RIGIDITY PROPERTIES OF P-BIHARMONIC MAPS AND P-BIHARMONIC SUBMANIFOLDS

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Abstract

We give some rigidity properties of a p-biharmonic map $u : (M, g) \to (N, h)$ between Riemannian manifolds (M^n, g) and (N^m, h). We first provide various sufficient conditions for p-biharmonic maps to be harmonic. Moreover, when the map u is an isometric immersion, by assuming that the $L^{\frac{n}{2}}$-norm of the sectional curvature on M is sufficiently small or if the fundamental tone of the p-biharmonic submanifold is sufficiently big, it is proved that M is minimal.

1 Introduction and results

Let $u : (M^n, g) \to (N^m, h)$ be a smooth map between Riemannian manifolds (M^n, g) and (N^m, h). The differential du can be considered as a section of the vector bundle $T^*M \otimes u^{-1}TN$. Given a local orthonormal frame $\{e_i\}$ on M, $|du|$ can be computed as

$$|du|^2 = \sum_{i=1}^{n} \langle du(e_i), du(e_i) \rangle,$$

where $|du|(x)$ is the Hilbert-Schmidt norm of $(du)(x)$ induced by the metrics g and h at a given point $x \in M$. If the map u is a critical point of the following energy functional

$$E(u) = \frac{1}{2} \int_M |du|^2$$

then we call the map u *harmonic*. It is well-known that the Euler-Lagrange equation [CL16] of the energy E is given by

$$\tau(u) := \sum_{i=1}^{n} \left[\widetilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \right] = 0,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on the pullback bundle $u^{-1}TN$ and $\tau(u)$ is called the *tension field* of u.

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On the other hand, in the study of higher-order elliptic problems, it is natural to consider the biharmonic maps which is the critical point of the bienergy functional

$$E_2(u) = \frac{1}{2}\int_M |\tau(u)|^2.$$

We note that the Euler-Lagrange equation [CL16] of $E_2(u)$ is given by

$$\tau_2(u) := \Delta\tau(u) - \sum_{i=1}^{n} R^N\left(\tau(u), du(e_i)\right) du(e_i) = 0.$$

In a general way, Hornung-Moser [HM14] (see also [HF14]) considered the p-bienergy ($p > 1$) functional as follows:

$$E_p(u) = \int_M |\tau(u)|^p.$$

The *p-bitension field* $\tau_p(u)$ is defined by

$$\tau_p(u) := \Delta\left(|\tau(u)|^{p-2}\tau(u)\right) - \sum_{i=1}^{n} R^N\left(|\tau(u)|^{p-2}\tau(u), du(e_i)\right) du(e_i).$$

The Euler-Lagrange equation for $E_p(u)$ is given by $\tau_p(u) = 0$ and a map u satisfying that $\tau_p(u) = 0$ is called a *p-biharmonic* map. One of the most interesting problems in the biharmonic theory is the following problem, which was proposed by Chen in 1988:

Conjecture (Chen's conjecture)**.** Any biharmonic submanifold in Euclidean space $\mathbb{R}^n$ is minimal.

More generally, Caddeo-Montaldo-Oniciuc [CMO01] proposed the generalized Chen's conjecture as follows.

Conjecture (Generalized Chen's conjecture)**.** Any biharmonic submanifold in a Riemannian manifold with nonpositive sectional curvature is minimal.

Chen's and the generalized Chen's conjectures have been intensively studied. For example, Chen [Chen91] and Jiang [Jia87] showed that Chen's conjecture is true for biharmonic surfaces in $\mathbb{R}^3$. Hasanis-Vlachos [HV95] and Fu-Hong-Zhan [FHZ21] gave an affirmative answer to Chen's conjecture in $\mathbb{R}^4$ and $\mathbb{R}^5$, respectively. Moreover, Ou-Tang [OT12] showed that the generalized Chen's conjecture is false. However it is interesting to find sufficient conditions for biharmonic submanifolds to be minimal. Nakauchi-Urakawa [NU11, NU13] proved the generalized Chen's conjecture when the L^2-norm of the mean curvature vector is finite. Motivated by this result, Luo [Luo15] extended their result under assumption on finiteness of the L^p-norm of the mean curvature vector for some $0 < p < \infty$. Furthermore, Nakauchi-Urakawa-Gudmundsson [NUG14] showed that the map is harmonic if the energy and bienergy of the domain manifold are finite and if the curvature of the target manifold is nonpositive. Recently, Seo-Yun [SY22] studied biharmonic maps and biharmonic submanifolds with small curvature integral. By assuming the domain manifold satisfies a Sobolev inequality, they gave many sufficient conditions for biharmonic maps to be harmonic and for biharmonic submanifolds to be minimal. We refer the readers to [AM13, BMO10, CMO01, CMO01b, CMO02, Chen91, Chen96, CM13, Def98, DIM92, Fu14, Jia86, Luo14, MAE14, NUG14, ONI02, OU10] for further discussion in this field. On the other direction, motivated by Chen's conjecture, Han [Han15] proposed the following conjecture for p-biharmonic submanifolds.

