$

In what folows, without lost of generality, we will assume that E and A are commutative. We recall the system (1)

$$\begin{aligned}E\dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0,\\ x(0) &= x^0,\end{aligned}$$

According to Theorem 2.6, we have decomposition $E = \widetilde{C} + \widetilde{N}$ with the properties of $\widetilde{C}$ and $\widetilde{N}$ as given there. We get the following lemma.

Lemma 2.8 (see [4])**.** *Equation* (1) *with $EA = AE$ is equivalent to the system*

$$\widetilde{C}\dot{x}_1(t) = Ax_1(t) + E^DEBu(t), \tag{13a}$$

$$\widetilde{N}\dot{x}_2(t) = Ax_2(t) + (I - E^DE)Bu(t), \tag{13b}$$

where

$$x_1(t) = E^DEx(t),\ x_2(t) = (I - E^DE)x(t), \tag{14}$$

and $\dot{x}_2(t_k)$ is understood by the Dini upper-right derivative of x_2 at t_k. Moreover, equation (13a) *is equivalent to the differential equation*

$$\dot{x}_1(t) = E^DAx_1(t) + E^DBu(t). \tag{15}$$

Remark 2.9. By the Lemma 2.8, a solution of the DAE (1) can be expressed by sum of a solution of the classical differential equation (15) and a solution of the DAE (13b) with the nilpotent leading matrix.

3. Event-triggered control

The goal of this section is to develop an event-triggered state feedback control law which has been introduced in [5] for positive descriptor systems as follow

$$u(t) = u(t_k) = -Kx(t_k) \text{ for } t \in [t_k, t_{k+1}) \tag{16}$$

where the sequence $\{t_k\}_{k\in\mathbb{N}}$ represents the instants at which (16) is re-computed and the actuator signals are updated. We refer to these instants as the *triggering* times.

Since, the inputs to be held constant in between the successive recomputations of (16), the closed-loop system is then written during the interval $[t_k, t_{k+1})$ by

$$\begin{aligned}E\dot{x}(t) &= Ax(t) + Bu(t_k), \quad t \in [t_k, t_{k+1}),\\ x(0) &= x^0,\end{aligned} \tag{17}$$

or

$$\begin{aligned}E\dot{x}(t) &= (A + BK)x(t) + BKe(t), \quad t \in [t_k, t_{k+1}),\\ x(0) &= x^0,\end{aligned} \tag{18}$$

Theorem 3.1. *Let the matrix pair $(E,A) \in (\mathbb{R}^{n\times n})^2$ be regular and commute. Then, for $t \in [t_k, t_{k+1})$, the solution of equation* (13b) *has only the form*

$$x_2(t) = -(I - E^DE)A^DBu(t_k), \tag{19}$$

and the consistent condition of $u(0), x^0$ for solvability of (17) *is*

$$(I - E^DE)\left(x^0 + A^DBu(0)\right) = 0.$$

Proof. By using Theorem 2.6, we obtain

$$\begin{aligned} A\widetilde{N} &= AE(I-E^DE) = E(I-E^DE)A = \widetilde{N}A, \\ \widetilde{N}(I-E^DE) &= E(I-E^DE)(I-E^DE) = E(I-E^DE) = \widetilde{N}. \end{aligned}$$

Since $\widetilde{N}$ is a nilpotent matrix of degree ν, from (13b) it follows that

$$\begin{aligned} 0 &= \widetilde{N}^{\nu}\dot{x}_2(t) = \widetilde{N}^{\nu-1}\widetilde{N}dx_2(t) \\ &= \widetilde{N}^{\nu-1}(Ax_2(t) + (I-E^DE)Bu(t)) \\ &= A\widetilde{N}^{\nu-1}x_2(t) + \widetilde{N}^{\nu-1}(I-E^DE)Bu(t). \end{aligned}$$

