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Practical Exponential Stability of Nonlinear Nonautonomous Differential Equations Under Perturbations

Cao Thanh Tinh
Email : tinhct@uit.edu.vn
Affiliationids : Aff1 Aff2

Do Duc Thuan
Email : thuan.doduc@hust.edu.vn
Affiliationids : Aff3, Correspondingaffiliationid : Aff3

Nguyen Khoa Son
Email : nkson@vast.vn
Affiliationids : Aff4

Le Trung Hieu
Email : lthieu@dthu.edu.vn
Affiliationids : Aff5

Aff1 Vietnam National University, Ho Chi Minh City, Vietnam

Aff2 Department of Mathematics and Physics, University of Information Technology, Ho Chi Minh City, Vietnam

Aff3 School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam

Aff4 Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam

Aff5 Department of Mathematics - Information Technology Teacher Education, Dong Thap University, Dong Thap, Vietnam

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Abstract

In this paper, we study the practical exponential stability of nonlinear nonautonomous differential equations under nonlinear perturbations. **AQ1** By introducing a new method, we obtain some explicit criteria for the practical exponential stability of these equations. Furthermore, several characterizations for the exponential stability of a class of nonlinear differential equations are also presented. **AQ2** The obtained results generalize some existing results in the literature. Applications to neutral networks are investigated. Some examples are given to illustrate the obtained results.

Keywords

Nonlinear nonautonomous differential equations
practical exponential stability
exponential stability
neutral networks

Mathematics Subject Classification

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1. Introduction and Preliminaries

Nonlinear differential equations are models for a variety of phenomena in the life sciences, physics and technology, chemistry and economics (see, e.g., [10, 15]). When studying these equations, stability analysis is always a central issue. In the literature, there are several works on the stability of nonlinear differential equations, for example some works in [2, 7, 12, 16, 17].

As its name indicates, the practical stability concept is motivated by engineering considerations: it is rarely needed for an industrial system to reach its target exactly (this means, asymptotic stability is a rather constraining aim), or to behave until a really infinite time (then, it may be possible to require finite time stability only). These considerations allow to relax some hypotheses usually needed in classical Lyapunov's stability (see, e.g., [1, 2, 4, 25]). The traditional approaches to practical stability of nonlinear differential equations are the Lyapunov's method and its variants (Razumikhin-type theorems, Lyapunov–Krasovskii functional techniques), (see, e.g., [1, 9, 11, 14, 23]). To the best of our knowledge, there are not many explicit criteria for the practical exponential stability of these equations.

In this paper, we will develop a new approach to the practical exponential stability of nonlinear differential equations. Our approach is based on the theory of comparison principle and nonnegative matrices, (see, e.g., [3]). This theory has been applied successfully to exponential stability and robust stability of some classes of differential and difference equations (see, e.g., [13, 17, 18, 21]). By using this theory, several explicit criteria for the practical exponential stability of some nonlinear differential equations will be given. Some applications to neutral networks will be investigated.

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let $\underline{m} := \{1, 2, \ldots, m\}$ and $\underline{m_0} := \{0, 1, 2, \ldots, m\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. For positive integers $l, q \geq 1, \mathbb{R}^l$ denotes the l-dimensional vector space over $\mathbb{R}$ and $\mathbb{R}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{R}$. Inequalities between real matrices or vectors will be understood componentwise, i.e., for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l, j = 1, \ldots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \ldots, l, j = 1, \ldots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors.

For $x \in \mathbb{R}^n$ and $P \in \mathbb{R}^{l\times q}$, we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{R}^n, |x| \leq |y|$. For example, the p-norm on $\mathbb{R}^n$ $(\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, 1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|)$ is monotonic.

For any matrix $M \in \mathbb{R}^{n\times n}$, the spectral abscissa of M is denoted by $\mu(M) := \max\{\Re\lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{\lambda \in \mathbb{C} : \det(\lambda I_n - M) = 0\}$ is the spectrum of M. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz stable if $\mu(M) < 0$. For an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$, the matrix measure of $M := (m_{ij}) \in \mathbb{R}^{n\times n}$ is defined by

$$s(M) := \lim_{\epsilon\to 0^+} \frac{\|I_n + \epsilon M\| - 1}{\epsilon},$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, see [8].

A matrix $M \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if all off-diagonal elements of M are nonnegative. We now summarize some properties of Metzler matrices which will be used in what follows.

Theorem 1 [21] Suppose that $M \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then,

(i) (Perron–Frobenius) $\mu(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$.

(ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.

(iii) $(tI_n - M)^{-1}$ exists and is nonnegative if and only if $t > \mu(M)$.

(iv) Given $B \in \mathbb{R}_+^{n\times n}, C \in \mathbb{C}^{n\times n}$. Then,

$$|C| \leq B \Rightarrow \mu(M + C) \leq \mu(M + B).$$

The following is immediate from Theorem 1 and is used in what follows.

Theorem 2 Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent:

(i) $\mu(M) < 0$;

(ii) $Mp \ll 0$ for some $p \in \mathbb{R}_+^n, p \gg 0$;

(iii) M is invertible and $M^{-1} \leq 0$;

(iv) for given $b \in \mathbb{R}^n, b \gg 0,$ there exists $x \in \mathbb{R}^n_+$, such that $Mx + b = 0$;

(v) for any $x \in \mathbb{R}^n_+ \backslash \{0\}$, the row vector $x^T M$ has at least one negative entry.

With a given matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we associate the Metzler matrix $M(A) := (\hat{a}_{ij}) \in \mathbb{R}^{n\times n}$, where

$$\hat{a}_{ii} := a_{ii},\ i \in \underline{n}; \quad \hat{a}_{ij} := |a_{ij}|,\ i \neq j, i, j \in \underline{n}.$$

2. Main Results

Consider a nonlinear nonautonomous differential system of the form

$$\dot{x}(t) = f(t, x(t)) + \omega(t, x(t)), \quad t \geq \sigma, \tag{1}$$

where $f, \omega : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and are locally Lipschitz in the second argument, uniformly in t on compact intervals of $\mathbb{R}$. This system is seen as a perturbation of the nominal system

$$\dot{x}(t) = f(t, x(t)), \quad t \geq \sigma. \tag{2}$$

The perturbation term $\omega(t, x)$ could result from modelling errors, aging or uncertainties and disturbances, which exist in any realistic problem.

