On a generalization of Steiner formula and its application to the conjunction probability of smooth stationary Gaussian fields

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Abstract: In this paper, we provide an explicit formula to calculate the volume of n-tuples $(t_1, \dots, t_n)$ satisfying the non-empty intersection condition $B(t_1, r_1) \cap \ldots \cap B(t_n, r_n) \cap S \neq \emptyset$ for given small enough radii $r_1, \dots, r_n$ and a set S of positive reach in $\mathbb{R}^d$. This formula can be seen as a generalization of the celebrated Steiner formula. As a consequence, through this formula, we derive an asymptotic expansion for the conjunction probability of smooth stationary Gaussian fields.

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1. Introduction

Let S be a convex body in $\mathbb{R}^d$. For a given $\epsilon > 0$, the ϵ-neighborhood of S, denoted by $S^{+\epsilon}$, is defined as

$$S^{+\epsilon} = \{t \in \mathbb{R}^d : \ \mathrm{dist}(t, S) \leq \epsilon \ (or) \quad B(t, \epsilon) \cap S \neq \emptyset\}.$$

Then the volume of $S^{+\epsilon}$ is provided by the celebrated *Steiner formula* [7] as a polynomial of the variable ϵ,

$$\lambda_d \left(S^{+\epsilon}\right) = \sum_{j=0}^{S} \omega_{d-j} \mu_j(S) \epsilon^{d-j}, \tag{1}$$

where ω_{d-j} is the volume of a $(d-j)$-dimensional unit ball (with respect to the usual $(d-j)$-dimensional Lebesgue measure), and $\mu_j(S)$'s are the geometric functionals of S. In particular, one has:

- $\mu_d(S)$ is the volume of S,
- $\mu_{d-1}(S)$ is half of the surface area (volume) of S,
- and $\mu_0(S) = 1$.

This formula is extended to the class of sets *with positive reach* by Federer [6]. For a subset $S \in \mathbb{R}^d$, the reach of S is defined as the supremum of the parameter

ϵ such that for any point t in $S^{+\epsilon}$, it has a unique projection on S. For this class of sets, the Steiner formula (1) becomes the *Weyl formula* that is valid for any ϵ smaller than the reach of S. Remark that, in the Weyl formula, $\mu_0(S)$ is the Euler-Poincare characteristic of S; and it can obtain any integer values, not only 1 as in (1). For example, for a planar compact set $S \in \mathbb{R}^2$, its Euler-Poincare characteristic $\mu_0(S)$ equals to the number of connected components minus the number of holes inside. For further progress of sets with positive reach, we refer the survey [10].

An other extension of the Steiner formula is to replace the Euclidean space $\mathbb{R}^d$ by other space with its own geodesic distance structure. For instance, Allendoerfer [2] considered the general sphere $\mathcal{S}^d$ and used the Gauss-Bonner formula to derive the Steiner-type formula.

The geometric functionals $\mu_j(S)$'s are also called Minkowski functionals or Killing-Lipschitz curvartures. Besides the above definition through the Steiner-Weyl formula, they can defined by other approaches. Let us recall the approach from theory of Geometric probability (or Integral geometry) [8] as follows. First, in low dimension (a line or a plane), the the geometric functionals can be defined and calculated easily in an intuitive way. Then for higher dimension, they can be calculated inductively through the ones of the cross sections in lower dimensional. In fact, the *Crofton formula* states that

$$\int_{\mathrm{Graff}(d,k)} \mu_j(S \cap V^*) d\lambda_k^d(V^*) = \frac{\omega_m}{\omega_n \omega_{m-n}} \binom{m}{n} \mu_{d-k+j}(S), \tag{2}$$

where Graff(d,k) is the affine Grassmannian of all k-dimensional affine subspaces of $\mathbb{R}^d$, and the measure λ_k^d is the invariant measure on Graff(d,k) under the group of Euclidean motions.

