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Robustness of exponential stability of a class of switched positive linear systems with time delays

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Summary

This paper investigates the robustness of exponential stability of a class of positive switched systems described by linear functional differential equations (FDE) under arbitrary switching or average dwell time switching. We will measure the stability robustness of such a system (which is considered as a nominal system) subject to parameter affine perturbations of its constituent subsystems matrices, by introducing the notion of structured stability radius. Some formulas for computing this radius, as well as estimating its lower bounds and upper bounds, are established. In the case of switched linear systems with multiple discrete time-delays or/and distributed time-delays, the obtained results yield tractably computable formulas or bounds for the stability radius. The extension of the obtained results to non-positive systems and the class of multi-perturbations has been presented. Examples are given to illustrate the proposed method.

KEYWORDS

robustness of stability, stability radius, structured perturbations, switched systems, time-delay systems

1 | INTRODUCTION

The *stability robustness* of dynamical systems subject to parameter perturbations or uncertainties has attracted significant attention from researchers for many years. The motivation comes from engineering practices. In order to study or control a real system with complicated dynamic behaviors the engineer usually considers a simplified mathematical model, which is called a nominal system. Then there arises the question of whether a desired property established for the nominal system, say asymptotic stability or controllability, is robust enough to be true when applied to the real system. Since a mathematical model never exactly represents the dynamics of a physical system, the robustness issue is not only important in the context of model reduction but is a fundamental problem for the application of systems and control theory in general.

One of the most effective approaches in measuring the system stability robustness is based on the concept of *stability radius*, see, for example, References 1,2. By definition, the structured stability radius of a given nominal asymptotically stable system $\dot{x}(t) = A_0x(t), t \geq 0$, is defined as the *maximal number* $\delta_0 > 0$ such that all the perturbed systems $\dot{x}(t) = \widetilde{A}_0x(t) = (A_0 + D\Delta E)x(t)$ are asymptotically stable whenever $\|\Delta\| < \delta_0$ (with some matrix norm $\|\cdot\|$) where D and E are given real matrices of appropriate dimensions defining the structure of perturbation and Δ is a unknown disturbance matrix. Depending on Δ being real or complex, one gets, correspondingly, the real stability radius $r_{\mathbb{R}}(A_0)$ or the complex stability radius $r_{\mathbb{C}}(A_0)$, which are, in general, distinct,[3] namely, $r_{\mathbb{C}}(A_0) < r_{\mathbb{R}}(A_0)$. Geometrically, when D and E are the identity matrix, the real stability radius is just the distance to instability of an asymptotically stable nominal system.

If the nominal system is real then the computation of $r_{\mathbb{R}}(A_0)$ is, of course, a more natural problem. However, this problem is much more difficult, compared with that of $r_{\mathbb{C}}(A_0)$, being reduced to a complicated global optimization problem for the parameterized structured singular value of A_0, see, for example, Reference 2. Problems of characterizing and calculating stability radii for different classes of dynamical systems and subject to different types of perturbations or uncertainties have been a topic of major interest in the system and control theory over the past several decades. The interested reader is referred to the monograph[3] where an extensive literature on this study can be found; see also Reference 4 for discrete-time systems and References 5,6 for *positive systems*. Recall that a dynamical system with state space $\mathbb{R}^n$ is called positive if any trajectory of the system starting at an initial state in the positive orthant $\mathbb{R}^n_+$ remains forever in $\mathbb{R}^n_+$. For instance, the linear system $\dot{x}(t) = A_0x(t),\ t \geq 0$ is positive if and only if A_0 is a Metzler matrix (i.e., all off-diagonal entries are nonnegative). Positive systems are used in many areas such as economics, population dynamics, and ecology, see, for example, Reference 7. The mathematical theory of positive systems is based on the theory of nonnegative matrices,[8] where the famous Perron–Frobenius theorem plays an essential role, particularly in the system stability analysis. It has been shown in Reference 5, for instance, that if the mentioned linear system is positive and D, E are nonnegative matrices, then its complex and the real stability radii coincide and are equal to the stability radius $r_{\mathbb{R}_+}(A_0)$, corresponding to the nonnegative perturbation $\Delta \geq 0$, and can be computed directly via a simple formula. In more recent years, robust stability problems have been considered intensively also for time-delay linear systems of the form $\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} A_ix(t-h_i),\ t \geq 0,\ h_i > 0$ (see, e.g., References 9,10) and, more generally, for linear functional differential equations (or shortly, FDEs) of the form $\dot{x}(t) = A_0x(t) + \int_{-h}^{0} d[\eta(\theta)]x(t+\theta), t \geq 0$ (see, e.g., References 11–13) where similar results have been proved for stability conditions and stability radii of positive systems.

Given the mentioned widespread popularity of stability radii in the robustness stability analysis, it is surprising that this approach has not been developed so far in the literature on the stability of *switched systems*. We recall that a switched system is a type of hybrid dynamical systems that consists of a family of subsystems and a rule called a switching signal that chooses an active subsystem from the family at every instant of time. For instance, a linear switched system can be represented in the form

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ t \geq 0,\ \sigma \in \Sigma, \tag{1}$$

where Σ is a set of switching signals which are piece-wise constant functions $\sigma : [0, +\infty) \to \underline{N} := \{1, 2, \ldots, N\}$ (satisfying some realistic assumptions). The study on switched systems has drawn considerable attention in the system and control community over the past few decades, due to the wide range of applications of this type of systems in the engineering practice. The reader is referred to the monographs,[14,15] survey papers,[16,17] and the references therein for more details on different problems regarding switched systems, in particular, on stability issues. It has been indicated, for instance, that the switched linear system (1) is exponentially stable *under arbitrary switching* σ if all constituent subsystems have a common quadratic Lyapunov function (CQLF, for short). Recently, similar problems have been considered intensively also for time-delay switched systems, for instance, the linear systems of the form $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-h), t \geq 0, \sigma \in \Sigma$, where different kinds of the so-called Lyapunov–Krasovskii functionals play a similar role (see, e.g., References 18,19, which contain the extensive literature on the topic). In the meantime, for the class of positive or compartmental switched linear systems, besides traditional quadratic Lyapunov functions, a more restrictive notion of common linear copositive Lyapunov functions, as well as the comparison principle, are exploited effectively in the study of stability problems (see e.g., References 20–22 and also References 23–25 for time-delay systems). Some recent development in stability problems for positive switched systems can be found, for instance, in References 26–29 and the references given therein. We would like to emphasize that the above references are all dedicated to the stability analysis of switched systems under arbitrary switching signals. Such a strong requirement is important when the switching mechanism is unknown, or too complicated to be useful in the stability analysis. On the other side, in engineering practices, switching laws must usually satisfy some technical restriction (for instance, the time between switching instances must be not smaller than a given number $\tau_D > 0$). In particular, the problem of stability and stabilization of switched systems, using the so-called *average dwell time switching* (or ADT switching, for short) has attracted considerable attention, see for example, References 14,30–36, for more recent contributions. It is emphasized that in Reference 36 based on the properties of positive systems and the comparison principle, we developed a unified approach to study exponential stability of nonlinear time-varying switched systems described by functional differential equations (or FDEs, for short), under arbitrary switching as well as ADT switching, which covers, as a particular case, the time-delay switched linear systems of the form

$$\dot{x}(t) = A^0_{\sigma(t)}x(t) + \int_{-h}^{0} d[\eta_{\sigma(t)}(\theta)]x(t+\theta), t \geq 0, \sigma \in \Sigma. \tag{2}$$

In contrast, little attention has been paid to the *stability robustness analysis* of switched systems. One of the challenges in this problem comes from the fact that instability of perturbed switched systems may be caused not only by the size of perturbations but also by the switching signals. The other obstacle is raised by the well-known fact that stability of all constituent subsystems may not be sufficient for the original switched system to be stable. There exist so far very few results dedicated to this topic in the existing literature, see, for example, References 14 (pp. 41–42) and 37,38 where some stability bounds of the delay-free switched linear system (1) have been calculated, with the underlying assumption that constituent subsystems have a *common quadratic Lyapunov function* (CQLF). While, it is well-known that the assumption on the existence of CQLF is rather conservative, being satisfied only for some classes of systems and, moreover, the CQLF is not easy to be constructed (see, e.g., Reference 16), the extension of stability results based on this approach to time-delay switched systems of the form (2) has not been available in the literature, to the best of our knowledge. This makes the problem of calculation of robust stability bounds for switched time-delay systems even more difficult.

The main purpose of this paper is to develop a different approach for studying the robust stability of time-delay switched systems of the form (2), under arbitrary switching or ADT switching, by making use of the exponential stability conditions obtained in Reference 36 for this class of positive systems. For this purpose, we introduce the notion of the system's stability radius and establish some formulas to estimate its bounds, when the subsystems matrices $A_k, \eta_k(\cdot), k \in \underline{N}$ are subjected to affine perturbations or multi-perturbations. Formulas will be derived and expressed in terms of coefficient matrices appearing in the initial nominal system's equations. The extensions of results are obtained also for non-positive systems. A similar approach was first introduced in our previous work,[38] but only for delay-free switched systems of the form (1), under simple affine perturbations of the subsystems matrices $A_k, k \in \underline{N}$, so that the method and the results of Reference 38 can not be applied to the general model considered in this paper. Additionally, it should be emphasized that the switched linear systems of the general form (2) cover systems with multiple discrete delays and those with distributed delays as particular cases, giving the obtained results a wider range of applications.

The remainder of the paper is organized as follows. In Section 2, we present the notation and the mathematical background, necessary to prove the main results of the paper, including some criteria for exponential stability of switched systems described by linear FDEs. Section 3 is devoted to presenting the main results of the paper. Definitions of the stability radii of switched linear FDEs subject to affine perturbations are given and some formulas for computing its bounds are established. In particular, for two-order positive switched linear systems with no time-delays, a formula for computing the unstructured stability radius is established. Examples are provided to illustrate the use of the obtained results and the extension to the class of multi-perturbations are given. Finally, in Section 4, we summarize the main contribution of the paper and give some remarks on future work.

2 | PRELIMINARIES AND MATHEMATICAL BACKGROUND

In this section, we introduce the main notation and present a number of previous results to be used in what follows. Throughout, $\mathbb{N}, \mathbb{R}$ stand, respectively, for the sets of positive integers and real numbers. For $r \in \mathbb{N}$, $\underline{r}$ denotes the set of numbers $\{1, 2, \dots, r\}$. $\mathbb{R}^n$ is the linear space of n-dimensional column vectors and $\mathbb{R}^{n\times m}$ is the space of $(n \times m)$-matrices (a_{ij}) with entries $a_{ij} \in \mathbb{R}$, I_n is the identity matrix of $\mathbb{R}^{n\times n}$. Inequalities between vectors and matrices are understood componentwise, so that for matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{n\times m}$, we write $A \geq B$ and $A \gg B$ iff $a_{ij} \geq b_{ij}$ and $a_{ij} > b_{ij}$ for $i \in \underline{n}, j \in \underline{m}$, respectively. $|A|$ stands for the matrix $(|a_{ij}|)$ and $A^\top$ is the transpose of A. Similar notation is applied for vectors $x = (x_i) \in \mathbb{R}^n$. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^{n\times m}_+ = \{A \in \mathbb{R}^{n\times m} : A \geq 0\}$. Without lost of generality, we assume that $\mathbb{R}^n$ is equipped with the ∞-norm: for $x = (x_i) \in \mathbb{R}^n$, $\|x\| = \max_{1\leq i\leq n} |x_i|$ and the norm in $\mathbb{R}^{n\times m}$ is the induced operator norm: $\|A\| = \max_{1\leq i\leq n} \sum_{j=1}^m |a_{ij}|$. The maximal real part of all eigenvalues of $A \in \mathbb{R}^{n\times n}$ is denoted by $\mu(A)$ while $\rho(A)$ stands for its spectral radius. If $\mu(A) < 0$ then A is said to be *Hurwitz stable*. Further, for $h > 0, C := C([-h, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \max_{\theta\in[-h,0]} \|\varphi(\theta)\|$ and $NBV([-h, 0], \mathbb{R})$ will stand for the linear space of all normalized functions of bounded variation $\psi : [-h, 0] \to \mathbb{R}$, which are left-side continuous on the interval $(-h, 0]$, $\psi(-h) = 0$ and have the bounded total variation $Var([-h, 0], \psi) = \sup_{P[-h,0]} \sum_k |\psi(\theta_k) - \psi(\theta_{k-1})| < \infty$, the supremum being taken over the set of all finite partitions of the interval $[-h, 0]$. Denote by $NBV([-h, 0], \mathbb{R}^{p\times q})$ the linear space of all matrix functions $\delta : [-h, 0] \to \mathbb{R}^{p\times q}$ such that $\delta_{ij}(\cdot) \in NBV([-h, 0], \mathbb{R}), \forall i \in \underline{p}, \forall j \in \underline{q}$, and define the the nonnegative matrix $V(\delta) \in \mathbb{R}^{p\times q}_+$, by setting

$$V(\delta) := \left(Var([-h, 0], \delta_{ij})\right) \geq 0. \tag{3}$$

Then it is easy to verify that $NBV([-h,0],\mathbb{R}^{p\times q})$ is a Banach space being equipped with the norm

$$\|\delta\| := \|V(\delta)\| = \max_{1\le i\le p}\sum_{j=1}^{q} Var([-h,0],\delta_{ij}). \tag{4}$$