Conjecture. Every complete p-biharmonic submanifolds in non-positively curved Riemannian manifold is minimal.

By using the method developed in [Luo15], Han [Han15] proved several results on the nonexistence of p-biharmonic submanifolds. Cao-Luo [CL16] studied the nonexistence result for general p-biharmonic submanifolds. Moreover, Han-Zhang [HZ15] investigated p-biharmonic maps and obtained the harmonicity of the biharmonic maps. They also obtained that any weakly convex p-biharmonic hypersurfaces in space form $N(c)$ with $c \leq 0$ is minimal. Inspired by these investigations, our aim in this paper is to study p-biharmonic maps and p-biharmonic submanifolds with small curvature integral. Firstly, if $L^{\frac{n}{2}}$-norm of $|R^N \circ u| \cdot |du|^2$ is sufficiently small, then we obtain the harmonicity of the p-biharmonic map as follows.

Theorem 1.1. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\||R^N \circ u| \cdot |du|^2\|_{L^{\frac{n}{2}}(M)} := \left(\int_M (|R^N \circ u| \cdot |du|^2)^{\frac{n}{2}} \right)^{\frac{2}{n}} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

We note that Theorem 1.1 recovers Theorem 2.2 and Theorem 2.3 in [SY22] (see Corollaries 3.1 and 3.3). When the product of L^n-norm of $|du|^2$ and L^n-norm of $|R^N \circ u|$ is sufficiently small, we get the harmonicity of the p-biharmonic map as follows.

Theorem 1.2. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|R^N \circ u\|_{L^n(M)} \||du|^2\|_{L^n(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Let Σ be a complete noncompact Riemannian manifold. Denote by $\lambda_1(\Omega)$ the first positive eigenvalue of the following eigenvalue problem for a bounded domain $\Omega \subset \Sigma$

$$\begin{cases} \Delta f + \lambda f = 0 \ \text{ in } \ \Omega, \\ f = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where Δ denotes the Laplace-Beltrami operator on Σ. Then the *fundamental tone* $\lambda_1(\Sigma)$ is defined by

$$\lambda_1(\Sigma) := \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all bounded domains in Σ. Replacing the condition on the Sobolev inequality by the condition on the fundamental tone of the domain manifold, we obtain a similar result as follows.

Theorem 1.3. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|R^N \circ u| \cdot |du|^2 \leq K$ for some constant $K > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2 K}{4(p-1)(Q+1-p)}$. Then u is harmonic.*

In particular, if $|du| \leq C$, $|R^N \circ u| \leq D$ and $p = Q = 2$, then we are able to obtain Theorem 2.4 in [SY22] (see Corollary 3.6). On the other hand, when the Weyl curvature tensor of the target manifold $W^N = 0$, we have the following result.

THEOREM 1.4. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\||Z^N \circ u| \cdot |du|^2\|_{L^{\frac{n}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

In the same way as Theorem 1.2, we have the following theorem.

THEOREM 1.5. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^n(M)} \||du|^2\|_{L^n(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Replacing the Sobolev inequality condition by a certain condition on the fundamental tone of the domain manifold M, we obtain a rigidity result for p-biharmonic as follows.

THEOREM 1.6. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$, $|Z^N \circ u| \cdot |du|^2 \leq K$ for some constant $K > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that $\lambda_1(M) > \frac{Q^2 K}{(m-2)(p-1)(Q+1-p)}$. Then u is harmonic.*

Recall that, if $u : (M^n, g) \to (N^m, h)$ is a p-biharmonic isometric immersion, then the map u is called *p-biharmonic*. Moreover, any 2-biharmonic submanifolds are called *biharmonic* simply. Using some condition on small integral of curvature and assuming that the Sobolev inequality (3.1) holds on M, we are able to prove the following rigidity result.

THEOREM 1.7. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic isometric immersion of a complete noncompact submanifold M into a Riemannian manifold N satisfying that $\int_M |\overrightarrow{H}|^Q < \infty$ for some constant $Q > p-1$, where $\overrightarrow{H}$ denotes the mean curvature vector field. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$||R^N \circ u||_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is minimal.

It should be mentioned that our approach is slight different from [SY22], where Seo-Yun considered two cases $m = n+1$ and $m > n+1$ separately. However, our approach here is applicable to both cases. Instead of using the Sobolev inequality, we obtain a similar result by using the fundamental tone of the domain manifold as follows.

THEOREM 1.8. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic isometric immersion of a complete noncompact submanifold M into a Riemannian manifold N satisfying that $|R^N \circ u| \leq K$ for some constant $K > 0$, and $\int_M |\overrightarrow{H}|^Q < \infty$ for some constant $Q > p-1$, where $\overrightarrow{H}$ denotes the mean curvature vector field. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2 K}{4(p-1)(Q+1-p)}$. Then u is minimal.*

Finally, if the target manifold (N^{n+1}, h) is Einstein, i.e., $\mathrm{Ric}^{\mathrm{N}} = \frac{scal^N}{n+1} h$, then we obtain the following rigidity property.

