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New results on stability and H_∞ filter design of linear singular time-varying delay systems

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Abstract This paper deals with the problem of stability and H_∞ filtering for a class of singular differential equations (SDEs) with time-variable delay. The time-variable delay is assumed to be continuous uniformly bounded and appears in both the observation measurement and the disturbance input. Based on singular value decomposition method and newly proposed lemmas on stability characterization of SDEs, we propose delay-dependent sufficient conditions for the existence of H_∞ filters such that the error singular equation is admissible with H_∞ norm bound. The conditions are established in terms of linear matrix inequalities (LMIs), which can be solved efficiently by LMI toolbox algorithm. The developed conditions are illustrated by a numerical example with simulations.

Keywords Stability · H_∞ filtering · singular systems · nondifferential delay · linear matrix inequalities.

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1 Introduction

The problem of estimation such as Kalman filter, H_∞ filtering has been widely studied and found many practical applications during the past decades. The Kalman filter addresses the minimization of filtering error covariance [1, 7], while the H_∞ filtering deals with the design of a filter making the error system asymptotically stable with a prescribed norm of the transfer function from the disturbance to the error output [16, 23, 24]. The H_∞ filtering problem has been paid much attention from researchers due to its theoretical and practical interests in control engineering. The problem has made substantive achievements by using various mathematical methods such as polynomial equation and interpolation approach, Lyapunov

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function and LMI methods [19, 20, 21, 26, 27, 30]. On the other hand, singular equations (or descriptor equations, implicit equations, differential-algebraic equations) are widespread concerned and great progress has been made in both theory and application [4, 28]. Moreover, time delays are the main causes of instability and poor performance of systems, and are frequently encountered in many control systems [6, 17]. Since the singular time-delay models describe a larger class of equations than usual state-space ones, great effort has been made to study stability and control of singular equations with delays (see [10, 11, 27] and the references therein). It should be pointed out that the H_∞ filtering problem for singular equations has been developed for a wide range of SDEs such as for time-invariant delay equations, time-variable delay equations, fuzzy SDEs with delays, stochastic SDEs with delays and impulse SDEs with delays [2, 29]. By the use of Lyapunov function method and LMI framework, H_∞ filtering problem was considered in [9, 15, 22, 24] for linear singular equations with constant delays, and in [3, 12, 14, 29, 31] for linear singular equations with time-variable delays. It is worth noting that the time-variable delay considered in the aforementioned papers is differentiable and its derivative is bounded, which reduce practical applications. Therefore, date and to the best of our knowledge, for SDEs with time-variable delay the problem of stability and H_∞ filtering has not been fully investigated yet.

This paper is interested in the stability and H_∞ filtering of SDEs with time-variable delay in such a novel manner that the delay function is non-differentiable, but continuous and bounded. Based on newly proposed stability characterization for SDEs with time-variable delay combined with augmented parameter-dependent Lyapunov–Krasovskii functionals, new sufficient conditions for the admissibility of the error singular equations are first established. Then, the suitable H_∞ filters will be designed through solving tractable LMIs. Comparing with the existing results, the main contributions of this paper lie in the following:

(i) The SDEs is subjected to non-differentiable delay, which appears in both the observation and the disturbance input.

(ii) The technical improvement (Proposition 2) is proposed to show the asymptotical stability for the SDEs. This allows us to exponentially estimate solutions of the delay differential and delay algebraic equations with a unified performance specification.

(iii) New delay-dependent sufficient conditions for designing H_∞ filters are established in terms of LMIs, which can be solved efficiently by LMI toolbox algorithm [5].

The organization of this paper is as follows. In Section 2, stability characterization with some auxiliary technical lemmas for SDEs with time-variable delay is formulated. The main result on the H_∞ filter design is presented in Section 3. An illustrative example is given in this section to demonstrate the effectiveness of the theoretical results

Notations. $\mathbb{Z}^+$ denotes the set of non-negative integers, $\mathbb{R}^n$ denotes the $n-$dimensional linear vector space. $\mathbb{R}^{n\times m}$ denotes the space of $n\times m$ matrices. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the set of maximal and minimal eigenvalues of A, respectively. $\|x_t\|$ denotes the norm of function $x(\cdot)$ on $[t-\tau,t]$ defined by $\|x_t\| = \sup_{s\in[-\tau,0]} \|x(t+s)\|$; $[M_{ij}]_{k\times k}$ denotes the square matrix with $(k\times k)-$dimensions. The symmetric term in a matrix is denoted by $*$.

2 Asymptotic stability

In this section, we provide asymptotic stability conditions for the following SDEs with delays

$$\begin{cases} \mathbb{E}\dot{x}(t) = \mathbb{A}x(t) + \mathbb{A}_d x(t-\beta(t)), \quad t \geq 0, \\ x(t) = \psi(t), \quad t \in [-\beta, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n, \mathbb{E} \in \mathbb{R}^{n\times n}$ is a singular matrix, rank $\mathbb{E} = r \leq n$; $\mathbb{A}, \mathbb{A}_d \in \mathbb{R}^{n\times n}$, the function $\psi(t)$ is continuous; the continuous delay function $\beta(t)$ satisfies $0 \leq \beta(t) \leq \beta, \quad t \geq 0$.

Definition 1 ([4]) Equation (1) is said to be admissible if it is regular, impulse-free and asymptotically stable.

Due to the singularity of matrix $\mathbb{E}$, there are invertible matrices F, K such that $\hat{E} = F\mathbb{E}K = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ and the equation (1) under the state transformation $u(t) = K^{-1}x(t) = [u_1(t), u_2(t)]$, is deduced to the equation

$$\hat{E}\dot{u}(t) = \hat{A}u(t) + A_d u(t - s(t)), \tag{2}$$

where

$$\hat{A} = F\mathbb{A}K = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \hat{A}_d = F\mathbb{A}_d K = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Equation (2) is written as differential-algebraic equations:

$$\begin{cases} \dot{u}_1(t) &= A_{11}u_1(t) + A_{12}u_2(t) + D_{11}u_1(t-\beta(t)) + D_{12}u_2(t-\beta(t)) \\ 0 &= A_{21}u_1(t) + A_{22}u_2(t) + D_{21}u_1(t-\beta(t)) + D_{22}u_2(t-\beta(t)) \end{cases} \tag{3}$$

Lemma 1 *([13]) Assume that $f \in C([-s,\infty), R^+)$ satisfies $f(t) \leq \eta\|f_t\| + M, t \geq 0$, where $M > 0, 0 < \eta < 1$. Then*

$$f(t) \leq \eta\|f_0\| + \frac{M}{1-\eta}, \quad t \geq 0,$$

where $f_t(\cdot) = f(t+\cdot)$ on $[-s,0]$, $\|f_t\| := \sup_{\xi\in[t-s,t]} \|f(\xi)\|$.