It is well known that for fixed $\sigma \in \mathbb{R}$ and given $x_0 \in \mathbb{R}^n$, there exists a unique local solution of (1), denoted by $x(\cdot; \sigma, x_0)$ satisfying the initial value condition

$$x(\sigma) = x_0, \tag{3}$$

see e.g., [10]. This solution is defined and continuous on $[\sigma, \gamma)$ for some $\gamma > \sigma$ and satisfies (1) for every $t \in [\sigma, \gamma)$, see e.g., [10]. Furthermore, if the interval $[\sigma, \gamma)$ is the maximum interval of existence of the solution $x(\cdot; \sigma, x_0)$, then $x(\cdot; \sigma, x_0)$ is said to be noncontinuable. The existence of a noncontinuable solution follows from Zorn's lemma and the maximum open interval of existence.

Definition 1 Equation (1) is said to be *practicallly exponentially stable* (shortly, PES) if there exist positive numbers K, β and $\Upsilon \geq 0,$ such that for each $\sigma \in \mathbb{R}$ and each $x_0 \in \mathbb{R}^n,$ the solution of (1)-(3) exists on $[\sigma, \infty)$ and furthermore satisfies

$$\|x(t, \sigma, x_0)\| \leq Ke^{-\beta(t-\sigma)}\|x_0\| + \Upsilon,\ \ \forall t \geq \sigma.$$

If $\Upsilon = 0,$ then equation (1) is said to be *exponentially stable* (shortly, ES).

Now, we consider

(H_1) There exists a continuous function $\theta(\cdot) : \mathbb{R} \to \mathbb{R}$ such that

$$s(J(t; x)) \leq \theta(t), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \tag{4}$$

where $J(t, x) := \left(\frac{\partial f_i}{\partial x_j}(t, x)\right) \in \mathbb{R}^{n\times n}, t \in \mathbb{R}, x \in \mathbb{R}^n,$ is the Jacobian matrix of $f(t, \cdot)$ at x.

(H_2) There exist a continuous function $\alpha(\cdot) : \mathbb{R} \to \mathbb{R}_+$ and a bounded function $h(\cdot; \cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\|\omega(t, u)\| \leq \alpha(t)\|u\| + h(t, u), \forall t \in \mathbb{R}, \forall u \in \mathbb{R}^n. \tag{5}$$

Theorem 3 Suppose that (H_1)-(H_2) hold and $f(t, 0)$ is bounded on $\mathbb{R}$. Then, (1) is PES if there exists $\delta_1 > 0$ such that

$$\theta(t) + \alpha(t) < -\delta_1, \forall t \in \mathbb{R}. \tag{6}$$

Moreover, if $f(t; 0) = 0, \forall t \in \mathbb{R}$ and $h(t, x) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n,$ then equation (1) is ES.

Proof Since (6), there exists $\Upsilon \in \mathbb{R}_+$ such that

$$\Big(\theta(t) + \alpha(t)\Big)\Upsilon \leq -\xi, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \tag{7}$$

where $\xi := \sup_{t\in\mathbb{R}, u\in\mathbb{R}^n}\{\|f(t, 0)\| + h(t, u)\}$. Let $\zeta > 0$ be arbitrary, but fixed and $x(t) := x(t; \sigma, x_0), t \in [\sigma, \gamma), x_0 \in \mathbb{R}^n,$ where the interval $[\sigma, \gamma)$ is the maximum interval of existence of the solution $x(\cdot; \sigma, x_0)$. Choose a positive number $K \geq 1$ such that $\|x_0\| \leq K\|x_0\|$. Define

$$\tilde{v}(t) := Ke^{-\delta_1(t-\sigma)}(\|x_0\| + \zeta) + \Upsilon, \quad t \in [\sigma, +\infty).$$

Clearly, $\|x(\sigma)\| < \tilde{v}(\sigma)$. We claim that $\|x(t)\| \leq \tilde{v}(t), \forall t \in [\sigma, \gamma)$.

Assume on the contrary that there exists $t^* > \sigma$ such that $\|x(t^*)\| > \tilde{v}(t^*)$. Set $t_b := \inf\{t \in (\sigma, \gamma) : \|x(t)\| > \tilde{v}(t)\}$. By continuity, $t_b > \sigma$ and

$$\|x(t)\| \leq \tilde{v}(t), \forall t \in [\sigma, t_b]; \quad \|x(t_b)\| = \tilde{v}(t_b); \quad \|x(\tau_k)\| > \tilde{v}(\tau_k), \tag{8}$$

for some $\tau_k \in (t_b, t_b + \frac{1}{k}), k \in \mathbb{N}$. Using the mean value theorem (see, e.g., [6]), we get the following estimates:

$$\begin{aligned} D^+\|x(t_b)\| &:= \limsup_{\epsilon\to 0^+} \frac{\|x(t_b+\epsilon)\| - \|x(t_b)\|}{\epsilon} \\ &= \limsup_{\epsilon\to 0^+} \frac{\|x(t_b) + \epsilon\dot{x}(t_b)\| - \|x(t_b)\|}{\epsilon} \\ &= \limsup_{\epsilon\to 0^+} \frac{\|x(t_b) + \epsilon(f(t_b, x(t_b)) + \omega(t_b, x(t_b)))\| - \|x(t_b)\|}{\epsilon} \\ &\le \limsup_{\epsilon\to 0^+} \frac{\|x(t_b) + \epsilon(f(t_b, x(t_b)) - f(t_b, 0))\| - \|x(t_b)\|}{\epsilon} \\ &\quad + \|f(t_b, 0)\| + \|\omega(t_b, x(t_b))\| \\ &= \limsup_{\epsilon\to 0^+} \frac{\|x(t_b) + \epsilon\Big(\int_0^1 J(t_b, sx(t_b))\mathrm{d}s\Big)x(t_b)\| - \|x(t_b)\|}{\epsilon} \\ &\quad + \|f(t_b, 0)\| + \|\omega(t_b, x(t_b))\| \\ &= \limsup_{\epsilon\to 0^+} \frac{\Big(\|I_n + \epsilon\int_0^1 J(t_b, sx(t_b))\mathrm{d}s\| - 1\Big)\|x(t_b)\|}{\epsilon} \\ &\quad + \|f(t_b, 0)\| + \|\omega(t_b, x(t_b))\| \\ &= s\Big(\int_0^1 J(t_b, sx(t_b))\mathrm{d}s\Big)\|x(t_b)\| + \|f(t_b, 0)\| + \|\omega(t_b, x(t_b))\|. \end{aligned}$$