Studying the Steiner formula and also the Minkowski functionals, as well as their generalizations plays a central role in Convex Geometry, Geometric Probability and other domains. It is surprising to see that one can find the appearance of this formula to derive the asymptotic expansion of the tail distribution of the maximum of stationary Gaussian fields $\{X(t): \ t \in S \subset \mathbb{R}^d\}$. Investigating this tail distribution is an interesting and challenging problem in Probability. Davies applied this distribution in practical statistical tests to determine the loci of gene. Worsley and Fristion applied in statistical tests to study the activations of human brains.

The first break through observation is provided by Sun [11]. Here the author assume that the random field has a finite Karhunen-Loeve expansion. It means that there exists the i.i.d random variables $Z_1, \dots, Z_k \sim \mathcal{N}(0,1)$ such that for any $t \in S$,

$$X(t) = \sum_{i=1}^k a_{t,i} Z_i = \langle a_t, Z \rangle,$$

with $a_{t,1}^2 + \ldots + a_{t,k}^2 = 1$. Then the statement $\langle a_t, Z \rangle \geq u$ is equivalent to $\langle a_t, U \rangle \geq \frac{u}{\|Z\|}$ for $U = \frac{Z}{\|Z\|} \sim Unif(\mathcal{S}^{k-1})$. Hence we obtain that $U \in S^{+\arccos(u/\|Z\|)}$

and we could apply the tubular Steiner formula on the sphere $\mathcal{S}^{k-1}$. See also [12].

A deeper connection is provided by Adler and Taylor [1] through the celebrated *Gaussian kinematic formula.* Here they considered the (random) excursion set for a given level u,

$$C_u = \{t \in S : \; X(t) \geq u\},$$

and calculated the expectation of the Euler characteristic of this set

$$\mathrm{E}(\mu_0(C_u)) = \sum_{i=0}^{d} \rho_i \mu_i(S), \tag{3}$$

where ρ_i's are the Euler characteristic densities defined as

$$\rho_0 = \overline{\Phi}(u) = \int_u^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx,$$

$$\rho_i = (2\pi)^{-(i+1)/2} H_{i-1}(u) e^{-u^2/2} = (2\pi)^{-i/2} H_{i-1}(u)\varphi(u), \forall i > 0,$$

with $\varphi(u) = e^{-u^2/2}/\sqrt{2\pi}$, and $H_j(x) = (-1)^n e^{\frac{x^2}{2}} \dfrac{d^j}{dx^j} e^{\frac{-x^2}{2}}$ is the Hermite polynomial of degree j. They also proved that for locally convex and tamed index set S, then this expectation can be used as a good approximation for the tail distribution of the maximum.

For planar non-convex index set S, Azais and Pham [3] proved that if S still has a Steiner-type expansion for the area of its ϵ-neighborhood, then one can derive the asymptotic formula for the tail distribution of the maximum with corresponding coefficients from the Steiner expansion. Their proof relies on a result of Azais and Wschebor [5] that for large threshold level u and with high probability, the shape of the excursion set with respect to level u is close to a ball centered at the unique (random) maximum point t_0 of the field with (random) radius r_0. Then by intuition, the maximum of the field indexed on S exceed the level u if and only if S has a non-empty intersection with this ball, or the maximum point t_0 is in the r_0-neighborhood of S.

In this paper, given a convex body S in $\mathbb{R}^d$ (or in general, a set with positive reach) and small enough radii $r_1, \dots, r_n$, we are interested in a formula for the volume of n-tuples $(t_1, \dots, t_n)$ satisfying the non-empty intersection condition

$$\lambda_{nd}\left((t_1, \dots, t_n) \in \mathbb{R}^{nd} : \bigcap_{1 \leq i \leq n} B(t_i, r_i) \cap S \neq \emptyset \right).$$

It is clear that for $n = 1$, we go back to the Steiner formula of the ϵ-neighborhood of S.

