Constrained stabilization of positive differential-difference equations with unbounded delay via LP approach

N. H. Sau[a], P. Niamsup[b] and V.N. Phat[c]

[a] Faculty of Fundamental Science, University of Industry, Bac Tu Liem District, Hanoi, Vietnam; [b] Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; [c]ICRTM, Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi 10307, Vietnam

ARTICLE HISTORY
Compiled May 8, 2023

ABSTRACT
In this paper, we propose an efficient approach based on linear programming to study constrained stabilization problem of linear positive differential-difference equations with unbounded delay. We first propose new characterizations of positivity and comparison solution principle, then sufficient conditions for designing state feedback controllers of such equations are established. The conditions are presented via solving linear programming problem. Numerical examples are provided to illustrate the validity and effectiveness of the theoretical results.

KEYWORDS
Differential-difference equations, Stabilization, Positivity, Linear programming, Unbounded delay, Bounded control.

1. Introduction

Differential-difference equations (DDEs) are becoming increasingly important in various technical fields as control engineering, power systems, aircraft control modeling, and so on [1-3]. Since DDEs with delay are not explicitly solvable or have hardly manageable solutions, which even need not to be unique, qualitative study of such system becomes more difficult. Stability control analysis of DDEs have attracted a lot of attention from scientists over the past decades [4-6]. Summarizing these results, the crucial method employed in the existing works is the use of Lyapunov function and linear matrix inequalities (LMIs) approaches. In [7,8], stability analysis of linear DDEs with delay with constant delays is studied by using the comparison principle. On the other hand, positive systems, where the variables are non-negative, arise naturally in many applications of physical systems such as pollutant transport, ecology, epidemiology, systems biology, industrial engineering [9,10]. Recently, extensive research on qualitative theory of positive systems has been devoted to stability and stabilization and many impressive results have been obtained by using various mathematical methods [11-14]. The result of [14] on constrained stabilization of linear positive delay systems was proposed for designing state feedback controllers via solving linear pro-

CONTACT V.N. Phat. Email: vnphat@math.ac.vn

gramming (LP) problem [15, 16]. Using input-output approaches, the author of [17] proposed stability criteria for linear positive systems with constant delays in terms of scaled small-gain theorems involving linear or semidefinite programs. For linear singular positive systems (LSPS) with delays, the problem of stability and stabilization has also gained considerable interest with many significant results; see, e.g. [10, 18-20] and the references therein. It is worth noting that most of the mentioned results are concerned with stability and stabilization for LSPS with bounded delay. It should be mentioned that for DDEs with time-varying delay, the stability method for the system with bounded delay cannot be applied for the system with unbounded delay. There are important results have been proposed in [21-23] for LSPS with time-varying delay using Lyapunov direct method, however the authors considered a discrete-time system where the time delay is uniformly bounded and the approach there can not be applied to singular systems with an unbounded delay. In [24] the author proposed some conditions for $l_\infty/L_\infty-$gain of positive linear systems with unbounded time-varying delays, but the system is considered without singularity performance. Proposing an analytical approach based on the bounding estimation technique, the authors of [25] provided sufficient conditions for asymptotical stability of positive DDEs with unbounded delays. The stabilization of positive DDEs with unbounded delay is considered in [26], where the stability and stabilization conditions are proposed in terms of LP problem, however, the system is considered without control constraints.

To the best of our ability, problem of constrained stabilization for DDEs with unbounded time-varying delays has not been fully studied so far. Therefore, the aim of this paper is to solve problem of constrained stabilization for linear positive DDEs with unbounded delay. Comparing with the existing results, our paper has the following novel features. (i) The innovation of research approach. In this paper, we attempt to develop an analytical approach based on linear programming to study constrained stabilization problem, which can be solved efficiently by convex optimization algorithms [26]. The proposed approach is the first trial in studying the constrained stabilization of DDEs with unbounded time-varying delay. (ii) The difficulty and generalization of the research result. The main drawbacks in control analysis of DDEs are the boundedness of delays and the unconstrained controllers. Our system model describes a wider class of DDEs, which subject to bounded control and unbounded delay. The mixture of the constrained control and unbounded delay gives rise to the difficulty in the control design due to limited research techniques. The main contributions of our paper lie in the following.

(i) The positive DDEs under consideration deal with constrained controls and unbounded delay.

(ii) New characterizations of positivity and comparison solution principle are proposed.

(iii) Sufficient conditions for designing admissible controllers are presented via LP problem [27, 28].

This work is organized as follows. In Section 2, we present problem formulation, notations and some auxiliary results needed in next sections. Section 3 presents main result on the constrained stabilization with numerical examples and simulation.

Notations. $\mathbb{R}^n$ denotes the vector space of real $n-$vectors; $x \succeq 0 (\succ 0)$ means $x_i \geq 0\,(> 0)\ \forall i = 1, 2, \ldots, n$, where $x = (x_1, x_2, \ldots, x_n)$. $(B)_i^T$ represents the ith row vector of matrix B. $\mathbb{R}^{m\times n}$ stands for the set of $(m\times n)-$matrices. $B \succeq 0$ (or $B \succ 0$) implies all its entries are nonnegative (or positive). $A \succeq B (A \succ B)$ means $A-B \succeq 0 (A-B \succ 0)$. $\mathbb{R}^n_{0,+}(\mathbb{R}^n_+)$ stands for space of nonnegative (positive) vectors of $\mathbb{R}^n$. $(M)_{(i,j)}$ denotes the ij entry of M. $\overline{1,m} = \{1,\ 2,\ \ldots,\ m\}$, where m is a positive integer. $PC\left([a,b], \mathbb{R}^m\right)$

stands for the set of piecewise continuous functions on $[a, b]$. $AC\left([a, b], \mathbb{R}^n\right)$ stands for the set of absolutely continuous functions on $[a, b]$.