THEOREM 1.9. *Let $u:(M^n,g)\to(N^{n+1},h)$ be a p-biharmonic isometric immersion of a complete noncompact hypersurface M into an Einstein manifold N with nonnegative constant scalar curvature S^N. Assume that $\int_M |\overrightarrow{H}|^Q<\infty$ for some constant $Q>p-1$ and $\lambda_1(M)>\frac{S^NQ^2}{4(n+1)(p-1)(Q+1-p)}$, where $\overrightarrow{H}$ denotes the mean curvature vector field. Then u is minimal.*

The rest of this paper is organized as follows: In Section 2, we recall some preliminary background of p-biharmonic maps. In Section 3, we prove many results for p-biharmonic maps with small curvature integral. Finally, the rigidity results of p-biharmonic submanifolds are proved in Section 4.

2 Preliminaries

Let $u:(M^n,g)\to(N^m,h)$ be a p-biharmonic map between Riemannian manifolds (M^n,g) and (N^m,h). Then the Bochner-Weitzenböck formula for $\tau(u)$ is given by

$$\frac{1}{2}\Delta|\tau(u)|^2=|\nabla\tau(u)|^2+\langle\Delta\tau(u),\tau(u)\rangle.$$

Therefore

$$\frac{1}{2}\Delta|\tau(u)|^{2(p-1)}=|\nabla\left(|\tau(u)|^{p-2}\tau(u)\right)|^2+\langle\Delta\left(|\tau(u)|^{p-2}\tau(u)\right),|\tau(u)|^{p-2}\tau(u)\rangle.$$

Since $\tau_p(u)=0$, we have

$$\begin{aligned}\frac{1}{2}\Delta|\tau(u)|^{2(p-1)}=&|\nabla\left(|\tau(u)|^{p-2}\tau(u)\right)|^2\\&+\langle\sum_{i=1}^n R^N\left(|\tau(u)|^{p-2}\tau(u),du(e_i)\right)du(e_i),|\tau(u)|^{p-2}\tau(u)\rangle.\end{aligned}\tag{2.1}$$

By the fact that $\Delta(fg)=f\Delta g+g\Delta f+2\langle\nabla f,\nabla g\rangle$ and the Kato inequality, the above inequality yields

$$\begin{aligned}|\tau(u)|^{p-1}\Delta|\tau(u)|^{p-1}&\geq\langle\sum_{i=1}^n R^N\left(|\tau(u)|^{p-2}\tau(u),du(e_i)\right)du(e_i),|\tau(u)|^{p-2}\tau(u)\rangle\\&\geq-|R^N\circ u||du|^2|\tau(u)|^{2(p-1)}.\end{aligned}$$

Therefore we have

$$|\tau(u)|\Delta|\tau(u)|^{p-1}\geq-|R^N\circ u||du|^2|\tau(u)|^p.$$

Fix a point $x_0\in M$. Denote $r(x)$ by the geodesic distance on M from x_0 to x. Choose $\varphi\in C_0^\infty(M)$ satisfying for $r>0$,

$$\varphi(x)=\begin{cases}1, & \text{if } r(x)\leq r\\ \in[0,1] \text{ and } |\nabla\varphi|(x)\leq\dfrac{2}{r}, & \text{if } r<r(x)\leq 2r\\ 0, & \text{if } r(x)>2r.\end{cases}\tag{2.2}$$

The above inequality yields

$$\int_M\varphi^2|\tau(u)|^{q+1}\Delta|\tau(u)|^{p-1}\geq-\int_M\varphi^2|R^N\circ u||du|^2|\tau(u)|^{p+q}.$$

Applying integration by parts for the term in the left hand side gives

$$\int_M\varphi^2|\tau(u)|^{q+1}\Delta|\tau(u)|^{p-1}=-\int_M\langle\nabla(\varphi^2|\tau(u)|^{q+1}),\nabla|\tau(u)|^{p-1}\rangle.$$

Therefore the above inequality yields

$$\int_M \varphi^2|R^N\circ u||du|^2|\tau(u)|^{p+q} \geq 2(p-1)\int_M \varphi|\tau(u)|^{p+q-1}\langle\nabla\varphi,\nabla|\tau(u)|\rangle + (p-1)(q+1)\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2.$$

Since

$$2\int_M \varphi|\tau(u)|^{p+q-1}\langle\nabla\varphi,\nabla|\tau(u)|\rangle \geq -\delta\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2 - \frac{1}{\delta}\int_M |\tau(u)|^{p+q}|\nabla\varphi|^2$$

for all $\delta > 0$, we have

$$[(p-1)(q+1)-\delta(p-1)]\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2 \leq \frac{p-1}{\delta}\int_M |\tau(u)|^{p+q}|\nabla\varphi|^2 + \int_M \varphi^2|R^N\circ u||du|^2|\tau(u)|^{p+q}. \tag{2.3}$$