Our motivation to study this problem is from the *conjunction probability* problem. Now, consider n independent copies $\{X_i(t); i = 1, 2, \dots, n\}$ of the smooth stationary Gaussian field X. For a level u, the conjunction probability is defined as

$$\mathrm{P}\left(\exists t \in S : \; X_i(t) \geq u, \; \forall i \in \overline{1, n}\right). \tag{4}$$

or equivalently,

$$\mathrm{P}\left(\sup_{t\in S}\min_{1\le i\le n} X_i(t)\ge u\right). \tag{5}$$

This problem is provided by Worsley and Friston [16] to study the statistical tests for comparing the activations of the brains between two genders male and female. Again, by Euler characteristic method, they provided the expectation of the conjunction set

$$\mathrm{P}(C_u\neq\emptyset)=\mathrm{P}\left(\sup_{t\in S}\min_{1\le i\le n} X_i(t)\ge u\right)\approx \mathrm{E}(\mu_0(C_u))=(1,0,\dots,0)R^n\mu(S), \tag{6}$$

where $\mu(S)=(\mu_0(S)b_0,\mu_1(S)b_1,\dots,\mu_d(S)b_d)$ is the column vector of the scaled Minkowski functionals of S, with $b_i=\Gamma((i+1)/2)/\Gamma(1/2)$ and R an upper-triangular Toeplitz matrix defined as

$$R=\begin{pmatrix}\rho_0/b_0 & \rho_1/b_1 & \dots & \rho_d/b_d\\ 0 & \rho_0/b_0 & \dots & \rho_{d-1}/b_{d-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \rho_0/b_0\end{pmatrix}. \tag{7}$$

However, it seems hard to prove that this expectation is a good approximation for the conjunction probability.

Continuing the ideas in Azais and Pham [3] and Azais and Wschebor [5], Pham [9] considered the maximum points $t_i,\ i=1,\dots,n$ of the fields; and argued that if the conjunction set is non-empty then intuitively, one has

$$\underset{1\le i\le n}{\cap} B(t_i,r_i)\cap S\neq\emptyset.$$

This implies the connection between the conjunction probability problem and our generalized Steiner formula. Then the author proved that for a fixed point t_1 in the parameter space and for $r_1,r_2,\dots,r_n>0$ small enough,

$$\lambda_{(n-1)d}\left((t_2,\dots,t_n)\in\mathbb{R}^{d(n-1)}:\ \underset{1\le i\le n}{\cap} B(t_i,r_i)\neq\emptyset\right)=T(r_1,\dots,r_n)$$
$$=\sum_{k_n=0}^{d}\ \sum_{k_{n-1}=d-k_n}^{d}\cdots\sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d} r_1^{(n-1)d-\sum_{i=2}^n k_i}\times$$
$$\prod_{i=2}^{n}\left(r_i^{k_i}\omega_{k_i}\right)\frac{\omega_d\,\omega_{(n-1)d-\sum_{i=2}^n k_i}}{\omega_{\sum_{i=2}^n k_i-(n-2)d}\prod_{i=2}^n\omega_{d-k_i}}\times\frac{d!}{[\sum_{i=2}^n k_i-(n-2)d]!.\prod_{i=2}^n(d-k_i)!}. \tag{8}$$

and

$$\lambda_{nd}\left((t_1,\dots,t_n)\in\mathbb{R}^{nd}:\ \underset{1\le i\le n}{\cap} B(t_i,r_i)\cap S\neq\emptyset\right)=\lambda_d(S)T(r_1,\dots,r_n)+O\Big(\sum_{\|\mathbf{m}\|=k+1}\mathbf{r}^{\mathbf{m}}\Big). \tag{9}$$

From this volume expansion, the author derived an one-term asymptotic formula for the conjunction probability of smooth Gaussian fields. This asymptotic formula coincides with the the first term of the heuristic approximation given by Worsley and Friston.

In this paper, we would like to develop the above results. Our main result is the following generalized Steiner formula.