The following proposition on the regularity and impulse-absence of (1) extends a result of [25] for the system with constant delay. The proof of this proposition is similar to the proof of Theorem 1 in [25], which is omitted herein.

Proposition 1 *Assume that there are matrices P, Q, R where P is non-singular, $Q > 0$ is symmetric, satisfying $\mathbb{E}^\top P^\top = P\mathbb{E} \geq 0$ and*

$$\begin{pmatrix} \mathbb{A}^\top P^\top + P\mathbb{A} + Q + R\mathbb{E} + (R\mathbb{E})^\top & P\mathbb{A}_d \\ * & -Q \end{pmatrix} < 0,$$

then system (1) is regular, impulse-free and $\|A_{22}^{-1}D_{22}\| < 1$.

The following result is an extension of [8] to the time-variable delay case.

Proposition 2 *Suppose that equation (2) is regular, impulse-free and $\|A_{22}^{-1}D_{22}\| < 1$. Equation (2) is asymptotically stable if there exist scalars $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$, a function $\mathscr{V}(.) : C([-s,0], R^n) \to R^+$ satisfying*

(i) $\alpha_1\|u_1(t)\|^2 \leq \mathscr{V}(x_t) \leq \alpha_2\|x_t\|^2$, $t \geq 0$,
(ii) $\dot{\mathscr{V}}(x_t) \leq -\alpha_3\|x(t)\|^2, \quad t \geq 0$.

Proof. Using (i)-(ii) we have

$$\alpha_1\|u_1(t)\|^2 \le \mathscr{V}(x_t) \le \mathscr{V}(x_0) \le \alpha_2\|x_0\|^2,$$

which gives

$$\exists\beta_1>0: \quad \|u_1(t)\| \le \beta_1\|x_0\|,\ \forall t\in[-s,\infty). \tag{4}$$

From (3) it follows that

$$u_2(t) = -A_{22}^{-1}[A_{21}u_1(t)+D_{21}u_1(t-\beta(t))] - A_{22}^{-1}D_{22}u_2(t-\beta(t))$$

and hence

$$\begin{aligned}\|u_2(t)\| \le \|A_{22}^{-1}\|\|[A_{21}u_1(t)+D_{21}u_1(t-\beta(t))]\|\\ +\|A_{22}^{-1}D_{22}\|\|u_2(t-\beta(t))\|.\end{aligned}$$

Using estimation (4) we can find a number $\beta_2>0$ satisfying

$$\|A_{22}^{-1}\|\|[A_{21}u_1(t)+D_{21}u_1(t-\beta(t))]\| \le \beta_2\|x_0\|,\, t\ge 0,$$

hence

$$\|u_2(t)\| \le \beta_2\|x_0\| + \eta\|u_2(t-\beta(t))\|,$$

where $\eta = \|A_{22}^{-1}D_{22}\| < 1$. Setting $f(t)=\|u_2(t)\|$, we have

$$f(t) \le \eta\|f_t(\cdot)\| + \beta_2\|x_0\|,\, t\ge 0,$$

Applying Lemma 1 gives

$$\begin{aligned}\|u_2(t)\| =& \|f(t)\| \le \eta\|f_0\| + \frac{\beta_2\|x_0\|}{1-\eta}\\ &\le \eta\|K\|\|x_0\| + \frac{\beta_2\|x_0\|}{1-\eta} \le \beta_3\|x_0\|,\end{aligned} \tag{5}$$

where $\beta_3 = \eta\|K\| + \dfrac{\beta_2}{1-\eta}$. From (4) and (5) it follows that

$$\|x(t)\| \le \|K\|\|u(t)\| \le \|K\|(\beta_1+\beta_3)\|x_0\|, \forall t\ge 0,$$

hence

$$\|x(t)\| \le \|K\|(\beta_1+\beta_3)\|x_0\|,\ \forall t\ge 0,$$

which shows that $x(t)$ is uniformly bounded. We now show that $\lim_{t\to\infty} x(t)=0$. Indeed, for any $\varepsilon>0$, we will show that there exists $c(\varepsilon)>0: \|x(t)\|\le\varepsilon, t\ge c(\varepsilon)$. Form the uniform stability of the solution, choosing $\delta=\delta(\varepsilon)>0$, such that $\|x_{t_0}\|\le\delta \Rightarrow \|x(t)\|\le\varepsilon,\ \forall t\ge t_0\ge 0$. If the number $c(\varepsilon)$ does not exist, we have $\|x_t\|>\delta=\delta(\varepsilon), \forall t\ge 0$. Hence, we will prove that $u_1(t)$ does not go to zero as $t\to\infty$ by contradiction. Assume that $\lim_{t\to\infty} u_1(t)=0$. From the second equation of (3), there is $\beta_0>0$ such that

$$\|u_{2,(n+1)s}\| \le \eta\|u_{2,ns}\| + \frac{\beta_0\max\{\|u_{1,ns}\|, \|u_{1,(n+1)s}\|\}}{1-\eta}.$$

For $M_1>0$, there is a $n_0>0$ such that $\|u_{1,ns}\|<M_1,\ \forall n\ge n_0$. Hence, for n large enough, we have

$$\|u_{2,(n+n_0)s}\| \le \eta^n\|u_{2,n_0\tau}\| + \frac{\beta_0}{1-\eta}M_1(1+\eta+\cdots+\eta^{n-1})$$

$$\leq M_1 + \frac{\beta_0}{1-\eta}\frac{M_1}{1-\eta},$$

which implies

$$\|x_{(n+n_0)s}\| \leq \|K\|\|u_{(n+n_0)ds}\| \leq \|K\|(2M_1 + \frac{\beta_0}{1-\eta}\frac{M_1}{1-\eta}) < \delta,$$

for M_1 is small enough. In the other words, there is $\delta_1 > 0$ and a sequence $\{t_k\}$, such that

$$\|u_1(t_k)\| > \delta_1 > 0,\ t_k > 0,\ t_{k+1} - t_k > 2d,\ \forall k = 1,2,...$$