Note that

$$s\Big(\int_0^1 J(t_b, sx(t_b))\mathrm{d}s\Big) \le \int_0^1 s\Big(J(t_b, sx(t_b))\Big)\mathrm{d}s,$$

(see, e.g., [8]). Then (H_1)-(H_2), and (6), (7) and (8) imply

$$\begin{aligned} D^+\|x(t_b)\| &\le \int_0^1 s\Big(J(t_b, sx(t_b))\Big)\mathrm{d}s\|x(t_b)\| + \|f(t_b, 0)\| + \|\omega(t_b, x(t_b))\| \\ &\overset{(H_1),(H_2)}{\le} \Big(\theta(t_b) + \alpha(t_b)\Big)\|x(t_b)\| + \|f(t_b, 0)\| + h(t_b, x(t_b)) \\ &\le \Big(\theta(t_b) + \alpha(t_b)\Big)\|x(t_b)\| + \xi \\ &\overset{(8)}{=} \Big(\theta(t_b) + \alpha(t_b)\Big)\tilde{v}(t_b) + \xi = \Big(\theta(t_b) + \alpha(t_b)\Big) \\ &\quad \times\Big(Ke^{-\delta_1(t_b-\sigma)}(\|x_0\| + \zeta) + \Upsilon\Big) + \xi \\ &\overset{(6),(7)}{<} -\delta_1 Ke^{-\delta_1(t_b-\sigma)}(\|x_0\| + \zeta) = \dot{\tilde{v}}(t_b). \end{aligned}$$

On the other hand, (8) implies that

$$\begin{aligned} D^+\|x(t_b)\| := \limsup_{t\to t_b^+} \frac{\|x(t)\| - \|x(t_b)\|}{t - t_b} &\ge \overline{\lim}_{k\to+\infty} \frac{\|x(\tau_k)\| - \|x(t_b)\|}{\tau_k - t_b} \\ \ge \overline{\lim}_{k\to+\infty} \frac{\tilde{v}(\tau_k) - \tilde{v}(t_b)}{\tau_k - t_b} &= \lim_{k\to+\infty} \frac{\tilde{v}(\tau_k) - \tilde{v}(t_b)}{\tau_k - t_b} = \dot{\tilde{v}}(t_b). \end{aligned}$$

This is a contradiction. Therefore,

$$\|x(t;\sigma, x_0)\| \le \tilde{v}(t) = Ke^{-\delta_1(t-\sigma)}(\|x_0\| + \zeta) + \Upsilon, \quad \forall t \in [\sigma, \gamma),\ \forall x_0 \in \mathbb{R}^n.$$

Letting ζ tend to zero, we obtain

$$\|x(t;\sigma, x_0)\| \le Ke^{-\delta_1(t-\sigma)}\|x_0\| + \Upsilon, \quad \forall t \in [\sigma, \gamma),\ \forall x_0 \in \mathbb{R}^n. \tag{9}$$

Now, we claim that $\gamma = \infty$ and so equation (1) is PES. Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (9) that $x(\cdot;\sigma, x_0)$ is bounded on $[\sigma, \gamma)$. Furthermore, this together with (1) implies that $\dot{x}(\cdot)$ is bounded on $[\sigma, \gamma)$. Thus, $x(\cdot)$ is uniformly continuous on $[\sigma, \gamma)$. This implies that $\lim_{t\to\gamma^-} x(t)$ exists and $x(\cdot)$ can be extended to a continuous differential function on $[\sigma, \gamma]$.

Therefore, one can find a solution of (1) through the point $(\gamma, x(\gamma))$ to the right of γ. This contradicts the noncontinuability hypothesis on $x(\cdot)$. Thus, γ must be equal to ∞.

Finally, it is easy to see that if $f(t, 0) = 0, \forall t \in \mathbb{R}$ and $h(t, x) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$ then $\Upsilon = 0$. Hence, (1) is ES. This completes the proof.

Remark 1 Based on a nonlinear inequality and Lyapunov's method, Makhlouf–Hammami [16, Example 2] show that

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + \frac{x_2(t)}{1+(x_1(t)+x_2(t))^2}e^{-t} + \sin(x_1^2(t)+x_2^2(t))\frac{\cos t}{1+t^2} \\ -x_2(t) + \frac{x_1(t)}{1+(x_1(t)+x_2(t))^2}e^{-t} + \sin(x_1^2(t)+x_2^2(t))\frac{\sin t}{1+t^2} \end{pmatrix}, t \geq 0 \tag{10}$$

is PES. Let $\mathbb{R}^2$ be endowed with 1-norm. It is easy to check that (10) is PES, by Theorem 3, where

$$f(t,x) := \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix},$$

and

$$\omega(t,x) = \begin{pmatrix} \frac{x_2}{1+(x_1+x_2)^2}e^{-t} + \sin(x_1^2+x_2^2)\frac{\cos t}{1+t^2} \\ \frac{x_1}{1+(x_1+x_2)^2}e^{-t} + \sin(x_1^2+x_2^2)\frac{\sin t}{1+t^2} \end{pmatrix}.$$

Definition 2 Equation (1) is said to be *ultimately practically exponentially stable* (shortly, UPES) if there exist a positive number β and vectors $\eta, v \geq 0$ such that for each $\sigma \in \mathbb{R}$ and each $x_0 \in \mathbb{R}^n$, the solution of (1)-(3) exists on $[\sigma, \infty)$ and furthermore satisfies

$$|x(t,\sigma,x_0)| \leq e^{-\beta(t-\sigma)}\|x_0\|\eta + v, \quad \forall t \geq \sigma.$$

If $v = 0$, then equation (1) is said to be *ultimately exponentially stable* (shortly, UES)).