It follows from the definition that if both $\mathbb{R}^p$ and $\mathbb{R}^q$ are provided with the norm $\|\cdot\|_\infty$ then, for any $\Delta\in\mathbb{R}^{p\times q}$, any $\delta,\delta_1,\delta_2\in NBV([-h,0],\mathbb{R}^{p\times q})$ and any constant matrices $D\in\mathbb{R}^{n\times p}, E\in\mathbb{R}^{q\times n}$, we have

$$\|\,|\Delta|\,\| = \|\Delta\|, \tag{5}$$

$$D\delta E\in NBV([-h,0],\mathbb{R}^{n\times n}) \quad \text{and} \quad V(D\delta E)\le |D|\;V(\delta)\;|E|, \tag{6}$$

$$V(\delta_1+\delta_2)\le V(\delta_1)+V(\delta_2);\;\; V(\alpha\delta)\le|\alpha|V(\delta),\;\; \forall\alpha\in\mathbb{R}. \tag{7}$$

Finally, recall that $A\in\mathbb{R}^{n\times n}$ is said to be a *Metzler matrix* if all off-diagonal elements of A are nonnegative: $a_{ij}\ge 0$, if $i\ne j$. For an arbitrary matrix $A\in\mathbb{R}^{n\times n}$ we can associate the Metzler matrix $\mathcal{M}(A)$, by defining,

$$\mathcal{M}(A) := (\hat{a}_{ij}),\;\; \hat{a}_{ii}=a_{ii},\forall i\in\underline{n},\;\text{ and }\;\hat{a}_{ij}=|a_{ij}|,\forall i\ne j\in\underline{n}. \tag{8}$$

It can easily be verified that

$$\mathcal{M}(A+B)\le\mathcal{M}(A)+|B|,\;\;\forall A,B\in\mathbb{R}^{n\times n}. \tag{9}$$

Some well-known properties of Metzler matrices are collected in the following lemma (see, e.g., Reference 8), which is a direct consequence of the Perron–Frobenius Theorem.

Lemma 1. *Let $A\in\mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent: (i) A is Hurwitz stable, that is, $\mu(A)<0$; (ii) $Ap\ll 0$ for some $p\in\mathbb{R}^n_+, p\gg 0$; (iii) A is invertible and $-A^{-1}\ge 0$.*

Consider a switched linear system, whose dynamics are described by the FDE of the form

$$(\mathcal{A},\Gamma,\sigma):\;\; \dot{x}(t)=A^0_{\sigma(t)}x(t)+\int_{-h}^{0} d[\eta_{\sigma(t)}(\theta)]x(t+\theta),\;\; t\ge 0,\;\; \sigma\in\Sigma. \tag{10}$$

Here, Σ is a set of *admissible switching signals* σ, which are assumed to be piece-wise constant and right-continuous functions $\sigma:[0,\infty)\to\underline{N}$, having on each bounded interval of $[0,+\infty)$ a finite number of discontinuities $\tau_k, k=1,2,\ldots$, known as the *switching instances*. For each $t\ge 0, A^0_{\sigma(t)}\in\mathcal{A}:=\left\{A^0_k, k\in\underline{N}\right\}\subset\mathbb{R}^{n\times n}$, a given family of N real matrices and $\eta_{\sigma(t)}\in\Gamma:=\{\eta_k(\cdot),k\in\underline{N}\}\subset NBV([-h,0],\mathbb{R}^{n\times n})$, a given family of N matrix functions with normalized bounded variation elements $\eta_{k,ij}(\cdot)$. In this paper, we shall assume that $\Sigma=\Sigma_+$ or Σ_{τ_a,N_0}, where Σ_+ is the class of all admissible switching signals σ for which the infimum of the time intervals between discontinuities of σ satisfies

$$\tau_{\min}(\sigma) := \inf_{k\in\mathbb{N}}(\tau_{k+1}-\tau_k)>0, \tag{11}$$

and Σ_{τ_a,N_0} is the class of all admissible switching signals σ for which the number of discontinuities of σ on the interval $(0,t)$ satisfies

$$N_\sigma(0,t)\le N_0+\frac{t}{\tau_a}, \tag{12}$$

where $N_0\ge 1$ and $\tau_a>0$ are given numbers, called the *chatter bound* and the *average dwell time* (or ADT, for short), respectively. To simplify the notation, in what follows, for an arbitrarily fixed chatter bound N_0, we will denote $\Sigma_{\tau_a}=\Sigma_{\tau_a,N_0}$. Note that such a class Σ_+ excludes, for example, any switching signal σ whose discontinuities have a finite accumulation point or occur at $\tau_{2j}=j, \tau_{2j+1}=j+\frac{1}{2j+2}, j=0,1,2,\ldots$ for which, clearly, $\tau_{\min}(\sigma)=0$. The concept of ADT switching was introduced in References 30,31 and has been proved to be a standard and effective tool in the stability analysis of switched systems.

Thus, each signal $\sigma \in \Sigma$ performs, by the system (10), the switching between the following N time-delay *constituent linear subsystems* (called sometimes *switched modes*):

$$\left(A_k^0, \eta_k\right) : \ \dot{x}(t) = A_k^0 x(t) + \int_{-h}^{0} d[\eta_k(\theta)]x(t+\theta), \ t \geq 0, \ k \in \underline{N}, \tag{13}$$

where the i-th component of the second term in (13), for each $k \in \underline{N}$ and $i = 1, \ldots, n$, is defined as

$$\left(\int_{-h}^{0} d[\eta_k(\theta)]x(t+\theta)\right)_i = \sum_{j=1}^{n} \int_{-h}^{0} d[\eta_{k,ij}(\theta)]x_j(t+\theta),$$

the integral terms on the right side being understood in the sense of Riemann–Stieltjes. In what follows, the switched system $(\mathcal{A}, \Gamma, \sigma)$, together with its constituent subsystems $\left(A_k^0, \eta_k\right), k \in \underline{N}$, is sometimes referred to as the system (10), (13) when the link between them needs to be specified.

For any $\varphi \in C([-h, 0], \mathbb{R}^n)$ and any switching signal $\sigma \in \Sigma_+$, the system (10) admits a unique solution $x(t) = x(t, \varphi, \sigma)$, $t \geq -h$, satisfying the initial condition $x(\theta) = \varphi(\theta), \ \theta \in [-h, 0]$. Note that the solution $x(t)$ is an absolutely continuous function on $[0, +\infty)$ and differentiable everywhere, except for the set of the switching instances $\{\tau_k\}$ of σ where $x(t)$ has only Dini right- and left-derivatives $D^+x(\tau_k), D^-x(\tau_k)$ which are generally different.

Definition 1. The switched system (10), (13) with $\Sigma = \Sigma_+$ or Σ_{τ_a}, is said to be *positive* if for any switching signal $\sigma \in \Sigma$ and any nonnegative initial function $\varphi \in C_+ := C([0, -h], \mathbb{R}^n_+)$, the corresponding solution $x(t) = x(t, \varphi, \sigma)$ of (10) satisfies $x(t) \geq 0, \forall t \geq 0$.

The following criterion of positivity of the FDE system (10), (13) is followed straightforwardly from the well-known results in the theory of positive operators, see, for example, Reference 39 and also References 11,12.

Proposition 1. *The switched system* (10), (13) *is positive if and only if, for each* $k \in \underline{N}, A_k^0$ *is a Metzler matrix and* $\eta_k(\cdot)$ *is non-decreasing on* $[-h, 0]$, *that is,* $0 = \eta_k(-h) \leq \eta_k(\theta_1) \leq \eta_k(\theta_2)$, *whenever* $-h \leq \theta_1 < \theta_2 \leq 0$.

Definition 2. Given $\tau_a > 0$, the switched system (10), (13), with $\Sigma = \Sigma_+$ or Σ_{τ_a}, is said to be globally exponentially stable (or shortly, GES) over Σ if, there exist real numbers $M > 0, \alpha > 0$ such that for any $\varphi \in C$ and any $\sigma \in \Sigma$, the solutions $x(t, \varphi, \sigma)$ of (10) satisfies

$$\|x(t, \varphi, \sigma)\| \leq Me^{-\alpha t}\|\varphi\|, \quad \forall t \geq 0. \tag{14}$$

The system (10), (13) is said to be GES over Σ_{ADT} if, there exists $\tau_a > 0$ such that it is GES over $\Sigma = \Sigma_{\tau_a}$. The number α satisfying (14) is called the exponential decay rate.

Remark 1. The concept 'GES over a set Σ of switching signals' in Definition 2 has been introduced and studied firstly in References 30,31. In many subsequent works, the equivalent concept "GES under arbitrary switching $\sigma \in \Sigma$" is used more frequently. In what follows, when saying that a switched system is GES under arbitrary switching, it is understood that the system is GES over the set of switching signals Σ_+. Similarly, saying that the system is GES under switching with ADT τ_a will amount to say that the system is GES over the set Σ_{τ_a}.

Clearly, if the switched system (10), (13) is GES over Σ_+ then it is GES over $\Sigma_{\tau_a}, \forall \tau_a > 0$. Since, for each $k \in \underline{N}$, the signal $\sigma(t) \equiv k$ belongs to Σ_+, it follows then that all of the constituent subsystems (13) are globally exponentially stable or, equivalently (see, e.g., Reference 40), all zeros of their *characteristic quasi-polynomials* have negative real parts. However, similarly to the case of switched systems with no delays (i.e., when $\eta_k \equiv 0, \forall k \in \underline{N}$), the last condition is not sufficient for (10) to be GES over Σ_+. On the other hand, it is easy to give a simple example of a second-order system, which is GES over Σ_{ADT} but is not GES over Σ_+ (see, e.g., Reference 14).

The following theorem collects some results obtained in Reference 36 (namely, Corollaries 2 and 6), which will be used in Section 3 for analyzing the stability robustness of the FDE switched systems.