2. Preliminaries

Consider linear DDEs with time-varying delay

$$\begin{cases} \dot{v}(t) = & Av(t) + Bz(t) + Mv(t-d(t)) + Nz(t-d(t)) + Hu(t), \\ 0 = & Cv(t) + Dz(t) + Pv(t-d(t)) + Qz(t-d(t)) + Ru(t), \end{cases} \tag{1}$$

where $v(t) \in \mathbb{R}^{n_1}, z(t) \in \mathbb{R}^{n_2}$ are the state vectors; $u(t) \in \mathbb{R}^m$ is the control vector; $A, M \in \mathbb{R}^{n_1 \times n_1}, B, N \in \mathbb{R}^{n_1 \times n_2}, C, P \in \mathbb{R}^{n_2 \times n_1}, D, Q \in \mathbb{R}^{n_2 \times n_2}, H \in \mathbb{R}^{n_1 \times m}, R \in \mathbb{R}^{n_2 \times m}$ are constant matrices. The admissible control function $u(t)$ satisfies the following constraint

$$\exists \bar{u} \in \mathbb{R}^m : \quad 0 \preceq u(t) \preceq u^*, \quad t \geq 0.$$

The delay function $d(t)$ satisfies the following condition:

$$\exists T > 0,\ \theta \in (0,1): \quad \sup_{t \geq T} \frac{d(t)}{t} \leq \theta. \tag{2}$$

Fom condition (2) it follows that $0 < (1-\theta)t \leq t - d(t), t \geq T$, setting $\tau = \max_{t \in [0,T]} d(t)$, we consider the following initial conditions $\varphi_i(.) \in PC([-\tau, 0], \mathbb{R}^n))$ of system (1):

$$v(s) = \varphi_1(s), \quad z(s) = \varphi_2(s), \quad s \in [-\tau, 0]. \tag{3}$$

For $v(\cdot) : [0, \infty) \to \mathbb{R}^{n_1}, z(\cdot) : [0, \infty) \to \mathbb{R}^{n_2}$, a pair of functions $(v(\cdot), z(\cdot))$ is said to be a solution of (1) if $(v(\cdot), z(\cdot)) \in AC\left([0, +\infty), \mathbb{R}^{n_1}\right) \times PC\left([0, +\infty), \mathbb{R}^{n_2}\right)$ satisfying (1) and (3).

Definition 2.1. ([9]). System (1) is positive if with non-negative $\varphi_i(t) \succeq 0$, the solution $v(t, \varphi_1, \varphi_2) \succeq 0, z(t, \varphi_1, \varphi_2) \succeq 0, t \geq 0$.

Definition 2.2. Equation (1) is stabilizable if there is an admissible control $u(t) = Kv(t) + Fz(t),\ K \in \mathbb{R}^{m \times n_1},\ F \in \mathbb{R}^{m \times n_2}$ such that the closed-loop equation

$$\begin{cases} \dot{v}(t) = & (A + HK)v(t) + (B + HF)z(t)\ + Mv(t-d(t)) + Nz(t-d(t)), \\ 0 = & (C + RK)v(t) + (D + RF)z(t) + Pv(t-d(t)) + Qz(t-d(t)), \end{cases}$$

is asymptotically stable.

Using the control $u(t) = Kv(t) + Fz(t)$ and denoting $A_K = A + HK, B_F = B + HF$, $C_K = C + RK, D_F = D + RF$, equation (1) is reduced to the closed-loop equation

$$\begin{cases} \dot{v}(t) = & A_K v(t) + B_F z(t) + Mx(t-d(t)) + Nz(t-d(t)), \\ 0 = & C_K v(t) + D_F y(t) + Pv(t-d(t)) + Qz(t-d(t)). \end{cases} \tag{4}$$

Lemma 2.3. *([9]). Assume that $M \in \mathbb{R}^{n\times n}$ is Metzler. The following conditions are equivalent.*

1) $\exists\ \zeta \in \mathbb{R}^n_+ : M\zeta \prec 0.$
2) $det(M) \neq 0$ *and* $M^{-1} \preceq 0.$
3) M *is Hurwitz.*

Lemma 2.4. *Let matrices A_K, D_F be Metzler and B_F, C_K, M, N, P, Q be non-negative. If there exist $\kappa \in \mathbb{R}^{n_1}_+,\ \ \nu \in \mathbb{R}^{n_2}_+$ such that*

$$(A_K + M)\kappa + (B_F + N)\nu \prec 0, \tag{5}$$

$$(C_K + P)\kappa + (D_F + Q)\nu \prec 0, \tag{6}$$

then
(i) For $w_1(t) \succeq 0, w_2(t) \succeq 0$:

$$\begin{cases} \dot{v}(t) = A_K v(t) + B_F v(t) + Mv(t-d(t)) + Nz(t-d(t)) + w_1(t) \\ z(t) = -D_F^{-1}C_K v(t) - D_F^{-1}Pv(t-d(t)) - D_F^{-1}Qz(t-d(t)) + w_2(t). \end{cases} \tag{7}$$

is positive.
(ii) For $\varphi_i(s) \succeq \phi_i(s), i = 1, 2,\ s \in [-\tau, 0)$:

$$v(t, \varphi_1, \varphi_2) \preceq v(t, \phi_1, \phi_2), \forall t \geq 0, \tag{8}$$

$$z(t, \varphi_1, \varphi_2) \preceq z(t, \phi_1, \phi_2), \quad \forall t \geq 0. \tag{9}$$

(iii) $\exists \mu \in (0, 1)$:

$$-D_F^{-1}\left(C_K + P\right)\kappa - D_F^{-1}Q\nu \prec (1-\mu)\nu. \tag{10}$$

$$-\left(D_F + Q\right)^{-1}\left(C_K + P\right)\kappa \prec (1-\mu)\nu. \tag{11}$$

Proof. (i) We see that $(C_K + P)\kappa \succeq 0$, and from (6) we get

$$(D_F + Q)\nu \prec 0. \tag{12}$$

Using $Q \succeq 0$ and (12) we have $D_F\nu \prec 0$ and from D_F Metzler and Lemma 2.3 it claims that D_F is invertible and $-D_F^{-1} \succeq 0$. Since A_K is Metzler, we derive that the matrices $B_F, M, N, -D_F^{-1}C_K, -D_F^{-1}P, -D_F^{-1}Q$ are non-negative, then the proof of (i) is similar to the one of [22, Lemma 2].
(ii) Employing the positivity and linearity of system (7), we have $v(t, \phi_1, \phi_2) - v(t, \varphi_1, \varphi_2) = v(t, \phi_1 - \varphi_1, \phi_2 - \varphi_2) \succeq 0, t \geq 0$, which implies inequality (8). Similarly, we can also derive inequality (9).
(iii) Taking Lemma 2.3 into account, Metzler matrix $D_F + Q$ and condition (12) into account, we have $-(D_F+Q)^{-1} \succeq 0$. Pre-multiplying both sides of (6) with $(-D_F^{-1}) \succeq 0$

gives

$$-D_F^{-1}(C_K+P)\kappa+(-D_F^{-1})Q\nu \prec \nu. \tag{13}$$

Similarly, pre-multiplying both sides of (6) with $-(D_F+Q)^{-1} \succeq 0$ gives

$$-(D_F+Q)^{-1}(C_K+P)\kappa \prec \nu. \tag{14}$$

Because the inequalities (13), (14) are strict, there exists $\mu \in (0,1)$ such that the conditions (10), (11) hold.