Take the Dini upper-right derivative on two sides, we obtain

$$0 = A\widetilde{N}^{\nu-1}\dot{x}_2(t) + \widetilde{N}^{\nu-1}(I-E^DE)B\dot{u}(t)$$

Since the Dini upper-right derivative $\dot{u}(t) = 0$ for all $t \geq t_0$, it implies that $A\widetilde{N}^{\nu-1}\dot{x}_2(t) = 0$ for all $t \geq t_0$. Multiplying this equation by $(I-E^DE)A^D$, we get

$$(I-E^DE)A^DA\widetilde{N}^{\nu-1}\dot{x}_2(t) = 0,$$

By Theorem 2.6 and $x_2(t) = (I-E^DE)x(t)$, we obtain

$$\begin{aligned} \widetilde{N}^{\nu-1}\dot{x}_2(t) &= \widetilde{N}^{\nu-1}(I-E^DE)\dot{x}_2(t) = (I-E^DE)\widetilde{N}^{\nu-1}\dot{x}_2(t) \\ &= (I-E^DE)A^DA\widetilde{N}^{\nu-1}\dot{x}_2(t) = 0. \end{aligned}$$

Applying this procedure continuously yields

$$\widetilde{N}\dot{x}_2(t) = 0$$

which implies that

$$Ax_2(t) + (I-E^DE)Bu(t) = 0.$$

Multiplying this equation by $(I-E^DE)A^D$ we obtain

$$(I-E^DE)A^DAx_2(t) + (I-E^DE)A^D(I-E^DE)Bu(t) = 0.$$

By Theorem 2.6 and $x_2(t) = (I-E^DE)x(t)$, it implies that

$$x_2(t) = -(I-E^DE)A^DBu(t) = -(I-E^DE)A^DBu(t_k),$$

for all $t_k \leq t \leq t_{k+1}$. For $t = t_0$, $x_2(t_0) = (I-E^DE)x^0$ and we obtain the consistent condition of $u(0), x^0$

$$(I-E^DE)(x^0 + A^DBu(0)) = 0.$$

□

Hence, the system (17) during the interval $[t_k, t_{k+1})$ is reduced to

$$\begin{aligned} \dot{x}_1(t) &= E^DAx_1(t) + E^DBu(t_k), \\ x_2(t) &= -(I-E^DE)A^DBu(t_k) \end{aligned} \tag{20}$$

We note that

$$u(t_k) = Kx(t_k) = K[x_1(t_k) + x_2(t_k)].$$

On the interval $[t_k, t_{k+1})$, we get that

$$x_2(t) = x_2(t_k) = -(I-E^DE)A^DBK[x_1(t_k) + x_2(t_k)].$$

This implies

$$x_2(t_k) = -\left(I + (I-E^DE)A^DBK\right)^{-1}(I-E^DE)A^DBKx_1(t_k).$$

Thereafter,

$$\begin{aligned} x(t_k) &= \left[I - \left(I + (I-E^DE)A^DBK\right)^{-1}(I-E^DE)A^DBK\right]x_1(t_k) \\ &= \left(I + (I-E^DE)A^DBK\right)^{-1}x_1(t_k). \end{aligned}$$

The state measurement error is defined by

$$\begin{aligned} e(t) &= e_1(t) + e_2(t) = x(t_k) - x(t) \\ &= [x_1(t_k) - x_1(t)] + [x_2(t_k) - x_2(t)], \end{aligned} \tag{21}$$

for $t \in [t_k, t_{k+1})$. We obtain the closed-loop system

$$\begin{aligned} \dot{x}_1(t) &= \left[E^DA + E^DBK\left(I + (I-E^DE)A^DBK\right)^{-1}\right]x_1(t) \\ &\quad + E^DBK\left(I + (I-E^DE)A^DBK\right)^{-1}e_1(t). \end{aligned} \tag{22}$$

The event-triggered condition is generated by

$$\|e_1\|^2 = \|E^DEe\|^2 \geq \sigma\|E^DEx\|^2 = \sigma\|x_1\|^2, \tag{23}$$

where σ is a constant satisfying $\sigma > 0$. The event-triggered mechanism means that the control input $u(t)$ is updated when the condition (23) holds. We note that the *triggering times* $\{t_k\}_{k\in\mathbb{N}}$ is implicitly defined by (23) as follow

$$t_0 = 0,\ t_{k+1} = \inf\left\{t > t_k \mid \|E^DEe\|^2 \geq \sigma\|E^DEx\|^2\right\}, \tag{24}$$

where $\sigma > 0$.