Theorem 1. *The time-delay switched linear system* (10), (13) *is GES over* Σ_{ADT} *if there exist vectors* $\xi_k \in \mathbb{R}^n, \xi_k \gg 0, k \in \underline{N}$ *satisfying*

$$\left(\mathcal{M}(A_k^0) + V(\eta_k)\right)\xi_k \ll 0, \ \forall k \in \underline{N}, \tag{15}$$

where the matrix of variations $V(\eta_k)$ and the Metzler matrix $\mathcal{M}(A_k^0)$ are defined, respectively, by (3), (8). If, moreover, $\xi_k = \xi,\ \forall k \in \underline{N}$ in (15), that is,

$$\left(\mathcal{M}\left(A_k^0\right) + V(\eta_k)\right)\xi \ll 0,\ \forall k \in \underline{N}, \tag{16}$$

then the switched linear system (10), (13) is GES over the set of switching signals Σ_+. In the particular case, when the system (10), (13) is positive, the condition (15) is replaced by

$$\left(A_k^0 + \eta_k(0)\right)\xi_k \ll 0,\ \forall k \in \underline{N}, \tag{17}$$

which becomes a necessary and sufficient condition for the system to be GES over Σ_{ADT}, while the condition (16) is replaced by

$$\left(A_k^0 + \eta_k(0)\right)\xi \ll 0,\ \forall k \in \underline{N}, \tag{18}$$

which gives a sufficient condition for the system to be GES over Σ_+.

Remark 2. Roughly speaking, the condition (16) means that the switched linear system (10), (13) is GES over Σ_+ if all *the dual* positive "upper bounding" linear subsystems $\dot{x} = (\mathcal{M}(A_k^0) + V(\eta_k))^\top x,\ k \in \underline{N}$ admit a common copositive linear Lyapunov function $L(x) := \xi^\top x$. On the other hand, if the nominal linear system is positive and delay-free then the condition (18) in Theorem 1 is reduced to the previously well-known result (see, e.g., Reference 22, Proposition 3.4). It is worth mentioning additionally that the conditions (18) and (16) can be verified directly via a finite procedure, due to the result obtained in References 21,41.

The most important particular case of (10), (13) is the class of switched linear system with *multiple discrete time-delays* and *distributed time-delays* of the form

$$\dot{x}(t) = A^0_{\sigma(t)}x(t) + \sum_{i=1}^{m} A^i_{\sigma(t)}x(t-h_i) + \int_{-h}^{0} B_{\sigma(t)}(\theta)x(t+\theta)d\theta,\ t \geq 0,\ \sigma \in \Sigma, \tag{19}$$

where $0 = h_0 < h_1 < \cdots < h_m = h$ and $\Sigma = \Sigma_+$ or Σ_{τ_a}, with some $\tau_a > 0$. Then, by Proposition 1, the system (19) is positive if and only if, for each $k \in \underline{N}, A_k^0$ are Metzler matrices and $A_k^i \geq 0,\ \forall i \in \underline{m},\ B_k(\theta) \geq 0, \forall \theta \in [-h, 0]$. Application of Theorem 1 to this class of systems yields the following verifiable condition of exponential stability, which will be used in the next Section 3.

Corollary 1. *The switched positive linear system with delay (19) is GES over Σ_{ADT} if and only if there exist vectors $\xi_k \in \mathbb{R}^n, \xi_k \gg 0, k \in \underline{N}$ such that*

$$\left(A_k^0 + \sum_{i=1}^{m} A_k^i + \int_{-h}^{0} B_k(s)ds\right)\xi_k \ll 0,\ \forall k \in \underline{N}. \tag{20}$$

If, moreover, $\xi_k = \xi, \forall k \in \underline{N}$ then the system (19) is GES over the set of switching signals Σ_+.

3 | MAIN RESULTS

Assume that *the nominal* time-delay switched linear system (10), (13) is GES over Σ, with $\Sigma = \Sigma_{\mathrm{ADT}}$ or Σ_+, as being defined by Definition 2. Let the matrices $A_k^0, \eta_k(\cdot),\ k \in \underline{N}$ of the constituent subsystems (13) be subjected to structured *real affine perturbations* of the form

$$A_k^0 \to \widetilde{A}_k^0 := A_k^0 + D_k^0 \Delta_k E_k^0,\ k \in \underline{N} \quad \text{and} \quad \eta_k(\cdot) \to \widetilde{\eta}_k(\cdot) := \eta_k(\cdot) + D_k^1 \delta_k(\cdot) E_k^1, k \in \underline{N}. \tag{21}$$

Here, for each $k \in \underline{N},\ D_k^0 \in \mathbb{R}^{n \times r_k},\ E_k^0 \in \mathbb{R}^{q_k \times n},\ D_k^1 \in \mathbb{R}^{n \times s_k},\ E_k^1 \in \mathbb{R}^{p_k \times n}$ are *given matrices* defining the structure of the perturbations, $\Delta_k \in \mathbb{R}^{r_k \times q_k}$ and $\delta_k \in NBV([-h, 0], \mathbb{R}^{s_k \times p_k})$ are unknown disturbances. These perturbations are said to be *nonnegative* if, moreover, $\Delta_k \geq 0$ and $\delta_k(\cdot)$ are non-decreasing on $[-h, 0], \forall k \in \underline{N}$. Then the perturbed switched system is described by

$$(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma) \;:\; \dot{x}(t) = \widetilde{A}^0_{\sigma(t)} x(t) + \int_{-h}^{0} d[\widetilde{\eta}_{\sigma(t)}(\theta)] x(t+\theta), t \geq 0, \;\sigma \in \Sigma. \tag{22}$$

As is well-known (see e.g., References 1,3 for the case of linear systems with no delays), by choosing appropriate structuring matrices $D^i_k, E^i_k, i = 0, 1,\; k \in \underline{N}$, one can represent the perturbations, which affect independently all elements of the k-th subsystem matrices $A^0_k, \eta_k(\cdot)$ or their individual rows, columns, or elements. In what follows, it is assumed that these structuring matrices are fixed in the perturbation models (21) and hence, if otherwise not stated, they will be dropped in all the definitions and notations related to the stability radius, for the sake of brevity.

The stability robustness question we are interested in is, given the perturbation model (21), how large disturbances $\Delta_k,\; \delta_k(\cdot),\; k \in \underline{N}$ are allowable without destroying the exponential stability property of the original nominal system (10), (13). To this end, let us measure the size of disturbances $\mathbf{\Delta} := \{[\Delta_k,\; \delta_k(\cdot)],\; k \in \underline{N}\}$ by the quantity

$$\|\mathbf{\Delta}\|_{\max} := \max_{k \in \underline{N}} (\|\Delta_k\| + \|\delta_k\|). \tag{23}$$

Then the robustness of exponential stability of the system (10), (13) can be quantified by the following definition.

Definition 3. Assume that the time-delay switched linear system (10), (13) is GES over Σ_{ADT} (resp., GES over Σ_+). Then its *structured stability radius* over Σ_{ADT} (respectively, over Σ_+), subject to affine perturbations of the form (21) is defined as

$$r^{str}_{\mathbb{R}}(\Sigma_{\text{ADT}}) := \inf \left\{ \|\mathbf{\Delta}\|_{\max} \;:\; \forall \tau_a > 0,\; \exists \sigma \in \Sigma_{\tau_a} \text{ s.t. the perturbed system } \; (\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma) \text{ is not GES} \right\}. \tag{24}$$

respectively as,

$$r^{str}_{\mathbb{R}}(\Sigma_+) := \inf \left\{ \|\mathbf{\Delta}\|_{\max} \;:\; \exists \sigma \in \Sigma_+ \text{ s.t. the perturbed system } \; (\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma) \text{ is not GES} \right\}. \tag{25}$$

If $r_k = s_k = q_k = p_k = n, D^i_k = I_n,\; E^i_k = I_n,\; i \in \{0, 1\},\; k \in \underline{N}$ in the perturbations model (21), then we get, by (24) and (25), the *unstructured stability radii* which are denoted respectively by $r^{unstr}_{\mathbb{R}}(\Sigma_{\text{ADT}})$ and $r^{unstr}_{\mathbb{R}}(\Sigma_+)$.

Remark 3. It follows from Definition 3 that, for any disturbance $\mathbf{\Delta} := \{[\Delta_k,\; \delta_k(\cdot)],\; k \in \underline{N}\}$ satisfying $\|\mathbf{\Delta}\|_{\max} = \max_{k \in \underline{N}}(\|\Delta_k\| + \|\delta_k\|) < r^{str}_{\mathbb{R}}(\Sigma_{\text{ADT}})$, the perturbed switched system (22) is GES over Σ_{ADT}, that is, there exists $\tau_a > 0$ such that $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is GES, for any $\sigma \in \Sigma_{\tau_a}$. The similar property holds also for $r^{str}_{\mathbb{R}}(\Sigma_+)$.

Remark 4. If the nominal system is positive then it is meaningful to restrict disturbances $\mathbf{\Delta}$ in (24) and (25) of Definition 3 to those perturbations which *preserve the positivity* of the system, for instance, by Proposition 1, to the class of *nonnegative* perturbations $\mathbf{\Delta} = \{[\Delta_k,\; \delta_k(\cdot)], k \in \underline{N}\}$ (so that $\Delta_k \geq 0$ and $\delta_k(\cdot)$ are non-decreasing on $[-h, 0]$), provided that the structuring matrices are all nonnegative: $D^i_k \geq 0, E^i_k \geq 0, i = 0, 1, k \in \underline{N}$. Denote by $r^{str}_{\mathbb{R}_+}(\Sigma_{\text{ADT}})$ and $r^{str}_{\mathbb{R}_+}(\Sigma_+)$ the stability radii corresponding to this class of perturbations, then we have obviously

$$r^{str}_{\mathbb{R}}(\Sigma_{\text{ADT}}) \leq r^{str}_{\mathbb{R}_+}(\Sigma_{\text{ADT}}) \text{ and } r^{str}_{\mathbb{R}}(\Sigma_+) \leq r^{str}_{\mathbb{R}_+}(\Sigma_+). \tag{26}$$

Obviously, if the switched system (10), (13) consists of only one subsystem (A^0_k, η_k), for a *fixed* $k \in \underline{N}$, then the above definitions of the two radii are identical and reduced to the well-known notion of the *real structured stability radius* $r_{\mathbb{R}}(A^0_k, \eta_k)$ of the time-delay subsystem (A^0_k, η_k) subject to structured affine perturbations

$$A^0_k \to \widetilde{A}^0_k := A^0_k + D^0_k \Delta_k E^0_k, \quad \eta_k \to \widetilde{\eta}_k := \eta_k + D^1_k \delta_k E^1_k, \tag{27}$$

that was studied in References 11,12, namely,

$$r_{\mathbb{R}}(A^0_k, \eta_k) = \inf \left\{ \|\Delta_k\| + \|\delta_k\| \;:\; (\widetilde{A}^0_k, \widetilde{\eta}_k) \;\text{ is not GES} \right\}. \tag{28}$$

It has been proved, in particular, that if the time-delay system (A_k^0, η_k) is *positive* (i.e., A_k^0 is Metzler and $\eta_k \in NBV([-h,0],\mathbb{R}^n)$ is non-decreasing on $[-h,0]$) and the structuring matrices $D_k^i, E_k^i, i=0,1$ are nonnegative then its structured stability radius $r_{\mathbb{R}}(A_k^0,\eta_k)$ coincides with $r_{\mathbb{R}_+}(A_k^0,\eta_k)$, the stability radius corresponding to nonnegative perturbations, and can be estimated by some computable bounds, which yield the explicit formula for computing the structured stability radius, if $D_k^0 = D_k^1$ or $E_k^0 = E_k^1$, namely,

$$r_{\mathbb{R}_+}\left(A_k^0, A_k^i\right) = r_{\mathbb{R}}\left(A_k^0, A_k^i\right) = \left[\max_{j\in M}\left\|E_k^j\left(A_k^0 + \eta_k(0)\right)^{-1} D_k^j\right\|\right]^{-1}, \quad M := \{0,1\}. \tag{29}$$

In the particular case of the positive linear systems with multiple discrete delays described by (19), with $B_k(\theta) \equiv 0$, the above result implies readily the following formula of the stability radius:

$$r_{\mathbb{R}_+}(A_k^0, A_k^i) = r_{\mathbb{R}}(A_k^0, A_k^i) = \left[\max_{j\in M}\left\|E_k^j\left(\sum_{i=0}^{m} A_k^i\right)^{-1} D_k^j\right\|\right]^{-1}, \quad M := \{0,1\}. \tag{30}$$

Remark 5. It has been shown in References 11,12 that, if the system is positive and $r_{\mathbb{R}_+}(A_k^0,\eta_k) < \infty$, then there exists a destabilizing perturbation of the form $[\widetilde{\Delta}_k, 0]$ or $[0, \widetilde{\delta}_k]$ such that $\widetilde{\Delta}_k \geq 0, \widetilde{\delta}_k(\cdot)$ is a step function $\widetilde{\delta}_k(-h) = 0, \widetilde{\delta}_k(\theta) = \widetilde{\Delta}_k^1 \geq 0, \theta \in (-h,0]$, both matrices $\widetilde{\Delta}_k, \widetilde{\Delta}_k^1$ being of rank-one and $\|\widetilde{\Delta}_k\| = r_{\mathbb{R}_+}\left(A_k^0,\eta_k\right)$ or $\|\widetilde{\delta}_k\| = \|\widetilde{\Delta}_k^1\| = r_{\mathbb{R}_+}\left(A_k^0,\eta_k\right)$.