Remark 2 It is easy to see that if equation (1) is UPES (UES, respectively), then it is PES (ES, respectively).

Now, we consider the assumptions:

(H_3) $f(t,\cdot)$ is continuously differentiable on $\mathbb{R}^n$ for any $t \in \mathbb{R}$ and there exists a matrix-valued continuous function $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that

$$M(J(t,x)) \leq A(t), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \tag{11}$$

where $J(t,x) := \left(\frac{\partial f_i}{\partial x_j}(t,x)\right) \in \mathbb{R}^{n\times n}, t \in \mathbb{R}, x \in \mathbb{R}^n,$ is the Jacobian matrix of $f(t,\cdot)$ at x.

(H_4) There exist a matrix-valued continuous function $B(\cdot) : \mathbb{R} \to \mathbb{R}_+^{n\times n}$ and a bounded function $g(\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+^n$ such that

$$|\omega(t,u)| \leq B(t)|u| + g(t,u), \forall t \in \mathbb{R}, \forall u \in \mathbb{R}^n. \tag{12}$$

(H_5) There exists a matrix-valued continuous function $C(\cdot) : \mathbb{R} \to \mathbb{R}_+^{n\times n}$ such that

$$|\omega(t,x) - \omega(t,y)| \leq C(t)|x-y|, \forall t \in \mathbb{R}, \forall x, y \in \mathbb{R}^n. \tag{13}$$

We are now in the position to state the second main result of this paper.

Theorem 4 Assume that (H_3) and (H_4) hold and $f(t, 0)$ is bounded on $\mathbb{R}$. If there exist $\beta > 0$ and $p := (p_1, p_2, \ldots, p_n)^T \in \mathbb{R}_+^n, p \gg 0$ such that

$$\Big(A(t) + B(t)\Big)p \ll -\beta p, \quad \forall t \in \mathbb{R}, \tag{14}$$

then equation (1) is UPES.

In addition, if $f(t,0) = 0, \forall t \in \mathbb{R}$ and $g(t,x) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$, then equation (1) is UES.

Proof Let $w := (w_1, w_2, \ldots, w_n)^T \in \mathbb{R}_+^n$ such that $w_i := \sup_{t\in\mathbb{R}, u\in\mathbb{R}^n}\{|f_i(t,0)| + g_i(t,u)\}$.

Since (14), there exists $v := (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$ such that

$$\Big(A(t) + B(t)\Big)v \leq -w, \forall t \in \mathbb{R}. \tag{15}$$

Let $A(t) := (a_{ij}(t)) \in \mathbb{R}^{n\times n}, t \in \mathbb{R};$ and $B(t) := (b_{ij}(t)) \in \mathbb{R}^{n\times n}, t \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary, but fixed. Define $x(t) := x(t;\sigma,x_0), t \in [\sigma,\gamma), x_0 \in \mathbb{R}^n$, where the interval $[\sigma,\gamma)$ is the maximum interval of existence of the solution $x(\cdot;\sigma,x_0)$. It follows from (3) that $|x(\sigma)| = |x_0| \leq \|x_0\|\frac{p}{\lambda}$, where $\lambda := \min_{i\in\underline{n}} p_i$. Define

$$u(t) := e^{-\beta(t-\sigma)}(\|x_0\| + \epsilon)\frac{p}{\lambda} + v, \quad t \in [\sigma, +\infty).$$

Clearly, $|x(\sigma)| = |x_0| \ll u(\sigma)$. We claim that $|x(t)| \leq u(t), \forall t \in [\sigma,\gamma)$. Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \not\leq u(t_0)$. Set $t_1 := \inf\{t \in (\sigma,\gamma) : |x(t)| \not\leq u(t)\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in \underline{n}$ such that

$$|x(t)| \leq u(t), \forall t \in [\sigma, t_1); |x_{i_0}(t_1)| = u_{i_0}(t_1), |x_{i_0}(\tau_k)| > u_{i_0}(\tau_k), \tag{16}$$

for some $\tau_k \in (t_1, t_1 + \frac{1}{k}), k \in \mathbb{N}$. By the mean value theorem (see, e.g., [6]), we have for each $t \in \mathbb{R}$ and for each $i \in \underline{n}$

$$\dot{x}_i(t) = f_i(t, x(t)) + \omega_i(t, x(t)) = (f_i(t, x(t)) - f_i(t, 0)) + f_i(t, 0) + \omega_i(t, x(t))$$
$$= \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial f_i}{\partial x_j}(t, sx(t))\mathrm{d}s \right) x_j(t) + f_i(t, 0) + \omega_i(t, x(t)).$$

Taking (H_3)-(H_4) into account, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x_i(t)| &= sgn(x_i(t))\dot{x}_i(t) = sgn(x_i(t)) \left(\sum_{j=1}^{n} \left(\int_0^1 \frac{\partial f_i}{\partial x_j}(t, sx(t))\mathrm{d}s \right) x_j(t) \right. \\ &\quad \left. + f_i(t, 0) + \omega_i(t, x(t)) \right) \\ &\le \left(\int_0^1 \frac{\partial f_i}{\partial x_i}(t, sx(t))\mathrm{d}s \right) |x_i(t)| + \sum_{j=1, j\neq i}^{n} \int_0^1 \left|\frac{\partial f_i}{\partial x_j}(t, sx(t))\right|\mathrm{d}s |x_j(t)| \\ &\quad + |f_i(t, 0)| + |\omega_i(t, x(t))|, \\ &\overset{(H_3),(H_4)}{\le} \sum_{j=1}^{n} a_{ij}(t)|x_j(t)| + \sum_{j=1}^{n} b_{ij}(t)|x_j(t)| \\ &\quad + |f_i(t, 0)| + g_i(t, x(t)) \\ &\le \sum_{j=1}^{n} a_{ij}(t)|x_j(t)| + \sum_{j=1}^{n} b_{ij}(t)|x_j(t)| + w_i,\end{aligned}$$

for almost any $t \in [\sigma, \gamma)$. It follows that for any $t \in [\sigma, \gamma)$,

