ON THE BOUNDARY BEHAVIOUR OF THE SQUEEZING FUNCTION NEAR WEAKLY PSEUDOCONVEX BOUNDARY POINTS

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ABSTRACT. The purpose of this article is twofold. The first aim is to prove that if there exist a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ and $a \in \Omega$ such that $\lim_{j\to\infty} \varphi_j(a) = \xi_0$ and $\lim_{j\to\infty} \sigma_\Omega(\varphi_j(a)) = 1$, where ξ_0 is a linearly convex boundary point of finite type, then ξ_0 must be strongly pseudoconvex. Then, the second aim is to investigate the boundary behaviour of the squeezing function of a general ellipsoid.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$ and $p \in \Omega$. Let us denote by $\mathrm{Aut}(D)$ the automorphism group of a domain D. For a holomorphic embedding $f\colon \Omega \to \mathbb{B}^n := \mathbb{B}(0;1)$ with $f(p) = 0$, we set

$$\sigma_{\Omega,f}(p) := \sup\{r > 0\colon B(0;r) \subset f(\Omega)\},$$

where $\mathbb{B}^n(z;r) \subset \mathbb{C}^n$ denotes the ball of radius r with center at z. Then the *squeezing function* $\sigma_\Omega : \Omega \to \mathbb{R}$ is defined as

$$\sigma_\Omega(p) := \sup_f \{\sigma_{\Omega,f}(p)\}.$$

(See Definition in [DGZ12].) Note that the squeezing function is invariant under biholomorphisms and $0 < \sigma_\Omega(z) \leqslant 1$ for any $z \in \Omega$. Moreover, by definition one sees that Ω is biholomorphically equivalent to the unit ball $\mathbb{B}^n$ if $\sigma_\Omega(z) = 1$ for some $z \in \Omega$.

It is well-known that if p is a strongly pseudoconvex boundary point, then $\lim\limits_{\Omega\ni z\to p\in\partial\Omega} \sigma_\Omega(z) = 1$ (cf. [DGZ16, DFW14, KZ16]). Conversely, motivated by Problem 4.1 in [FW18], let us consider the following problem.

Problem 1. If Ω is a bounded pseudoconvex domain with smooth boundary, and if $\lim\limits_{j\to\infty} \sigma_\Omega(q_j) = 1$ for some sequence $\{q_j\} \subset \Omega$ converging to $p \in \partial\Omega$, then is the boundary of Ω strongly pseudoconvex at p?

In the case that $\partial\Omega$ is pseudoconvex of D'Angelo finite type near ξ_0, the answer to this problem is affirmative for the following cases:

- $\{q_j\} \subset \Omega$ converges to ξ_0 along the inner normal line to $\partial\Omega$ at ξ_0 (for details, see [JK18] for $n = 2$ and [MV19] for general case).
- $\{q_j\} \subset \Omega$ converges nontangentially to ξ_0 (see [Nik18]).
- $\{q_j\} \subset \Omega$ converges $\left(\frac{1}{m_1}, \dots, \frac{1}{m_{n-1}}\right)$-nontangentially to an h-extendible boundary point ξ_0 (see [NN20, Definition 3.4]), where $(1, m_1, \dots, m_{n-1})$ is the *multitype*

Date: October 9, 2023.
2020 *Mathematics Subject Classification.* Primary 32H02; Secondary 32M05, 32F18.
Key words and phrases. Hyperbolic complex manifold, exhausting sequence, h-extendible domain.

of $\partial\Omega$ *at* ξ_0 and the *h-extendiblility at* ξ_0 means that the Catlin multitype and D'Angelo multitype of $\partial\Omega$ at ξ_0 coincide (see [Yu95, Definition 3.3]).