The following theorem gives a formula for computing the stability radius of positive switched systems over Σ_{ADT}.

Theorem 2. *Let the positive switched linear system (10), (13) be GES over Σ_{ADT} and subjected to affine perturbations (21), with nonnegative structuring matrices $D_k^i, E_k^i, i \in \{0,1\}, k \in \underline{N}$. Then its stability radius $r_{\mathbb{R}_+}^{str}(\Sigma_{\mathrm{ADT}})$ is calculated as*

$$r_{\mathbb{R}_+}^{str}(\Sigma_{\mathrm{ADT}}) = \min_{k\in\underline{N}} r_{\mathbb{R}_+}(A_k^0,\eta_k). \tag{31}$$

If, moreover, $D_k^0 = D_k^1$ or $E_k^0 = E_k^1$, $k \in \underline{N}$, then we get

$$r_{\mathbb{R}_+}^{str}(\Sigma_{\mathrm{ADT}}) = \left[\max_{k\in\underline{N},\ i\in M}\left\|E_k^i\left(A_k^0 + \eta_k(0)\right)^{-1} D_k^i\right\|\right]^{-1}, \ M := \{0,1\}. \tag{32}$$

In particular, the unstructured stability radius over Σ_{ADT} is calculated by

$$r_{\mathbb{R}_+}^{unstr}(\Sigma_{\mathrm{ADT}}) = \left[\max_{k\in\underline{N}}\left\|\left(A_k^0 + \eta_k(0)\right)^{-1}\right\|\right]^{-1}. \tag{33}$$

Proof. To prove the upper bound, assume that $\min_{k\in\underline{N}} r_{\mathbb{R}_+}\left(A_k^0,\eta_k\right) = r_{\mathbb{R}_+}\left(A_{k_0}^0,\eta_{k_0}\right)$, for some $k_0 \in \underline{N}$. By Remark 5, there exists a rank-one nonnegative perturbation $[\widetilde{\Delta}_{k_0}, \widetilde{\delta}_{k_0}]$ such that

$$\|\widetilde{\Delta}_{k_0}\| + \|\widetilde{\delta}_{k_0}\| = r_{\mathbb{R}_+}\left(A_{k_0}^0,\eta_{k_0}\right)$$

and the time-delay positive perturbed system $\left(\widetilde{A}_{k_0}^0, \widetilde{\eta}_{k_0}\right)$ $\left(\text{with } \widetilde{A}_{k_0}^0 := A_{k_0}^0 + D_{k_0}^0\widetilde{\Delta}_{k_0}E_{k_0}^0,\ \widetilde{\eta}_{k_0}(\cdot) := \eta_{k_0}(\cdot) + D_{k_0}^1\widetilde{\delta}_{k_0}(\cdot)E_{k_0}^1\right)$ is not GES. This implies, however, that the perturbed switched linear system (22) associated with perturbation $\widetilde{\boldsymbol{\Delta}} := \{[\Delta_k,\ \delta_k(\cdot)],\ k \in \underline{N}\}$ with $[\Delta_k,\ \delta_k(\cdot)] = 0, \forall k \neq k_0$ and $[\Delta_k,\ \delta_k(\cdot)] = [\widetilde{\Delta}_{k_0},\ \widetilde{\delta}_{k_0}(\cdot)]$ for $k = k_0$ is not GES under the switching signal $\sigma_{k_0}(t) \equiv k_0,\ t \geq 0$. On the other hand, it is obvious that $\sigma_{k_0} \in \Sigma_{\tau_a}, \forall \tau_a > 0$, therefore, by the definition of the stability radius, we get

$$r_{\mathbb{R}_+}^{str}(\Sigma_{\mathrm{ADT}}) \leq \|\widetilde{\boldsymbol{\Delta}}\|_{\max} = \|\widetilde{\Delta}_{k_0}\| + \|\widetilde{\delta}_{k_0}\| = r_{\mathbb{R}_+}(A_{k_0}^0,\eta_{k_0}), \tag{34}$$

as to be shown. To prove the lower bound, let $\mathbf{\Delta} := \{[\Delta_k,\ \delta_k(\cdot)],\ k \in \underline{N}\}$ be an arbitrary nonnegative disturbance such that $\|\mathbf{\Delta}\|_{\max} := \max_{k\in\underline{N}}(\|\Delta_k\| + \|\delta_k\|) < \min_{k\in\underline{N}} r_{\mathbb{R}_+}\left(A_k^0, \eta_k\right) = r_{\mathbb{R}_+}\left(A_{k_0}^0, \eta_{k_0}\right)$. Then for each $k \in \underline{N}$, we have

$$\|\Delta_k\| + \|\delta_k\| < r_{\mathbb{R}_+}\left(A_k^0, \eta_k\right),$$

and, hence, by definition of the stability radius of the subsystem (A_k^0, η_k), all the positive perturbed systems $(\widetilde{A}_k^0, \widetilde{\eta}_k), k \in \underline{N}$ are GES. It follows, by theorem 4.1 of Reference 13 that there exist strictly positive vectors $\xi_k \gg 0, k \in \underline{N}$, such that $(\widetilde{A}_k^0 + \widetilde{\eta}_k(0))\xi_k \ll 0, k \in \underline{N}$ which, by Theorem 1, implies that the perturbed switched system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is GES over $\Sigma_{\rm ADT}$. Indeed, it has been shown by corollary 6 in Reference 36 that the positive switched system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is then GES over Σ_{τ_a} for any $\tau_a > \tau_* = \frac{\ln\gamma}{\alpha}$, where $\gamma := \max\{\xi_{k,i}/\xi_{l,i},\ k, l \in \underline{N},\ i \in \underline{n}\} > 1$ and $\alpha > 0$ is chosen to satisfy $\left(\widetilde{A}_k^0 + e^{\alpha h}\widetilde{\eta}_k(0)\right)\xi_k \le -\alpha\xi_k,\ k \in \underline{N}$. Therefore, by the definition of the stability radius, we get $r_{\mathbb{R}_+}^{str}(\Sigma_{\rm ADT}) \ge \min_{k\in\underline{N}} r_{\mathbb{R}_+}\left(A_k^0, \eta_k\right)$, which, together with (34) and (29) implies (31) and (32), completing the proof. ■

Example 1. Consider the time-delay switched positive linear system (10) in $\mathbb{R}^2$ with $h = 1, N = 2$,

$$A_1^0 = \begin{bmatrix} -5.0221 & 0.2531 \\ 1.0103 & -3.0105 \end{bmatrix},\ A_2^0 = \begin{bmatrix} -4.1023 & 0.2517 \\ 0.5314 & -2.4531 \end{bmatrix},$$

and, for $k = 1, 2$,

$$\eta_k(\theta) = \begin{cases} 0 & \text{if } \theta = -1; \\ A_k^1 \in \mathbb{R}^{2\times 2} & \text{if } \theta \in (-1,\ 0], \end{cases}$$
$$A_1^1 = \begin{bmatrix} 0.6321 & 0.3507 \\ 1.0315 & 0.2403 \end{bmatrix},\ A_2^1 = \begin{bmatrix} 1.103 & 0.5041 \\ 0.7013 & 0.1102 \end{bmatrix}.$$

Choosing positive vectors $\xi_1 = [0.5\ 1]^\top$ and $\xi_2 = [1\ 0.8]^\top$, it is readily verified that the condition (20) holds and therefore the system is GES over $\Sigma_{\rm ADT}$. Now, assume that the second rows of the above subsystems matrices are subjected to unknown disturbances so that the perturbed subsystems take the form

$$\dot{x}(t) = \widetilde{A}_k^0 x(t) + \widetilde{A}_k^1 x(t-1), t \ge 0, k = 1, 2, \tag{35}$$

where

$$\widetilde{A}_1^0 = \begin{bmatrix} -5.0221 & 0.2531 \\ 1.0103 + \omega_{1,1}^0 & -3.0105 + \omega_{1,2}^0 \end{bmatrix},\quad \widetilde{A}_1^1 = \begin{bmatrix} 0.6321 & 0.3507 \\ 1.0315 + \omega_{1,1}^1 & 0.2403 + \omega_{1,2}^1 \end{bmatrix},$$
$$\widetilde{A}_2^0 = \begin{bmatrix} -4.1023 & 0.2517 \\ 0.5314 + \omega_{2,1}^0 & -2.4531 + \omega_{2,2}^0 \end{bmatrix},\quad \widetilde{A}_2^1 = \begin{bmatrix} 1.103 & 0.5041 \\ 0.7013 + \omega_{2,1}^1 & 0.1102 + \omega_{2,2}^1 \end{bmatrix},$$

and $\omega_k^i, k = 1, 2, i = 0, 1$ are unknown parameters. Then, by defining the structuring matrices $D_1^0 = D_2^0 = D_1^1 = D_2^1 = [0\ 1]^\top \in \mathbb{R}^{2\times 1}$ and $E_1^0 = E_2^0 = E_1^1 = E_2^1 = I_2 \in \mathbb{R}^{2\times 2}$ (the identity matrix) and setting $\mathbf{\Delta} = [\Delta_k^0, \Delta_k^1]$ with $\Delta_k^i = [\omega_{k,1}^i\ \omega_{k,2}^i] \in \mathbb{R}_+^{1\times 2},\ i = 0, 1$ and $k = 1, 2$ we get the affine perturbation model (21). By Theorem 2 and formula (30), the stability radius of the switched system under consideration is

$$r_{\mathbb{R}_+}^{str}(\Sigma_{\rm ADT}) = \min_{k\in\underline{N}} r_{\mathbb{R}_+}(A_k^0, A_k^i) = 2.0323.$$

It follows, by definition, that the perturbed switched system having (35) as subsystems is GES over $\Sigma_{\rm ADT}$, for any disturbances $\omega_{k,j}^i \ge 0$ satisfying $\max_{k=1,2}\left\{\omega_{k,1}^0 + \omega_{k,2}^0 + \omega_{k,1}^1 + \omega_{k,2}^1\right\} < 2.0323$.

Now, let the switched system (10), (13) be GES over Σ_+ and the constituent subsystems be subjected to affine perturbations (21). Then, by the same argumentation as in the proof of Theorem 2, it can be shown that

$$r_{\mathbb{R}}^{str}(\Sigma_+) \leq \min_{k\in\underline{N}} r_{\mathbb{R}}(A_k, \eta_k). \tag{36}$$

However, it is easy to construct an example to shown that a formula similar to (31) is not true for this radius $r_{\mathbb{R}}^{str}(\Sigma_+)$, even for the unstructured perturbations case.

Finding a formula for computing or estimating the stability radius of a switched system over Σ_+ is a difficult problem. So far, this problem was considered only for linear delay-free systems (see, e.g., References 14,37,38) where some lower bounds for the stability radius were obtained. In the case of positive systems in $\mathbb{R}^2$, with two switched modes, based on the necessary and sufficient conditions of stability obtained in References 42,43, we have a formula for computing the unstructured stability radius, by the following theorem.

