SPECTRAL CRITERIA FOR THE ASYMPTOTIC CONSTANCY OF SOLUTIONS TO IMPLICIT DIFFERENCE EQUATIONS

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Abstract. This paper deals with spectral criteria for the asymptotic constancy of solutions to the implicit difference equation $Cx(n+1) = Tx(n)+y(n)$ in a Banach space $\mathbb{X}$, where the bounded sequence $\{y(n)\}_n$ is asymptotically constant. The main result states that, if 1 is either not in $\sigma_\Gamma(C,T)$, or is its isolated element, then the implicit difference equation has an asymptotic solution that is asymptotically constant, provided it has a bounded asymptotic solution. In the case of $\sigma_\Gamma(C,T) \subset \{1\}$ we prove that every asymptotic solution is asymptotically constant. Furthermore, we give an application of the result to periodic evolution equations associated with C-semigroups.

1. Introduction

In this paper, we study the asymptotic behavior of the implicit difference equation

$$Cx(n+1) = Tx(n) + y(n), \quad n \in \mathbb{N}, \tag{1}$$

where $x(n) \in \mathbb{X}$, $\mathbb{X}$ is a Banach space, T is a bounded linear operator acting in $\mathbb{X}$, C is an injective operator in $\mathcal{L}(\mathbb{X})$, and $y(n) \in \mathbb{X}$ is a bounded asymptotically constant sequence in $\mathbb{X}$.

Historically, Katznelson–Tzafriri [11] studied the asymptotic behavior of the sequence $\{T^n\}_n$, where T is a power bounded operator in a Banach space $\mathbb{X}$, that is, $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. A famous result of that paper is the following theorem:

Theorem A (Katznelson–Tzafriri [11, Theorem 1]). *Let T be a linear contraction on a Banach space $\mathbb{X}$. Then,*

$$\lim_{n\to\infty} (T^{n+1} - T^n) = 0 \quad \textit{if (and only if)} \quad \sigma_\Gamma(T) \subset \{1\}.$$

We can refer to Vu [27] for a short proof of the Theorem A, see also [1, 15, 17, 26] for discussions related to this result. Recent developments related to Theorem A can be found in [2, 3, 4, 7, 16, 23, 28]. Theorem A can be expressed as the spectral criterion for all solutions to the homogeneous equation $x(n + 1) = Tx(n)$ to be asymptotically constant.

Date: November 16, 2023.
2020 *Mathematics Subject Classification.* Primary: 39A06, 39A30, 47B39; Secondary: 39A99, 47B48.
Key words and phrases. Katznelson-Tzafriri theorem, Massera theorem, asymptotic constancy asymptotic solution, implicit difference equation.
This work was completed while the authors were visiting the Vietnam Institute of Advanced Study in Mathematics (VIASM). The authors would like to thank VIASM for its hospitality. The authors thank Prof. Nguyen Van Minh (University of Arkansas at Little Rock) for various supports, carefully reading the manuscript and for his suggestions to improve the presentation.

For the case of explicit difference equation

$$x(n+1) = Tx(n) + y(n), \quad n \in \mathbb{N}, \tag{2}$$

that is, implicit difference equation (1) in the case of C is the identity operator I, by using the spectral decomposition technique in [8, 20], Minh–Matsunaga–Huy–Luong [17] proved in 2022 that if $\sigma_\Gamma(C,T) \subset \{1\}$ and $\{y(n)\}_n$ is asymptotically constant, then every bounded solution of (2) is asymptotically constant. Furthermore, Minh–Matsunaga–Huy–Luong [17] introduced the concept of *asymptotic solution* (see Definition 3.3) to study the case when the condition $\sigma_\Gamma(T) \subset \{1\}$ may not hold. The result states that if 1 is either not in $\sigma_\Gamma(T)$, or is an isolated point of $\sigma_\Gamma(T)$, then (2) has an asymptotic solution that is asymptotically constant, provided it has a bounded solution (see Theorem 3.8). A result of this type is often referred to as a Massera-type theorem. Such results play a very important role in studying the periodicity of differential and difference equations. The reader is referred to [6, 8, 12, 14, 18, 19, 20, 21, 22, 29, 32], see also [10, 13, 17, 24, 25] for more recent developments.