$$\begin{aligned}D^+|x_i(t)| &:= \limsup_{\epsilon \to 0^+} \frac{|x_i(t+\epsilon)| - |x_i(t)|}{\epsilon} = \limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{d}{ds}|x_i(s)|\mathrm{d}s \\ &\le \sum_{j=1}^{n} a_{ij}(t)|x_j(t)| + \sum_{j=1}^{n} b_{ij}(t)|x_j(t)| + w_i,\end{aligned}$$

where D^+ denotes the Dini upper-right derivative. In particular, it follows from (14), (15) and (16) that

$$\begin{aligned}D^+|x_{i_0}(t_1)| &\le \sum_{j=1}^{n} a_{i_0 j}(t_1)|x_j(t_1)| + \sum_{j=1}^{n} b_{i_0 j}(t_1)|x_j(t_1)| + w_{i_0}, \\ &\overset{(16)}{\le} \left(\sum_{j=1}^{n} a_{i_0 j}(t_1)p_j + \sum_{j=1}^{n} b_{i_0 j}(t_1)p_j \right) e^{-\beta(t_1-\sigma)} \frac{\|x_0\| + \epsilon}{\lambda} \\ &\quad + \left(\sum_{j=1}^{n} a_{i_0 j}(t_1)v_j + \sum_{j=1}^{n} b_{i_0 j}(t_1)v_j \right) + w_{i_0} \\ &\overset{(14),(15)}{<} -\beta e^{-\beta(t_1-\sigma)}(\|x_0\| + \epsilon)\frac{p_{i_0}}{\lambda} = D^+ u_{i_0}(t_1).\end{aligned}$$

On the other hand, (16) implies that

$$\begin{aligned}D^+|x_{i_0}(t_1)| &:= \limsup_{t \to t_1^+} \frac{|x_{i_0}(t)| - |x_{i_0}(t_1)|}{t - t_1} \ge \overline{\lim}_{k \to +\infty} \frac{|x_{i_0}(\tau_k)| - |x_{i_0}(t_1)|}{\tau_k - t_1} \\ &\ge \overline{\lim}_{k \to +\infty} \frac{u_{i_0}(\tau_k) - u_{i_0}(t_1)}{\tau_k - t_1} = \lim_{k \to +\infty} \frac{u_{i_0}(\tau_k) - u_{i_0}(t_1)}{\tau_k - t_1} \\ &= D^+ u_{i_0}(t_1).\end{aligned}$$

This is a contradiction. Therefore,

$$|x(t; \sigma, x_0)| \le u(t) = e^{-\beta(t-\sigma)}(\|x_0\| + \epsilon)\frac{p}{\lambda} + v, \quad \forall t \in [\sigma, \gamma),\ \forall x_0 \in \mathbb{R}^n.$$

Letting ϵ tend to zero, we obtain

$$|x(t; \sigma, x_0)| \le e^{-\beta(t-\sigma)}\|x_0\|\frac{p}{\lambda} + v, \quad \forall t \in [\sigma, \gamma),\ \forall x_0 \in \mathbb{R}^n. \tag{17}$$

Now similar to the final part in the proof of Theorem 3, we imply $\gamma = \infty$ and so equation (1) is UPES. Finally, if $f(t, 0) = 0, \forall t \in \mathbb{R}$ and $g(t, x)) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$, then $v = 0$ and equation (1) is UES. This completes the proof.

Corollary 1 Suppose (H_3) and (H_4) hold and $f(t, 0)$ is bounded on $\mathbb{R}$. Then equation (1) is UPES if one of the following conditions is satisfied:

(i) There exists a Hurwitz stable matrix $B_0 \in \mathbb{R}^{n\times n}$ such that

$$A(t) + B(t) \le B_0, \quad \forall t \in \mathbb{R}. \tag{18}$$

(ii) There exist $p, q \in \mathbb{R}^n_+, p, q \gg 0$ such that

$$\Big(A(t) + B(t)\Big)p \le -q, \quad \forall t \in \mathbb{R}. \tag{19}$$

In addition, if $f(t, 0) = 0, \forall t \in \mathbb{R}$ and $g(t, x) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$, then (1) is UES.

Proof (i) Assume that (i) holds. It remains to show that (14) of Theorem 4 holds. Note that B_0 is a Metzler matrix. Since B_0 is Hurwitz stable, there exists $p \in \mathbb{R}^n_+, p \gg 0$ so that $B_0 p \ll 0$, by Theorem 2. By continuity, this implies that

$$B_0 p \ll -\beta p, \tag{20}$$

for some sufficiently small $\beta > 0$. Therefore,

$$\Big(A(t) + B(t)\Big)p \le B_0 p \overset{(20)}{\ll} -\beta p, \quad \forall t \in \mathbb{R}.$$

Thus, (14) holds.

(ii) Suppose that (ii) holds. We show that (14) of Theorem (4) holds. Let $p := (p_1, p_2, \ldots, p_n)^T, q := (q_1, q_2, \ldots, q_n)^T \in \mathbb{R}^n$ with $p_i, q_i > 0, \forall i \in \underline{n}$. Fix $i \in \underline{n}$ and $t \in \mathbb{R}, x \in \mathbb{R}^n$, and consider the function

$$F_i(\beta) = \beta p_i + \sum_{j=1}^{n} (a_{ij}(t) + b_{ij}(t))p_j,$$

with $\beta \in \mathbb{R}_+$. Clearly, $F_i(\beta)$ is continuous in β on $\mathbb{R}_+$ and $\lim_{\beta \to +\infty} F_i(\beta) = +\infty$,

$$F_i(0) = \sum_{j=1}^{n} (a_{ij}(t) + b_{ij}(t))p_j \overset{(19)}{\le} -q_i < 0,$$

and

$$\frac{dF_i}{d\beta} = p_i > 0.$$

Therefore, $F_i(\beta)$ is strictly increasing on $\mathbb{R}_+$. Therefore, there is a unique positive number, say $\beta_i(t)$, such that