Theorem 1.1. *Let S be a convex body in $\mathbb{R}^d$. Then for every $r_1, \ldots, r_n$ we have*

$$\begin{aligned}&\lambda_{nd}((t_1, \ldots, t_n): \ B(t_1, r_1) \cap \ldots \cap B(t_n, r_n) \cap S \neq \emptyset) \\ &= \sum_{k_n=0}^{d} \sum_{k_{n-1}=d-k_n}^{d} \cdots \sum_{k_1=(n-1)d-(k_n+k_{n-1}+\ldots+k_2)}^{d} \mu_{nd-\sum k_i}(S) \times \\ &\prod_{i=1}^{n} \left(r_i^{k_i} \omega_{k_i} \begin{bmatrix} (d-k_n) + \ldots + (d-k_i) \\ d-k_i \end{bmatrix} \right).\end{aligned}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{\omega_m}{\omega_n \omega_{m-n}} \binom{m}{n}.$$

The detailed proof of the main theorem is presented in Section 2. The proof relies on the classic Steiner-Weyl formula and Crofton formula.

2. Proof of the main theorem

The following lemma is the key ingredient of our proof of the main theorem. We believe that it has its own interest.

Lemma 2.1. *Let T be a convex body in $\mathbb{R}^d$. Then for any r and $i = 0, 1 \ldots, d$,*

$$\int_{\mathbb{R}^d} \mu_{d-i}(T \cap B(t, r)) dt = \sum_{k=d-i}^{d} r^k \omega_k \begin{bmatrix} d-i+d-k \\ d-k \end{bmatrix} \mu_{d-i-(d-k)}(T).$$

Proof. By Crofton formula,

$$\mu_{d-i}((T \cap B(t, r))) = \int_{\mathrm{Graff}(d,i)} \mu_0 \left(T \cap B(t, r) \cap V*\right) d\lambda_i^d(V*).$$

Given a k-dimensional affine subspaces $V^* \in \mathrm{Graff}(d, k)$. One can define its corresponding couple (V, p) where V is a translation of V^* to be a linear subspace containing the origin; and p is the intersection point between V^* and $V^{*,\perp} = V^\perp$ the maximal linear subspace of $\mathbb{R}^d$ orthogonal to V^* and containing the origin.

Then we can rewrite the above integral as follows

$$\int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mu_0 \left(T \cap B(t,r) \cap (V+p)\right) dp$$
$$= \int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mathbb{I}_{\left\{p \in T \cap B(t,r)|_{V^\perp}\right\}} dp$$

By the Fubini theorem,

$$\int_{\mathbb{R}^d} \mu_{d-i}(T \cap B(t,r)) dt$$
$$= \int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mathbb{I}_{\{p \in T|_{V^\perp}\}} dp \int_{\mathbb{R}^d} \mathbb{I}_{\{B(t,r) \cap (T \cap (V+p)) \neq \emptyset\}} dt$$
$$= \int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mathbb{I}_{\{p \in T|_{V^\perp}\}} dp \lambda_d \left(T \cap (V+p)\right)^{+r}$$

It is clear that $T \cap (V+p)$ is an i-dimensional convex set, then apply the Weyl formula for the volume of the tube around the d-dimensional space, we have

$$\int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mathbb{I}_{\{p \in T|_{V^\perp}\}} \sum_{k=d-i}^{d} r^k \omega_k \mu_{d-k} \left(T \cap (V+p)\right) dp$$
$$= \sum_{k=d-i}^{d} r^k \omega_k \int_{\mathrm{Gr}(d,i)} d\nu_i^d(V) \int_{V^\perp} \mu_{d-k} \left(T \cap (V+p)\right) dp.$$

Here we use again the Crofton formula to complete the proof.

□

Now we are ready to present the detailed **proof the main theorem**.