3 Rigidity results for p-biharmonic maps

In this section, we study p-biharmonic maps under the condition that the domain manifold (M^n, g) satisfies the following Sobolev inequality:

$$\left(\int_M \varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s\int_M |\nabla\varphi|^2 \text{ for all } \varphi\in C_0^1(M). \tag{3.1}$$

Proof of Theorem 1.1. By the Hölder inequality and (3.1), we have

$$\begin{aligned}
&\int_M \varphi^2|R^N\circ u||du|^2|\tau(u)|^{p+q}\\
&\leq \left[\int_M \left(|R^N\circ u|\cdot|du|^2\right)^{\frac{n}{2}}\right]^{\frac{2}{n}}\left[\int_M \left(\varphi^2|\tau(u)|^{p+q}\right)^{\frac{n}{n-2}}\right]^{\frac{n-2}{n}}\\
&\leq C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}\int_M |\nabla\left(\varphi|\tau(u)|^{\frac{p+q}{2}}\right)|^2\\
&= C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}\left[\frac{(p+q)^2}{4}\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2 + \int_M |\tau(u)|^{p+q}|\nabla\varphi|^2\right]\\
&\quad + C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}(p+q)\int_M \varphi|\tau(u)|^{p+q-1}\langle\nabla|\tau(u)|,\nabla\varphi\rangle\\
&\leq C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}\left(\frac{(p+q)^2}{4}+\frac{\alpha(p+q)}{2}\right)\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2\\
&\quad + C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}\left(1+\frac{p+q}{2\alpha}\right)\int_M |\tau(u)|^{p+q}|\nabla\varphi|^2
\end{aligned}$$

for all $\alpha > 0$. Here we used the Cauchy-Schwarz inequality in the last inequality. Combining the above inequality and (2.3), we obtain

$$A\int_M \varphi^2|\tau(u)|^{Q-2}|\nabla|\tau(u)||^2 \leq B\int_M |\tau(u)|^Q|\nabla\varphi|^2, \tag{3.2}$$

where the constants A, B, and Q are defined by

$$A = (p-1)(Q+1-p) - \delta(p-1) - C_s\||R^N\circ u|\cdot|du|^2\|_{L^{\frac{n}{2}}(M)}\left(\frac{Q^2}{4}+\frac{\alpha Q}{2}\right),$$

$$B = \frac{p-1}{\delta} + C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{Q}{2\alpha}\right),$$
$$Q = p + q.$$

Moreover, since $\| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s}$ by our assumption, we have

$$(p-1)(Q+1-p) - C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \cdot \frac{Q^2}{4} > 0.$$

Thus, for δ and α small enough, we see that $A > 0$. Therefore (3.2) yields

$$\begin{aligned} A \int_{B_{x_0}(r)} |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 &\leq A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \\ &\leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2 \\ &\leq \frac{4B}{r^2} \int_{B_{x_0}(2r)} |\tau(u)|^Q. \end{aligned}$$

Letting r tend to ∞, we see that $|\tau(u)|^{Q-2} ||\nabla |\tau(u)||^2 = 0$ on M, which implies that $|\tau(u)| = $ constant. Since the volume of M is infinite, we conclude that $\tau(u) = 0$. □

By the proof of Theorem 1.1, when $L^{\frac{n}{2}}$-norm of the sectional curvature of the image $u(M) \subset N$ is sufficiently small or L^n-norm of $|du|$ is sufficiently small, we can obtain two consequences as follows.

Corollary 3.1. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|du| \leq C$ for some constant $C > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$||R^N \circ u||_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Corollary 3.2. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h), with $|R^N \circ u| \leq D$ for some constant $D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$||du||^2_{L^n(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 D C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By the Hölder inequality and (3.1), we have

$$\begin{aligned} &\int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q} \\ &\leq \left[\int_M \left(|R^N \circ u|\right)^n\right]^{\frac{1}{n}} \left[\int_M \left(|du|^2\right)^n\right]^{\frac{1}{n}} \left[\int_M \left(\varphi^2 |\tau(u)|^{p+q}\right)^{\frac{n}{n-2}}\right]^{\frac{n-2}{n}} \\ &\leq C_s \|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \int_M |\nabla \left(\varphi |\tau(u)|^{\frac{p+q}{2}}\right)|^2 \\ &= C_s \|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \left[\frac{(p+q)^2}{4} \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 + \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2\right] \end{aligned}$$

$$
\begin{aligned}
&+ C_s\|R^N\circ u\|_{L^n(M)}\||du|^2\|_{L^n(M)}\int_M \varphi|\tau(u)|^{p+q-1}\langle\nabla|\tau(u)|,\nabla\varphi\rangle\\
&\le C_s\|R^N\circ u\|_{L^n(M)}\||du|^2\|_{L^n(M)}\left(\frac{(p+q)^2}{4}+\frac{\alpha(p+q)}{2}\right)\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2\\
&+ C_s\|R^N\circ u\|_{L^n(M)}\||du|^2\|_{L^n(M)}\left(1+\frac{p+q}{2\alpha}\right)\int_M |\tau(u)|^{p+q}|\nabla\varphi|^2
\end{aligned}
$$