$$\beta_i(t)p_i + \sum_{j=1}^{n} (a_{ij}(t) + b_{ij}(t))p_j = 0.$$

For each $i \in \underline{n}$, let us define

$$\beta_i^* := \inf_{t \in \mathbb{R}} \{\beta_i(t) > 0 : F(\beta_i(t)) = 0\}.$$

Obviously, $\beta_i^* \ge 0$. We show that $\beta_i^* > 0$. Suppose this is not true. Let $0 < \epsilon_i < \frac{q_i}{p_i}$. Then there exists $t^* \in \mathbb{R}$ such that $\beta_i(t^*) < \epsilon_i$ and

$$\beta_i(t^*)p_i + \sum_{j=1}^{n} (a_{ij}(t^*) + b_{ij}(t^*))p_j = 0.$$

Then,

$$\begin{aligned} 0 =& \beta_i(t^*)p_i + \sum_{j=1}^{n} (a_{ij}(t^*) + b_{ij}(t^*))p_j < \epsilon_i p_i + \sum_{j=1}^{n} (a_{ij}(t^*) + b_{ij}(t^*))p_j \\ <& q_i + \sum_{j=1}^{n} (a_{ij}(t^*) + b_{ij}(t^*))p_j \overset{(19)}{<} q_i - q_i = 0, \end{aligned}$$

which is a contradiction. Therefore, $\beta_i^* > 0$, for all $i \in \underline{n}$. Let $0 < \beta < \min_{i \in \underline{n}} \{\beta_i^*\}$. It follows that

$$\beta p_i + \sum_{j=1}^{n} (a_{ij}(t) + b_{ij}(t))p_j < 0,$$

for all $t \in \mathbb{R}$, and for all $i \in \underline{n}$. Hence, (14) holds. This completes the proof. □

Remark 3 It follows from the condition (i) of Corollary 1 that we can choose $v = -(B_0)^{-1} w$ such that (15) is satisfied.

The following follows from Theorem 4 and Corollary 1.

Theorem 5 Assume that (H_3), (H_5) hold and $f(t,0), \omega(t,0)$ are bounded on $\mathbb{R}$. Then equation (1) is UPES if one of the following conditions is satisfied:

(i) There exist $\beta > 0$ and $p := (p_1, p_2, \ldots, p_n)^T \in \mathbb{R}^n_+, p \gg 0$ such that

$$\Big(A(t) + C(t)\Big)p \ll -\beta p, \quad \forall t \in \mathbb{R}. \tag{21}$$

(ii) There exists a Hurwitz stable matrix $B_0 \in \mathbb{R}^{n \times n}$ such that

$$A(t) + C(t) \le B_0, \quad \forall t \in \mathbb{R}. \tag{22}$$

(iii) There exist $p, q \in \mathbb{R}^n_+, p, q \gg 0$ such that

$$\Big(A(t)+C(t)\Big)p \le -q, \quad \forall t \in \mathbb{R}. \tag{23}$$

In addition, if $f(t,0)+\omega(t,0)=0, \forall t \in \mathbb{R}$, then equation (1) is UES.

Proof Since (H_5), we have

$$|\omega(t,x)| \le C(t)|x| + |\omega(t,0)|, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n.$$

Then (H_4) holds with $g(t,x) := |\omega(t,0)|$. Thus, the conclusion of Theorem 5 is straightforward from Theorem 4 and Corollary 1.

Example 1 Consider the differential equation:

$$\dot{x}(t) := (-4-2\cos^2 t)x(t) + 8\sin(t^2+0.001x(t)). \tag{24}$$

Clearly, (24) is of the form (1) with $f(t,x) := (-4-2\cos^2 t)x + 8\sin(t^2+0.001x), t \in \mathbb{R}$. Furthermore, it is easy to see that $f(t, x)$ is local Lipschitz continuous with respect to x on each compact subset of $\mathbb{R}\times\mathbb{R}$ and

$$\frac{\partial f}{\partial x}(t,x) = -4-2\cos^2 t + 0.008\cos(0.001x+t^2).$$

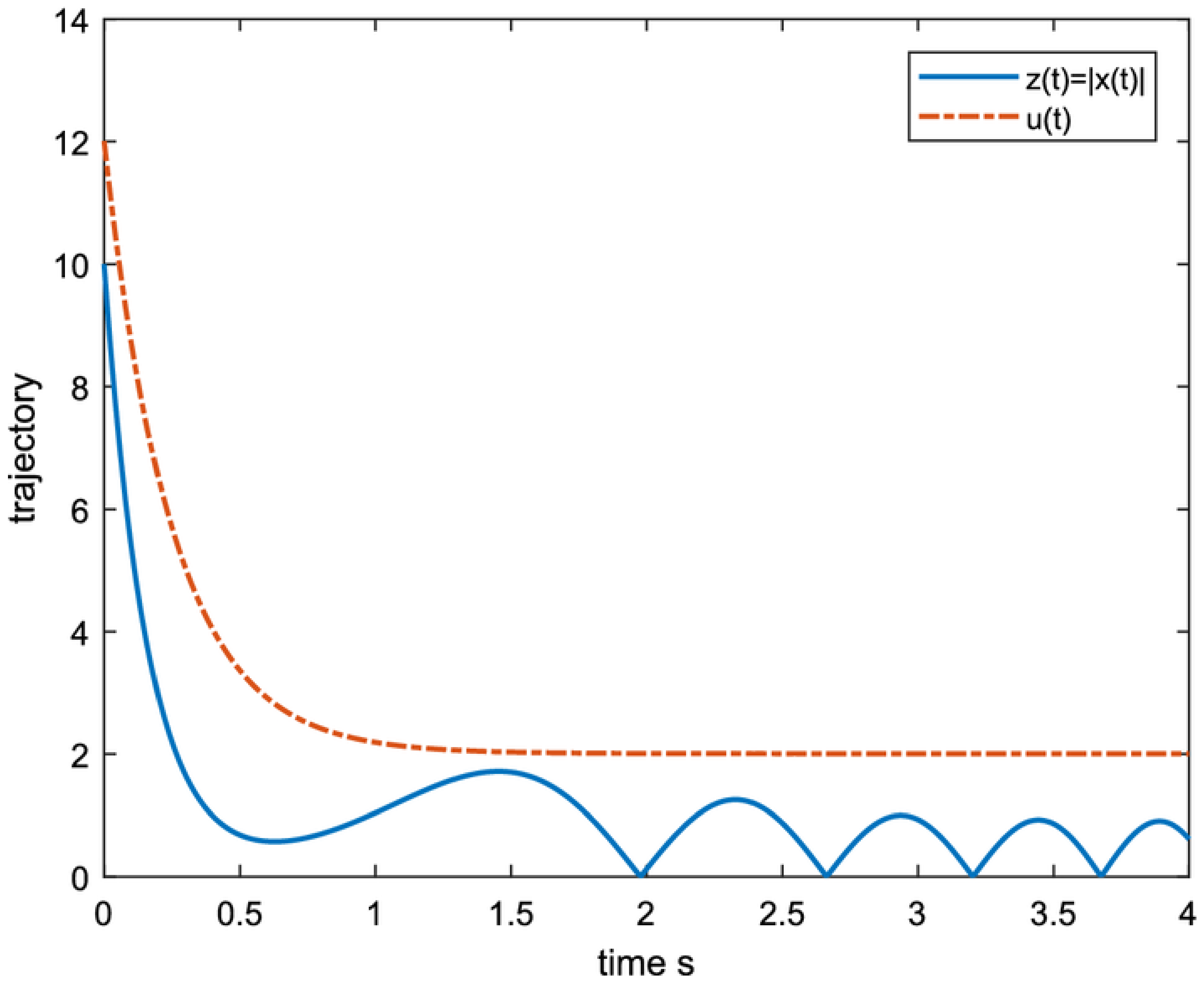
Let $\beta_1 = 3.991, p = 1$. It is clear that