Lemma 2.5. *Let $\bar{v} \in \mathbb{R}_+^{n_1}$, $\bar{z} \in \mathbb{R}_+^{n_2}$ be upper bounds of the initial conditions of system (3) such that $0 \preceq \varphi_1(s) \preceq \bar{v}$, $0 \preceq \varphi_2(s) \preceq \bar{z}, s \in [-\tau, 0]$. Moreover, assume that A_K, D_F are Metzler, B_F, C_K, M, N, P, Q are non-negative. If*

$$\begin{aligned} &(A_K+M)\bar{v}+(B_F+N)\bar{z} \prec 0,\\ &(C_K+P)\bar{v}+(D_F+Q)\bar{z} \prec 0, \end{aligned} \tag{15}$$

then $0 \preceq v(t) \preceq \bar{v}$, $0 \preceq z(t) \preceq \bar{z}$, $t \geq 0$.

Proof. By using the second inequality of (15), $(C_K+P)\bar{v} \succeq 0$ and $Q\bar{z} \succeq 0$ we have $D_F\bar{y} \prec 0$. Since D_F is Metzler, D_F is Hurwitz and $-D_F^{-1} \succeq 0$. Setting $v_1(t) := \bar{v}-v(t)$, $v_2(t)=\bar{z}-z(t)$ gives

$$\begin{aligned} \dot{v}_1(t) &= A_Kv_1(t)+B_Fv_2(t)+Mv_1(t-d(t))+Nv_2(t-d(t))\\ &\quad-\left((A_K+M)\bar{x}+(B_F+N)\bar{y}\right),\\ v_2(t) &= -D_F^{-1}C_Kv_1(t)-D_F^{-1}Pv_1(t-d(t))-D_F^{-1}Qv_2(t-d(t))\\ &\quad+D_F^{-1}\left((C_K+P)\bar{x}+Q\bar{y}\right)+\bar{y} \end{aligned}$$

On the other hand, taking (15) into account, we obtain that

$$\begin{aligned} &-\left((A_K+M)\bar{v}+(B_F+N)\bar{z}\right) \succ 0,\\ &D_F^{-1}\left((C_K+P)\bar{v}+Q\bar{z}\right)+\bar{z} \succeq 0. \end{aligned}$$

Therefore, using Lemma 2.4-(i), (ii) we get $v_1(t) \succeq 0$, $v_2(t) \succeq 0, t \geq 0$. Moreover, the matrices $-\left((A_K+M)\bar{v}+(B_F+N)\bar{z}\right)$, $D_F^{-1}\left((C_K+P)\bar{x}+Q\bar{z}\right)+\bar{z}$ are non-negative inputs, we have $0 \preceq v(t) \preceq \bar{v}$, and $0 \preceq z(t) \preceq \bar{z}$, $\forall t \geq 0$. □

3. CONSTRAINED STABILIZATION

Consider linear DDEs (1), where the time-varying delay satisfies the unbounded condition (2). Sufficient conditions for positivity and constrained stabilization are presented in the following theorem.

Theorem 3.1. *Assume that matrices M, N, P, Q are non-negative. The system (1) with initial condition $0 \preceq \varphi_1(t) \preceq \bar{v} = (\lambda_1, ..., \lambda_{n_1})^T, 0 \preceq \varphi_2(t) \preceq \bar{z} = (\lambda_{n_1+1}, ..., \lambda_{n_1+n_2})^T$, $t \in [-\tau, 0]$, is positive and stabilizable if the LP problem is feasible in the variables $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n_1+n_2})$, $\lambda_i \in \mathbb{R}_+$, $k_l \in \mathbb{R}_{0,+}^m$, $l \in \{1, 2, \ldots, n_1 +$*

$n_2\}$, $\sum_{j=1}^{n_1+n_2} k_i \preceq u^*$:

$$\begin{aligned} &\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{(i,j)} \lambda_j + \begin{pmatrix} H \\ R \end{pmatrix}_i^T k_j \geq 0,\, i,j \in \overline{1, n_1+n_2}, i \neq j, \\ &\begin{pmatrix} A+M & B+N \\ C+P & D+Q \end{pmatrix} \lambda + \begin{pmatrix} H \\ R \end{pmatrix} \sum_{i=1}^{n_1+n_2} k_i \prec 0. \end{aligned} \tag{16}$$

Moreover, the admissible controller is defined by

$$u(t) = \begin{bmatrix} \frac{k_1}{\lambda_1} & \frac{k_2}{\lambda_2} & \cdots & \frac{k_{n_1}}{\lambda_{n_1}} \end{bmatrix} v(t) + \begin{bmatrix} \frac{k_{n_1+1}}{\lambda_{n_1+1}} & \frac{k_{n_1+2}}{\lambda_{n_1+2}} & \cdots & \frac{k_{n_1+n_2}}{\lambda_{n_1+n_2}} \end{bmatrix} z(t), \quad t \geq 0.$$

Proof. *1. Positivity.* We have $K = \begin{bmatrix} \frac{k_1}{\lambda_1} & \cdots & \frac{k_{n_1}}{\lambda_{n_1}} \end{bmatrix}$, $F = \begin{bmatrix} \frac{k_{n_1+1}}{\lambda_{n_1+1}} & \cdots & \frac{k_{n_1+n_2}}{\lambda_{n_1+n_2}} \end{bmatrix}$. It is clear that $\sum_{j=1}^{n_1+n_2} k_j = K\bar{x} + F\bar{y} \preceq u^*$. Using $\lambda_j > 0, j = 1, 2, \ldots, n_1+n_2$, and (16) we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{(i,j)} + \begin{pmatrix} H \\ R \end{pmatrix}_i^T \frac{k_j}{\lambda_j} \geq 0,\ i,j \in \overline{1, n_1+n_2},\ j \neq i,$$

which gives $\begin{pmatrix} A+HK & B+HF \\ C+RK & D+RF \end{pmatrix}_{(i,j)} \geq 0, \quad j \neq i, \quad i \in \overline{1, n_1+n_2},\ j \in \overline{1, n_1+n_2}$. Hence, A_K, D_F are Metzler matrices and $B_F, C_K \succeq 0$. Besides, we see that

$$\begin{pmatrix} A+M & B+N \\ C+P & D+Q \end{pmatrix} \lambda + \begin{pmatrix} H \\ R \end{pmatrix} (K\bar{v} + F\bar{z}) \prec 0,$$

which implies

$$\begin{aligned} &(A_K+M)\bar{v} + (B_F+N)\bar{z} \prec 0, \\ &(C_K+P)\bar{v} + (D_F+Q)\bar{z} \prec 0. \end{aligned} \tag{17}$$