The aims of this paper are to extend the methods and results of Minh–Matsunaga–Huy–Luong [17] to the case of *implicit* difference equation (1) and to further study the relationship of the spectrum of a bounded sequence and spectrum of a bounded asymptotic solution (see the first inclusion in Lemma 3.4). More specifically, we will give a spectral criterion for the asymptotic constancy of solutions to implicit difference equation (1). Our technique is to use the spectral definition of the sequences in Definition 2.1 to establish the desired criterion. Hence, the main difficulty when transitioning to the case of *implicit* difference equation is proving the analyticity of the function $\rho(C,T) \ni \lambda \mapsto R(\lambda, C, T) := (\lambda C - T)^{-1}$. This difficulty is resolved in Lemma 3.2 which gives the openness of the resolvent set $\rho(C,T)$ in the complex plane and the analyticity of the function $\lambda \mapsto R(\lambda, C, T)$ on the resolvent set.

With Lemma 3.2, our main results are contained in Lemma 3.4, Theorem 3.6, Theorem 3.7, Theorem 3.8, and Theorem 4.3. We also prove that the spectrum of a bounded sequence is a subset of the spectrum of a bounded asymptotic solution, see inclusion "$\sigma(y) \subset \sigma(x)$" in Lemma 3.4. Note that, the result "$\sigma(x) \subset \sigma(y) \cup \sigma_\Gamma(C,T)$" in Lemma 3.4 is analogous to Naito–Minh–Shin [20, Lemma 3.2] for the implicit difference equations, see Remark 3.5.

This paper is organized as follows: In Section 2, we first list some notations used in the paper. Then, we recall some background materials on spectral theory. Section 3 contains the main results, which begins with Lemma 3.2 on the analyticity of the resolvent $R(\lambda, C, T) := (\lambda C - T)^{-1}$. Using Lemma 3.4, we describe a necessary condition for implicit difference equation (1) to have an asymptotic constant solution in Theorem 3.6. A spectral criterion for any bounded asymptotic solution to be asymptotically constant is stated in Theorem 3.7. Theorem 3.8 builds a spectral criterion to infer the existence of asymptotically constant from the existence of a bounded asymptotic solution. Finally, Section 4 discusses an application to periodic evolution equations associated with C-semigroups.

2. Preliminaries

Notations. For a complex Banach space $\mathbb{X}$, the space of all bounded linear operators acting in $\mathbb{X}$ is denoted by $\mathcal{L}(\mathbb{X})$; $\rho(C,T)$ and $\sigma(C,T)$ denote the resolvent set and spectrum of linear operator pencil $\{C,T\}$, respectively. For $\lambda \in \rho(C,T)$, we denote $R(\lambda, C, T) := (\lambda C - T)^{-1}$. A sequence in $\mathbb{X}$ will be denoted by $\{x(n)\}_n$.

Consider the Banach spaces of sequences

$$l^\infty(\mathbb{X}) := \left\{ x = \{x(n)\}_n \subset \mathbb{X} : \sup_{n\in\mathbb{N}} \|x(n)\| < \infty \right\}, \tag{3}$$

and

$$c_0(\mathbb{X}) := \left\{ x = \{x(n)\}_n \subset \mathbb{X} : \lim_{n\to\infty} x(n) = 0 \right\} \tag{4}$$

are equipped with sup-norm, $\|x\| := \sup_{n\in\mathbb{N}} \|x(n)\|$. The *shift operator* S acts in $l^\infty(\mathbb{X})$ as

$$Sx(n) = x(n+1), \qquad n \in \mathbb{N}, \quad x \in l^\infty(\mathbb{X}).$$

The operator S is a contraction (see Minh–Matsunaga–Huy–Luong [17]).