From the first equation of (4), there is $L > 0$ such that $\|\dot{u}_1\| \leq L,\ \forall t \geq 0$. Hence,

$$\|u_1(t)\| \geq \delta_1/2 > 0,\ \forall t \in [t_k - \frac{\delta_1}{2L}, t_k + \frac{\delta_1}{2L}].$$

Choosing L large enough and for $t_k > \frac{\delta_1}{2L}$, the intervals $[t_k - \frac{\delta_1}{2L}, t_k + \frac{\delta_1}{2L}]$ are not overlapped, and then we have

$$\begin{aligned}\mathscr{V}(x_{\left(t_k+\frac{\delta_1}{2L}\right)}) - \mathscr{V}(x_{\left(t_k-\frac{\delta_1}{2L}\right)}) = & \dot{\mathscr{V}}(x_\xi)\frac{\delta_1}{L} \\ & \leq -\lambda_3\|x(\xi)\|^2\frac{\delta_1}{L} \leq -\lambda_3\|y(\xi)\|^2\frac{\delta_1}{L\|K^{-1}\|^2} \\ & \leq -\|y_1(\xi)\|^2\frac{\lambda_3\delta_1}{L\|K^{-1}\|^2} \leq -\frac{\lambda_3\delta_1^3}{4L\|K^{-1}\|^2},\end{aligned}$$

where $\xi \in (t_k - \frac{\delta_1}{2L}, t_k + \frac{\delta_1}{2L})$. Moreover,

$$\mathscr{V}(x_{\left(t_k+\frac{\delta_1}{2L}\right)}) - \mathscr{V}(x_0) \leq -(k-1)\frac{\lambda_3\delta_1^3}{4L\|K^{-1}\|^2}.$$

For k enough larger, we obtain that $\mathscr{V}(x_{\left(t_k+\frac{\delta_1}{2L}\right)}) < 0$, which leads to a contradiction such that the number $c(\varepsilon)$ exists. The proof is completed.

3 H_∞ filtering design

Consider the following SDEs with time-variable delay in the measurement and the observation

$$\begin{cases}\mathbb{E}\dot{x}(t) = \mathbb{A}x(t) + \mathbb{A}_d x(t-\beta(t)) + \mathbb{B}\omega(t), \quad t \geq 0, \\ y(t) = Cx(t) + C_d x(t-\beta(t)), \\ z(t) = Nx(t) + N_d x(t-\beta(t)), \\ x(t) = \psi(t), \quad t \in [-\beta, 0],\end{cases} \tag{6}$$

where $x(t)$ is the state, $y(t)$ is the observation, $z(t)$ is the measurement signal, $w(t)$ is the disturbance; $\mathbb{B}, C, C_d, N, N_d$ are constant matrices of appropriate dimensions.

Associated with equation (6), consider the following filtering equation

$$\begin{cases} \mathbb{E}_f\dot{\hat{x}}(t) = \mathbb{A}_f\hat{x}(t) + \mathbb{B}_f y(t), \\ \hat{z}(t) = C_f\hat{x}(t) + G_f y(t), \end{cases} \tag{7}$$

where $\mathbb{E}_f, \mathbb{A}_f, \mathbb{B}_f, C_f, G_f$ are the filters to be constructed. Defining $v(t) = (x(t), \hat{x}(t))$, $e(t) = z(t) - \hat{z}(t)$, the error equation for (7) is

$$\begin{cases} \bar{E}\dot{v}(t) = \bar{A}v(t) + \bar{A}_d v(t-\beta(t)) + \bar{B}\omega(t), \\ e(t) = \bar{C}v(t) + \bar{C}_d v(t-\beta(t)), \\ v(t) = [\psi(t), 0],\, t \in [-\beta, 0], \end{cases} \tag{8}$$

where $\bar{C} = [N - G_f C, -C_f]$, $\bar{C}_d = [N_d - G_f C_d, 0]$ and

$$\bar{E} = \begin{pmatrix} \mathbb{E} & 0 \\ 0 & \mathbb{E}_f \end{pmatrix}, \bar{A} = \begin{pmatrix} \mathbb{A} & 0 \\ \mathbb{B}_f C & \mathbb{A}_f \end{pmatrix}, \bar{A}_d = \begin{pmatrix} \mathbb{A}_d \\ \mathbb{B}_f C_d \end{pmatrix}[I, 0], \bar{B} = \begin{pmatrix} \mathbb{B} \\ 0 \end{pmatrix}.$$

Definition 2 The H_∞ filtering problem for equation (6) is solvable if for $\gamma > 0$ there exist the filters (7) such that equation (8) is admissible and the H_∞ performance

$$\int_0^\infty \|e(t)\|^2 dt \le \gamma \int_0^\infty \|\omega(t)\|^2 dt \tag{9}$$

holds for all zero initial conditions and no-zero $\omega \in L_2[0, +\infty)$.

The purpose is to find filters $\mathbb{E}_f, \mathbb{A}_f, \mathbb{B}_f, C_f, G_f$ for solving the H_∞ filtering problem of equation (6).

Let us first introduce the following matrix notations for brief presentation.