From $(C_K+P)\bar{v} \succeq 0$ and (17) it follows that $(D_F+Q)\bar{z} \prec 0$. Combining this with Lemma 2.3 and $(D_F+Q)-$Metzler implies that D_F+Q is a Hurwitz matrix. Since D_F+Q is a Hurwitz matrix and D_F is Metzler, we get D_F is Hurwitz and $-D_F^{-1} \succeq 0$. Note that system (4) is reduced to the system

$$\begin{cases} \dot{v}(t) = A_K v(t) + B_F z(t) + Mv(t-d(t)) + Nz(t-d(t)), \\ z(t) = -D_F^{-1} C_K v(t) - D_F^{-1} P v(t-d(t)) - D_F^{-1} Q z(t-d(t)), \end{cases} \tag{18}$$

which shows the positivity of systems (4) by using Lemma 2.3.
2. Asymptotic stability. Let $\bar{v} := \kappa; \quad \bar{z} := \nu$. We first show that there are numbers $t_1 > 0, \delta \in (0, \mu)$ and $t_1 > 0$ satisfying

$$v(t, \kappa, \nu) \preceq (1-\delta)\kappa.\ \forall t \geq t_1, \tag{19}$$

$$z(t,\kappa,\nu) \preceq (1-\delta)\nu. \ \forall t \geq t_1, \tag{20}$$

where μ is defined in (iii) of Lemma 2.4. Setting $\upsilon_1(t) := \kappa - v(t,\kappa,\nu)$, $\upsilon_2(t) = \nu - z(t,\kappa,\nu)$, we have

$$\begin{aligned}\dot{\upsilon}_1(t) =& A_K\upsilon_1(t) + B_F\upsilon_2(t) + M\upsilon_1(t-d(t)) \\ &+ N\upsilon_2(t-d(t)) - \left((A_K+M)\kappa + (B_F+N)\nu\right),\end{aligned} \tag{21}$$

$$\begin{aligned}\upsilon_2(t) =& - D_F^{-1}C_K\upsilon_1(t) - D_F^{-1}P\upsilon_1(t-d(t)) \\ &- D_F^{-1}Q\upsilon_2(t-d(t)) + D_F^{-1}\left((C_K+P)\kappa + Q\nu\right) + \nu.\end{aligned} \tag{22}$$

Taking conditions $(i),(ii)$ of Lemma 2.3 into account we get $\upsilon_1(t) \succeq 0$, $\upsilon_2(t) \succeq 0, t \geq 0$. By using $-\left((A_K+M)\kappa + (B_F+N)\nu\right)$ and $D_F^{-1}\left((C_K+P)\kappa + Q\nu\right) + \nu$ as a non-negative input, we have

$$v(t,\kappa,\nu) \preceq \kappa, \ \forall t \geq 0, \tag{23}$$

$$z(t,\kappa,\nu) \preceq \nu, \ \forall t \geq 0. \tag{24}$$

Considering the second equation of (18) gives

$$z(t) = -D_F^{-1}C_Kv(t) - D_F^{-1}Pv(t-d(t)) - D_F^{-1}Qz(t-d(t)). \tag{25}$$

Using (10), (24)-(26) and (25) we have

$$z(t,\kappa,\nu) \prec (1-\mu)\nu. \tag{26}$$

Condition (17) and $M\kappa + (B_F+N)\nu \succeq 0$ give $A_K\kappa \prec 0$, and hence according to Lemma 1, A_K is invertible and $-A_K^{-1} \succeq 0$. Moreover, using the first inequality of (17) gives

$$-A_K^{-1}M\kappa - A_K^{-1}\left(B_F+N\right)\nu \prec \kappa. \tag{27}$$

Further, we show that

$$\lim_{t\to\infty} v(t,\kappa,\nu) \preceq -A_K^{-1}M\kappa - A_K^{-1}\left(B_F+N\right)\nu. \tag{28}$$

For this, taking the conditions (17), (23), (24) combining with the condition (18) implies $\dot{v}(t,\kappa,\nu) \prec 0, t \geq 0$, which shows that $v(t,\kappa,\nu)$ is decreasing on $[0,+\infty)$. Considering system

$$\dot{s}(t) = A_Ks(t) + M\kappa + (B_F+N)\nu, \ t \geq 0, \tag{29}$$

and using $(23),(24)$ we obtain that

$$v(t,\kappa,\nu) \preceq s(t,\kappa), \ t \geq 0. \tag{30}$$

Moreover, note that $\dot{w}(t) = A_K w(t)$, where $w(t) := A_K^{-1} M\kappa + A_K^{-1}(B_F + N)\nu + s(t)$. In addition, since A_K is Metzler and Hurwitz, equation $\dot{w}(t) = A_K w(t)$ is positive and asymptotically stable. From (27) it follows that

$$\kappa + A_K^{-1} M\kappa + A_K^{-1}(B_F + N)\nu \succ 0,$$

and hence

$$w\left(t, \kappa + A_K^{-1} M\kappa + A_K^{-1}(B_F + N)\nu\right) \succeq 0, \quad t \geq 0,$$

and $\lim_{t\to\infty} w\left(t, p + A_K^{-1} M\kappa + A_K^{-1}(B_F + N)\nu\right) = 0$ which implies that

$$\lim_{t\to\infty} s(t, \kappa) = -A_K^{-1} M\kappa - A_K^{-1}(B_F + N)\nu. \tag{31}$$

Combining (30) and (31) gives (28). Moreover, note that the inequality (27) is strict, there are a number $\delta \in (0, \mu)$ satisfying

$$-A_K^{-1} M\kappa - A_K^{-1}(B_F + N)\nu \prec (1 - \delta)\kappa \prec \kappa. \tag{32}$$

From (28) and (32) it follows that the condition (19) holds for some $t_1 > 0$. Combining (26) and $\delta < \mu$ we obtain (20).

Further, we show that there exists an increasing sequence $\{T_i,\}i = 1, 2, ..., 0 = T_0, T_i < T_{i+1}$ such that

$$v(t, \kappa, \nu) \preceq (1 - \delta)^n \kappa, \ \forall t \in [T_n, T_{n+1}], \tag{33}$$

$$z(t, \kappa, \nu) \preceq (1 - \delta)^n \nu, \ \forall t \in [T_n, T_{n+1}]. \tag{34}$$

Indeed, by assumption (2), setting $h_0 = 0, h_1 = T, h_{m+1} = \left\lceil \frac{h_m}{1-\theta} \right\rceil$, $m = 1,\ 2,\ 3, \ldots,$ we have
(i) h_i is a strict increasing sequence, $h_i \xrightarrow{i\to+\infty} +\infty$,
(ii) $\forall\ k > 0$ we get $t - r(t) \geq h_k$, $\forall\ t \geq h_{k+1}$.