Consider the quotient Banach space $\mathbb{Y} := l^\infty(\mathbb{X})/c_0(\mathbb{X})$ with the induced norm. The equivalent class of $x \in l^\infty(\mathbb{X})$ will be denoted by $\bar{x}$. Since S leaves $c_0(\mathbb{X})$ invariant it induces a bounded linear operator $\bar{S}$ acting in $\mathbb{Y}$. Similarly, each operator $T \in \mathcal{L}(\mathbb{X})$ induces an operator $\bar{T} \in \mathcal{L}(\mathbb{Y})$. Moreover, one notes that $\bar{S}$ is a surjective isometry. As a consequence, $\sigma(\bar{S}) \subset \Gamma$, where Γ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. We put

$$\sigma_\Gamma(C,T) := \sigma(C,T) \cap \{z \in \mathbb{C} : |z| = 1\}. \tag{5}$$

2.1. **Spectral Theory.** In this subsection, we present some materials on spectral theory, see detail in [15, 17].

Firstly, the resolvent of the isometry $\bar{S}$ satisfies

$$\left\|R\left(\lambda, \bar{S}\right)\right\| \le \frac{1}{||\lambda| - 1|}, \qquad \text{for all } |\lambda| \neq 1. \tag{6}$$

Definition 2.1. The *spectrum* of $\bar{x} \in \mathbb{Y}$, denoted by $\sigma(\bar{x})$, is defined to be the set of all non-removable singular points of the complex function

$$g(\lambda) := R(\lambda, \bar{S})\bar{x}. \tag{7}$$

If $x = \{x(n)\}_n \in l^\infty(\mathbb{X})$, then, its spectrum, denoted by $\sigma(x)$, is said to be $\sigma(\bar{x})$.

From the definition of spectrum of a bounded sequence x it follows that $\sigma(x)$ is a closed subset of $\mathbb{C}$.

Lemma 2.2 (Minh [15, Lemma 2.2])**.** *Assume that $\bar{x} \in \mathbb{Y}$, and ξ_0 is an isolated point of $\sigma(\bar{x})$. Then, ξ_0 is a pole of first order of the complex function $g(\lambda) := R(\lambda, \bar{S})\bar{x}$.*

Definition 2.3. A sequence $\{x_n\}_n$ is said to be *asymptotically constant* if

$$\lim_{n\to\infty} [x(n+1) - x(n)] = 0.$$

Proposition 2.4 (see Minh–Matsunaga–Huy–Luong [17])**.** *The following assertions are valid:*

1. *Let $x \in l^\infty(\mathbb{X})$. Then, $\sigma(x) = \emptyset$ if and only if $x \in c_0(\mathbb{X})$;*
2. *Let $x \in l^\infty(\mathbb{X})$. Then $\sigma(x) \subset \{1\}$ if and only if x is asymptotically constant;*
3. *Let Λ be a closed subset of Γ, and $\mathbb{Y}_\Lambda := \{\bar{x} \in \mathbb{Y} : \sigma(\bar{x}) \subset \Lambda\}$. Then, $\mathbb{Y}_\Lambda$ is a closed subspace of $\mathbb{Y}$;*
4. *Let $\Lambda = \Lambda_1 \sqcup \Lambda_2$, where Λ_1, Λ_2 are disjoint closed subsets of Γ. Then, $\mathbb{Y}_\Lambda = Y_{\Lambda_1} \oplus Y_{\Lambda_2}$. Moreover, the projection associated with this direct sum commutes with the shift operator $\bar{S}$ and the operator $\bar{T}$.*

3. Main Results

In this section, we present the main results of the paper on spectral criteria for the asymptotic constancy of solutions to implicit difference equations in a Banach space. We start with the concepts of resolvent set and spectrum of a linear operators pencil (see Gohberg–Goldberg–Kaashoek [9, Chapter IV] and references therein for more information on the matter).

Definition 3.1. A linear operator pencil $\{C, T\}$, where C, $T \in \mathcal{L}(\mathbb{X})$, is said to be *regular* if there exists $\lambda \in \mathbb{C}$ such that the linear operator $\lambda C - T$ is invertible. We set

$$R(\lambda, C, T) := (\lambda C - T)^{-1}, \quad \text{for all } \lambda \in \rho(C, T). \tag{8}$$

The complement of such $\lambda \in \mathbb{C}$ is called the *spectrum*, and is denoted by $\sigma(C, T)$.