Remark 3.2. If index$(E,A) = 0$ or E is invertible, then $E^D = E^{-1}$. This implies that $I - E^DE = 0$. Thereafter, the system (22) simplify to

$$\dot{x}(t) = E^{-1}(A + BK)x(t) + E^{-1}BKe(t).$$

Furthermore, the triggered condition reduce to

$$\|e\| \geq \sigma\|x\|.$$

This becomes exactly the linear case of the problem mentioned in [5].

4. Main results

Before showing the main results of this paper, we give a following technical lemma which is useful in implementing our controller gain.

Lemma 4.1. *Suppose that $M \in \mathbb{R}^{n\times m}$ and $N \in \mathbb{R}^{m\times n}$ then we have $(I_n - MN)$ is invertible if and only if $(I_m - NM)$ is invertible.*

Proof. We prove that if $(I_n - MN)$ is invertible, then $(I_m - NM)$ is invertible. Seeking a contradiction, suppose that $(I_m - NM)$ is singular. Then there exists a vector $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v} \in \mathbb{R}^m$ such that

$$(I_m - NM)\mathbf{v} = \mathbf{0}.$$

This leads to

$$NM\mathbf{v} = \mathbf{v}.$$

Then

$$MN(M\mathbf{v}) = M\mathbf{v} \text{ or } (I - MN)(M\mathbf{v}) = \mathbf{0}.$$

Refer to $(I_n - MN)$ is non-singular, this implies that $M\mathbf{v} = \mathbf{0}$. Then $\mathbf{v} = NM\mathbf{v} = \mathbf{0}$, a contradiction. Hence, $(I_m - NM)$ is non-singular. The reverted statement is proved by similarity. In conclusion, $(I_n - MN)$ is non-singular if and only if $(I_m - NM)$ is non-singular. □

Lemma 4.2. *Provided that there exists a symmetric semi-positive definite matrix Q satisfied $\mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$, then there exists a symmetric positive definite matrix P such that*

(i) $PQ = QP$.

(ii) QP *is a projection, i.e,* $(QP)^2 = QP$.

(iii) $\mathrm{im}(QP) = \mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$.

Proof. Since Q is the symmetric semi-positive definite matrix, we can diagonalize Q as follow

$$Q = T \begin{bmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} T^\top$$

with T is a orthogonal matrix. Thereafter, we set

$$P = T \begin{bmatrix} \lambda_1^{-1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_k^{-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} T^\top.$$

Then, we have

$$PQ = QP = T \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} T^\top.$$

This also implies that QP is a projection. Moreover, $\mathrm{im}(QP) = \mathrm{im}(Q)$. Hence, it follows that $\mathrm{im}(QP) = \mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$. □

We mention a well-known lemma that is also useful for the proof of our results.

Lemma 4.3 (Schur complement lemma)**.** *For any matrices X, Y, Z of appropriate dimensions, $X = X^\top$, $Y = Y^\top > 0$, then*

$$\begin{bmatrix} X & Z^\top \\ Z & -Y \end{bmatrix} \leq 0 \tag{25}$$

if and only if the so-called Schur complement $X + Z^\top Y^{-1} Z$ *is semi-positive definite. Furthermore, if the strict inequality holds for* (25) *then the Schur complement is positive definite.*

Now, we are in the position to introduce the main results of this paper.