Now we consider the case that $\{q_j\} \subset \Omega$ is a sequence converging $\left(\frac{1}{m_1}, \ldots, \frac{1}{m_{n-1}}\right)$-nontangentially to ξ_0. Then, the condition that $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ ensures that the unit ball $\mathbb{B}^n$ is biholomorphically equivalent to some model M_P given by

$$M_P = \{z \in \mathbb{C}^n : \mathrm{Re}(z_n) + P(z') < 1\},$$

where P is a $\left(\frac{1}{m_1}, \ldots, \frac{1}{m_{n-1}}\right)$-homogeneous polynomial on $\mathbb{C}^{n-1}$ (see [Yu95, Definition 3.1]). Therefore, $m_1 = m_2 = \cdots = m_{n-1} = 1$, or ξ_0 is strongly pseudoconvex ([NN20]). Unfortunately, the point ξ_0 may not be strongly psudoconvex when $\{q_j\} \subset \Omega$ does not converge $\left(\frac{1}{m_1}, \ldots, \frac{1}{m_{n-1}}\right)$-nontangentially to ξ_0. For instance, the following example points out that $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ for some $\{q_j\} \subset \Omega$ converging to a weakly pseudoconvex boundary point (see also Example 4.1 for general case).

Example 1.1. Let $E_{1,2} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^4 < 1\}$. Consider the sequence $a_n = \left(\sqrt[4]{\dfrac{2}{n} - \dfrac{2}{n^2}}, 1 - \dfrac{1}{n}\right) \to (0,1)$ as $n \to \infty$. Denote by $\rho(z) := |z_2|^2 - 1 + |z_1|^4$ a defining function for $E_{1,2}$ and denote by $\sigma(z_1) = |z_1|^4$ a $(\frac{1}{4})$-weighted homogeneous polynomial. Then, a computation shows that

$$\rho(a_n) = \left|1 - \frac{1}{n}\right|^2 - 1 + \left|\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}\right|^4 = -\frac{2}{n} + \frac{1}{n^2} + \frac{2}{n} - \frac{2}{n^2} = -\frac{1}{n^2} < 0$$

Therefore, $\mathrm{dist}(a_n, \partial E_{1,2}) \approx |\rho(a_n)| = \dfrac{1}{n^2}$, $|\mathrm{Re}(a_{n2}) - 1| = \left|-\dfrac{1}{n}\right| = \dfrac{1}{n}$, and $\sigma(a_{n1}) = \sigma(\sqrt[4]{\dfrac{2}{n} - \dfrac{2}{n^2}}) = \left(\sqrt[4]{\dfrac{2}{n} - \dfrac{2}{n^2}}\right)^4 = \dfrac{2}{n} - \dfrac{2}{n^2} \approx \dfrac{2}{n}$. This implies that $\{a_n\}$ does not converge $(\frac{1}{4})$-nontangentially to the boundary point $p = (0,1)$.

Let us consider the automorphism $\psi_n \in \mathrm{Aut}(E_{1,2})$, given by

$$\psi_n(z) = \left(\frac{(1 - |a_{n2}|^2)^{1/4}}{(1 - \bar{a}_{n2} z_2)^{1/2}} z_1, \frac{z_2 - a_{n2}}{1 - \bar{a}_{n2} z_2}\right),$$

and hence $\psi_n(a_n) = (b_n, 0)$, where $b_n = \dfrac{a_{n1}}{(1 - |a_{n2}|^2)^{1/4}} = \dfrac{\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}}{\sqrt[4]{\frac{2}{n} - \frac{1}{n^2}}} \to 1$ as $n \to \infty$. Since $\psi_n(a_n)$ converges to the strongly pseudoconvex boundary point $(1,0)$ of $\partial E_{1,2}$, by [KZ16, Theorrem 3.1] it follows that $\sigma_{E_{1,2}}(a_n) = \sigma_{E_{1,2}}(\psi_n(a_n)) \to 1$ as $n \to \infty$. However, the point $(0,1)$ is weakly pseudoconvex. □

To give a statement of our result, let us recall that $\partial\Omega$ is *linearly convex near* $\xi_0 \in \partial\Omega$ if there exists a neighbourhood U of ξ_0 such that, for all $z \in \partial\Omega \cap U$, the intersection

$$(z + T_z^{10}\partial\Omega) \cap (\Omega \cap U) = \varnothing.$$

We note that in [Ni09] the first author proved a characterization of linearly convex domains in $\mathbb{C}^n$ by their noncompact automorphism groups.