$$\begin{aligned}\Big(\frac{\partial f}{\partial x}(t,x)\Big)p &= -4-2\cos^2 t + 0.008\cos(0.001x+t^2)\\ &\le -3.992 < -3.991 = -\beta_1 p, \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}.\end{aligned}$$

Therefore, (24) is PES, by Theorem 4. For a visual simulation, if we choose $x(0)=10$, then the trajectory of system (24) is given in Fig. 1. AQ3

Fig. 1

XXX



In the particular case, if equation (1) is not perturbed (i.e., $\omega(t,x)\equiv 0$), then the following follows directly from Theorem 5.

Corollary 2 Suppose that $\omega(t,x)\equiv 0$, (H_3) holds and $f(t;0)=0, \forall t \in \mathbb{R}$. Then equation (1) is UES if one of the following conditions is satisfied:

(i) There exist $\beta > 0$ and $p := (p_1, p_2, \ldots, p_n)^T \in \mathbb{R}^n_+, p \gg 0$ such that

$$A(t)p \ll -\beta p, \quad \forall t \in \mathbb{R}. \tag{25}$$

(ii) There exists a Hurwitz stable matrix $B_0 \in \mathbb{R}^{n\times n}$ such that

$$A(t) \le B_0, \quad \forall t \in \mathbb{R}. \tag{26}$$

(iii) There exist $p, q \in \mathbb{R}^n_+, p, q \gg 0$ such that

$$A(t)p \le -q, \quad \forall t \in \mathbb{R}. \tag{27}$$

Remark 4 The result in Corollary 2 includes the well-known result in [18, Theorem 2.2]. Furthermore, in this paper, we have shown that equation (2) is globally exponentially stable for all $x_0 \in \mathbb{R}^n$, while Theorem 2.2 in [18] proved that equation (2) is locally exponentially stable with $x_0 \in B_r = \{x :\in \mathbb{R}^n, \|x\| < r\}$.

Remark 5 Our approach may be more easy to check UPES and UES in some cases than other results.

(i) Song–Lib–Wang [22, Example 4.1, page 1309] show that

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - 2 & k \\ k & \sin\ln(t+1) + \cos\ln(t+1) - 2 \end{pmatrix} \times \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{28}$$

is ES provided $k < \frac{1}{2}$. It is easy to check that (28) is UES provided $k < 2 - \sqrt{2}$, by Corollary 2.

(ii) Errebii–Ellouze–Hammami [7, Example 3.1, page 170] show that the scalar differential equation with delay

$$\dot{x}(t) = -x(t) + \frac{1}{1 + x^2(t)} e^{-t}, t \ge 0 \tag{29}$$

is PES. This is immediate from Theorem 4.

(iii) A similar result has been found in [1, Example 1, page 60]. More precisely, the differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + \frac{x_1(t)}{1+x_1^2(t)} e^{-x_1^2(t)} + \frac{1}{1+t^2} x_2(t) \\ -x_2(t) + e^{-x_2(t)} \end{pmatrix} \tag{30}$$

is UPES. Once again, it is easy to see that this assertion follows from Theorem 4 with $x = (x_1, x_2)^T \in \mathbb{R}^2$

$$f(t, x) := \begin{pmatrix} -x_1 + \frac{1}{1+t^2} x_2 \\ -x_2 + e^{-x_2} \end{pmatrix},$$

and

$$\omega(t, x) = \begin{pmatrix} \frac{x_1}{1+x_1^2} e^{-x_1^2} \\ 0 \end{pmatrix}.$$

3. Applications to Neural Networks

Consider the cellular neural network described by

$$\dot{x}_i(t) = -c_i(t) x_i(t) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t)) + I_i(t), \quad i \in \underline{n}, \tag{31}$$

where n corresponds to the number of units in the neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $c_i(t)$ represents the rate at which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $g_j(x_j(t))$ denotes the output of the jth unit at the time $t, a_{ij}(t)$ denotes the strength of the jth unit on the ith unit at time t and $I_i(t)$ denotes the external bias on the ith unit at the time t.

Let $g_i(\cdot), c_i(\cdot), I_i(\cdot), a_{ij}(\cdot)$ be continuous functions. Assume that $a_{ij}(\cdot)$ is bounded.

Suppose that

(A_1) For each $j \in \underline{n}$, there exists $L_j \ge 0$ so that $|g_j(u_j) - g_j(v_j)| \le L_j |u_j - v_j|, \ \forall u_j, v_j \in \mathbb{R}$.