Theorem 4.4. *Assume that, for a given $\sigma > 0$, and symmetric semi-positive definite matrix Q satisfied* $\mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$, *there exits a matrix $Z \in \mathbb{R}^{m \times n}$ such that following conditions are all satisfied*

$$\big(I - (I - E^D E)A^D BZ\big) \text{ is invertible}, \tag{26a}$$

$$\begin{bmatrix} Q(E^D A + E^D BZ)^\top + (E^D A + E^D BZ)Q & E^D BZQ & Q \\ QZ^\top B^\top (E^D)^\top & -2\varepsilon Q + \varepsilon^2 I & 0 \\ Q & 0 & -\sigma^{-1} I \end{bmatrix} \leq 0. \tag{26b}$$

Then, the closed-loop system (18) *is asymptotically stable, where the control gain K is given by*

$$K = \big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z. \tag{27}$$

Proof. Using Lemma (4.1), we can see the fact that if condition (26a) holds, then $\big(I_m - Z(I - E^D E)A^D B\big)$ is invertible or $\big(I_m - Z(I - E^D E)A^D B\big)^{-1}$ exists.

In order to get presentation of Z, we need to show that $\big[I + (I - E^D E)A^D BK\big]$ is invertible. Indeed, we have

$$K = \big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z.$$

So we need to prove that

$$\Big[I + (I - E^D E)A^D B\big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z\Big] \text{ is invertible.}$$

Refer to Lemma (4.1), it equivalent to

$$\Big[I_m + \big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z(I - E^D E)A^D B\Big] \text{ is invertible.}$$

On the other side,

$$I_m + \big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z(I - E^D E)A^D B = \big(I_m - Z(I - E^D E)A^D B\big)^{-1}.$$

This means that

$$\Big[I_m + \big(I_m - Z(I - E^D E)A^D B\big)^{-1} Z(I - E^D E)A^D B\Big] \text{ is invertible.}$$

Therefore, we deduce

$$Z = K\left(I + (I - E^D E)A^D BK\right)^{-1}.$$

In order to prove that the system (18) is asymptotically stable, we will first show that the system (22) is asymptotically stable. Seeking of simplicity, let us denote $A_1 = E^D A$, $B_1 = E^D B$. Then we can rewrite the system (22) as follows

$$\dot{x}_1(t) = [A_1 + B_1 Z]x_1(t) + B_1 Z e_1(t). \tag{28}$$

Since the Lemma 4.2, then there exists a symmetric positive definite matrix P such that

(i) $PQ = QP$.

(ii) QP is a projection, i.e, $(QP)^2 = QP$.

(iii) $\mathrm{im}(QP) = \mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$.

Now, we consider the Lyapunov function $V(x_1) = x_1^\top P x_1$ with the note that $x_1 = E^D E x$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_1) = x_1^\top\Big[(A_1 + B_1 Z)^\top P + P(A_1 + B_1 Z)\Big]x_1 + e_1^\top (B_1 Z)^\top P x_1 + x_1^\top P B_1 Z e_1$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_1) = \begin{pmatrix} x_1^\top & e_1^\top \end{pmatrix} \begin{bmatrix} (A_1 + B_1 Z)^\top P + P(A_1 + B_1 Z) & PB_1 Z \\ (B_1 Z)^\top P & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ e_1 \end{pmatrix}. \tag{29}$$

On the other hand, we can rewrite the condition (26b) by the term of A_1 and B_1 as following

$$\begin{bmatrix} Q(A_1 + B_1 Z)^\top + (A_1 + B_1 Z)Q & B_1 Z Q & Q \\ Q(B_1 Z)^\top & -2\varepsilon Q + \varepsilon^2 I & 0 \\ Q & 0 & -\sigma^{-1} I \end{bmatrix} \le 0.$$

By using Schur complement lemma (4.3), this is equivalent to

$$\begin{bmatrix} Q(A_1 + B_1 Z)^\top + (A_1 + B_1 Z)Q + \sigma Q^2 & B_1 Z Q \\ Q(B_1 Z)^\top & -2\varepsilon Q + \varepsilon^2 I \end{bmatrix} \le 0.$$