Setting $k_1 = \min\{k \in \mathbb{N} : t_1 \leq h_k\}$ for $n = 0$ and choosing $T_1 = h_{k_1}$, from (23) and (24) it follows that (33), (34) hold for $n = 0$. For the case $n = 1$, we consider the equation

$$\begin{cases} \dot{v}_1(t) = A_K v_1(t) + B_F z_1(t) + M v_1(t - d(t)) + N z_1(t - d(t)), \ t \geq h_{k_1+1} \\ z_1(t) = -D_F^{-1} C_K v_1(t) - D_F^{-1} P v_1(t - d(t)) - D_F^{-1} Q z_1(t - d(t)). \end{cases} \tag{35}$$

Similar part (ii) of Lemma 2.4, we can show the following inequalities

$$v_1(t, \phi_1, \phi_2) \succeq v_1(t, \varphi_1, \varphi_2), \forall t \geq h_{k_1+1}, \tag{36}$$

$$z_1(t, \phi_1, \phi_2) \succeq z_1(t, \varphi_1, \varphi_2), \forall t \geq h_{k_1+1}, \tag{37}$$

hold provided $\varphi_i(s) \preceq \phi_i(s)$, $i = 1, 2$, $s \in [h_{k_1}, h_{k_1+1})$. Setting

$$\kappa_1 = (1-\delta)\kappa,\ \nu_1 = (1-\delta)\nu, \tag{38}$$

and taking (19), (20), (36) and (37) into account, we get

$$v(t,\kappa,\nu) \preceq v_1(t,\kappa_1,\nu_1), \forall t \geq h_{k_1+1}, \tag{39}$$

$$z(t,\kappa,\nu) \preceq z_1(t,\kappa_1,\nu_1), \forall t \geq h_{k_1+1}. \tag{40}$$

Moreover, we see that conditions (10)-(11) are satisfied with $\kappa = \kappa_1$ and $\nu = \nu_1$. By using similar way as in the proof of *(a)* for (35), there is $t_2 > h_{k_1+1}$ satisfying

$$v_1(t,\kappa_1,\nu_1) \preceq (1-\delta)\kappa_1.\ \forall t \geq t_2, \tag{41}$$

$$z_1(t,\kappa_1,\nu_1) \preceq (1-\delta)\nu_1.\ \forall t \geq t_2. \tag{42}$$

Taking (38)-(42) gives

$$v(t,\kappa,\nu) \preceq (1-\delta)^2\kappa.\ \forall t \geq t_2, \tag{43}$$

$$z(t,\kappa,\nu) \preceq (1-\delta)^2\nu.\ \forall t \geq t_2. \tag{44}$$

Setting $k_2 := \min\{k \in \mathbb{N} : h_k \geq t_2\}$ and taking $h_{k_2} = T_2$, from (43) and (44) it follows that conditions (33), (34) are satisfied with $n = 1$. Further, we can find $T_2 < T_3 < \cdots$ such that (33) and (34) hold for $n = 2, 3, \ldots$.

Finally, we are now in position to show the asymptotic stability of the closed-loop system (4). For $\epsilon > 0$ we set $\chi = \max\{\|\kappa\|_\infty, \|\nu\|_\infty\}$ and find $\delta = \frac{\epsilon}{\chi}\min\{\kappa_{\min}, \nu_{\min}\}$, where $\kappa_{\min} = \min\limits_{1\leq i\leq r} \kappa_i, \nu_{\min} = \min\limits_{1\leq j\leq n-r} \nu_j$. Thus, for initial functions φ_i, $i = 1, 2$ satisfying $\|\varphi_i\|_\infty < \delta$, $i = 1, 2$ we have

$$\varphi_1(s) < \frac{\epsilon}{\chi}\kappa,\ \varphi_2(s) < \frac{\epsilon}{\chi}\nu, \quad s \in [-\tau, 0).$$

Using (23), (24) gives

$$v(t,\varphi_1,\varphi_2) \preceq \frac{\epsilon}{\chi}v(t,\kappa,\nu) \preceq \frac{\epsilon}{\chi}\kappa,\ \forall t \geq 0, \tag{45}$$

$$z(t,\varphi_1,\varphi_2) \preceq \frac{\epsilon}{\chi}z(t,\kappa,\nu) \preceq \frac{\epsilon}{\chi}\nu,\ \forall t \geq 0, \tag{46}$$

which implies that $\|v(t,\varphi_1,\varphi_2)\|_\infty \leq \epsilon$ and $\|z(t,\varphi_1,\varphi_2)\|_\infty \leq \epsilon$. Then, from (33), (34) it follows that $\lim\limits_{t\to\infty} x(t,\kappa,\nu) = 0$ and $\lim\limits_{t\to\infty} z(t,\kappa,\nu) = 0$. Using (45), (46) we have $\lim\limits_{t\to\infty} v(t,\varphi_1,\varphi_2) = 0$ and $\lim\limits_{t\to\infty} y(t,\varphi_1,\varphi_2) = 0$. Moreover, by using Lemma 2.5 and

(17), the solutions $v(t)$ and $z(t)$ of (4) satisfy $0 \preceq v(t) \preceq \bar{v}$ and $0 \preceq z(t) \preceq \bar{z}$ with the initial conditions $0 \preceq v(s) \preceq \bar{v},\ 0 \preceq z(s) \preceq \bar{z},\ s \in [-\tau, 0]$. Finally, since

$$\sum_{i=1}^{n_1+n_2} k_i = K\bar{v} + F\bar{z}, \bar{v} \in \mathbb{R}_+^{n_1},\ \bar{z} \in \mathbb{R}_+^{n_2},\ k_j \in \mathbb{R}_{0,+}^m,\ j \in \overline{1, n_1+n_2},$$

and $\sum_{j=1}^{n_1+n_2} k_j \preceq u^*$, we have

$$0 \preceq u(t) = Kv(t) + Fz(t) \preceq K\bar{v} + F\bar{z} = \sum_{j=1}^{n_1+n_2} k_j \preceq u^*,\ \forall t \geq 0,$$

which implies $0 \preceq u(t) \preceq u^*$ for all $t \geq 0$. □

In the following corollary, constrained stabilization conditions will be derived for the system with single-input control, i.e. $m = 1$. For this case, H and R are column vectors and the gain matrix $K,\ F$ are row vectors.