for all $\alpha > 0$. Therefore (2.3) yields

$$
A\int_M \varphi^2|\tau(u)|^{Q-2}|\nabla|\tau(u)||^2 \le B\int_M |\tau(u)|^Q|\nabla\varphi|^2,
$$

where the constants A, B, and Q are defined by

$$
\begin{aligned}
&A=(p-1)(Q+1-p)-\delta(p-1)-C_s\|R^N\circ u\|_{L^n(M)}\cdot\||du|^2\|_{L^n(M)}\left(\frac{Q^2}{4}+\frac{\alpha Q}{2}\right),\\
&B=\frac{p-1}{\delta}+C_s\|R^N\circ u\|_{L^n(M)}.\||du|^2\|_{L^n(M)}\left(1+\frac{Q}{2\alpha}\right),\\
&Q=p+q.
\end{aligned}
$$

In the same manner as in the proof of Theorem 1.1, we get the conclusion. □

In particular, when $p=Q=2$, we obtain the following harmonicity of the biharmonic.

Corollary 3.3. *Let $u:(M^n,g)\to(N^m,h)$ be a biharmonic map from a complete noncompact Riemannian manifold (M^n,g) into a Riemannian manifold (N^m,h) with $\int_M|\tau(u)|^2<\infty$. Assume that (M^n,g) satisfies the Sobolev inequality (3.1) and*

$$
\|R^N\circ u\|_{L^n(M)}\||du|^2\|_{L^n(M)}<\frac{1}{C_s},
$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Now we give a proof of Theorem 1.3.

Proof of Theorem 1.3. By the assumption $|R^N\circ u|\cdot|du|^2\le K$, we have

$$
\begin{aligned}
&\int_M \varphi^2|R^N\circ u|\cdot|du|^2|\tau(u)|^{p+q}\\
&\le K\int_M \varphi^2|\tau(u)|^{p+q}\\
&\le \frac{K}{\lambda_1(M)}\int_M |\nabla\left(\varphi|\tau(u)|^{\frac{p+q}{2}}\right)|^2\\
&\le \frac{K}{\lambda_1(M)}\left[\frac{(p+q)^2}{4}\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2+\int_M|\tau(u)|^{p+q}|\nabla\varphi|^2\right]\\
&\quad+\frac{K}{\lambda_1(M)}(p+q)\int_M\varphi|\tau(u)|^{p+q-1}\langle\nabla|\tau(u)|,\nabla\varphi\rangle\\
&\le \frac{K}{\lambda_1(M)}\left(\frac{(p+q)^2}{4}+\frac{\alpha(p+q)}{2}\right)\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2\\
&\quad+\frac{K}{\lambda_1(M)}\left(1+\frac{p+q}{2\alpha}\right)\int_M|\tau(u)|^{p+q}|\nabla\varphi|^2
\end{aligned}
$$

for all $\alpha>0$. Hence (2.3) yields

$$
A\int_M \varphi^2|\tau(u)|^{p+q-2}|\nabla|\tau(u)||^2 \le B\int_M |\tau(u)|^{p+q}|\nabla\varphi|^2,
$$

where the constants A and B are given by

$$A = (p-1)(q+1) - \delta(p-1) - \frac{K}{\lambda_1(M)}\left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2}\right),$$
$$B = \frac{p-1}{\delta} + \frac{K}{\lambda_1(M)}\left(1 + \frac{p+q}{2\alpha}\right).$$

By repeating the arguments in Theorem 1.1, we get the conclusion. □

Using Theorem 1.3, we have the following.

COROLLARY 3.4. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|du| \le C$, $|R^N \circ u| \le D$ for some constants $C > 0, D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2C^2D}{4(p-1)(Q+1-p)}$. Then u is harmonic.*

On the other hand, if the Weyl curvature tensor W^N vanishes, then we have the following decomposition

$$R^N = \frac{S^N}{2m(m-1)} h \owedge h + \frac{1}{2} Z^N \owedge h,$$

where S^N denotes the scalar curvature and $Z^N = \mathrm{Ric}^N - \frac{S^N}{m}h$ denotes the traceless Ricci tensor of (N^m, h). Thus we have

$$\begin{aligned}
&\sum_{i=1}^n \langle R^N(du(e_i), \tau(u))du(e_i), \tau(u)\rangle \\
=&\frac{S^N}{m(m-1)}\left(|du|^2|\tau(u)|^2 - \sum_{i=1}^n \langle du(e_i), \tau(u)\rangle_h\right) \\
&+ \frac{1}{m-2}\left(\sum_{i=1}^n Z^N(du(e_i), du(e_i)|\tau(u)|^2 + Z^N(\tau(u), \tau(u)|du|^2\right) \\
&- \frac{2}{m-2}\left(\sum_{i=1}^n Z^N(du(e_i), \tau(u))\langle du(e_i, \tau(u)\rangle_h\right) \\
\le&\frac{S^N}{m(m-1)}\left(|du|^2|\tau(u)|^2 - \sum_{i=1}^n \langle du(e_i, \tau(u)\rangle_h\right) + \frac{4}{m-2}|Z^N \circ u||du|^2|\tau(u)|^2.
\end{aligned}$$