Theorem 3. *Consider the positive linear switched system in $\mathbb{R}^2$, with two stable modes,*

$$\dot{x} = A^0_{\sigma(t)}x(t), t \geq 0, \sigma(t) \in \{1, 2\}. \tag{37}$$

Assume that the system is GES over Σ_+. Then the unstructured stability radius of this system is calculated by

$$r_{\mathbb{R}}^{unstr}(\Sigma_+) = \min_{\alpha\in[0,\ 1]} r_{\mathbb{R}}\left(\alpha A_1^0 + (1-\alpha)A_2^0\right) := \gamma, \tag{38}$$

where $r_{\mathbb{R}}(A)$ denotes the real unstructured stability radius of a Hurwitz matrix A.

Proof. Since the switched system (37) is positive and GES over Σ_+, it follows that all matrices of the *matrix pencil* $A_\alpha := \alpha A_1^0 + (1-\alpha)A_2^0, \alpha \in [0,\ 1]$ are Metzler and, moreover, Hurwitz, by Corollary 2.3 of Reference 14. Due to the continuity of the stability radius (see, e.g., Reference 3), there exists $\alpha_0 \in [0, 1]$ such that

$$\min_{\alpha\in[0,\ 1]} r_{\mathbb{R}}(A_\alpha) = r_{\mathbb{R}}(A_{\alpha_0}) = \gamma$$

We proof first that

$$r_{\mathbb{R}}^{unstr}(\Sigma_+) \leq \gamma. \tag{39}$$

Obviously, (39) holds if $\alpha_0 = 0$ or 1. Assume to the contrary that $r_{\mathbb{R}}(A_{\alpha_0}) = \gamma < r_{\mathbb{R}}^{unstr}(\Sigma_+)$, for some $\alpha_0 \in (0, 1)$. Then, by the definition (25), the perturbed systems $(\widetilde{\mathcal{A}}, \sigma)$ are GES for any signal $\sigma \in \Sigma_+$ and any perturbations $\mathbf{\Delta} = [\Delta_1, \Delta_2]$ such that $\|\mathbf{\Delta}\|_{\max} := \max\{\|\Delta_1\|, \|\Delta_2\|\} \leq \gamma < r_{\mathbb{R}}^{unstr}(\Sigma_+)$, implying, again by Corollary 2.3 of Reference 14, that all convex combination

$$\widetilde{A}_\alpha := \alpha(A_1^0 + \Delta_1) + (1-\alpha)(A_2^0 + \Delta_2), \alpha \in [0, 1] \text{ are Hurwitz.} \tag{40}$$

On the other hand, it follows from the theory of stability radius (see, e.g., References 3,5), that there exists a *minimal destabilizing perturbation* $\Delta_{\alpha_0} \in \mathbb{R}^{2\times 2}$ such that $\|\Delta_{\alpha_0}\| = r_{\mathbb{R}}(A_{\alpha_0}) = \gamma$ and the perturbed matrix

$$\widetilde{A}_{\alpha_0} = A_{\alpha_0} + \Delta_{\alpha_0} = \alpha_0(A_1^0 + \Delta_{\alpha_0}) + (1-\alpha_0)(A_2^0 + \Delta_{\alpha_0})$$

is not Hurwitz, contradicting to (40). Thus (39) is proved. Further, let $\mathbf{\Delta} = [\Delta_1, \Delta_2]$ be an arbitrary perturbation such that $\Delta_i \geq 0, i = 1, 2$ and $\|\mathbf{\Delta}\|_{\max} < \gamma = r_{\mathbb{R}}(A_{\alpha_0}) = r_{\mathbb{R}_+}(A_{\alpha_0})$, the last equality between stability radii being proved in Reference 5, for any Metzler and Hurwitz matrix. Then, defining, for any $\alpha \in [0, 1]$, the pertubation $\Delta_\alpha := \alpha\Delta_1 + (1-\alpha)\Delta_2$ we have, obviously, $\|\Delta_\alpha\| \leq \max_{i=1,2} \|\Delta_i\| < \gamma = r_{\mathbb{R}}(A_{\alpha_0}) \leq r_{\mathbb{R}}(A_\alpha)$ and, therefore, the perturbed matrix $\widetilde{A}_\alpha := A_\alpha + \Delta_\alpha$ is Hurwitz, while being Metzler. It follows, by a result in References 42 and 43, that the perturbed *positive* switched system $(\widetilde{\mathcal{A}}, \sigma)$ is GES for arbitrary switching $\sigma \in \Sigma_+$ (the perturbed modes $\dot{x} = (A_k^0 + \Delta_k)x, k = 1, 2$, having, moreover, a common quadratic Lyapunov function). Since this was proved for an arbitrary nonnegative perturbation $\mathbf{\Delta}$ satisfying $\|\mathbf{\Delta}\|_{\max} < r_{\mathbb{R}}(A_{\alpha_0})$ it implies, by the definition (25) and Remark 3, that $r_{\mathbb{R}}^{unstr}(\Sigma_+) \geq r_{\mathbb{R}}(A_{\alpha_0}) = \gamma$. This, together with (39), yields (38), completing the proof. ∎

Remark 6. Note that the proof of the upper bound (39) is valid without the assumption on positivity of the system. Further, it follows from the above proof that, actually, the equality holds in (26), for the class of two-dimensional positive switched systems (37). Remark, furthermore, that the real unstructured stability radius of a Hurwitz matrix $A \in \mathbb{R}^{2\times 2}$ can be calculated straightforwardly as $r_{\mathbb{R}}(A) = \min\{s_2(A); |\mathrm{Tr}(A)|/2\}$ where $s_2(A)$ is the smallest singular value of A and Tr denotes its trace, provided that the Euclidean vector norm in $\mathbb{R}^2$ is used (see, e.g., Reference 3). On the other hand, if the matrix $A \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz, then its real unstructured stability radius is given by $r_{\mathbb{R}}(A) = \left[\|A^{-1}\|\right]^{-1}$, for any monotonic vector norm of $\mathbb{R}^n$, in particular for ∞-norm (see, e.g., Reference 5). As a consequence, the problem of calculation of the unstructured stability radius of switched linear system (37) is reduced, by (38), to a minimization problem with respect to the parameter $\alpha \in [0, 1]$. The extension of Theorem 3 to higher-order systems is not valid, because the proof of the lower bound of the stability radius $r_{\mathbb{R}}^{unstr}(\Sigma_+)$ is based on the result of Reference 42, which was shown to be false for $n = 3$, by a counterexample in Reference 44. On the other hand, based on Reference 45, a result similar to Theorem 3 can be obtained for a more general switched systems in $\mathbb{R}^2$ of the form $\dot{x} = A_\sigma^0 x(t), t \geq 0, \sigma(t) \in \{1, 2, \dots, N\}$, with $N > 2$, namely, we have

$$r_{\mathbb{R}}^{unstr}(\Sigma_+) = \min_{i,j\in \underline{N}} \min_{\alpha\in[0,\ 1]} r_{\mathbb{R}}\left(\alpha A_i^0 + (1-\alpha)A_j^0\right). \tag{41}$$

The following example illustrates the use of Theorem 3.

Example 2. Consider the switched positive linear system (19) in $\mathbb{R}^2$ with $h = 0, m = 0, N = 2$, and with the two stable modes

$$A_1^0 = \begin{bmatrix} -1 & 0.2 \\ 0.5 & -1 \end{bmatrix},\ A_2^0 = \begin{bmatrix} -1 & 0.4 \\ 1 & -1 \end{bmatrix}.$$

It is readily verified that vector $\xi = [1\ 2]^\top$ satisfies $A_k^0\xi \ll 0, k = 1, 2$ so that, by Corollary 1, this switched system is GES over Σ_+. Define the matrix pencil

$$A_\alpha = \alpha A_1^0 + (1-\alpha)A_2^0 = \begin{bmatrix} -1 & 0.4 - 0.2\alpha \\ 1 - 0.5\alpha & -1 \end{bmatrix},\ \alpha \in [0, 1].$$

Then, we have

$$r_{\mathbb{R}}(A_\alpha) = \frac{1}{\|A_\alpha^{-1}\|} = \frac{6 + 4\alpha - \alpha^2}{20 - 5\alpha},\ \alpha \in [0, 1].$$

By a simple calculation, we obtain $\min_{\alpha\in[0,1]} r_{\mathbb{R}}(A_\alpha) = r_{\mathbb{R}}(A_2^0) = 0.3$. Therefore, by Theorem 3, $r_{\mathbb{R}}^{unstr}(\Sigma_+) = 0.3$.

Below we make use of the approach developed in Reference 38 and Theorem 1 to establish some bounds for the structured stability radius of the system (10), (13) over Σ_+.

Assume that the condition (18) holds for the switched time-delay linear system (10), (13) or, equivalently, the open convex cone

$$\mathcal{G}_{\mathcal{A},\Gamma} = \left\{\xi \in \mathbb{R}_+^n : \xi \gg 0, \left(\mathcal{M}(A_k^0) + V(\eta_k)\right)\xi \ll 0, k \in \underline{N}\right\} \tag{42}$$

is non-empty. Then, by Theorem 1, system (10), (13) is GES over Σ_+. Denote, for each $\xi \in \mathcal{G}_{\mathcal{A},\Gamma}$ and each $k \in \underline{N}$,

$$\left(\mathcal{M}(A_k^0) + V(\eta_k)\right)\xi = -\left(\beta_1^k(\xi), \beta_2^k(\xi), \dots, \beta_n^k(\xi)\right)^\top, \tag{43}$$

and define

$$\beta(\xi) := \min_{k\in\underline{N}, i\in\underline{n}} \beta_i^k(\xi), \tag{44}$$

then, clearly, $\beta(\xi) > 0$. The following theorem gives computable estimates for the structured stability radius of the system (10) over Σ_+. The proof is similar to the case of delay-free systems considered in Reference 38. We give the details for the completeness of the presentation.

Theorem 4. *Let the switched time-delay linear system (10), (13) be GES over Σ_+. Assume, moreover, that the condition (16) holds or, equivalently, $\mathcal{G}_{\mathcal{A},\Gamma} \neq \emptyset$. Then the structured stability radius of the switched linear system (10) over Σ_+, subject to structured affine perturbations (21), satisfies the inequality*

$$\frac{1}{M_0} \sup_{\xi \in \mathcal{G}_{\mathcal{A},\Gamma}} \frac{\beta(\xi)}{\|\xi\|} \leq r_{\mathbb{R}}^{str}(\Sigma_+) \leq \min_{k \in \underline{N}} r_{\mathbb{R}}\left(A_k^0, \eta_k\right), \tag{45}$$

where

$$M_0 := \max_{k \in \underline{N}} \left\{ \|D_k^0\| \ \|E_k^0\|; \|D_k^1\| \ \|E_k^1\| \right\}.$$

Proof. The upper bound in (45) is proved similarly as in Theorem 2. To prove the lower bound, first we have, by (9), (43), and (44), for each $k \in \underline{N}, \xi \in \mathcal{G}_{\mathcal{A},\Gamma}$ and arbitrary disturbances $\Delta_k \in \mathbb{R}^{r_k \times q_k}$, $\delta_k(\cdot) \in NBV([-h,0], \mathbb{R}^{s_k \times p_k})$,

$$\begin{aligned} \left(\mathcal{M}(\widetilde{A}_k^0) + V(\widetilde{\eta}_k)\right)\xi &\leq \left(\mathcal{M}(A_k^0)\xi + V(\eta_k)\xi + \left(|D_k^0 \Delta_k E_k^0| + V\left(D_k^1 \delta_k E_k^1\right)\right)\xi\right. \\ &\leq -\beta(\xi)\mathbf{1}_n + \left(|D_k^0 \Delta_k E_k^0| + V\left(D_k^1 \delta_k E_k^1\right)\right)\xi, \end{aligned} \tag{46}$$

where $\mathbf{1}_n := (1, 1, \ldots, 1)^\top$. Using (5), (6) and the definition of M_0 we get easily, for any $k \in \underline{N}$,

$$\left\| \left(|D_k^0 \Delta_k E_k^0| + V\left(D_k^1 \delta_k E_k^1\right)\right)\xi \right\| \leq M_0(\|\Delta_k\| + \|\delta_k\|)\|\xi\|. \tag{47}$$