The considering volume can be rewritten as

$$I = \lambda_{nd} \left((t_1, \ldots, t_n) \in \mathbb{R}^{dn} : \bigcap_{1 \le i \le n} B(t_i, r_i) \cap S \neq \emptyset \right)$$
$$= \int_{\mathbb{R}^{(n-1)d}} \mathbb{I}_{\{S \cap \bigcap_{1 \le i \le n-1} B(t_i,r_i) \neq \emptyset\}} dt_1 \ldots dt_{n-1} \int_{\mathbb{R}^d} \mathbb{I}_{\left\{B(t_n,r_n) \cap \left(S \cap \bigcap_{1 \le i \le n-1} B(t_i,r_i)\right) \neq \emptyset\right\}} dt_n$$
$$= \int_{\mathbb{R}^{(n-1)d}} \mathbb{I}_{\{S \cap \bigcap_{1 \le i \le n-1} B(t_i,r_i) \neq \emptyset\}} \lambda_d \left(S \cap \left(\bigcap_{1 \le i \le n-1} B(t_i, r_i) \right)^{+r_n} \right) dt_1 \ldots dt_{n-1}$$

By Steiner-Weyl tube formula,

$$\lambda_d \left(S \cap \left(\bigcap_{1 \le i \le n-1} B(t_i, r_i) \right)^{+r_n} \right) = \sum_{k_n=0}^{d} r_n^{k_n} \omega_{k_n} \mu_{d-k_n} \left(S \cap \bigcap_{1 \le i \le n-1} B(t_i, r_i) \right).$$

Therefore

$$I = \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n} \int_{\mathbb{R}^{(n-1)d}} \mu_{d-k_n}\left(S \cap \underset{1\le i\le n-1}{\cap} B(t_i,r_i)\right) dt_1\dots dt_{n-1}$$
$$= \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n} \int_{\mathbb{R}^{(n-2)d}} dt_1\dots dt_{n-2}\int_{\mathbb{R}^d} \mu_{d-k_n}\left(S \underset{1\le i\le n-2}{\cap} B(t_i,r_i)\cap B(t_{n-1},r_{n-1})\right) dt_{n-1}.$$

We can apply Lemma 2.1 for $t = t_{n-1}$ and $T = S \cap \underset{1\le i\le n-2}{\cap} B(t_i, r_i)$ to obtain that

$$I = \sum_{k_n=0}^{d} r_n^{k_n}\omega_{k_n} \sum_{k_{n-1}=d-k_n}^{d} r_{n-1}^{k_{n-1}}\omega_{k_{n-1}} \begin{bmatrix} d-k_n+d-k_{n-1} \\ d-k_{n-1}\end{bmatrix}$$
$$\int_{\mathbb{R}^{(n-3)d}} \mu_{d-(k_n+k_{n-1}-d)}\left(S \cap \underset{1\le i\le n-2}{\cap} B(t_i,r_i)\right) dt_1\dots dt_{n-2}.$$

It means that each time we reduces the number of variables to integrate. Then using this argument repeatedly until the last time for $t = t_1$ and $T = S$,

$$I = \sum_{k_n=0}^{d}\sum_{k_{n-1}=d-k_n}^{d}\cdots\sum_{k_1=(n-1)d-(k_n+k_{n-1}+\ldots+k_2)}^{d} \mathcal{L}_{nd-\sum k_i}(S)\times$$
$$\prod_{i=1}^{n}\left(r_i^{k_i}\omega_{k_i}\begin{bmatrix}(d-k_n)+\ldots+(d-k_i) \\ d-k_i\end{bmatrix}\right).$$

The proof completes.

Remark. By checking the proof carefully, the key ingredients are Steiner-Weyl and Crofton formulas, and a trick by using the Fubini theorem. Then we can also state the same result for a subset S of $\mathbb{R}^d$ with positive reach r_0. Then in this case we have to restrict the radii $r_1, \dots, r_n$ smaller than r_0 to apply the Steiner-Weyl formula. In the original proof, we use the fact that the intersection of two convex sets (S and a ball, or S and a linear space) is also a convex set. To apply for the positive reach case, we need the same property. It is proven to be true in [10].

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