$$\begin{aligned}
&\bar{P} \quad = \mathrm{diag}\{P_1, P_2\},\, \bar{S} = [Z, 0],\\
&\bar{M}_{11} = \bar{P}\bar{A} + \bar{A}^\top\bar{P}^\top + \bar{Q}_3\bar{E} + \bar{E}^\top\bar{Q}_3^\top + \bar{Q}_5,\\
&\bar{R}_{11} = \beta\bar{Q}_1 - \bar{Q}_5 + \bar{P}\bar{A} + \bar{A}^\top\bar{P}^\top, \bar{R}_{13} = \bar{A}^\top\bar{P}^\top - \bar{P}\\
&\bar{R}_{12} = \beta\bar{Q}_2 - \bar{Q}_3\bar{E} + \bar{E}^\top\bar{Q}_5^\top + \bar{P}\bar{A}_d + \bar{A}^\top\bar{P}^\top,\\
&\bar{R}_{1j} = 0, j = 7, 8, 10, \bar{R}_{14} = A^\top\bar{S}^\top, \bar{R}_{15} = \bar{R}_{16} = \bar{P}\bar{B},\\
&\bar{R}_{19} = [N - V_2C, -V_1]^\top, \bar{R}_{2j} = 0, j = 5, 6, 8, 9,\\
&\bar{R}_{22} = \beta\bar{Q}_4 - \bar{Q}_5\bar{E} - \bar{E}^\top\bar{Q}_5^\top + \bar{P}\bar{A}_d + \bar{A}_d^\top\bar{P}^\top + \bar{Q}_5,\\
&\bar{R}_{23} = \bar{A}_d^\top\bar{P}^\top - \bar{P}, \bar{R}_{24} = \bar{A}_d^\top\bar{S}^\top, \bar{R}_{33} = \tau\bar{U}_5 - \bar{P} - \bar{P}^\top,\\
&\bar{R}_{27} = \bar{P}\bar{B}, \bar{R}_{2,10} = [N_d - V_2C_d, 0]^\top,\\
&\bar{R}_{34} = \bar{S}^\top, \bar{R}_{38} = \bar{P}\bar{B}, \bar{R}_{3j} = 0, j = 5, 6, 7, 9, 10\\
&\bar{R}_{44} = \bar{S}\bar{B} + \bar{B}^\top\bar{S}^\top, R_{4j} = 0, j = 5, 6, .., 10,\\
&R_{5j} = 0, j = 6, 7, .., 10, R_{6j} = 0, j = 7, 8, .., 10,\\
&R_{7j} = 0, j = 8, 9, 10, R_{9,10} = 0, \bar{R}_{ii} = -\frac{\gamma}{4}I, i = 5, ..., 10.
\end{aligned}$$

Theorem 1 *The H_∞ filtering problem for equation (6) is solvable if there exist an invertible matrix $\bar{P}$: $\bar{P}\bar{E} = \bar{E}^\top\bar{P}^\top \geq 0$, matrices $\bar{Q}_i, i = 1,2,...,5$: $[\bar{Q}_i]_{5\times5} > 0$, and free-weighting matrices X,Y,Z,V_1,V_2 such that the following LMIs hold:*

$$\begin{pmatrix} \bar{M}_{11} & \bar{P}\bar{A}_d \\ * & -\bar{Q}_5 \end{pmatrix} < 0, \tag{10}$$

$$[\bar{R}_{ij}]_{10\times10} < 0. \tag{11}$$

The filters are defined by

$$\mathbb{E}_f = \mathbb{E}, \mathbb{A}_f = P_2^{-1}X, \mathbb{B}_f = P_2^{-1}Y, C_f = V_1, G_f = V_2.$$

Proof. We first show that singular equation (8) is regular and impulse-free. Using Proposition 1, it is enough to show that there exist a symmetric matrix $\bar{Q} \in R^{2n} > 0$ and a matrix $\bar{R} \in R^{2n}$ such that $\bar{P}\bar{E} = \bar{E}^\top\bar{P}^\top \geq 0$ and

$$\begin{pmatrix} \bar{P}\bar{A} + \bar{A}^\top\bar{P}^\top + \bar{Q} + \bar{R}\bar{E} + \bar{E}^\top\bar{R}^\top & \bar{P}\bar{A}_d \\ * & -\bar{Q} \end{pmatrix} < 0. \tag{12}$$

We can see that the condition (12) is equivalent to LMI (10) by taking $\bar{R} = \bar{Q}_3, \bar{Q} = \bar{Q}_5$, hence the equation is regular and impulse-free. Moreover, the following condition can be obtained

$$\|\bar{A}_{22}^{-1}\bar{D}_{22}\| < 1, \tag{13}$$

where $\bar{A}_{22}^{-1}, \bar{D}_{22}$ are the block matrices in the decomposed differential-algebraic equations of the equation (6) defined by

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \bar{A}_d = \begin{pmatrix} \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_{22} \end{pmatrix}.$$

Consider the Lyapunov function $\mathscr{V}(v_t) = \sum_{i=1}^{3} \mathscr{V}_i(v_t)$: where

$$\begin{aligned}
\mathscr{V}_1(v_t) =& v^\top(t)\bar{P}\bar{E}v(t), \\
\mathscr{V}_2(v_t) =& \int_{-\beta}^{0}\int_{t+s}^{t} \dot{v}^\top(\theta)\bar{E}^\top\bar{Q}_5\bar{E}\dot{v}(\theta)d\theta ds, \\
\mathscr{V}_3(v_t) =& \int_{0}^{t}\int_{\theta-\beta(\theta)}^{\theta} s^\top(\tau,\theta)\bar{Q}s(\tau,\theta)d\tau d\theta,
\end{aligned}$$

where

$$s^\top(\tau,\theta) = [v(\theta)^\top, v(\theta - s(\theta))^\top, (\bar{E}\dot{v}(\tau))^\top], \bar{Q} = [Q_i]_{5\times5}.$$

Let $\hat{\bar{P}} = \bar{K}^\top\bar{P}\bar{F}^{-1} = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix}$, where matrices $\bar{F}, \bar{K}$ are invertible such that $\hat{\bar{E}} = \bar{F}\bar{E}\bar{K} = \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix}$. Using condition $\bar{P}\bar{E} = \bar{E}^\top\bar{P}^\top \geq 0$, we get $\hat{\bar{P}}\hat{\bar{E}} = \hat{\bar{E}}^\top\hat{\bar{P}}^\top$. Due to the regularity of $\hat{\bar{P}}$, we get $\bar{P}_{21} = 0, \bar{P}_{11} = \bar{P}_{11}^\top > 0$, and then $\hat{\bar{P}}\hat{\bar{E}} = \begin{pmatrix} \bar{P}_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, $\exists\alpha_1 > 0, \alpha_2 > 0$ such that

$$\alpha_1\|\bar{u}_1(t)\|^2 \leq \mathscr{V}(v_t) \leq \alpha_2\|v_t\|^2, \quad t \geq 0, \tag{14}$$

where $\bar{u}(t) = \bar{K}^{-1}v(t) = [\bar{u}_1(t), \bar{u}_2(t)], \bar{u}_1(t) \in R^{2r},\ \bar{u}_2(t) \in R^{2n-2r}$. We have