It follows that, for any disturbance $\mathbf{\Delta} := \{[\Delta_k, \delta_k(\cdot)], k \in \underline{N}\}$ satisfying

$$\|\mathbf{\Delta}\|_{\max} := \max_{k \in \underline{N}} (\|\Delta_k\| + \|\delta_k\|) < \frac{1}{M_0} \frac{\beta(\xi)}{\|\xi\|}, \tag{48}$$

we have $\left\| \left(|D_k^0 \Delta_k E_k^0| + V\left(D_k^1 \delta_k E_k^1\right)\right)\xi \right\| < \beta(\xi)$. Therefore every component of the vector $\left(|D_k^0 \Delta_k E_k^0| + V\left(D_k^1 \delta_k E_k^1\right)\right)\xi$ is strictly smaller than $\beta(\xi)$. It follows, by (46), that

$$\left(\mathcal{M}(\widetilde{A}_k^0) + V(\widetilde{\eta}_k)\right)\xi \ll 0, \ \forall k \in \underline{N}.$$

Therefore, by Theorem 1, the perturbed system (22) is GES over Σ_+. Since this is proved for any disturbance $\mathbf{\Delta}$ satisfying (48), we have, by definition, $r_{\mathbb{R}}^{str}(\Sigma_+) \geq \frac{1}{M_0} \frac{\beta(\xi)}{\|\xi\|}$, for any $\xi \in \mathcal{G}_{\mathcal{A},\Gamma}$, yielding the lower bound in (45) and completing the proof. ∎

Remark 7. We note that the calculation of the lower bound of stability radius in (45) requires solving a system on N linear inequalities to form, by (42), the open convex cone $\mathcal{G}_{\mathcal{A},\Gamma}$ of all strictly positive solutions and then solving the optimization problem $\sup \left\{ \frac{\beta(\xi)}{\|\xi\|} : \xi \in \mathcal{G}_{\mathcal{A},\Gamma} \right\}$. Moreover, since $\beta(\xi)/\|\xi\| = \beta(\xi/\|\xi\|)$ the last problem is obviously reduced to finding the maximum of the function $\beta(\cdot)$ over the compact set $\mathrm{cl}\mathcal{G}_{\mathcal{A},\Gamma} \cap S_1$ where $S_1 = \{\xi \in \mathbb{R}^n : \|\xi\| = 1\}$, the unit ball of $\mathbb{R}^n$. In this regard, it is an interesting question to find out a particular class of time-delay switched linear systems for which the estimates (45) yields actually a formula for calculation of the stability radius (see Reference 38; Corollary 3, for the case of delay-free switched systems).

In the following example, a numerical simulation in $\mathbb{R}^2$ is given to illustrate Theorem 4.

Example 3. Consider the time-delay switched positive linear system (10) in $\mathbb{R}^2$ with $h = 1, N = 2$,

$$A_1^0 = \begin{bmatrix} -4.0122 & 0.1501 \\ 1.1102 & -4.0215 \end{bmatrix}, \ A_2^0 = \begin{bmatrix} -5.2102 & 0.3125 \\ 0.2102 & -1.2135 \end{bmatrix},$$

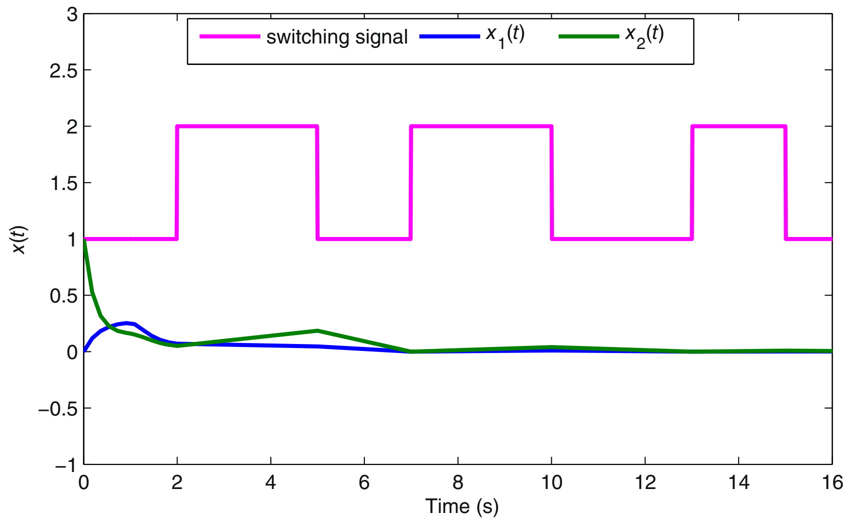

FIGURE 1 The solution trajectory of the system a switching $\sigma_0 \in \Sigma_+$.

and, for $k = 1, 2$,

$$\eta_k(\theta) = \begin{cases} 0 & \text{if} \quad \theta = -1; \\ B_k \in \mathbb{R}^{2\times 2} & \text{if } \theta \in (-1,\ 0], \end{cases}$$

$$B_1 = \begin{bmatrix} 0.2312 & 1.0102 \\ 0.1205 & 0.3112 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2123 & 0.6125 \\ 0.4102 & 0.5213 \end{bmatrix}.$$

Choosing $\xi_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top \gg 0$, we verify readily that ξ_0 satisfies (43). Therefore, this time-delay switched linear system is exponentially stable. For instance, if we choose the switching signal $\sigma_0 \in \Sigma_+$ defined as $\sigma_0(t) = 1$, for t in the intervals $[0, 2); [5, 7); [10, 13); [15, 16)$ and $\sigma_0(t) = 2$, for t in the intervals $[2, 5); [7, 10); [13, 15)$, and the initial condition given by the function $\varphi(\theta) = (|\sin(\theta)|\ |\cos(\theta)|)^\top,\ \theta \in [-1, 0]$, then the solution's trajectory of the time-delay switched linear system converges exponentially to zero, as shown in Figure 1.

Assume that the system's matrices are subjected to structured perturbations so that the perturbed subsystems take the form

$$\dot{x}(t) = \widetilde{A}_k^0 x(t) + \widetilde{B}_k x(t-1), t \geq 0, k = 1, 2, \tag{49}$$

where

$$\widetilde{A}_1^0 = \begin{bmatrix} -4.0122 & 0.1501 \\ 1.1102 + \delta_1 & -4.0215 + \delta_2 \end{bmatrix}, \widetilde{B}_1 = \begin{bmatrix} 0.2312 + \gamma_1 & 1.0102 + \gamma_2 \\ 0.1205 & 0.3112 \end{bmatrix},$$
$$\widetilde{A}_2^0 = \begin{bmatrix} -5.2102 + \delta_3 & 0.3125 + \delta_4 \\ 0.2102 & -1.2135 \end{bmatrix}, \widetilde{B}_2 = \begin{bmatrix} 0.2123 & 0.6125 \\ 0.4102 + \gamma_3 & 0.5213 + \gamma_4 \end{bmatrix},$$

and $\delta_k, \gamma_k, k \in \underline{4}$ are unknown disturbances. Then, taking the structuring matrices $D_1^0 = D_1^1 = [0\ 1]^\top,\ D_2^0 = D_2^1 = [1\ 0]^\top$ and $E_1^0 = E_2^0 = E_1^1 = E_2^1 = I_2$ (the identity matrix in $\mathbb{R}^{2\times 2}$) we can represent this perturbation model in the form (21). Since all subsystems are positive and

$$A_1^0 + \eta_1(0) = A_1^0 + B_1 = \begin{bmatrix} -3.7810 & 1.1603 \\ 1.2307 & -3.7103 \end{bmatrix},$$
$$A_2^0 + \eta_2(0) = A_2^0 + B_2 = \begin{bmatrix} -4.9979 & 0.9250 \\ 0.6204 & -0.6922 \end{bmatrix},$$

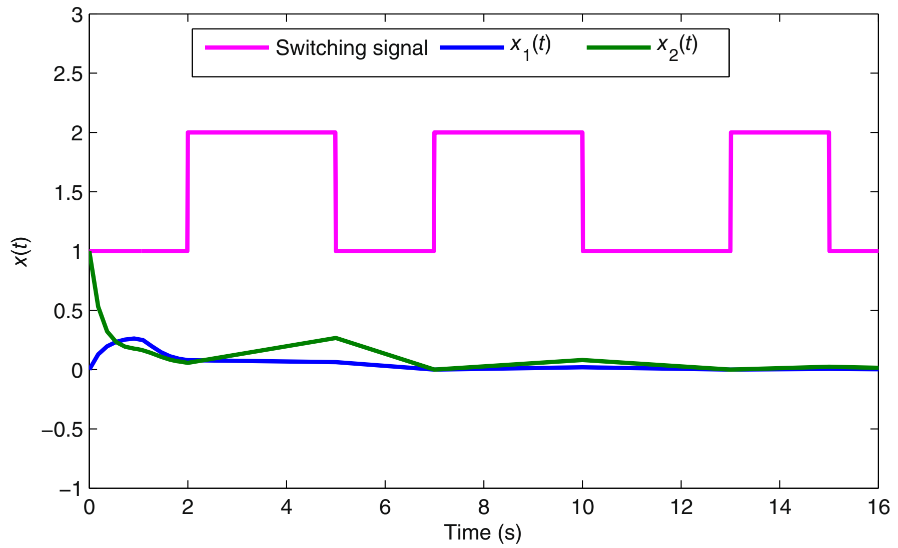

FIGURE 2 All perturbed systems are exponentially stable under switching $\sigma_0 \in \Sigma_+$ if $\|\mathbf{\Delta}\|_{\max} < 0.2547$.

we can use (29) to compute their real stability radii and get the upper bound in (45) as

$$\min_{k\in\underline{N}} r_{\mathbb{R}}(A_k^0, \eta_k) = 2.1984.$$

By (43) and (44), we get readily $\beta(\xi_0) = \min_{k\in\underline{N},k\in\underline{n}} \beta_i^k(\xi_0) = 0.7640$. Moreover, clearly, $M_0 = \max_{k=1,2}\{\|D_k^0\|\|E_k^0\|, \|D_k^1\|\|E_k^1\|\} = 1$. Therefore, by Theorem 4, we obtain the following lower bound for the stability radius of the time-delay switched linear system under consideration:

$$r_{\mathbb{R}}^{str}(\Sigma_+) \geq \frac{1}{M_0} \sup_{\xi\in\mathcal{G}_{A,\Gamma}} \frac{\beta(\xi)}{\|\xi\|} \geq \frac{\beta(\xi_0)}{\|\xi_0\|} = 0.2547.$$

It follows, by definition, that the perturbed time-delay switched linear system associated with (73) is exponentially stable, under arbitrary switching $\sigma \in \Sigma_+$, for any disturbance $\mathbf{\Delta}$ satisfying $\|\mathbf{\Delta}\| = \max_{k\in\underline{4}}\{|\delta_k| + |\gamma_k|\} < 0.2547$. Thus, if we choose randomly disturbance parameters $\delta_k, \gamma_k, k \in \underline{4}$ satisfying this condition and the same switching $\sigma_0 \in \Sigma_+$ as above, then the trajectory of the perturbed switched system (simulated by MATLAB toolbox) decays exponentially to zero as t tends to the infinity, as shown in Figure 2.

Below we will make use of Lemma 1 and Theorem 1 in Section 2 to get another lower bound of $r_{\mathbb{R}}^{str}(\Sigma_+)$, which is explicitly expressed in terms of the system's data.