Since,

$$Q^2 \ge 2\varepsilon Q - \varepsilon^2 I,$$

it implies that

$$\begin{bmatrix} Q(A_1 + B_1 Z)^\top + (A_1 + B_1 Z)Q + \sigma Q^2 & B_1 Z Q \\ Q(B_1 Z)^\top & -Q^2 \end{bmatrix} \le 0.$$

Then we have

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} Q(A_1 + B_1 Z)^\top + (A_1 + B_1 Z)Q + \sigma Q^2 & B_1 Z Q \\ Q(B_1 Z)^\top & -Q^2 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \le 0,$$

or

$$\begin{bmatrix} PQ(A_1+B_1Z)^\top P + P(A_1+B_1Z)QP + \sigma PQ^2P & PB_1ZQP \\ PQ(B_1Z)^\top P & -PQ^2P \end{bmatrix} \le 0$$

Note that $PQ = QP$, $(PQ)^2 = PQ$, $\mathrm{im}(QP) = \mathrm{im}(Q) \supseteq \mathrm{im}(E^D E)$ and $\mathrm{im}(E^D) = \mathrm{im}(E^D E)$, we have

$$QPx_1 = QPE^D E x = E^D E x = x_1,\; QPe_1 = QPE^D E e = E^D E e = e_1.$$

Then, we obtain

$$\begin{pmatrix} x_1^\top & e_1^\top \end{pmatrix} \begin{bmatrix} PQ(A_1+B_1Z)^\top P + P(A_1+B_1Z)QP + \sigma PQ^2P & PB_1ZQP \\ PQ(B_1Z)^\top P & -PQ^2P \end{bmatrix} \begin{pmatrix} x_1 \\ e_1 \end{pmatrix}$$
$$= \begin{pmatrix} x_1^\top & e_1^\top \end{pmatrix} \begin{bmatrix} (A_1+B_1Z)^\top P + P(A_1+B_1Z) + \sigma I & PB_1Z \\ (B_1Z)^\top P & -I \end{bmatrix} \begin{pmatrix} x_1 \\ e_1 \end{pmatrix} \le 0.$$

To cooperate with triggered condition (23), it leads to

$$\begin{pmatrix} x_1^\top & e_1^\top \end{pmatrix} \begin{bmatrix} (A_1 + B_1 Z)^\top P + P(A_1 + B_1 Z) & PB_1 Z \\ (B_1 Z)^\top P & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ e_1 \end{pmatrix}$$
$$\le \begin{pmatrix} x_1^\top & e_1^\top \end{pmatrix} \begin{bmatrix} -\sigma I & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} x_1 \\ e_1 \end{pmatrix} = \|e_1\|^2 - \sigma\|x_1\|^2 \le 0,$$

To combine with (29), we obtain that the system (22) is asymptotically stable. Moreover, for all $t \in [t_k, t_{k+1})$

$$x_2(t) = x_2(t_k) = -\left(I + (I - E^D E)A^D BK\right)^{-1} (I - E^D E)A^D BK x_1(t_k).$$

So,

$$\|x_2(t)\| \le \gamma \|x_1(t_k)\|,$$

where $\gamma = \|\left(I + (I - E^D E)A^D BK\right)^{-1}(I - E^D E)A^D BK\|$. Hence, x_2 is bounded and $\lim_{t\to\infty}\|x_2(t)\| = 0$. Therefore, $x(t) = x_1(t) + x_2(t)$ is asymptotically stable. This means that the system (18) is asymptotically stable. ☐

Remark 4.5.