$$\begin{aligned}
\dot{\mathscr{V}}_1(v_t) =& 2v^\top(t)\bar{P}\bar{E}\dot{v}(t)\\
=&\eta(t)^\top \begin{pmatrix} \bar{P}\bar{A}+\bar{A}^\top\bar{P}^\top & \bar{P}\bar{A}_d\\ \bar{A}_d^\top\bar{P}^\top & 0\end{pmatrix}\eta(t) + 2v^\top(t)\bar{P}\bar{B}\omega(t),\\
\dot{\mathscr{V}}_2(v_t) =& \beta\dot{v}^\top(t)\bar{E}^\top\bar{Q}_5\bar{E}\dot{v}(t) - \int_{t-s}^{t}\dot{v}^\top(\tau)\bar{E}^\top\bar{Q}_5\bar{E}\dot{v}(s)d\tau,\\
\dot{\mathscr{V}}_3(v_t) \le& \int_{t-\beta(t)}^{t} s^\top(\tau,t)\bar{Q}s(\tau,t)d\tau\\
=&\beta(t)\eta^\top(t)\hat{X}\eta(t) + 2\eta^\top(t)\begin{pmatrix}\bar{Q}_3\\ \bar{U}_5\end{pmatrix}[\bar{E}v(t) - \bar{E}v(t-\beta(t))]\\
&+\int_{t-\beta(t)}^{t}\dot{v}^\top(s)\bar{E}^\top\bar{Q}_5\bar{E}\dot{v}(s)ds\\
\le&\beta\eta^\top(t)\hat{X}\eta(t) + 2[v(t)^\top\bar{Q}_3 + v(t-\beta(t))^\top\bar{Q}_5][\bar{E}v(t) - \bar{E}v(t-\beta(t))]\\
&+\int_{t-\beta}^{t}\dot{v}^\top(\tau)\bar{E}^\top\bar{Q}_5\bar{E}\dot{v}(\tau)d\tau,
\end{aligned}$$

where $\eta(t) = [v(t)\, v(t-\beta(t))]$ and $\hat{X} = \begin{pmatrix}\bar{Q}_1 & \bar{Q}_2\\ * & \bar{Q}_4\end{pmatrix}$. Therefore, we have

$$\begin{aligned}
\dot{\mathscr{V}}(v_t) \le& \eta(t)^\top \begin{pmatrix} \bar{P}\bar{A}+\bar{A}^\top\bar{P}^\top & \bar{P}\bar{A}_d\\ \bar{A}_d^\top\bar{P}^\top & 0\end{pmatrix}\eta(t)\\
&+\beta\dot{v}^\top(t)\bar{E}^\top\bar{U}_5\bar{E}\dot{v}(t) + 2v^\top(t)\bar{P}\bar{B}\omega(t) + \beta\eta^\top(t)\hat{X}\eta(t)\\
&+2\Big[v(t)^\top\bar{Q}_3 + v(t-\beta(t))^\top\bar{Q}_5\Big]\Big[\bar{E}v(t) - \bar{E}v(t-\beta(t))\Big].
\end{aligned}$$

Multiplying both sides of equation (3) by $-2\dot{v}^\top(t)\bar{E}^\top\bar{P}$, $-2v^\top(t)\bar{P}$, $-2v^\top(t-\beta(t))\bar{P}$, and $-2\omega^\top(t)\bar{S}$, respectively, the resulting zero-value terms are added in the derived above inequality, then applying the following inequality

$$0 \le -\|e(t)\|^2 + 2v(t)^\top\bar{C}^\top\bar{C}v(t) + 2v(t-\beta(t))^\top\bar{C}_d^\top\bar{C}_d v(t-\beta(t)),$$

where $\bar{C} = [N - V_2C, -V_1], \bar{C}_d = [N_d - V_2C_d, 0]$, we obtain that

$$\dot{\mathscr{V}}(v_t) \le \eta^\top\mathscr{W}_1\eta + \mu^\top\mathscr{W}_2\mu - \|e(t)\|^2 + \gamma\|\omega(t)\|^2, \tag{15}$$

where $\mu(t)^\top = [v(t)^\top, v(t-\beta(t))^\top, (\bar{E}\dot{v}(t))^\top,\ \omega(t)^\top]$ and

$$\begin{aligned}
&\mathscr{W}_1 = \begin{pmatrix}\bar{M}_{11} & \bar{P}\bar{A}_d\\ * & -U_5\end{pmatrix}, \mathscr{W}_2 = [N_{ij}]_{4\times4},\\
&\bar{M}_{11} = \bar{P}\bar{A} + \bar{A}^\top\bar{P}^\top + \bar{U}_3\bar{E} + \bar{E}^\top\bar{Q}_3^\top + \bar{Q}_5,\\
&N_{11} = \beta\bar{Q}_1 - \bar{Q}_5 + \bar{P}\bar{A} + \bar{A}^\top\bar{P}^\top + \frac{4}{\gamma}\bar{P}\bar{B}\bar{B}^\top\bar{P}^\top + \frac{4}{\gamma}\bar{P}\bar{B}\bar{B}^\top\bar{P}^\top + 2\bar{C}^\top\bar{C},\\
&N_{12} = \beta\bar{Q}_2 - \bar{U}_3\bar{E} + \bar{E}^\top\bar{U}_5^\top + \bar{P}\bar{A}_d + \bar{A}^\top\bar{P}^\top,\\
&N_{13} = \bar{A}^\top\bar{P}^\top - \bar{P}, N_{14} = \bar{A}^\top\bar{S}^\top, N_{23} = \bar{A}_d^\top\bar{P}^\top - \bar{P}, N_{24} = \bar{A}_d^\top\bar{S}^\top\\
&N_{22} = \beta\bar{Q}_4 - \bar{Q}_5\bar{E} - \bar{E}^\top\bar{U}_5^\top + \bar{P}\bar{A}_d + \bar{A}_d^\top\bar{P}^\top + \bar{Q}_5 + \frac{4}{\gamma}\bar{P}\bar{B}\bar{B}^\top\bar{P}^\top + 2\bar{C}_d^\top\bar{C}_d,\\
&N_{44} = \bar{S}\bar{B} + \bar{B}^\top\bar{S}^\top, N_{33} = \beta\bar{U}_5 - \bar{P} - \bar{P}^\top + \frac{4}{\gamma}\bar{P}\bar{B}\bar{B}^\top\bar{P}^\top, N_{34} = \bar{S}^\top.
\end{aligned}$$