Corollary 3.2. *Assume that matrices M, N, P, Q are non-negative. The system* (1) *with initial condition* $0 \preceq \varphi_1(t) \preceq \bar{v} = (\lambda_1, ..., \lambda_{n_1})^T, 0 \preceq \varphi_2(t) \preceq \bar{z} = (\lambda_{n_1+1},\ ....,\ \lambda_{n_1+n_2})^T$, $t \in [-\tau, 0]$, *is positive, stabilizable if the LP problem is feasible in the variables* $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n_1+n_2}\}$, $\lambda_i \in \mathbb{R}_+$, $k_l \in \mathbb{R}_{0,+}$, $l \in \{1, 2, \ldots, n_1+n_2\}$, $\sum_{j=1}^{n_1+n_2} k_i \preceq u^*$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{(i,j)} \lambda_j + \begin{pmatrix} H \\ R \end{pmatrix}_i^T k_j \geq 0,\ i, j \in \overline{1, n_1+n_2}, i \neq j,$$

$$\begin{pmatrix} A+M & B+N \\ C+P & D+Q \end{pmatrix} \lambda + \begin{pmatrix} H \\ R \end{pmatrix} \sum_{i=1}^{n_1+n_2} k_i \prec 0.$$

Moreover, the admissible controller is defined by

$$u(t) = \begin{bmatrix} \frac{k_1}{\lambda_1} & \frac{k_2}{\lambda_2} & \cdots & \frac{k_{n_1}}{\lambda_{n_1}} \end{bmatrix} v(t) + \begin{bmatrix} \frac{k_{n_1+1}}{\lambda_{n_1+1}} & \frac{k_{n_1+2}}{\lambda_{n_1+2}} & \cdots & \frac{k_{n_1+n_2}}{\lambda_{n_1+n_2}} \end{bmatrix} z(t), \quad t \geq 0.$$

Remark 1. The authors of paper [17] studied asymptotic stability of coupled differential-difference equations, the problem of constrained stabilization is not considered there. Moreover, the DDE considered in [17] is a special case of our DDEs (1) (when $B = 0, M = 0,\ D = -I,\ P = 0$,) and the derived stability conditions in [17] are less effective than ours. For example, Theorem 4 in [17] show that if A is a Metzler matrix, N, C, Q are non-negative, Q is a Schur matrix and $s(A+N(I-Q)^{-1}C) < 0$, then system is asymptotically stable. However, in the Example 1 of our paper we consider DDEs, where B, M, P are both non-zero matrices, which is asymptotically stable by Theorem 1, while Theorem 4 in [17] can not be applied to get asymptotical stability. In paper [26], the author studied unconstrained stabilization problem and the DDEs is also a special case of our DDEs (1) (when $B = 0, M = 0,\ D = -I,\ P = 0$).

Remark 2. Note that the constrained stabilization of linear positive DDEs was studied in [6, 11, 15, 22, 24], however the obtained results are limited to the system with bounded delay such that the approach proposed there can not be applied to our system with unbounded delay. Moreover, the stability and positivity of linear DDEs studied in [25, 26] are based on the singular value decomposition approach and the regularity and impulse-free assumptions without designing admissible controllers. Moreover, Theorem 3.1 provides sufficient conditions for constrained stabilization of positive DDEs with unbounded delays via LP problem.

Remark 3. It is worth noting that Theorem 3.1 provides sufficient conditions for the constrained stabilization problem, which are not necessary. It is possible to make some of the stability conditions necessary and sufficient, for example the authors of [15] proposed some necessary and sufficient conditions for stability and performance of DDEs with bounded time-varying delay. For the stabilization problem, it is usually difficult to get necessary and sufficient conditions due to the dependently designed feedback controllers. However, we can establish an independent necessary condition for the constrained stabilization in the following theorem.

Theorem 3.3. *Consider system (1). Assume that M, N, P, Q are nonnegative matrices. If there exists a controller $u(t) = Kx(t) + Fy(t)$ such that the system (4) is positive and asymptotically stable then the LP problem is feasible in the variables $\Lambda_1 \in \mathbb{R}^{n_1}_{0,+}$, $\Lambda_2 \in \mathbb{R}^{n_2}_{0,+}$:*

$$\begin{aligned} &(A_K + M)\Lambda_1 + (B_F + N)\Lambda_2 \preceq 0, \\ &(C_K + P)\Lambda_1 + (D_F + Q)\Lambda_2 \preceq 0. \end{aligned} \tag{47}$$

Proof. Consider the system (4) with constant delay

$$\begin{cases} \dot{v}(t) = A_K v(t) + B_F y(t) + Mv(t-\tau) + Nz(t-\tau), \\ 0 = C_K v(t) + D_F z(t) + Pv(t-\tau) + Qz(t-\tau), \\ v(s) = \varphi_1(s), \ s \in [-\tau, \ 0], \\ z(s) = \varphi_2(s), \ s \in [-\tau, \ 0), \end{cases} \tag{48}$$

where $\tau = \max_{t\in[0,T]} d(t)$. By the assumption, the system (4) is positive and asymptotically stable for the unbounded delay $d(t)$, then the system (48) with constant delay τ is also positive and asymptotically stable. Integrating the equation (48) in the interval $[0, T]$ we obtain

$$\begin{aligned} v(T) - v(0) &= \int_0^T A_K v(t) + B_F z(t)dt + \int_0^T Mv(t-\tau) + Nz(t-\tau)dt \\ &= \int_0^T A_K v(t) + B_F z(t)dt + \int_{-\tau}^{T-\tau} Mv(t) + Nz(t)dt \\ 0 &= \int_0^T C_K v(t) + D_F z(t)dt + \int_0^T Pv(t-\tau) + Qz(t-\tau)dt \\ &= \int_0^T C_K v(t) + D_F z(t)dt + \int_{-\tau}^{T-\tau} Pv(t) + Qz(t)dt. \end{aligned} \tag{49}$$

On the other hand, we have

$$\begin{aligned}\int_{-\tau}^{T-\tau} Mv(t)+Nz(t)dt &= \left(\int_{-\tau}^{0} Mv(t)+Nz(t)dt + \int_{0}^{T} Mv(t)+Nz(t)dt\right)\\ &\quad - \int_{T-\tau}^{T} Mx(t)+Ny(t)dt,\\ \int_{-\tau}^{T-\tau} Pv(t)+Qz(t)dt &= \left(\int_{-\tau}^{0} Pv(t)+Qz(t)dt + \int_{0}^{T} Pv(t)+Qz(t)dt\right)\\ &\quad - \int_{T-\tau}^{T} Pv(t)+Qz(t)dt.\end{aligned} \tag{50}$$

With initial condition $v(s)=\varphi_1(s)$, $s\in[-\tau,0]$, $z(s)=\varphi_2(s)$, $s\in[-\tau,0)$, since $v(t)\to 0$, $z(t)\to 0$ as $t\to\infty$, then there exists a large enough σ such that

$$\begin{aligned} &x(\sigma)-v(0)-\int_{-\tau}^{0} Mv(t)+Nz(t)dt+\int_{\sigma-\tau}^{\sigma} Mv(t)+Nz(t)dt \preceq 0,\\ &-\int_{-\tau}^{0} Pv(t)+Qz(t)dt+\int_{\sigma-\tau}^{\sigma} Pv(t)+Qz(t)dt \preceq 0.\end{aligned} \tag{51}$$

Fom (49), (50) and (51) we get

$$\begin{aligned}(A_K+M)\int_0^{\sigma} v(t)dt+(B_F+N)\int_0^{\sigma} z(t)dt &= x(\sigma)-v(0)-\int_{-\tau}^{0} Mv(t)+Nz(t)dt\\ &\quad+\int_{\sigma-\tau}^{\sigma} Mx(t)+Ny(t)dt \preceq 0,\\ (C_K+P)\int_0^{\sigma} v(t)dt+(D_F+Q)\int_0^{\sigma} z(t)dt &= -\int_{-\tau}^{0} Pv(t)+Qz(t)dt\\ &\quad+\int_{\sigma-\tau}^{\sigma} Pv(t)+Qz(t)dt\\ &\preceq 0.\end{aligned}$$

This implies that (47) holds for $\Lambda_1=\int_0^{\sigma} v(t)dt$, $\Lambda_2=\int_0^{\sigma} z(t)dt$, which completes the proof. □

Remark 4. The following procedure for designing the admissible controllers can be applied.