Theorem 5. *Assume that the switched linear system (10), (13) is GES over Σ_+ and subjected to affine perturbations of the form (21). Assume, moreover, that the condition (16) holds. Then the real structured stability radius of the switched linear system (10) over Σ_+, subject to affine perturbations (21), satisfies the following estimates:*

$$r_0 := \left[\max_{k\in\underline{N};i,j\in M} \left\| \, |E_k^i| \left(-\left(\mathcal{M}(A_k^0)\right) + V(\eta_k)\right)^{-1} \right) |D_k^j| \, \right\| \right]^{-1} \leq r_{\mathbb{R}}^{str}(\Sigma_+) \leq \min_{k\in\underline{N}} r_{\mathbb{R}}(A_k^0, \eta_k), \tag{50}$$

where $M := \{0, 1\}$.

Proof. The upper bound in (50) is proved similarly as in Theorem 2. In order to prove the lower bound, by Definition 3, it suffices to show that, for any switching signal $\sigma \in \Sigma_+$ and any perturbation $\mathbf{\Delta} := \{[\Delta_k, \delta_k(\cdot)], k \in \underline{N}\}$ such that

$$\|\mathbf{\Delta}\|_{\max} = \max_{k\in\underline{N}}(\|\Delta_k\| + \|\delta_k\|) < r_0, \tag{51}$$

the perturbed system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ described by (22) is GES, where $\widetilde{A}_k^0 = A_k^0 + D_k^0 \Delta_k E_k^0$ and $\widetilde{\eta}_k = \eta_k + D_k^1 \delta_k E_k^1$. By using (6) and (9), we have, for all $k \in \underline{N}$,

$$\begin{aligned} \mathcal{M}\big(\widetilde{A}_k^0\big) + V(\widetilde{\eta}_k) &\leq \mathcal{M}\big(A_k^0\big) + \big|D_k^0 \Delta_k E_k^0\big| + V(\eta_k) + V\big(D_k^1 \delta_k E_k^1\big) \\ &\leq \mathcal{M}\big(A_k^0\big) + V(\eta_k) + \big|D_k^0\big|\,|\Delta_k|\,\big|E_k^0\big| + \big|D_k^1\big|\,V(\delta_k)\,\big|E_k^1\big|. \end{aligned} \tag{52}$$

Further, by the condition (16) there exists $\xi_0 \gg 0$ such that

$$\big(\mathcal{M}(A_k^0) + V(\eta_k)\big)\,\xi_0 \ll 0, \ \forall k \in \underline{N}. \tag{53}$$

We will show that

$$\Big(\mathcal{M}(\widetilde{A}_k^0) + V(\widetilde{\eta}_k)\Big)\,\xi_0 \ll 0, \ \forall k \in \underline{N}, \tag{54}$$

which would imply, by the second part of Theorem 1, that the perturbed system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is GES. Assume, to the contrary, that (54) does not hold, or, equivalently, there exists $k_0 \in \underline{N}$ such that $0 \leq \Big(\mathcal{M}(\widetilde{A}_{k_0}^0) + V(\widetilde{\eta}_{k_0})\Big)\,\xi_0$. Then, by (52), we have

$$-\Big(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\Big)\,\xi_0 \leq |D_{k_0}^0|\,|\Delta_{k_0}|\,|E_{k_0}^0|\xi_0 + |D_{k_0}^1|\,V(\delta_{k_0})\,|E_{k_0}^1|\,\xi_0. \tag{55}$$

By Lemma 1 (iii), it follows from (53) that the matrix $\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})$ has the negative inverse. Therefore, (55) implies

$$\xi_0 \leq -\Big(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\Big)^{-1}\Big(|D_{k_0}^0|\,|\Delta_{k_0}|\,|E_{k_0}^0|\xi_0 + |D_{k_0}^1|\,V(\delta_{k_0})\,|E_{k_0}^1|\,\xi_0\Big). \tag{56}$$

Let $p \in M := \{0, 1\}$ be an index such that $\||E_{k_0}^p|\,\xi_0\| = \max\{\||E_{k_0}^0|\,\xi_0\|; \||E_{k_0}^1|\,\xi_0\|\}$ then $\||E_{k_0}^p|\,\xi_0\| > 0$ because, otherwise, $|E_{k_0}^i|\,\xi_0 = 0, i = 0, 1$ and (56) would imply $\xi_0 = 0$, a contradiction. Left multiplying the inequality (56) by $|E_{k_0}^p|$ and taking the norm of the both sides of the resulted inequality we get

$$\begin{aligned} 0 < \|\,|E_{k_0}^p|\xi_0\| &\leq \left\||E_{k_0}^p|\,\left(-\Big(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\Big)^{-1}\right)\,|D_{k_0}^0|\,|\Delta_{k_0}|\,|E_{k_0}^0|\xi_0\right\| + \\ &\quad + \left\||E_{k_0}^p|\,\left(-\Big(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\Big)^{-1}\right)\,|D_{k_0}^1|\,V(\delta_{k_0})\,|E_{k_0}^1|\,\xi_0\right\| \\ &\leq \max_{i,j \in M}\left\||E_{k_0}^i|\,\left(-\Big(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\Big)^{-1}\right)\,|D_{k_0}^j|\right\|\,\big(\|\Delta_{k_0}\| + \|\delta_{k_0}\|\big)\,\,\||E_{k_0}^p|\,\xi_0\| \\ &\leq \max_{k \in \underline{N}; i,j \in M}\left\||E_k^i|\,\left(-\big(\mathcal{M}(A_k^0) + V(\eta_k)\big)^{-1}\right)\,|D_k^j|\right\|\,\|\mathbf{\Delta}\|_{\max}\,\||E_k^p|\,\xi_0\|, \end{aligned}$$

which implies $\|\mathbf{\Delta}\|_{\max} \geq r_0$, conflicting with (51). The proof is completed. ■

It is worth mentioning that Theorems 4 and 5 can be applied to give the lower bounds for stability radii of switched linear systems with discrete multi-delays and/or distributed delay of the form (19), as particular cases. For example, we can formulate the following consequence of Theorem 5, which gives calculable bounds for unstructured stability radius over Σ_+.

Corollary 2. *Assume that the positive linear system with delays*

$$\dot{x}(t) = A_0^0 x(t) + \sum_{i=1}^{m} A_0^i x(t - h_i) + \int_{-h}^{0} B_0(\theta) x(t + \theta) d\theta, t \geq 0, \tag{57}$$

(where $0 < h_1 < h_2 < \cdots < h_m = h$) is GES. Then, for any triples $(A_k^0, A_k^i, B_k(\cdot)) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times C([-h, 0],$ $\mathbb{R}^{n \times n}), i \in \underline{m}, k \in \underline{N}, \theta \in [-h, 0]$ satisfying

$$\mathcal{M}(A_k^0) \leq A_0^0, \ |A_k^i| \leq A_0^i, \ |B_k(\theta)| \leq B_0(\theta), \tag{58}$$

the switched linear system with delay

$$\dot{x}(t) = A^0_{\sigma(t)}x(t) + \sum_{i=1}^{m} A^i_{\sigma(t)}x(t-h_i) + \int_{-h}^{0} B_{\sigma(t)}(\theta)x(t+\theta)d\theta, t \geq 0, \tag{59}$$

is GES over Σ_+. Moreover, for each of the switched systems (59) satisfying (58), the real unstructured stability radius over Σ_+ subject to perturbations

$$A^0_k \to \widetilde{A}^0_k = A^0_k + \Delta^0_k;\ A^i_k \to \widetilde{A}^i_k = A^i_k + \Delta^i_k,\ i \in \underline{m}\ ;\ B_k(\theta) \to \widetilde{B}_k(\theta) = B_k(\theta) + C_k(\theta), \theta \in [-h, 0], k \in \underline{N}, \tag{60}$$

(with unknown disturbance matrices $\Delta^0_k, \Delta^i_k \in \mathbb{R}^{n\times n}$ and unknown disturbance functions $C_k(\cdot) \in C([-h, 0], \mathbb{R}^n)$) satisfies the following lower bound:

$$r^{unstr}_{\mathbb{R}}(\Sigma_+) \geq \left[\left\| \left(\sum_{i=0}^{m} A^i_0 + \int_{-h}^{0} B_0(\theta)d\theta \right)^{-1} \right\| \right]^{-1}. \tag{61}$$

If, moreover, A^0_k are Metzler, $A^i_k \geq 0$ and $B_k(\theta) \geq 0, \forall\theta \in [-h, 0], \forall k \in \underline{N}, i \in \underline{m}$, then the following upper bound estimate holds:

$$r^{unstr}_{\mathbb{R}}(\Sigma_+) \leq \left[\max_{k\in\underline{N}} \left\| \left(\sum_{i=0}^{m} A^i_k + \int_{-h}^{0} B_k(\theta)d\theta \right)^{-1} \right\| \right]^{-1}. \tag{62}$$

Now we consider the case when the nominal system's matrices of (10), (13) are subjected to *multi-perturbations* of the form

$$A^0_k \to \widetilde{A}^0_k := A^0_k + \sum_{i=1}^{N_0} \alpha^i_k A^{0,i}_k;\ \ \eta_k(\cdot) \to \widetilde{\eta}_k(\cdot) := \eta_k(\cdot) + \sum_{j=1}^{N_1} \beta^j_k \eta^j_k(\cdot),\ k \in \underline{N}, \tag{63}$$

where, for each $k \in \underline{N}, A^{0,i}_k \in \mathbb{R}^{n\times n}$, and $\eta^j_k \in NBV([-h, 0], \mathbb{R}^{n\times n})$ are given, $\alpha^i_k, \beta^j_k \in \mathbb{R},\ i \in \underline{N_0},\ j \in \underline{N_1}$ are unknown disturbance parameters. In the case of non switched systems, such a class of parameter perturbations was considered in References 4,6 for delay-free systems and in Reference 12 for FDE systems. To measure the robustness of stability under this perturbation model, let us denote

$$\begin{aligned} \mathbf{\Delta} &:= \{\alpha^i_k, \beta^j_k,\ k \in \underline{N},\ i \in \underline{N_0},\ j \in \underline{N_1}, \};\ \|\mathbf{\Delta}\| := \max\{\overline{\alpha}, \overline{\beta}\}; \\ \overline{\alpha} &:= \max\{|\alpha^i_k|,\ k \in \underline{N},\ i \in \underline{N_0}\},\ \ \overline{\beta} := \max\{|\beta^j_k|,\ k \in \underline{N},\ j \in \underline{N_1}\} \end{aligned} \tag{64}$$

and define the following notion of stability radius.