Taking the inequalities (14), (15) into account and using the Schur complement lemma, we get $\mathscr{W}_i < 0, i = 1,2$, it follows that

$$\exists \lambda_3 > 0: \quad \dot{\mathscr{V}}(v_t) \le \eta^\top \mathscr{W}_1 \eta + \mu^\top \mathscr{W}_2 \mu < -\lambda_3 \|v(t)\|^2. \tag{16}$$

Applying Proposition 2 with the derived conditions (13), (14), (16), we obtain that the solution $v(t)$ is asymptotically stable. Finally, we show the H_∞ performance condition (9). For this, note that from the derived inequality (15) and $\mathscr{W}_i < 0, i = 1,2$, it follows that

$$\int_0^t \|e(s)\|^2 - \gamma \|\omega(s)\|^2 ds \le -\int_0^t \dot{\mathscr{V}}(v_s) ds$$
$$= \mathscr{V}(v_0) - \mathscr{V}(v_t) \le \mathscr{V}(x_0) = 0.$$

Letting $s \to \infty$, we have

$$\int_0^\infty \|e(s)\|^2 ds \le \gamma \int_0^\infty \|\omega(s)\|^2 ds,$$

which implies the condition (9).

Remark 1 Note that the conditions (10), (11) are LMIs, since for $\mathbb{A}_f = P_2^{-1} X, \mathbb{B}_f = P_2^{-1} Y$, we have

$$\bar{P}\bar{A} = \begin{bmatrix} P_1\mathbb{A} & 0 \\ YC & X \end{bmatrix}, \ \bar{P}\bar{A}_d = \begin{bmatrix} P_1\mathbb{A}_d \\ YC_d \end{bmatrix} H, \ \bar{P}\bar{B} = \begin{bmatrix} P_1\mathbb{B} \\ 0 \end{bmatrix},$$
$$\bar{S}\bar{A} = \begin{bmatrix} Z\mathbb{A} & 0 \end{bmatrix}, \ \bar{S}\bar{A}_d = \begin{bmatrix} Z\mathbb{A}_d & 0 \end{bmatrix}, \ \bar{S}\bar{B} = Z\mathbb{B}.$$

Remark 2 Theorem 1 provides new delay-dependent H_∞ filtering conditions for SDEs with non-differentiable delay. Different from the existing works, a set of improved Lyapunov-Krasovsii functions $\mathscr{V}_i(.), i = 1,2,3$ are constructed to avoid the smooth assumption on $s(t)$. The proposed method gives less conservative stability conditions as it does for singular equations with time-variable delay studied in [3, 12, 14, 29, 31], where the differentiability is needed.

Example 1 Consider SDEs (1), where $\beta(t) = 0.1 + 0.4|sin(t)|, \ \gamma = 0.01, \beta = 0.5$ and

$$\mathbb{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbb{A} = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix}, \ \mathbb{A}_d = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathbb{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}, C_d = \begin{bmatrix} -1 & 0.1 \\ 1 & -0.1 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.01 & 0.1 \\ 0.01 & 0.01 \end{bmatrix}, \ N_d = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

We see that the delay $\beta(t)$ is non-differentiable and the method used in papers [3, 12, 14, 29, 31] can not be applicable for this system. The solutions of LMIs (10), (11) are defined as

$$\bar{P} = \begin{bmatrix} 0.0031 & 0 & 0 & 0 \\ 0 & 0.0027 & 0 & 0 \\ 0 & 0 & 0.0755 & 0 \\ 0 & 0 & 0 & 0.0227 \end{bmatrix},$$

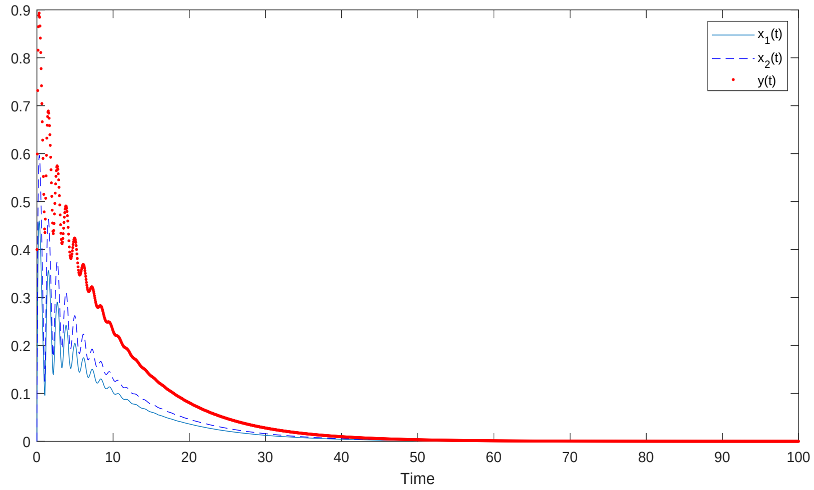

Fig. 1 The state x_1 and $\hat{x}_1$

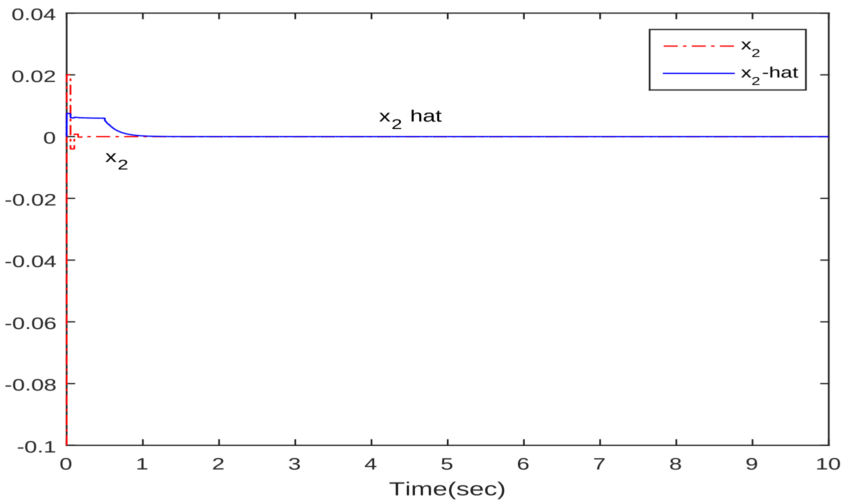

Fig. 2 The state x_2 and $\hat{x}_2$

$$X = \begin{bmatrix} -0.1165 & 0 \\ 0 & -0.0332 \end{bmatrix}, Y = \begin{bmatrix} -0.1165 & -0.0004 \\ -0.0004 & 0.0014 \end{bmatrix},$$