- Step 1: Input the system matrices $A,\ B,\ C,\ D,\ M,\ N,\ P,\ Q,\ H, R$, where $M,\ N,\ P,\ Q$ are nonnegative matrices.
- Step 2: Find feasible solutions $\lambda=\{\lambda_1,\lambda_2,...,\lambda_{n_1+n_2}\}$, $\lambda_i\in\mathbb{R}_+$, $k_j, j\in\{1,2,\dots,n_1+n_2\}$ satisfying the condition (16) by solving LP toolbox [27, 28].
- Step 3: Compute the controller gain matrices K, F by

$$K=\begin{bmatrix}\frac{k_1}{\lambda_1} & \cdots & \frac{k_{n_1}}{\lambda_{n_1}}\end{bmatrix},\ F=\begin{bmatrix}\frac{k_{n_1+1}}{\lambda_{n_1+1}} & \cdots & \frac{k_{n_1+n_2}}{\lambda_{n_1+n_2}}\end{bmatrix}.$$

- Step 4: Define admissible controllers $u(t)=Kv(t)+Fz(t), t\geq 0$.

Example 3.4. Consider system (1), where

$$d(t) = \begin{cases} 0.9 & t \in [0,\ 0.9] \\ \frac{8}{9}t + 0.1 & t \geq 0.9 \end{cases}, \quad 0 \preceq u(t) \preceq \begin{pmatrix} 4 \\ 6 \end{pmatrix},\ t \geq 0,$$

$$A = \begin{bmatrix} -10 & 4 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} -0.9 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 \end{bmatrix},\ D = \begin{bmatrix} -6 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.1 & 0.2 \\ -4 & 1 \end{bmatrix}, M = \begin{bmatrix} 3 & 1.2 \\ 1 & 4.5 \end{bmatrix}, N = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ P = \begin{bmatrix} 3 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 \end{bmatrix},\ R = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

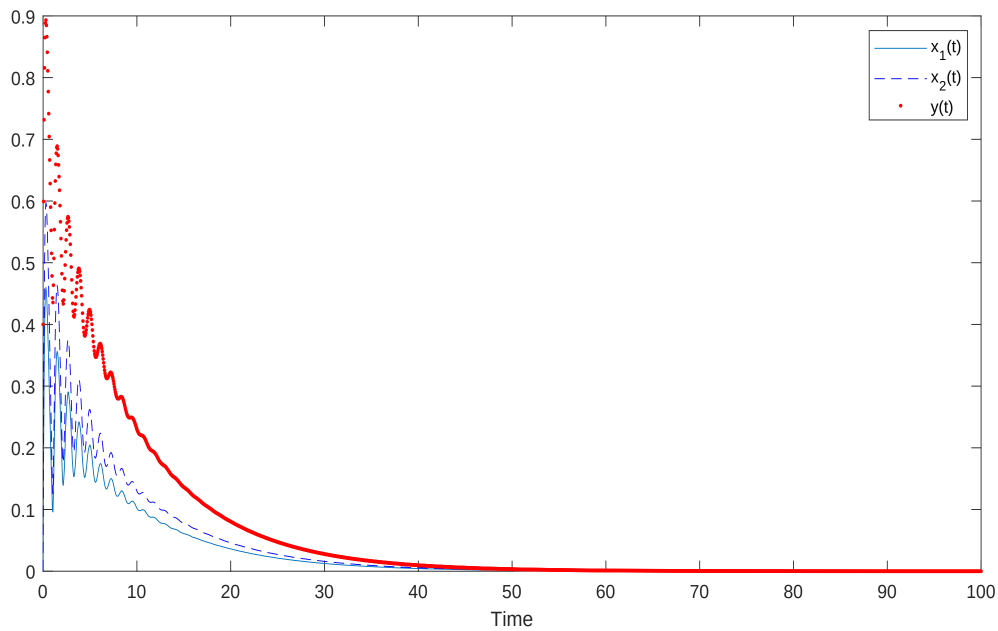

Figure 1. Solution response of the closed-loop system

Note that, the delay function $d(t)$ satisfies condition (2) with $T = \frac{8.1}{8}$ and $\theta = \frac{8}{8.1}$, we get $\tau = \max_{t \in [0, \frac{8.1}{8}]} d(t) = 1$. Solving the LP problem for (16) we have

$$\lambda = (1.1,\ 1,\ 1),\ k_1 = (1, 2),\ k_2 = (2, 0),\ k_3 = (1, 4),$$

$$K = \begin{bmatrix} \frac{10}{11} & 2 \\ \frac{20}{11} & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Therefore, using Theorem 3.1 the closed-loop system is positive and asymptotically stable with the admissible controllers:

$$u(t) = \begin{bmatrix} \frac{10}{11} & 2 \\ \frac{20}{11} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 4 \end{bmatrix} y(t), \quad t \geq 0.$$

Figure 1 shows the state response of the closed-loop system with initial values

$$\varphi_1(s) = (s^2,\ -s), s \in [-1, 0] \quad \varphi_2(s) = \frac{1}{3}(s + 1.2), \quad s \in [-1, 0).$$

Example 3.5. Consider the following single-input DDEs with delays:

$$\begin{cases} \dot{v}(t) = & Av(t) + By(t) + Mv(t - d(t)) + Nz(t - d(t)) + Hu(t), \\ 0 = & Cv(t) + Dz(t) + Pv(t - d(t)) + Qz(t - d(t)) + Ru(t), \end{cases} \tag{52}$$

where

$$A = \begin{pmatrix} -5/2 & 1 & 2 \\ 1/2 & -3 & 3/10 \\ 11/20 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1/5 & 1/10 \\ 1/10 & 1/5 \\ 1/10 & 1/10 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 5 \\ 3 & 3/2 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} -3 & 1/10 \\ 3/10 & -4 \end{pmatrix}, \quad M = \begin{pmatrix} 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 \\ 1/10 & 1/100 & 1/10 \end{pmatrix}, \quad N = \begin{pmatrix} 1/20 & 3/100 \\ 3/50 & 1/50 \\ 1/100 & 1/10 \end{pmatrix},$$

$$P = \begin{pmatrix} 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/5 \\ 1/10 & 1/20 \end{pmatrix} \quad ,$$

$$H = \begin{pmatrix} 1/2 & 1 & -9/2 \end{pmatrix}^T, \quad R = \begin{pmatrix} -1/10 & -1/10 \end{pmatrix}^T, \quad 0 \preceq u(t) \preceq 2,\ t \geq 0.$$