Lemma 3.2. *The following assertions are valid:*

(1) *The resolvent set*

$$\rho(C, T) := \left\{\lambda \in \mathbb{C} : \exists (\lambda C - T)^{-1},\ (\lambda C - T)^{-1} \in \mathcal{L}(\mathbb{X})\right\} \tag{9}$$

is an open subset of $\mathbb{C}$.

(2) *The function*

$$\rho(C, T) \ni \lambda \mapsto R(\lambda, C, T) := (\lambda C - T)^{-1} \in \mathcal{L}(\mathbb{X}) \tag{10}$$

is analytic.

Proof.

(1) We have

$$\lambda C - T = (\lambda - \lambda_0)C + (\lambda_0 C - T). \tag{11}$$

Therefore, if $\lambda_0 \in \rho(C, T)$, then $A := (\lambda_0 C - T)^{-1}$ exists as an element of $\mathcal{L}(\mathbb{X})$. It is easily seen that if $A \in \mathcal{L}(\mathbb{X})$ is invertible, then there exists an open neighborhood of A consisting of all invertible operators. That means, if $\lambda - \lambda_0$ is sufficiently small, $(\lambda - \lambda_0)C + (\lambda_0 C - T)$.

(2) Let $\lambda_0 \in \rho(C, T)$ and $\lambda \in B_\varepsilon(\lambda_0)$ with sufficiently small $\varepsilon > 0$. We have

$$\begin{aligned}
&(\lambda_0 C - T)^{-1}(\lambda C - T) \\
&= (\lambda - \lambda_0)(\lambda_0 C - T)^{-1}C + (\lambda_0 C - T)^{-1}(\lambda_0 C - T) \\
&= (\lambda - \lambda_0)(\lambda_0 C - T)^{-1}C + I.
\end{aligned}$$

Therefore, for sufficiently small ε, say,

$$\varepsilon < \frac{1}{\|(\lambda_0 C - T)^{-1}C\|},$$

by Neumann series formula, the operator

$$\begin{aligned}
&\left[(\lambda - \lambda_0)(\lambda_0 C - T)^{-1}C + I\right]^{-1} \\
&= \sum_{k=0}^{\infty} (-1)^k (\lambda - \lambda_0)^k (\lambda_0 C - T)^{-k} C^k.
\end{aligned}$$

That means,

$$\left[(\lambda_0 C - T)^{-1}(\lambda C - T)\right]^{-1} = (\lambda C - T)^{-1}(\lambda_0 C - T)$$

exists and is an analytic function of λ in a small neighborhood of $\lambda_0 \in \rho(C, T)$. Since $(\lambda_0 C - T)$ is invertible, this yields that $(\lambda C - T)^{-1}$ is well defined and analytic in a small neighborhood of λ_0.

The lemma is proved. □

Definition 3.3 (see Minh–Matsunaga–Huy–Luong [17, Definition 2.2])**.** Let $u = \{u(n)\}_n$ be a bounded sequence in $\mathbb{X}$. Then, u is said to be an *asymptotic solution* to (1) if

$$Cu(n+1) = Tu(n) + y(n) + \varepsilon(n), \quad n \in \mathbb{N},$$

where the sequence $\{\varepsilon(n)\}_n$ satisfies $\lim_{n\to\infty} \varepsilon(n) = 0$.

Lemma 3.4. *Let $x = \{x(n)\}_n$ be a bounded asymptotic solution to* (1) *and $y = \{y(n)\}_n$ is any bounded sequence. Then,*

$$\sigma(y) \subset \sigma(x) \subset \sigma(y) \cup \sigma_\Gamma(C,T). \tag{12}$$

Proof.