$$Z = 10^{-3} \begin{bmatrix} -0.2953 & -0.0750 \\ -0.0750 & -0.0321 \end{bmatrix}, V_1 = \begin{bmatrix} 0.0001 & 0 \\ -0.0012 & 0.0005 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.0023 & 0.0996 \\ -0.0357 & 0.0244 \end{bmatrix}, \bar{Q}_1 = \begin{bmatrix} 0.0501 & -0.0087 & 0.0036 & -0.0001 \\ -0.0087 & -0.0054 & -0.0001 & -0.0001 \\ 0.0036 & -0.0001 & 0.0671 & 0.0000 \\ -0.0001 & -0.0001 & 0.0000 & 0.0467 \end{bmatrix},$$

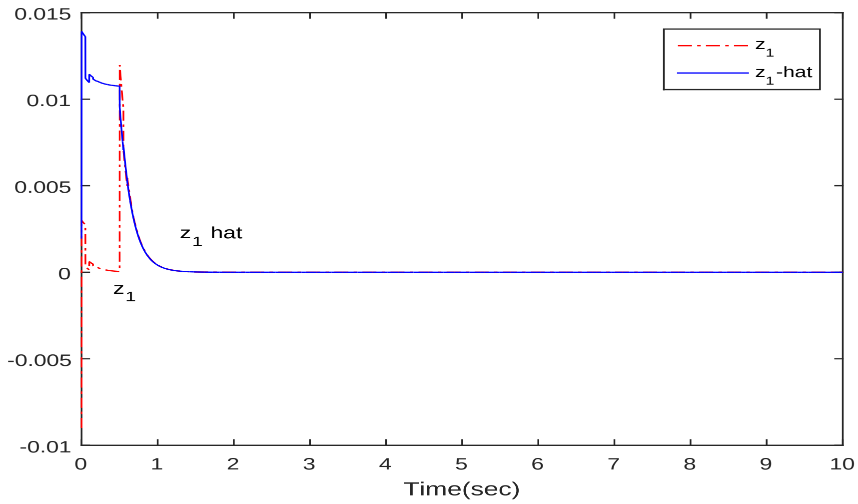

Fig. 3 The signal z_1 and the estimated signal $\hat{z}_1$

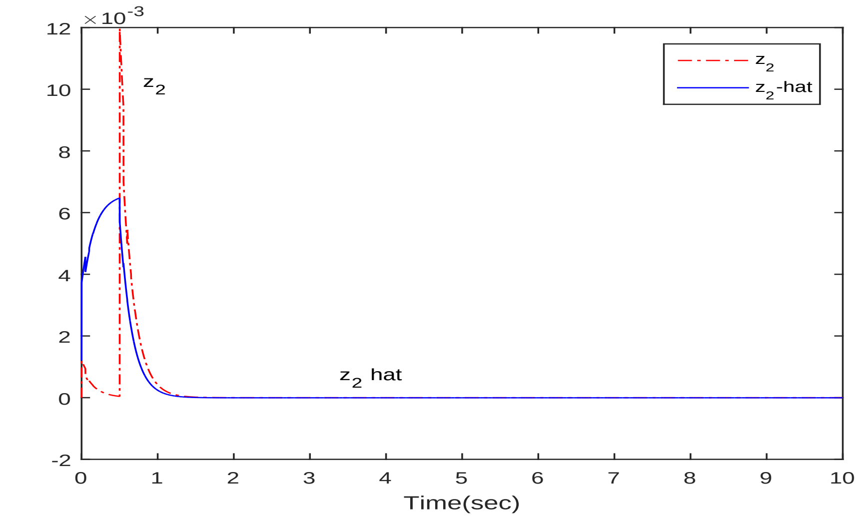

Fig. 4 The signal z_2 and the estimated signal $\hat{z}_2$

$$\bar{Q}_2 = \begin{bmatrix} -0.0392 & -0.0048 & -0.0039 & 0.0006 \\ 0.0014 & 0.0191 & -0.0004 & -0.0014 \\ -0.0060 & 0.0005 & 0.0047 & -0.0000 \\ -0.0008 & 0.0000 & 0.0001 & 0.0214 \end{bmatrix},$$

$$\bar{Q}_3 = \begin{bmatrix} -0.0112 & -0.0024 & -0.0011 & 0.0001 \\ 0.0032 & 0.0062 & -0.0004 & 0.0002 \\ -0.0018 & -0.0001 & 0.0410 & 0.0000 \\ -0.0003 & 0.0004 & 0.0001 & 0.0008 \end{bmatrix},$$

$$\bar{Q}_4 = \begin{bmatrix} 0.0579 & -0.0005 & 0.0038 & -0.0015 \\ -0.0005 & 0.0137 & -0.0005 & -0.0006 \\ 0.0038 & -0.0005 & 0.0586 & -0.0000 \\ -0.0015 & -0.0006 & -0.0000 & 0.0088 \end{bmatrix},$$

$$\bar{Q}_5 = \begin{bmatrix} 0.0240 & 0.0024 & -0.0010 & -0.0002 \\ 0.0024 & 0.0101 & 0.0001 & 0.0004 \\ -0.0010 & 0.0001 & 0.0582 & 0.0000 \\ -0.0002 & 0.0004 & 0.0000 & 0.0011 \end{bmatrix}.$$

The H_∞ filtering problem is solvable due to Theorem 1, and the filters are defined by

$$\mathbb{E}_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{A}_f = \begin{bmatrix} -0.01291 & 0 \\ 0 & -0.0037 \end{bmatrix}, \mathbb{B}_f = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0002 \end{bmatrix},$$

$$C_f = \begin{bmatrix} 0.0001 & 0 \\ -0.0012 & 0.0005 \end{bmatrix}, G_f = \begin{bmatrix} 0.0023 & 0.0996 \\ -0.0357 & 0.0244 \end{bmatrix}.$$

Figures 1-4 show the response states $x = [x_1, x_2], \hat{x} = [\hat{x}_1, \hat{x}_2], z = [z_1, z_2]$ and estimated signal $\hat{z} = [\hat{z}_1, z_2]$ of the system with the initial conditions $\psi(t) = [0.1, -0.1]$.