The first aim of this paper is the following theorem.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary. Assume that ξ_0 is a boundary point of Ω of D'Angelo finite type such that $\partial\Omega$ is linearly convex at ξ_0 and there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $q_j := \varphi_j(a) \to \xi_0$ as $j \to \infty$ for some $a \in \Omega$. If $\lim\limits_{j\to\infty} \sigma_\Omega(q_j) = 1$, then $\partial\Omega$ is strongly pseudoconvex at ξ_0.*

Remark 1.1. Thanks to the linear convexity of $\partial\Omega$ near a boundary orbit accumulation point ξ_0 and the condition that $\lim\limits_{j\to\infty} \sigma_\Omega(q_j) = 1$, the scaling method can be applied to implies that $\mathbb{B}^n$ is biholomorphically equivalent to a model M_P, where is a real nondegenerate plurisubharmonic polynomial of degree less than or equal to the type of $\partial\Omega$ at ξ_0. Moreover, since $q_j = \varphi_j(a)$ for some $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ and $a \in \Omega$, the scaling method yields Ω is biholomorphically equivalent to a model M_P (see [Ni09, Theorem 1.1]), that is, Ω is biholomorphically equivalent to the unit ball $\mathbb{B}^n$. Consequently, the point ξ_0 is strongly pseudoconvex, as desired.

Now we move to the second part of this paper. First of all, let us fix positive integers $m_1, \ldots, m_{n-1}$ and let $P(z')$ be a $(1/m_1, \ldots, 1/m_{n-1})$-homogeneous polynomial given by

$$P(z) = \sum_{wt(K)=wt(L)=1/2} a_{KL} {z'}^K \bar{z}'^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \neq 0$. Here and in what follows, $z' := (z_1, \ldots, z_{n-1})$ and $wt(K) := \sum_{j=1}^{n-1} \frac{k_j}{2m_j}$ denotes the weight of any multi-index $K = (k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}$ with respect to $\Lambda := (1/m_1, \ldots, 1/m_{n-1})$. Then the general ellipsoid D_P in $\mathbb{C}^n$ $(n \geqslant 1)$, defined in [NNTK19] by

$$D_P := \{(z', z_n) \in \mathbb{C}^n \colon |z_n|^2 + P(z') < 1\}.$$

We note that

$$P(a^{1/m_1} z_1, a^{1/m_2} z_2, \ldots, a^{1/m_{n-1}} z_{n-1}) = P(z'),\ \forall z' \in \mathbb{C}^{n-1}, \forall a \in \mathbb{C}\backslash\{0\}. \tag{1}$$

Therefore, $\mathrm{Aut}(D_P)$ contains the automorphisms $\phi_a \in \mathrm{Aut}(D_P)$, $a \in \Delta$, defined by

$$(z', z_n) \mapsto \left(\frac{(1-|a|^2)^{1/2m_1}}{(1+\bar{a}z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1+\bar{a}z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1 + \bar{a}z_n} \right). \tag{2}$$

These automorphisms play a crucial role in the proofs of Theorem 1.2 and Theorem 1.4 below.

Throughout this paper, we assume that the domain D_P is a WB-domain, i.e., D_P is strongly pseudoconvex at every boundary point outside the set $\{(0', e^{i\theta}) \colon \theta \in \mathbb{R}\}$ (cf. [AGK16]).