(1) We prove that

$$\sigma(x) \subset \sigma(y) \cup \sigma_\Gamma(C,T). \tag{13}$$

Applying the transformation (7) to (1), we have

$$R\left(\lambda, \bar{S}\right) \bar{C}\bar{S}\bar{x} = R\left(\lambda, \bar{S}\right) \bar{T}\bar{x} + R\left(\lambda, \bar{S}\right) \bar{y}, \qquad \text{for all } |\lambda| \neq 1. \tag{14}$$

Since $R\left(\lambda, \bar{S}\right) \bar{S}\bar{x} = \lambda R\left(\lambda, \bar{S}\right) \bar{x} - \bar{x}$, we obtain that

$$\lambda R\left(\lambda, \bar{S}\right) \bar{C}\bar{S}\bar{x} - \bar{C}\bar{x} = R\left(\lambda, \bar{S}\right) \bar{T}\bar{x} + R\left(\lambda, \bar{S}\right) \bar{y}. \tag{15}$$

Suppose that $\Gamma \ni \lambda_0 \notin \sigma(y) \cup \sigma_\Gamma(C,T)$. Then, $R\left(\lambda, \bar{C}, \bar{T}\right)$, by Lemma 3.2, and $R(\lambda, \bar{S})\bar{y}$ are extendable to an analytic function in a neighborhood of λ_0. Therefore,

$$\left(\lambda\bar{C} - \bar{T}\right) R\left(\lambda, \bar{S}\right) \bar{x} = \bar{C}\bar{x} + R\left(\lambda, \bar{S}\right) \bar{y}, \quad |\lambda| \neq 1, \tag{16}$$

thus,

$$R\left(\lambda, \bar{S}\right) \bar{x} = R\left(\lambda, \bar{C}, \bar{T}\right) \left(\bar{C}\bar{x} + R(\lambda, \bar{S})\bar{y}\right), \quad |\lambda| \neq 1. \tag{17}$$

Therefore, $R\left(\lambda, \bar{S}\right) \bar{x}$ is analytic in a neighborhood of λ_0, hence $\lambda_0 \notin \sigma(x)$. This proves (13).

(2) We now prove that

$$\sigma(y) \subset \sigma(x). \tag{18}$$

From (16), we have

$$\left(\lambda\bar{C} - \bar{T}\right) R\left(\lambda, \bar{S}\right) \bar{x} - \bar{C}\bar{x} = R\left(\lambda, \bar{S}\right) \bar{y}, \quad |\lambda| \neq 1. \tag{19}$$

If $R\left(\lambda, \bar{S}\right) \bar{x}$ an be extended to an analytic function in a neighborhood of $\lambda_0 \in \Gamma$, then so is the left-hand side of (19). Therefore, $R\left(\lambda, \bar{S}\right) \bar{y}$ is also extendable to an analytic function in a neighborhood of $\lambda_0 \in \Gamma$, that is, $\lambda_0 \notin \sigma(y)$. This implies $\sigma(y) \subset \sigma(x)$, finishing the proof.

The proof is completed. □

Remark 3.5. The part "$\sigma(x) \subset \sigma(y) \cup \sigma_\Gamma(C,T)$" in Lemma 3.4 is an analog to Naito–Minh–Shin [20, Lemma 3.2] for the implicit difference equations.

Below we will apply Lemma 3.4 to study the asymptotic constancy of solutions to implicit differece equation (1), where y is assumed to be a bounded sequence.

Theorem 3.6. *The necessary condition for* (1)*, where y is assumed to be a bounded sequence, to have an asymptotic constant solution is*

$$\sigma(y) \subset \{1\}. \tag{20}$$

That is y must be asymptotic constant itself.

Proof. Since Lemma 3.4, we have $\sigma(y) \subset \sigma(x)$. If x is an asymptotic constant, then $\sigma(x) \subset \{1\}$. This finishes the proof. □

Theorem 3.7. *Suppose that*

(1) $\{y(n)\}_n \subset l^\infty(\mathbb{X})$ *is asymptotically constant, and*
(2) $\sigma_\Gamma(C,T) \subset \{1\}$.