Solving the LP problem we obtain the following solutions:

$$\lambda = (6.5, \quad 4, \quad 2, \quad 15, \quad 10), k_1 = 0.5,$$

$$k_2 = 0.3, k_3 = 1, k_4 = 0.1, k_5 = 0.1.$$

Therefore, the control gain matrices are given by

$$K = \begin{bmatrix} 1/13 & 3/40 & 1/2 \end{bmatrix}, \quad F = \begin{bmatrix} 1/150 & 1/100 \end{bmatrix}.$$

By using Corollary 3.2, the closed-loop systems of system (52) is positive and asymptotic stable.

4. Conclusions

We have investigated constrained stabilization for positive DDEs with unbounded delay via LP approach. From the approach, sufficient conditions for designing admissible controllers have been proposed for positive DDEs with unbounded delay. A necessary condition for the problem has been proposed for DDEs with bounded delay. A desired feedback controllers can be determined by solving LP problem. Finally, numerical examples with simulation are given to demonstrate the proposed results.

Acknowledgments

The authors sincerely thank Vietnam Institute for Advance Study in Mathematics (VIASM) for supporting and providing a fruitful research environment and hospitality for them during the research visit. The research of the first author is supported by the Faculty of Science, Chiang Mai University, Thailand. The research of the second and third author is supported by the National Foundation of Science and Technology Development, Vietnam (NAFOSTED 101.01-2021.01).

Disclosure statement

No potential conflict of interest was reported by the author(s)

References

[1] Lam J., Xu S., Robust Control and Filtering of Singular Systems. Springer, Berlin, 2006.

[2] Dai L., Singular Control Systems, Springer, Berlin, 1989.

[3] Kunkel P., Mehrmann V., Differential-Algebraic Equations. Analysis and Numerical Solution, EMS Publishing House, Zurich, Switzerland, 2006.

[4] Feng Q., Nguang S.K., Dissipative delay range analysis of coupled differential-difference delay systems with distributed delays. Systems & Control Letters, 2018, 116:56–65.

[5] Wang F., Liang J., Wang F., Optimal control for discrete-time singular stochastic systems with input delay, Optim. Contr. Appl. Methods, 2016, 37(6): 1282–1313.

[6] Gu K., Zhang Y., Xu Z., Small gain problem in coupled differential-difference equations, time-varying delays, and direct Lyapunov method. Int. J. of Robust Nonl. Control, 2011, 21:429–451.

[7] Ngoc P.H.A. , Stability of coupled functional differential-difference equations. Int. J. Control, 93(2020), 1920–1930.

[8] Sau N.H. , Niamsup P., Phat V.N., Thuan M.V., State bounding for linear positive singular discrete systems with unbounded time-varying delay, Journal of the Franklin Institute, 2022, 359: 4587–4604.

[9] Farina L., Rinaldi S., Positive Linear Systems, Theory and Applications. Wiley, New York, 2000.

[10] Virnik E., Analysis of Positive Descriptor Systems: Topics in Systems and Control Theory. Springer, Berlin, 2008.

[11] Liu X., Yu W., Wang L¿, Stability analysis for continuous-time positive systems with time-varying delays. IEEE Trans. Autom. Contr., 2010, 55: 1024–1028.

[12] Liu G., Zhao P., Ruonan L., Stabilization of positive coupled differential-difference equations with unbounded time-varying delays. Optim. Control Appl. Meth., 42(2021), 42: 481–95.

[13] Domoshnitsky A., Shklyar R., Positivity for non-Metzler systems and its applications to stability of time-varying delay systems. Systems & Control letters, 2018, 18: 44–51.

[14] Liu X., Constrained control of positive systems with delays. IEEE Trans. Autom. Contr., 2009, 54: 1596–1600.

[15] Briat C., Stability and performance analysis of linear positive systems with delays using input-output methods. Int. J. Control, 2018, 91: 1669–1692.

[16] Tardella F., The fundamental theorem of linear programming: extensions and applications, Optimization, 2011, 60: 283–301.

[17] Rami A.M., Tadeo F., Controller synthesis for positive linear systems with controls. IEEE Trans. TCAS II, Express Briefs, 2007, 54: 5151–155.

[18] Phat V.N. , Sau N.H. , Exponential stabilization of positive singular linear discrete-time delay systems with bounded control. IET Control Theory & Applications, 2018, 13: 905–911.

[19] Cui Y., Shen J., Chen Y., Stability analysis for positive singular systems with distributed delays. Automatica, 2018, 94: 170–177.

[20] Elloumi W., Makni S., Mehdi D., Chrifi-Alaoui L., Chaabane M., Mohamed Bahloul, Exponential stability and stabilization of positive T-S fuzzy delayed systems, Optimal Control Applications and Methods, 2022, Doi.org/10.1002/oca.2942.

[21] Liu G., Ma Q., Lu J., Li Y., Xu S., Mixed H_∞ and passive filtering for a class of singular systems with interval time-varying delays, *Optim. Contr. Appl. Methods,* 2017, 39(1): 377-392.

[22] De Iuliis V., Germani A., Manes C., Internally positive representations and stability analysis of coupled differential-difference systems with time-varying delays. IEEE Trans. Auto. Control, 2019, 64: 2514–2521.

[23] Shen J., Zheng M.W., Positivity and stability of coupled differential-difference equations with time-varying delays. Automatica, 2015, 57: 123–127.

[24] Shen J., Lam J., $l_\infty/L_\infty-$gain analysis for positive linear systems with unbounded time-varying delays. IEEE Trans. Autom. Contr., 2015, 60: 857–862.

[25] Pathirana N.P., Nam P.T., Trinh H.T., Stability of positive coupled differential-difference equations with unbounded time-varying delays. Automatica, 2018, 92: 259–263.

[26] Liu G., Zhao P., Li R., Stabilization of positive coupled differential-difference equations with unbounded time-varying delays, Optim. Contr. Appl. Meth., 2021, 42: 81–95.

[27] Ploskas N., Samaras N., Linear Programming Using MATLAB. Springer, New York, 2017.

[28] Shao L., Ehrgott M., Primal and dual multi-objective linear programming algorithms for linear multiplicative programmes, Optimization, 2016, 65: 415–431.