For any $s, r \in (0,1]$ and $\alpha \in [0,2)$, as in [NNN23] we define D_P^s, $D_{P,r}^s$, $D_{P,r}$, and $D_P^s(\alpha)$ respectively by

$$D_P^s := \{z \in \mathbb{C}^n \colon |z_n - b|^2 + sP(z') < s^2\};$$
$$D_{P,r}^s := \{z \in \mathbb{C}^n \colon |z_n - b|^2 + \frac{s}{r}P(z') < s^2\};$$
$$D_{P,r} := D_{P/r} = \Big\{z \in \mathbb{C}^n \colon |z_n|^2 + \frac{1}{r}P(z') < 1\Big\};$$
$$D_P^s(\alpha) = \Big\{z \in \mathbb{C}^n \colon \Big|z_n + \frac{(1-s)\alpha}{2s(1-\alpha)+\alpha}\Big|^2 + \frac{s(2-\alpha)}{2s(1-\alpha)+\alpha}\, P(z') < \frac{2s-\alpha}{2s(1-\alpha)+\alpha} + \Big|\frac{(1-s)\alpha}{2s(1-\alpha)+\alpha}\Big|^2\Big\},$$

where $s = 1-b$. We note that $D_{P,r}^s \subset D_P^s \subset D_P$ (cf. [NNN23]), $D_P^s(0) = D_P$, and $D_{P,1}^s = D_P^s$.

In what follows, let us denote by Δ the unit disc in $\mathbb{C}$ and for a sequence $\{a_j\} \subset \Delta$ converging to $1 \in \partial\Delta$ we always denote by $x_j := 1 - \mathrm{Re}(a_j)$ and $y_j := \mathrm{Im}(a_j)$ for $j \geqslant 1$. Suppose that $\{q_j = (q_j', a_j)\} \subset D_P^s$ for some $0 < s < 1$. Then one sees that

$$|a_j - 1|^2 + 2s\mathrm{Re}(a_j - 1) + sP(q_j') < 0,$$

which implies that

$$|a_j - 1|^2 < -2s\mathrm{Re}(a_j - 1), \quad \text{for } j \geqslant 1,$$

or equivalently $x_j^2 + y_j^2 < 2sx_j$, for $j \geqslant 1$. Therefore, passing to a subsequence if necessary, we can assume that there exists

$$0 \leqslant \alpha := \lim_{j\to\infty} \frac{y_j^2}{x_j} \leqslant 2s < 2.$$

In addition, to each sequence $\{a_j\} \subset \Delta$ we associate a sequence $\psi_j := \phi_{a_j} \in \mathrm{Aut}(D_P)$, i.e,

$$\psi_j(z,w) = \left(\frac{(1-|a_j|^2)^{1/2m_1}}{(1+\bar{a}_j z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a_j|^2)^{1/2m_{n-1}}}{(1+\bar{a}_j z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a_j}{1+\bar{a}_j z_n}\right),\ j \geqslant 1. \tag{3}$$

To state our second result, we need the following definition.

Definition 1.1. We say that $\{q_j\} \subset D_P \cap U$ converges Λ^α-*nontangentially* to $p = (0',1)$ if there exists $0 < r < 1$ such that $q_j \in D_{P,r}$ for all $j \geqslant 1$, $\lim_{j\to\infty} q_j = (0',1)$, and $\lim_{j\to\infty} \frac{y_j^2}{x_j} = \alpha \in [0,2)$, where $q_{jn} = 1 - x_j + iy_j$, $j \geqslant 1$.

Indeed, we prove the following theorem.

Theorem 1.2. *Let Ω be a subdomain of D_P such that $D_P^s \subset \Omega \subset D_P$ for some $s \in (0,1]$. Let $\{q_j\} \subset D_{P,r}^s$ be a sequence that converges Λ^α-nontangentially to $(0',1)$ in D_P for some $0 < r < 1$. Then, there exists $\gamma_1 > 0$ depending on s, α, P, r such that*

$$\liminf_{j\to\infty} \sigma_\Omega(q_j) \geqslant \gamma_1.$$

Remark 1.2. Let $\{q_j = (q'_j, q_{nj})\} \subset D^s_{P,r}$ be as in the statement of Theorem 1.2. Then Lemma 2.1 in Section 2 ensures that $\lim\limits_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D^s_{P,r}(\alpha)$ and $\lim\limits_{j\to\infty} \psi_j^{-1}(D^s_P) = D^s_P(\alpha)$. Therefore, the proof of Theorem 1.2 follows from the invariance of the squeezing function under biholomorphisms.