Then, every bounded asymptotic solution to (1) *is asymptotically constant.*

Proof. By Lemma 3.4, we have

$$\sigma(x) \subset \sigma(y) \cup \sigma_\Gamma(C,T) \subset \{1\}$$

for any asymptotic solution $x \in l^\infty(\mathbb{X})$ to (1). Therefore, x is asymptotically constant according to Proposition 2.4. □

Theorem 3.8. *Suppose that*

(1) *sequence* $\{y(n)\}_n \subset l^\infty(\mathbb{X})$ *is asymptotically constant;*
(2) *if either* $1 \notin \sigma_\Gamma(C,T)$*, or* 1 *is an isolated point of* $\sigma_\Gamma(C,T)$*;*
(3) *Equation* (2) *has a bounded asymptotic solution.*

Then, (2) *has an asymptotic solution that is asymptotically constant.*

Proof. Set $\Lambda_1 := \{1\}$ and $\Lambda_2 = \sigma_\Gamma(C,T)\backslash\{1\}$. Then Λ_1 and Λ_2 are both closed and disjoint by the assumption. By Proposition 2.4, we have

$$\bar{x} = \bar{x}_1 + \bar{x}_2, \tag{21}$$

where $\bar{x}_1 := P\bar{x}$, $\bar{x}_2 = (I-P)\bar{x}$, and P is the Riesz spectral project corresponding to the splitting $\mathbb{Y}_\Lambda = \mathbb{Y}_{\Lambda_1} \oplus \mathbb{Y}_{\Lambda_2}$. Since x is an asymptotic solution to (1),

$$CSx = Tx + y + \varepsilon, \quad \text{where } \varepsilon = \{\varepsilon(n)\}_n \in c_0(\mathbb{X}), \tag{22}$$

so

$$P\bar{C}\bar{S}\bar{x} = P\bar{T}\bar{x} + P\bar{y}. \tag{23}$$

As $\sigma(\bar{y}) \subset \{1\}$, we have

$$P\bar{C}\bar{S}\bar{x} = P\bar{T}\bar{x} + \bar{y}. \tag{24}$$

Therefore

$$\bar{C}\bar{S}P\bar{x} = \bar{T}P\bar{x} + \bar{y}$$

since P commutes with $\bar{S}$ and $\bar{T}$. If we choose z as a representative of $P\bar{x}$, then z satisfies the equation

$$Cz(n+1) = Tz(n) + y(n) + \varepsilon'(n), \tag{25}$$

where $\{\varepsilon'(n)\}_n \in c_0(\mathbb{X})$. Obviously, $\{z(n)\}_n$ is an asymptotic solution to (1), and is asymptotically constant since $\sigma(z) \subset \{1\}$. □

4. An Application to Evolution Equations Associated with C-Semigroups

This section presents an application to evolution equations associated with C-semigroups. The concept of C-semigroups introduced to approach a larger class of evolution equations which are ill-posed (see, e.g., [5, 30, 31] and the references therein for more information).

Definition 4.1. A family of bounded linear operators $(T(t))_{t\geq 0}$ from a Banach space $\mathbb{X}$ to itself is called 1-*periodic* C*-semigroup* if the following conditions are satisfied

(1) $T(0) = C$;
(2) $T(t+s) = T(t)T(s)C$ for all $t,\ s \geq 0$;
(3) for each $x \in \mathbb{X}$, the map $t \mapsto T(t)x$ is continuous;
(4) $T(t+1) = T(t)$ for all $t \geq 0$;
(5) $\|T(t)\| \leq Ne^{\omega t}$ for some positive constants N and ω independent of $t \geq 0$.

A function $f \in C(\mathbb{R}^+, \mathbb{X})$ is said to be *asymptotically* 1*-periodic* if

$$\lim_{t\to\infty} [f(t+1) - f(t)] = 0. \tag{26}$$

Consider the following linear 1-periodic evolution equation

$$x' = Ax + f(t), \quad t \geq 0, \tag{27}$$

where $x \in \mathbb{X}$, f is an asymptotically 1-periodic function. The homogeneous evolution equation

$$x' = Ax, \quad t \geq 0, \tag{28}$$

of Equation (27) is said to be C*-well-posed* if it generates a 1-periodic C-semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}$.