4 Conclusions

In this paper, the problem of the stability and H_∞ filtering for linear singular differential equations with time-variable delay has been investigated. Firstly, by introducing new augmented Lyapunov-Krasovskii functionals and using singular value theory, sufficient conditions for the admissibility of the filter error equations are firstly established. Then, based on the derived the stability conditions, the H_∞ filters are designed in terms of tractable LMIs. Finally, a numerical example with simulations is given to show the validity and effectiveness of the theoretical results.

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References

1. Bernstein D.S, Haddad W.M., Steady-state Kalman filtering with an H_∞ error bound, Syst. Control Lett., **12**, 9–16 (1989).
2. Chang X.H. , Robust Output Feedback H_∞ Control and Filtering for Uncertain Linear Systems, Springer, New York, 2014.
3. Chen W., Xu S. , Lu J., Li Y., Chu Y., Zhang Z., Further results on delay-dependent H_∞ filtering for singular systems with interval time-varying delays, Optim. Control Appl. Meth., 2021, https://doi.org/10.1002/oca.2711
4. Dai L., Singular Control Systems, Springer, Berlin, 1981.
5. Gahinet P., Nemirovskii A., Laub A.J., Chilali M., LMI Control Toolbox For use with MATLAB, Massachusetts, The MathWorks, Inc., 1995 bibitem 6 Gu K., Kharitonov V.L., Chen J., Stability of Time-Delay Systems. Birkhauser, Boston, 2003.

6. Eubank R.L., A Kalman Filter Primer, Chapman and Hall/CRC, London, 2005.
7. Fridman E., Stability of linear descriptor system with delay: a Lyapunov based approach, J. Math. Analysis Appl., **273**, 24–44 (2002).
8. Fridman E., Shaked U., An improved delay-dependent H_∞ filtering of linear neutral systems, IEEE Trans. Signal Proces., **52**, 668–673 (2004).
9. Fu X., Sheng Z., C. Lin, Chen B., New results on admissibility and dissipativity analysis of descriptor time-delay systems, Appl. Math. Comput., 419(2022), 126860.
10. Haidar A., Boukas E.K., Exponential stability of singular systems with multiple time-varying delays, Automatica, **45**, 539–545 (2009).
11. He Y., Wang Q.G., Lin C., An improved H_∞ filter design for systems with time-varying interval delay, IEEE Trans. CAS II: Express Briefs, **53**, 1235–1239 (2006).
12. Li F., Zhang X., A delay-dependent bounded real lemma for singular LPV systems with time-variant delay, Int. J. Robust. Nonl. Contr., **22**, 559–574 (2012).
13. Liu L., Zhai D., Lu A., Zhang Q., H_∞ fuzzy filtering for nonlinear singular systems with time-varying delay, Advances in Difference Equations, **92**, 1–19 (2015).
14. Lu R., Xu Y., Xue A., H_∞ filtering for singular systems with communication delays, Signal Processing, **90**, 240–1248 (2010).
15. Mahmoud M., Robust Control and Filtering for Time-Delay Systems. Marcel Dekker, New York, 2000.
16. Park J.H., Lee T.H., Liu Y., Chen J., Dynamic Systems with Time Delays: Stability and Control, Springer, Singapore, 2019.
17. Park J.H., Kwon O., Won S., LMI approach to robust H_∞ filtering for neutral delay differential systems, Appl. Math. Comput., **150**, 235–244 (2004).
18. Phat V.N., Thanh N.T., Trinh H., New results on H_∞ filtering for nonlinear large-scale systems with interconnected time-varying delays, Optim. Cont. Appl. Methods, **37**, 958–964 (2016).
19. Qi Y., Zhao X., Huang J., H_∞ filtering for switched systems subject to stochastic cyber attacks: A double adaptive storage event-triggering communication, Appl. Math. Comput., **394**, 125789 (2021).
20. Revathi V.M., Balasubramaniam P., Park J.H., Lee T.H., H_∞ filtering for sample data systems with stochastic sampling and Markovian jumping parameters, Nonlinear Dynam., **78**, 813–830 (2014).
21. Shao H., Delay-range-dependent robust H_∞ filtering for uncertain stochastic systems with mode-dependent time delays and Markovian jump parameters. Journal on Mathematical Analysis and Applications, **342**, 1084–1095 (2008).
22. Saberi A., Stoorvoge A.A., Ali K.S., Filtering Theory with Applications to Fault Detection, Isolation and Estimation, Birkhauser, Berlin, 2007.
23. Xu S., Lam J., Robust Control and Filtering of Singular Systems, Springer, Berlin, 2006.
24. S. Xu, P. Van Dooren, R. Stefan, J. Lam, Robust stability and stabilization for singular systems with state delay and parameter uncertainty. IEEE Trans. Autom. Contr., 47(2002), 1122–1128.
25. Tuan L. A., Phat V.N., Robust finite-time stability and H_∞ control of linear discrete-time delay systems with norm-bounded disturbances, *Acta Math Vietnamica,* **41**, 481-493 (2016).
26. Xua Q., Zhang Y., Qi W., Xiao S., Event-triggered mixed and passive H_∞ filtering for discrete-time networked singular Markovian jump systems, Appl. Math.Comput., **368**, 124803 (2020).
27. Wu Z., Su H., Analysis and Synthesis of Singular Systems with Time-Delays, Springer, Berlin, 2013.
28. Wu Z., Su H., Chu J., H_∞ filtering for singular systems with time-varying delay, Int. J. Robust Nonl. Contr., **20**, 1269–1284 (2010).
29. Zhang Z.M., Han Q.L., Robust H_∞ filtering for a class of uncertain linear systems with time-varying delay, Automatica, **44**, 157–166 (2008).
30. Zhu X., Wang Y., Gan Y., H_∞ filtering for continuous-time singular systems with time-varying delay, Int. J. Adapt. Contr. Sign. Proces., **25**, 137–154 (2011).