Now let us denote the cone with vertex at $p = (0', 1)$ by

$$\Gamma_c := \{(z', z_n) \in \mathbb{C}^n \colon |\mathrm{Im}(z_n)| \leqslant c|1 - \mathrm{Re}(z_n)|\},$$

for some $c > 0$. Then for any sequence $\{q_j\} \subset D^s_{P,r} \cap \Gamma_c$ converging to $(0', 1)$, we always have $\alpha = \lim\limits_{j\to\infty} \dfrac{y_j^2}{x_j} = 0$. Therefore, again by Lemma 2.1, $\lim\limits_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D_{P,r}$ for any $0 < r \leqslant 1$. Moreover, we obtain the following corollary, which is a generalization of [NNC21, Theorem 1.3].

Corollary 1.3. *Let Ω be a subdomain of D_P such that $D^s_P \subset \Omega \subset D_P$ for some $s \in (0, 1]$. Then, for any $r \in (0, 1), c > 0$ there exist $\epsilon_0, \gamma_2 > 0$ depending on r and c such that*

$$\sigma_\Omega(q) \geqslant \gamma_2, \ \forall\, q \in D^s_{P,r} \cap \Gamma_c \cap B(p, \epsilon_0).$$

In contrast to the Λ^α-nontangential convergence ($0 \leqslant \alpha < 2$), we have the following definition.

Definition 1.2. We say that $\{q_j\} \subset D_P \cap U$ converges Λ-*tangentially* to $p = (0', 1)$ if $\lim\limits_{j\to\infty} q_j = (0', 1)$ and for any $0 < r < 1$ there exists $j_r \in \mathbb{N}$ such that $q_j \notin D_{P,r}$ for all $j \geqslant 1$.

With the notion of Λ-tangential convergence, we have the following theorem.

Theorem 1.4. *Let $\{\Omega_j\}$ be a sequence of subdomains of D_P such that $\Omega_j \cap U = D_P \cap U$, $j \geqslant 1$, for a fixed neighborhood U of the origin in $\mathbb{C}^n$. Let $\{q_j\} \subset D_P \cap U$ be a sequence that converges Λ-tangentially to $(0', 1)$ in D_P. Then, $\lim_{j\to\infty} \sigma_{\Omega_j}(q_j) = 1$.*

We note that D_P is holomorphically homogeneous regular (cf. [NNC21, Theorem 1.1]). In addition, Proposition 4.1 in Section 4 gives the uniform lower bound for the squeezing function near $(0', 1) \in \partial D_P$.

The organization of this paper is as follows: In Sections 2, we introduce several technical lemmas needed later. Next, in Section 3 we give a proof of Theorem 1.1. Finally, the proofs of Theorem 1.2 and Theorem 1.4 are given in Section 4.

2. Several technical lemmas

In this section, we prove the following lemma.

Lemma 2.1. *Let $\{a_j = 1 - x_j + iy_j\} \subset \Delta$ be a given sequence satisfying that $\lim\limits_{j\to\infty} a_j = 1$ and $\lim\limits_{j\to\infty} \dfrac{y_j^2}{x_j} = \alpha \in [0, 2)$. Then, for any $s \in (0, 1)$ we have that $\psi_j^{-1}(D^s_P)$ converges to $D^s_P(\alpha)$, where the sequence $\{\psi_j\}$ is given in (3).*

Remark 2.1. In the case that $\alpha = 0$, one sees that $D^s_P(0) = D_P$ and therefore $\psi_j^{-1}(D^s_P)$ converges to D_P.

To give a proof of Lemma 2.1, we need the following lemma.