Definition 4.2. A function $u \in C(\mathbb{R}^+, \mathbb{X})$ is a *mild asymptotic solution* of Equation (27) if there is a function $\varepsilon \in C(\mathbb{R}^+, \mathbb{X})$ such that

$$\lim_{t\to\infty} \varepsilon(t) = 0, \tag{29}$$

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-\xi)[f(\xi) + \varepsilon(\xi)]\mathrm{d}\xi, \quad t \geq 0. \tag{30}$$

We denote P the monodromy operator $T(1)$ of the semigroup $(T(t))_{t\geq 0}$. As an application of Theorem 3.7 we have the following:

Theorem 4.3. *Assume that the homogeneous evolution equation* (28) *generates a* 1*-periodic C-semigroup $(T(t))_{t\geq 0}$ in a Banach space $\mathbb{X}$ such that*

$$\sigma_\Gamma(C, P) \subset \{1\}. \tag{31}$$

Then, every bounded asymptotic solution to (27) *satisfies*

$$\lim_{t\to\infty} \|C(u(t+1) - u(t))\| = 0. \tag{32}$$

Proof. If u is a mild asymptotic solution to (27), then

$$Cu(n+1) = T(1)u(n) + \int_n^{n+1} T(n+1-\xi)[f(\xi) + \varepsilon(\xi)]\mathrm{d}\xi. \tag{33}$$

Put

$$y(n) := \int_n^{n+1} T(n+1-\xi)[f(\xi) + \varepsilon(\xi)]\mathrm{d}\xi. \tag{34}$$

From the 1-periodicity of the C-semigroup $(T(t))_{t\geq 0}$ we have $T(1) = P$. Then, (33) becomes

$$Cu(n+1) = Pu(n) + y(n), \quad n \in \mathbb{N}. \tag{35}$$

We will show that $\{y(n)\}_n$ is asymptotically constant. By computing directly, we can obtain that

$$y(n+1) - y(n)$$

$$= \int_{n+1}^{n+2} T(n+2-\xi)[f(\xi)+\varepsilon(\xi)]\mathrm{d}\xi - \int_{n}^{n+1} T(n+1-\xi)[f(\xi)+\varepsilon(\xi)]\mathrm{d}\xi$$
$$= \int_{n}^{n+1} T(n+1-\xi)[f(\xi+1)+\varepsilon(\xi+1)-f(\xi)-\varepsilon(\xi)]\mathrm{d}\xi.$$

Since f is an asymptotically 1-periodic function and the limit $\lim_{t\to\infty}\varepsilon(t)=0$ it follows that
$$\lim_{n\to\infty}[y(n+1)-y(n)]=0.$$
In other words, $\{u(n)\}_n$ is an asymptotic solution to (35). Under the assumption in Theorem 4.3, all conditions of Theorem 3.7 are satisfied. Hence, every bounded asymptotic solution to (35) is asymptotic constant,
$$\lim_{n\to\infty}[u(n+1)-u(n)]=0. \tag{36}$$
Denoting $n=[t]$, the integer part of t, we have
$$\begin{aligned} &C(u(t+1)-u(t)) \\ &= T(t-n)[u(n+1)-u(n)] + \int_{n+1}^{t+1} T(t+1-\xi)[f(\xi)+\varepsilon(\xi)]\mathrm{d}\xi \\ &\quad - \int_{n}^{t} T(t-\xi)[f(\xi)+\varepsilon(\xi)]\mathrm{d}\xi \\ &= T(t-n)[u(n+1)-u(n)] \\ &\quad + \int_{n}^{t} T(t-\xi)[(f(\xi+1)+\varepsilon(\xi+1))-(f(\xi)+\varepsilon(\xi))]\mathrm{d}\xi. \end{aligned}$$
By Definition 4.1, we have the following estimate
$$\begin{aligned} &\|C(u(t+1)-u(t))\| \\ &\le Ne^{\omega}\|u(n+1)-u(n)\| + Ne^{\omega}\int_{n}^{t} \|f(\xi+1)+\varepsilon(\xi+1)-f(\xi)-\varepsilon(\xi)\|\mathrm{d}\xi. \end{aligned}$$
Since $\lim_{t\to\infty}\varepsilon(t)=0$, the asymptotical 1-periodicity of f and the asymptotic constancy of $\{u(n)\}_n$, it follows that
$$\lim_{t\to\infty}\|C(u(t+1)-u(t))\|=0.$$
The theorem is proved. □

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