Definition 4. Assume that the time-delay switched linear system (10), (13) is GES over Σ_+. Then its structured stability radius over Σ_+ subject to multi-perturbations of the form (63) is defined as

$$r^{mstr}_{\mathbb{R}}(\Sigma_+) := \inf \left\{ \|\mathbf{\Delta}\|\ :\ \exists\sigma \in \Sigma_+ \text{ s.t. the perturbed system } \ (\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma) \text{ is not GES} \right\}. \tag{65}$$

Theorem 6. *Assume that the time-delay switched linear system (10), (13) is GES over Σ_+ and is subjected to multi-perturbations of the form (63). Assume, moreover, that the condition (16) holds or, equivalently, $\mathcal{G}_{\mathcal{A},\Gamma} \neq \emptyset$. Then the real structured stability radius of the switched linear system (10) over Σ_+, subject to multi-perturbations (63), satisfies the following estimates:*

$$r_1 := \left[\max_{k\in\underline{N}} \left\| -\left(\mathcal{M}(A^0_k) + V(\eta_k)\right)^{-1} \left(\sum_{i=1}^{N_0} |A^{0,i}_k| + \sum_{j=1}^{N_1} V(\eta^j_k) \right) \right\| \right]^{-1} \leq r^{mstr}_{\mathbb{R}}(\Sigma_+) \leq \min_{k\in\underline{N}} r_{\mathbb{R}}\left(A^0_k, \eta_k\right). \tag{66}$$

Proof. We have to prove only the lower bound in (66). The proof is partly similar to that of Theorem 5. Assume to the contrary that there exists a perturbation $\mathbf{\Delta}$ of the form (64) and a switching signal $\sigma \in \Sigma_+$ such that

$$\|\mathbf{\Delta}\| = \max\{\overline{\alpha}, \overline{\beta}\} < r_1 \tag{67}$$

and the perturbed system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is not GES, where the matrices of perturbed constituent systems are given by (63). Then, for all $k \in \underline{N}$, we have, by (9) and (7),

$$\begin{aligned}\mathcal{M}(\widetilde{A}_k^0) + V(\widetilde{\eta}_k) &\le \mathcal{M}(A_k^0) + \left|\sum_{i=1}^{N_0} \alpha_k^i A_k^{0,i}\right| + V(\eta_k) + V\left(\sum_{j=1}^{N_1} \beta_k^j \eta_k^j\right) \\ &\le \mathcal{M}(A_k^0) + \sum_{i=1}^{N_0} |\alpha_k^i|\, |A_k^{0,i}| + V(\eta_k) + \sum_{j=1}^{N_1} |\beta_k^j| V(\eta_k^j) \\ &\le \mathcal{M}(A_k^0 + V(\eta_k) + \|\mathbf{\Delta}\| \left(\sum_{i=1}^{N_0} |A_k^{0,i}| + \sum_{j=1}^{N_1} V(\eta_k^j)\right).\end{aligned} \tag{68}$$

Further, since $\mathcal{G}_{\mathcal{A},\Gamma} \neq \emptyset$, there exists $\xi_0 \gg 0$ such that $\left(\mathcal{M}(A_k^0 + V(\eta_k)\right) \xi_0 \ll 0$. Since the perturbed system $(\widetilde{\mathcal{A}}, \widetilde{\Gamma}, \sigma)$ is not GES, we have, by Theorem 1, that there exists $k_0 \in \underline{N}$ such that

$$0 \le \left(\mathcal{M}(\widetilde{A}_{k_0}^0) + V(\widetilde{\eta}_{k_0})\right) \xi_0$$

It follows, by (68), that

$$\xi_0 \le -\left(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\right)^{-1} \|\mathbf{\Delta}\| \left(\sum_{i=1}^{N_0} |A_{k_0}^{0,i}| + \sum_{j=1}^{N_1} V(\eta_{k_0}^j)\right) \xi_0.$$

Taking the norm of both sides, we get

$$\|\xi_0\| \le \left\| -\left(\mathcal{M}(A_{k_0}^0) + V(\eta_{k_0})\right)^{-1} \left(\sum_{i=1}^{N_0} |A_{k_0}^{0,i}| + \sum_{j=1}^{N_1} V(\eta_{k_0}^j)\right)\right\| \|\mathbf{\Delta}\|\ \|\xi_0\|,$$

which implies

$$\|\mathbf{\Delta}\| \ge \left[\max_{k \in \underline{N}} \left\| -\left(\mathcal{M}(A_k^0) + V(\eta_k)\right)^{-1} \left(\sum_{i=1}^{N_0} |A_k^{0,i}| + \sum_{j=1}^{N_1} V(\eta_k^j)\right)\right\|\right]^{-1} = r_1,$$

a contradiction to (67). The proof is completed. ∎

As an application of the above result, consider the switched linear system with a single delay

$$\dot{x} = A_{\sigma(t)} x(t) + B_{\sigma(t)} x(t-h), \ \sigma \in \Sigma_+, \ \sigma(t) \in \underline{N}, \ t \ge 0. \tag{69}$$

Assume that the switched system (69) is *positive* (or, equivalently, A_k is Metzler and $B_k \ge 0$, for each $k \in \underline{N}$, by Proposition 1) and is GES over Σ_+ (for instance, if $(A_k + B_k)\xi_0 \ll 0$, $\forall k \in \underline{N}$, for some positive vector $\xi_0 \gg 0$, by Corollary 1). Assume that the constituent systems are subjected to multi-perturbation of the form

$$A_k \to \widetilde{A}_k = A_k + \sum_{i=1}^{N_0} \alpha_k^i A_k^i, \ \ B_k \to \widetilde{B}_k = B_k + \sum_{j=1}^{N_1} \beta_k^j B_k^j, \ k \in \underline{N}, \tag{70}$$

where $A_k^i = \left(a^i_{k,pq}\right) \geq 0, B_k^j = \left(b^j_{k,pq}\right) \geq 0, k \in \underline{N}, i \in \underline{N_0}, j \in \underline{N_1}$ are given structuring matrices and $\alpha_k^i, \beta_k^j, i \in \underline{N_0}, j \in \underline{N_1}$ are unknown perturbation parameters. Define "upper bounding" $(n \times n)$-matrices $A_0 = (a_{0,pq}), B_0 = (b_{0,pq})$ by setting

$$a_{0,pq} = \max\{a_{k,pq}, k \in \underline{N}\}; b_{0,pq} = \max\{b_{k,pq}, k \in \underline{N}\}. \tag{71}$$

Then, by Theorem 6, we get the following result.

Corollary 3. *Assume that the time-delay switched positive linear system (69) is GES over Σ_+ and the Meztler matrix $A_0 + B_0$ defined by (71) is Hurwitz stable. Then the stability radius of system (69), subject to multi-perturbation (70), satisfies the estimates*

$$\left[\max_{k\in\underline{N}}\left\|(-A_0 - B_0)^{-1}\left(\sum_{i=1}^{N_0} A_k^i + \sum_{j=1}^{N_1} B_k^j\right)\right\|\right]^{-1} \leq r_{\mathbb{R}}^{mstr}(\Sigma_+) \leq \min_{k\in\underline{N}} r_{\mathbb{R}}\left(A_k, B_k, A_k^i, B_k^j\right), \tag{72}$$

where, for each $k \in \underline{N}$, $r_{\mathbb{R}}\left(A_k, B_k, A_k^i, B_k^j\right)$ denotes the stability radius of the constituent system $\dot{x} = A_k x(t) + B_k x(t-h), t \geq 0$ subject to multi-perturbation (70).

Note additionally that, in this case, the upper bound of $r_{\mathbb{R}}^{mstr}(\Sigma_+)$ in (72) can be calculated explicitly, due to a result in Reference 12, as

$$\min_{k\in\underline{N}} r_{\mathbb{R}}\left(A_k, B_k, A_k^i, B_k^j\right) = \left[\max_{k\in\underline{N}} \rho\left[(-A_k - B_k)^{-1}\left(\sum_{i=1}^{N_0} A_k^i + \sum_{j=1}^{N_1} B_k^j\right)\right]\right]^{-1},$$

where $\rho(A)$ denotes the spectral radius of a nonnegative matrix $A \in \mathbb{R}_+^{n\times n}$.

Example 4. Consider the time-delay switched positive linear system (69) in $\mathbb{R}^2$ with $h = 1, N = 2$,

$$A_1 = \begin{bmatrix} -5 & 2 \\ 0.2 & -4 \end{bmatrix}, A_2 = \begin{bmatrix} -4 & 1.3 \\ 0.1 & -3 \end{bmatrix}, B_1 = \begin{bmatrix} 1.1 & 0.2 \\ 0.3 & 1.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4 & 1 \\ 0.5 & 0.1 \end{bmatrix}.$$

We have that condition $(A_k + B_k)\xi_0 \ll 0, k = 1, 2$ is satisfied with $\xi_0 = [2\ 1]^\top$ and hence, by Corollary 20, the time-delay switched linear system (69) is GES over Σ_+. Assume that the system's matrices are subjected to perturbations so that the perturbed subsystems take the form

$$\dot{x}(t) = \widetilde{A}_k x(t) + \widetilde{B}_k x(t-1),\ \ t \geq 0,\ \ k = 1, 2, \tag{73}$$

where

$$\widetilde{A}_1 = \begin{bmatrix} -5+\alpha_1 & 2 \\ 0.2 & -4+\alpha_1 \end{bmatrix}, \widetilde{B}_1 = \begin{bmatrix} 1.1 & 0.2+\beta_1 \\ 0.3+\beta_1 & 1.2 \end{bmatrix},$$
$$\widetilde{A}_2 = \begin{bmatrix} -4 & 1.3+\alpha_2 \\ 0.1+\alpha_2 & -3 \end{bmatrix}, \widetilde{B}_2 = \begin{bmatrix} 0.4+\beta_2 & 1 \\ 0.5 & 0.1+\beta_2 \end{bmatrix},$$

and $\alpha_k,\ \beta_k \in \mathbb{R},\ k = 1, 2$ are unknown disturbances. It is important to mention that the above perturbations cannot be represented in the form of the affine perturbation model (21). Then, defining the structuring matrices

$$A_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; B_1^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_2^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we can represent the perturbed matrices $\widetilde{A}_k, \widetilde{B}_k$ in the form of multi-perturbation model (70), namely

$$A_k \to \widetilde{A}_k = A_k + \alpha_k A_k^1,\ B_k \to \widetilde{B}_k = B_k + \beta_k B_k^1, k = 1, 2. \tag{74}$$

By using (71) we get

$$A_0 = \begin{bmatrix} -4 & 2 \\ 0.2 & -3 \end{bmatrix},\ B_0 = \begin{bmatrix} 1.1 & 1 \\ 0.5 & 1.2 \end{bmatrix}.$$

Then, since $(A_0 + B_0)\xi_0 \ll 0$ with $\xi_0 = [2\ 1]^{\top}$, the Metzler matrix $A_0 + B_0$ is Hurwitz stable, by Lemma 1. Therefore, by Corollary 3, the stability radius of this system, subject to multi-perturbation (74), satisfies the estimates

$$\left[\max_{k=1,2} \|(-A_0 - B_0)^{-1}\left(A_k^1 + B_k^1\right)\|\right]^{-1} = 0.3250 \le r_{\mathbb{R}}^{mstr}(\Sigma_+) \le \left[\max_{k\in \underline{N}} \rho\left[(-A_k - B_k)^{-1}\left(A_k^1 + B_k^1\right)\right]\right]^{-1} = 0.9638.$$

4 | CONCLUDING REMARKS

We have presented a unified approach to study the robustness of exponential stability, under arbitrary switching or ADT switching, for the class of time-delay switched linear systems, described by linear functional differential equations, by making use of the notion of the structured stability radius with respect to real affine perturbations of the subsystem's matrices. As the main contribution of this paper, we obtained a number of new results on computation and estimation of this radius, including: (a) the formula for computing the stability radius under the average dwell time switching; (b) the formulas for estimating the bounds of the system's stability radius, under arbitrary switching, with respect to affine perturbations and multi-perturbations, which are expressed explicitly in terms of the constituent subsystems matrices; (c) the formula for computing the stability radius of two-order delay-free switched linear systems. The results are established mainly for positive switched systems, but the extension to non-positive systems has also been given, whenever possible. In the particular case of switched linear systems with multiple discrete delays and/or distributed delays, the obtained general results yield, as the consequence, easily verifiable formulas for calculating or estimating the system's stability radius, without using common quadratic Lyapunov functions as in most of the previous works. Some numerical examples are given to illustrate the use of the obtained results. To the best of our knowledge, such kinds of results on robustness of stability for switched systems with time-delays have not been available so far in the literature and are given for the first time in this paper. We believe that the approach developed in this paper is applicable for addressing similar problems under less restrictive assumptions and more general types of parameter perturbations or uncertainties. This, in turn, is expected to yield better and less conservative estimates for the system's stability radius. Of course, this approach is also applicable to studying similar problems for discrete-time systems. Moreover, for classes of switched systems where necessary and sufficient conditions of GES under arbitrary switching/or ADT switching are available, see for example, References 42,43,45, our approach can be used to establish the formula for computation of the stability radius, as has been shown in this paper for the case of two-order delay-free switched systems. It is worthy to note additionally that the results similar to those of the above mentioned works are still lacking in the literature for switched linear systems with delays.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interest for this article.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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