- If $\mathrm{index}(E, A) = 0$ or E is invertible, then $E^D = E^{-1}$. This implies that $I - E^D E = 0$. It follows that the condition (26a) is trivial.
- The condition (26b) can be implemented as follow LMI condition,

$$\begin{bmatrix} QA_1^\top + A_1 Q + Y^\top B_1^\top + B_1 Y & B_1 Y & Q \\ Y^\top B_1^\top & -2\varepsilon Q + \varepsilon^2 I & 0 \\ Q & 0 & -\sigma^{-1} I \end{bmatrix} \le 0, \tag{30}$$

with variables Y, and $Q \ge 0$. Then the matrix Z is recovered by solving the matrix equation $ZQ = Y$.

We will show that the Zeno behavior does not happen, which mean that there exists a time $\tau > 0$ such that $t_{k+1} - t_k > \tau$ for any $k \in \mathbb{N}$. The technique use to prove follow theorem is conventional and has been mentioned in many paper written about even-triggered control, namely but a few [5, 3, 1].

Theorem 4.6. *Given system* (1) *and the controller gain K which is given in Theorem 4.4, then there exists a time $\tau > 0$ such that for any consistent initial value x^0, the inter-execution time $\{t_{k+1} - t_k\}_{k\in\mathbb{N}}$ implicitly defined by the execution rule* (24) *are lower bounded by τ.*

Proof. Recall the system (22)

$$\dot{x}_1(t) = \left[E^D A + E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1}\right] x_1(t) + E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1} e_1(t).$$

Therefore,

$$\|\dot{x}_1\| \le \left\|E^D A + E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1}\right\| \|x_1\| + \left\|E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1}\right\| \|e_1\|.$$

Or

$$\|\dot{x}_1\| \le a\|x_1\| + b\|e_1\|, \tag{31}$$

where $a = \left\|E^D A + E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1}\right\|$, and $b = \left\|E^D BK\left(I + (I - E^D E)A^D BK\right)^{-1}\right\|$.

We can bound the inter-event time by looking at the dynamic of $\frac{\|e_1\|}{\|x_1\|}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|e_1\|}{\|x_1\|} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{(e_1^\top e_1)^{1/2}}{(x_1^\top x_1)^{1/2}} \\
&= -\frac{e_1^\top \dot{x}_1}{\|e_1\|\|x_1\|} - \frac{x_1^\top \dot{x}_1}{\|x_1\|^2}\frac{\|e_1\|}{\|x_1\|} \quad (\text{ by } \dot{x}_1 = -\dot{e}_1) \\
&\le \frac{\|e_1\|\|\dot{x}_1\|}{\|e_1\|\|x_1\|} + \frac{\|x_1\|\|\dot{x}_1\|}{\|x_1\|^2}\frac{\|e_1\|}{\|x_1\|} \\
&= \left(1 + \frac{\|e_1\|}{\|x_1\|}\right)\frac{\|\dot{x}_1\|}{\|x_1\|} \\
&\le a + (a+b)\frac{\|e_1\|}{\|x_1\|} + b\left(\frac{\|e_1\|}{\|x_1\|}\right)^2 \quad (\text{ by (31)}).
\end{aligned}$$

Consequently, the inter-event times are lower bounded by time τ satisfying

$$\phi(\tau, 0) = \sigma,$$

where $\phi(t, \phi_0)$ is the solution of

$$\dot{\phi} = a + (a+b)\phi + b\phi^2$$

satisfying $\phi(0, \phi_0) = \phi_0$. As a result, $\tau = \dfrac{1}{a-b}\ln\dfrac{a + a\sigma}{a + b\sigma} > 0$, and $t_{k+1} - t_k \ge \tau$, for all $k \in \mathbb{N}$. □

5. Simulation

To demonstrate the application of our controller, we consider an example of a simple RLC electrical circuit in Figure 1 which has been shown in [4, Example 2.52]. Seeking the simplicity, we choose the resistance $R = 1$, inductance $L = 1$, and capacitance $C = 1$. The corresponding voltage drops are denoted by v_R, v_L, and v_C, respectively, and I denotes the current. We obtain the circuit equation