Hence, if we assume that N has nonpositive scalar curvature, i.e., $S^N \le 0$, then we have

$$\sum_{i=1}^n \langle R^N(du(e_i), \tau(u))du(e_i), \tau(u)\rangle \le \frac{4}{m-2}|Z^N \circ u||du|^2|\tau(u)|^2.$$

This implies

$$\sum_{i=1}^n \langle R^N(du(e_i), |\tau(u)|^{p-2}\tau(u))du(e_i), |\tau(u)|^{p-2}\tau(u)\rangle \le \frac{4}{m-2}|Z^N \circ u||du|^2|\tau(u)|^{2(p-1)}.$$

Combining the above inequality and (2.1), we have

$$\frac{1}{2}\Delta|\tau(u)|^{2(p-1)} \ge |\nabla\left(|\tau(u)|^{p-2}\tau(u)\right)|^2 - \frac{4}{m-2}|Z^N \circ u||du|^2|\tau(u)|^{2(p-1)}.$$

Applying the Kato inequality, the above inequality yields

$$|\tau(u)|\Delta|\tau(u)|^{p-1} \ge -\frac{4}{m-2}|Z^N \circ u||du|^2|\tau(u)|^p.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2 |\tau(u)|^{q+1} \Delta |\tau(u)|^{p-1} \geq -\frac{4}{m-2} \int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q}.$$

Using the divergence theorem and Young's inequality, we get

$$\begin{aligned} &[(p-1)(q+1) - \delta(p-1)] \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\ \leq & \frac{p-1}{\delta} \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 + \frac{4}{m-2} \int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q} \end{aligned} \tag{3.3}$$

for any $\delta > 0$. Using (3.3), we are able to prove Theorem 1.4.

Proof of Theorem 1.4. By the Hölder inequality and (3.1), we have

$$\begin{aligned} &\int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q} \\ &\leq C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \int_M |\nabla \left(\varphi |\tau(u)|^{\frac{p+q}{2}} \right)|^2 \\ &\leq C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\ &\quad + C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \end{aligned}$$

for all $\alpha > 0$. Combining the above inequality and (3.3), we obtain

$$A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2,$$

where the constants A, B, and Q are defined by

$$\begin{aligned} A &= (p-1)(Q+1-p) - \delta(p-1) - \frac{4}{m-2} C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right), \\ B &= \frac{p-1}{\delta} + \frac{4}{m-2} C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{Q}{2\alpha} \right), \\ Q &= p + q. \end{aligned}$$

Using the same argument in the proof of Theorem 1.1, we get the conclusion. □

On the other hand, it is known that the Weyl curvature tensor $W^N = 0$ when $m = 3$ or $m \geq 4$ and (N^m, h) is locally conformally flat. Thus Theorem 1.4 implies the following.

Corollary 3.5. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and assume that either*

(i) $m = 3$ or

(ii) $m \geq 4$ and N is locally conformally flat.

If

$$\| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

By the proof of Theorem 1.4, it is easy to prove two following corollaries.

Corollary 3.6. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0$, $|du| \leq C$ for some constant $C > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^{\frac{n}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2C^2C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Corollary 3.7. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0$, $|Z^N \circ u| \leq D$ for some constant $D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|du\|^2_{L^n(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2DC_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Using the general Hölder inequality for three functions in the terms of curvature of (3.3) as in the proof of Theorem 1.2, we have the following result.

Theorem 3.8. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^n(M)} \||du|^2\|_{L^n(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

In particular, if $p = Q = 2$, then we immediately obtain the following.

Corollary 3.9. *Let $u : (M^n, g) \to (N^m, h)$ be a biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h), with $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^2 < \infty$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^n(M)} \||du|^2\|_{L^n(M)} < \frac{m-2}{4C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Proof of Theorem 1.6. By using (3.3) and repeating the arguments in Theorem 1.3, we complete the proof of Theorem 1.6. □

Corollary 3.10. *Let $u : (M^n, g) \to (N^m, h)$ be a p-biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0$, $|du| \leq C$, $|Z^N \circ u| \leq D$ for some constants $C > 0, D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that $\lambda_1(M) > \frac{C^2DQ^2}{(m-2)(p-1)(Q+1-p)}$. Then u is harmonic.*

4 Rigidity results for p-biharmonic submanifolds

We recall that if $u : (M^n, g) \to (N^m, h)$ is an isometric immersion, then u is called a *p-biharmonic* submanifold. In particular, a 2-biharmonic submanifold is called a *biharmonic* submanifold. The second fundamental form $B : TM \times TM \to T^{\perp}M$ is defined by:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

for $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ is the Levi-Civita connection on N and ∇ is the Levi-Civita connection on M. The Weingarten formula is given by