Lemma 2.2. *Let $\{a_j\}$ be a sequence in Δ such that* $\lim_{j\to\infty} \dfrac{(\mathrm{Im}(a_j))^2}{1-\mathrm{Re}(a_j)} = \alpha \in [0,2)$ *and* $\lim_{j\to\infty} a_j = 1$. *Then we have*

$$\text{(i)}\ \lim_{j\to\infty} \frac{1-\mathrm{Re}(a_j)}{1-|a_j|^2} = \frac{1}{2-\alpha};$$
$$\text{(iii)}\ \lim_{j\to\infty} \frac{(1-\bar{a}_j)^2}{1-|a_j|^2} = \frac{-\alpha}{2-\alpha}.$$
$$\text{(iii)}\ \lim_{j\to\infty} \frac{|1-a_j|^2}{1-|a_j|^2} = \frac{\alpha}{2-\alpha}.$$

Proof. We have $x_j \to 0^+$, $y_j \to 0$, and $y_j^2/x_j \to \alpha$ as $j \to \infty$, where $x_j := 1-\mathrm{Re}(a_j), y_j := \mathrm{Im}(a_j)$. Moreover, a direct calculation yields that

$$\frac{1-\mathrm{Re}(a_j)}{1-|a_j|^2} = \frac{x_j}{1-(1-x_j)^2-y_j^2} = \frac{x_j}{2x_j - x_j^2 - y_j^2} = \frac{1}{2-x_j-y_j^2/x_j};$$
$$\frac{(1-\bar{a}_j)^2}{1-|a_j|^2} = \frac{(x_j+iy_j)^2}{1-(1-x_j)^2-y_j^2} = \frac{x_j^2 - y_j^2 + 2ix_jy_j}{2x_j - x_j^2 - y_j^2} = \frac{x_j - y_j^2/x_j + 2iy_j}{2-x_j-y_j^2/x_j};$$
$$\frac{|1-a_j|^2}{1-|a_j|^2} = \frac{x_j^2+y_j^2}{1-(1-x_j)^2-y_j^2} = \frac{x_j^2+y_j^2}{2x_j - x_j^2 - y_j^2} = \frac{x_j + y_j^2/x_j}{2-x_j-y_j^2/x_j},\ \forall j \geqslant 1.$$

Therefore, the assertions follow since $x_j \to 0^+$ and $y_j^2/x_j \to \alpha$ as $j \to \infty$. □

Proof of Lemma 2.1. The proof of this lemma is given in [NNN23]. However, for the convenience of the reader we give a detailed proof. Indeed, recall that $b = 1 - s$ or

$s = 1 - b \in (0,1)$. Then, by the property (1) a straightforward calculation shows that

$$\begin{aligned}
&\left|\frac{z_n + a_j}{1+\bar{a}_j z_n} - b\right|^2 + sP\left(\frac{(1-|a_j|^2)^{1/2m_1}}{(1+\bar{a}_j z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a_j|^2)^{1/2m_{n-1}}}{(1+\bar{a}_j z_n)^{1/m_{n-1}}} z_{n-1}\right) < s^2 \\
\Leftrightarrow &\left|\frac{z_n + a_j}{1+\bar{a}_j z_n} - b\right|^2 + s\frac{1-|a_j|^2}{|1+\bar{a}_j z_n|^2} P(z') < s^2 \\
\Leftrightarrow &\left|\frac{z_n + a_j - b(1+\bar{a}_j z_n)}{1+\bar{a}_j z_n}\right|^2 + s\frac{1-|a_j|^2}{|1+\bar{a}_j z_n|^2} P(z') < s^2 \\
\Leftrightarrow &|z_n + a_j - b(1+\bar{a}_j z_n)|^2 + s(1-|a_j|^2)P(z') < s^2|1+\bar{a}_j z_n|^2 \\
\Leftrightarrow &|z_n(1-\bar{a}_j b) + a_j - b|^2 + s(1-|a_j|^2)P(z') < s^2|1+\bar{a}_j z_n|^2 \\
\Leftrightarrow &|z_n|^2|1-\bar{a}_j b|^2 + 2\mathrm{Re}\left[(\bar{a}_j - b)(1-\bar{a}_j b)z_n\right] + |a_j - b|^2 + (1-b)(1-|a_j|^2)P(z') \\
&< s^2\left(|a_j|^2|z_n|^2 + 2\mathrm{Re}[\bar{a}_j z_n] + 1\right) \\
\Leftrightarrow &|z_n|^2\left(|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2\right) + 2\mathrm{Re}\left[\left((\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j\right) z_n\right] \\
&+ (1-b)(1-|a_j|^2)P(z') < (1-b)^2 - |a_j - b|^2 \\
\Leftrightarrow &|z_n|^2 + 2\mathrm{Re}\left[\frac{(\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} z_n\right] \\
&+ \frac{(1-b)(1-|a_j|^2)}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} P(z') < \frac{(1-b)^2 - |a_j - b|^2}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} \\
\Leftrightarrow &\left|z_n + \frac{(\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2}\right|^2 + \frac{(1-b)(1-|a_j|^2)}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} P(z') \\
&< \frac{(1-b)^2 - |a_j - b|^2}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} + \left|\frac{(\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2}\right|^2.
\end{aligned}$$