(A_2) For each $j \in \underline{n}$, there exists $M_j \ge 0$ so that $0 \le \dot{g}_j(u_j) \le L_j, \ \forall u_j \in \mathbb{R}$.

Define $[h(t)]^+ = \max\{h(t), 0\}$, for every $t \in \mathbb{R}$. Let $x(t) := x(t, x_0)$ be the solution of (31).

Corollary 3 Assume that (A_1) holds and $I_i(\cdot)$ is bounded. Then equation (31) is UPES if one of the following conditions holds:

(i) There exist a scalar $\beta > 0$ and positive numbers $p_1, p_2, \ldots, p_n$ such that

$$-c_i(t)p_i + \sum_{j=1}^{n} |a_{ij}(t)|L_j p_j < -\beta p_i, \forall t \in \mathbb{R}, \forall i \in \underline{n}.$$

(ii) There exists a Hurwitz stable matrix $B := (b_{ij}) \in \mathbb{R}^{n\times n}$ such that for each $i, j \in \underline{n}, i \neq j,$

$$-c_i(t) + |a_{ii}(t)|L_i \le b_{ii}, \forall t \in \mathbb{R}; |a_{ij}(t)|L_j \le b_{ij}, \forall t \in \mathbb{R}.$$

(iii) There exist positive numbers $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ such that

$$-c_i(t)p_i + \sum_{j=1}^{n} |a_{ij}(t)|L_j p_j \le -q_i, \forall t \in \mathbb{R}, \forall i \in \underline{n}.$$

Proof Let

$$f(t,x) := (f_1(t,x), f_2(t,x), \ldots, f_n(t,x))^T,$$
$$w(t,x) := (w_1(t,x), w_2(t,x), \ldots, w_n(t,x))^T$$

with $f_i(t,x) := -c_i(t)x_i(t), w_i(t,x) := \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + I_i(t), i \in \underline{n}$. It is not hard to see that the equation (31) is of the form (1). Then, the conclusions of Corollary 3 are straightforward from Theorem 5.

Corollary 4 Assume that (A_2) holds and $I_i(\cdot)$ is bounded. Then equation (31) is UPES if one of the following conditions holds:

(i) There exist a scalar $\beta > 0$ and positive numbers $p_1, p_2, \ldots, p_n$ such that

$$-c_i(t)p_i + [a_{ii}(t)]^+ L_i p_i + \sum_{j=1, j\neq i}^{n} |a_{ij}(t)|L_j p_j < -\beta p_i, \forall t \in \mathbb{R}, \forall i \in \underline{n}.$$

(ii) There exists a Hurwitz stable matrix $B := (b_{ij}) \in \mathbb{R}^{n\times n}$ such that for each $i, j \in \underline{n}, i \neq j,$

$$-c_i(t) + [a_{ii}(t)]^+ L_i \le b_{ii}, \forall t \in \mathbb{R}; |a_{ij}(t)|L_j \le b_{ij}, \forall t \in \mathbb{R}, j \neq i.$$

(iii) There exist positive numbers $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ such that

$$-c_i(t)p_i + [a_{ii}(t)]^+ L_i p_i + \sum_{j=1, j\neq i}^{n} |a_{ij}(t)|L_j p_j \le -q_i, \forall t \in \mathbb{R}, \forall i \in \underline{n}.$$

Proof Let

$$f(t,x) := (f_1(t,x), f_2(t,x), \ldots, f_n(t,x))^T,$$
$$w(t,x) := (w_1(t,x), w_2(t,x), \ldots, w_n(t,x))^T$$

with $f_i(t,x) := -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + I_i(t), w_i(t,x) := 0, i \in \underline{n}$. Clearly, the conclusions of Corollary 4 are straightforward from Theorem 5.

Corollary 5 Assume that (A_2) holds and $I_i(\cdot)$ is bounded. Then equation (31) is UPES if one of the following conditions holds:

(i) There exists a scalar $\beta_1 > 0$ such that

$$-c_j(t) + [a_{jj}(t)]^+ L_j + \sum_{i=1, i\neq j}^{n} |a_{ij}(t)|L_i < -\beta_1, \forall t \in \mathbb{R}, \forall j \in \underline{n},$$

or

$$-c_i(t) + [a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ij}(t)|]^+ L_i < -\beta_1, \forall t \in \mathbb{R}, \forall i \in \underline{n}.$$

(ii) There exists a scalar $\beta_2 > 0$ such that

$$-c_j(t) + [a_{jj}(t)]^+ L_j + \frac{1}{2}\sum_{i=1, i\neq j}^{n} (|a_{ij}(t)|L_i + |a_{ji}(t)|L_j) < -\beta_2, \forall t \in \mathbb{R}, \forall j \in \underline{n}.$$

Proof Let

$$f(t,x) := (f_1(t,x), f_2(t,x), \ldots, f_n(t,x))^T,$$
$$w(t,x) := (w_1(t,x), w_2(t,x), \ldots, w_n(t,x))^T$$

with $f_i(t,x) := -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + I_i(t), w_i(t,x) := 0, i \in \underline{n}$. It is easy to see that the conclusions of Corollary 5 are straightforward from Theorem 3.

Assume that $x^* := (x_1^*, x_2^*, \ldots, x_n^*)^T \in \mathbb{R}^n$ is an equilibrium of (31). It is obvious that $u(\cdot) := x(\cdot) - x^*$ satisfies

$$\dot{u}_i(t) = -c_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)s_j(u_j(t)), \quad i \in \underline{n},$$

where $s_j(u_j) := g_j(u_j + x_j^*) - g_j(x_j^*), j \in \underline{n}$.

The following is immediate from Corollary 2 and Theorem 3.

Corollary 6 The equilibrium x^* of (31) is ES if one of the following conditions holds:

(i) (A_1) and one of conditions (i), (ii), (iii) of Corollary 3 are satisfied.

(ii) (A_2) and either one of conditions (i), (ii), (iii) of Corollary (4) or one of the conditions (i), (ii) of Corollary (5) are satisfied.

Remark 6 Corollary 6 includes some existing criteria for the exponential stability of the cellular neural network in the literature as special cases (see, e.g., [5, 8, 19, 20, 24]).

Publisher's Note

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My manuscript has no associated data.

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