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi$$

for $X \in \Gamma(TM)$, where A_ξ is called the Weingarten map with respect to $\xi \in T^{\perp}M$ and $\nabla^{\perp}$ denotes the normal connection on the normal bundle of M in N. For any $x \in M$, the mean curvature vector field H of M at x is

$$\vec{H} = \frac{1}{n}\sum_{i=1}^{n} B(e_i, e_i).$$

Now suppose $u : (M^n, g) \to (N^m, h)$ is a p-biharmonic isometric immersion of a Riemannian manifold M with mean curvature vector $\overrightarrow{H}$ into a Riemannian manifold N. Then we have

$$\tau(u) = n\overrightarrow{H}.$$

Moreover, the p-biharmonic submanifold u satisfies the following equation:

$$\tau_p(u) := \Delta(|\overrightarrow{H}|^{p-2}\overrightarrow{H}) - \sum_{i=1}^{n} R^N(|\overrightarrow{H}|^{p-2}\overrightarrow{H}, e_i)e_i = 0.$$

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Let B be the second fundamental form of M and A be the Weingarten map. Then, by the Bochner-Weitzenböck formula, we have (see [CL16, Han15] for example)

$$\begin{aligned}\Delta|\overrightarrow{H}|^{2p-2} &= 2|\nabla(|\overrightarrow{H}|^{p-2}\overrightarrow{H})|^2 + 2\langle\Delta(|\overrightarrow{H}|^{p-2}\overrightarrow{H}), |\overrightarrow{H}|^{p-2}\overrightarrow{H}\rangle \\ &= 2|\nabla(|\overrightarrow{H}|^{p-2}\overrightarrow{H})|^2 + 2\langle\sum_{i=1}^{n} B(A_{|\overrightarrow{H}|^{p-2}\overrightarrow{H}}e_i, e_i), |\overrightarrow{H}|^{p-2}\overrightarrow{H}\rangle \\ &\quad + 2\sum_{i=1}^{n}\left\langle R^N(e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H})e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H}\right\rangle. \end{aligned} \tag{4.1}$$

On the other hand, we have (see [Han15] for instance)

$$\left\langle\sum_{i=1}^{n} B(A_{|\overrightarrow{H}|^{p-2}\overrightarrow{H}}e_i, e_i), |\overrightarrow{H}|^{p-2}\overrightarrow{H}\right\rangle \geq n|\overrightarrow{H}|^{2p}, \tag{4.2}$$

and

$$\sum_{i=1}^{n}\left\langle R^N(e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H})e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H}\right\rangle \geq -|\overrightarrow{H}|^{2p-2}|R^N \circ u|. \tag{4.3}$$

Combining (4.1)-(4.3), we obtain

$$\Delta|\overrightarrow{H}|^{2p-2} \geq 2|\nabla(|\overrightarrow{H}|^{p-2}\overrightarrow{H})|^2 + 2n|\overrightarrow{H}|^{2p} - 2|\overrightarrow{H}|^{2p-2}|R^N \circ u|.$$

By the Kato inequality, we have

$$|\overrightarrow{H}|\Delta|\overrightarrow{H}|^{p-1} \geq n|\overrightarrow{H}|^{p+2} - |\overrightarrow{H}|^{p}|R^N \circ u|.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2|\overrightarrow{H}|^{q+1}\Delta|\overrightarrow{H}|^{p-1} \geq n\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} - \int_M \varphi^2|\overrightarrow{H}|^{p+q}|R^N \circ u|.$$

Using integration by parts, the above inequality implies

$$\begin{aligned}\int_M \varphi^2|\overrightarrow{H}|^{p+q}|R^N \circ u| \geq{}& [(p-1)(q+1) - \delta(p-1)]\int_M \varphi^2|\overrightarrow{H}|^{p+q-2}|\nabla|\overrightarrow{H}||^2 \\ &+ n\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} - \frac{p-1}{\delta}\int_M |\overrightarrow{H}|^{p+q}|\nabla\varphi|^2\end{aligned} \tag{4.4}$$

for all $\delta > 0$. Here we used the Cauchy-Schwarz inequality in (4.4). By the Hölder inequality and the Sobolev inequality (3.1), we get

$$\begin{aligned}\int_M \varphi^2|\overrightarrow{H}|^{p+q}|R^N \circ u| &\leq C_s\|R^N \circ u\|_{L^{\frac{n}{2}}(M)}\int_M |\nabla(\varphi|\overrightarrow{H}|^{\frac{p+q}{2}})|^2 \\ &\leq C_s\|R^N \circ u\|_{L^{\frac{n}{2}}(M)}\left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2}\right)\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} \\ &\quad + C_s\|R^N \circ u\|_{L^{\frac{n}{2}}(M)}\left(1 + \frac{p+q}{2\alpha}\right)\int_M |\overrightarrow{H}|^{p+q}|\nabla\varphi|^2.\end{aligned}$$