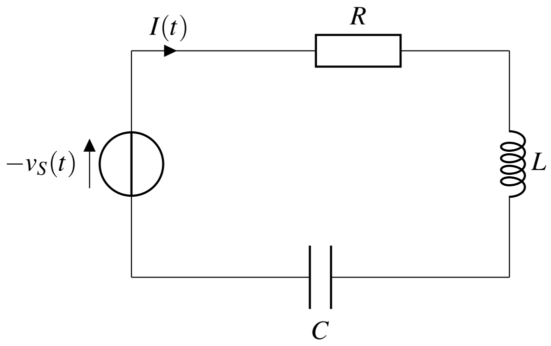

Figure 1: A simple RLC circuit.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{v}_C \\ \dot{v}_R \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_S. \tag{32}$$

We read that the system (32) is regular, and has index $= 2$ with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

However, E, A does not commute. Thereafter, we multiple both sides of (32) with matrix $(E+A)^{-1}$ in order to get a new systems having commuting condition.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{v}_C \\ \dot{v}_R \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} v_S. \tag{33}$$

With the system (33), we denote

$$E_{new} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad A_{new} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}.$$

It is easy to check that $E_{new}A_{new} = A_{new}E_{new}$ and $\text{index}(E_{new}, A_{new}) = 2$.

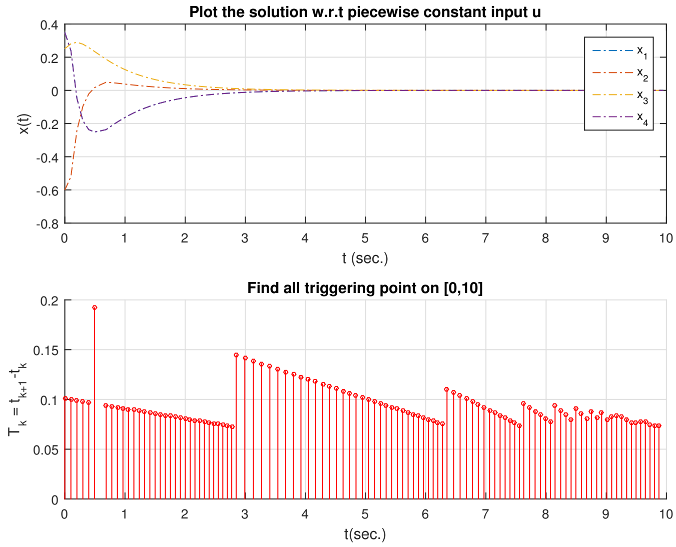

Figure 2: Solution $x(t)$ and triggering point on $[0,10]$.

By solving the LMI (30) we obtain

$$Y = \begin{bmatrix} -65.4198 & 71.5116 & -6.0801 & -65.4267 \end{bmatrix},$$
$$Q = \begin{bmatrix} 17.6651 & -13.9796 & -3.6824 & 17.6620 \\ -13.9796 & 15.1144 & -1.1277 & -13.9819 \\ -3.6824 & -1.1277 & 4.8121 & -3.6796 \\ 17.6620 & -13.9819 & -3.6796 & 17.6663 \end{bmatrix},$$
$$Z = \begin{bmatrix} -1.7505 & 2.9472 & -1.9298 & -0.0228 \end{bmatrix},$$
$$K = \begin{bmatrix} -1.75052.9472 - 1.9298 - 0.0228 \end{bmatrix}.$$

The solution $x(t)$ to system (1) is constructed by solving the coupled-system (22) and take $x(t) = x_1(t) + x_2(t)$. We notice, that the dimension of the couple system (22) is two times bigger than the dimension of system (1). With the initial condition $x_0 = \begin{bmatrix} 0.3500 & 0.1966 & 0.2511 & 0.6160 \end{bmatrix}^\top$, the numerical solution is illustrated in Figure 2. Here we choose $\varepsilon = 20$ and $\sigma = 0.01$.

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