Therefore (4.4) implies

$$C\int_M \varphi^2|\overrightarrow{H}|^{Q-2}.|\nabla|\overrightarrow{H}||^2 + n\int_M \varphi^2|\overrightarrow{H}|^{Q+2} \leq D\int_M |\overrightarrow{H}|^{Q}|\nabla\varphi|^2$$

where the constants C, D, and Q are defined by

$$\begin{aligned}C &= (p-1)(Q+1-p) - \delta(p-1) - C_s\|R^N \circ u\|_{L^{\frac{n}{2}}(M)}\left(\frac{Q^2}{4} + \frac{\alpha Q}{2}\right), \\ D &= \frac{p-1}{\delta} + C_s\|R^N \circ u\|_{L^{\frac{n}{2}}(M)}\left(1 + \frac{Q}{2\alpha}\right), \\ Q &= p+q.\end{aligned}$$

As before, we can conclude that $|\overrightarrow{H}| = 0$. □

Proof of Theorem 1.8. By using (4.4) and applying the same argument as in the proof of Theorem 1.3, we are able to prove Theorem 1.8. □

Finally we prove Theorem 1.9.

Proof of Theorem 1.9. Let ν be the unit normal vector field of M. Then we have

$$\tau(u) = n\overrightarrow{H} = n|\overrightarrow{H}|\nu.$$

Since (N^{n+1}, h) is Einstein, we have

$$\begin{aligned}\sum_{i=1}^n \langle R^N(e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H})e_i, |\overrightarrow{H}|^{p-2}\overrightarrow{H}\rangle &= \sum_{i=1}^n |\overrightarrow{H}|^{2p-2}\langle R^N(e_i,\nu)e_i,\nu\rangle \\ &= -|\overrightarrow{H}|^{2p-2}\mathrm{Ric^N}(\nu,\nu)\end{aligned}$$

$$= -\frac{S^N}{n+1}|\overrightarrow{H}|^{2p-2}.$$

Combining inequalities (4.1), (4.2) and the above equality, we obtain

$$\Delta|\overrightarrow{H}|^{2p-2} \geq 2|\nabla(|\overrightarrow{H}|^{p-2}\overrightarrow{H})|^2 + 2n|\overrightarrow{H}|^{2p} - 2\frac{S^N}{n+1}|\overrightarrow{H}|^{2p-2}.$$

By the Kato inequality, we have

$$|\overrightarrow{H}|\Delta|\overrightarrow{H}|^{p-1} \geq n|\overrightarrow{H}|^{p+2} - \frac{S^N}{n+1}|\overrightarrow{H}|^{p}.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2|\overrightarrow{H}|^{q+1}\Delta|\overrightarrow{H}|^{p-1} \geq n\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} - \frac{S^N}{n+1}\int_M \varphi^2|\overrightarrow{H}|^{p+q}.$$

Using integration by parts and the Cauchy-Schwarz inequality, the above inequality implies

$$\begin{aligned} \frac{S^N}{n+1}\int_M \varphi^2|\overrightarrow{H}|^{p+q} \geq{}& [(p-1)(q+1)-\delta(p-1)]\int_M \varphi^2|\overrightarrow{H}|^{p+q-2}|\nabla|\overrightarrow{H}||^2 \\ &+ n\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} - \frac{p-1}{\delta}\int_M |\overrightarrow{H}|^{p+q}|\nabla\varphi|^2 \end{aligned} \tag{4.5}$$

for all $\delta > 0$. Moreover, we have

$$\begin{aligned} \frac{S^N}{n+1}\int_M \varphi^2|\overrightarrow{H}|^{p+q} &\leq \frac{S^N}{(n+1)\lambda_1(M)}\int_M |\nabla(\varphi|\overrightarrow{H}|^{\frac{p+q}{2}})|^2 \\ &\leq \frac{S^N}{(n+1)\lambda_1(M)}\left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2}\right)\int_M \varphi^2|\overrightarrow{H}|^{p+q+2} \\ &\quad + \frac{S^N}{(n+1)\lambda_1(M)}\left(1 + \frac{p+q}{2\alpha}\right)\int_M |\overrightarrow{H}|^{p+q}|\nabla\varphi|^2. \end{aligned}$$

Therefore (4.5) implies

$$C\int_M \varphi^2|\overrightarrow{H}|^{Q-2}|\nabla|\overrightarrow{H}||^2 + n\int_M \varphi^2|\overrightarrow{H}|^{Q+2} \leq D\int_M |\overrightarrow{H}|^{Q}|\nabla\varphi|^2$$

where the constants C, D, and Q are defined by

$$\begin{aligned} C &= (p-1)(Q+1-p) - \delta(p-1) - \frac{S^N}{(n+1)\lambda_1(M)}\left(\frac{Q^2}{4} + \frac{\alpha Q}{2}\right), \\ D &= \frac{p-1}{\delta} + \frac{S^N}{(n+1)\lambda_1(M)}\left(1 + \frac{Q}{2\alpha}\right), \\ Q &= p+q. \end{aligned}$$

Using the same argument as in the proof of Theorem 1.1, we can obtain that $|\overrightarrow{H}| = 0$, which implies that u is minimal. □

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