Moreover, by a computation one obtains the following

$$\begin{aligned}
(\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j &= \bar{a}_j - b - \bar{a}_j^2 b + \bar{a}_j b^2 - \bar{a}_j + 2\bar{a}_j b - \bar{a}_j b^2 = -b(1-\bar{a}_j)^2; \\
(1-b)^2 - |a_j - b|^2 &= 1 - 2b + b^2 - |a_j|^2 + 2b\mathrm{Re}(a_j) - b^2 \\
&= 1 - |a_j|^2 - 2b(1 - \mathrm{Re}(a_j)); \\
|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2 &= 1 - 2\mathrm{Re}(a_j b) + |a_j|^2 b^2 - |a_j|^2 + 2b|a_j|^2 - b^2|a_j|^2 \\
&= 1 - |a_j|^2 - 2b\left(\mathrm{Re}(a_j) - |a_j|^2\right) \\
&= 1 - |a_j|^2 - 2b\left(\mathrm{Re}(a_j) - 1 + 1 - |a_j|^2\right) \\
&= (1-|a_j|^2)\left[1 - 2b\Big(1 - \frac{1-\mathrm{Re}(a_j)}{1-|a_j|^2}\Big)\right].
\end{aligned}$$

Hence, by Lemma 2.2 yields that

$$\lim_{j\to\infty} \frac{(\bar{a}_j - b)(1-\bar{a}_j b) - (1-b)^2\bar{a}_j}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} = \frac{b\alpha}{(1-b)(2-\alpha)+b\alpha} = \frac{(1-s)\alpha}{2s(1-\alpha)+\alpha};$$
$$\lim_{j\to\infty} \frac{(1-b)(1-|a_j|^2)}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} = \frac{(1-b)(2-\alpha)}{(1-b)(2-\alpha)+b\alpha} = \frac{s(2-\alpha)}{2s(1-\alpha)+\alpha};$$
$$\lim_{j\to\infty} \frac{(1-b)^2 - |a_j - b|^2}{|1-\bar{a}_j b|^2 - (1-b)^2|a_j|^2} = \frac{2-\alpha-2b}{(1-b)(2-\alpha)+b\alpha} = \frac{2s-\alpha}{2s(1-\alpha)+\alpha}.$$

Therefore, this implies that $\psi_j^{-1}(D_P^s) \to D_P^s(\alpha)$ as $j \to \infty$, as desired. □

3. Squeezing function for linearly convex domains

Throughout this section, the domain $\Omega \subset \mathbb{C}^n$ and the boundary point $\xi_0 \in \partial\Omega$ are assumed to satisfy the hypothesis of Theorem 1.1, namely $\partial\Omega$ is linearly convex, of finite type $2m$ near a point ξ_0 of $\partial\Omega$. We may also assume that $\xi_0 = 0$. There exists a neighbourhood U of $\xi_0 = 0$ in $\mathbb{C}^n$ such that $\Omega \cap U$ is linearly convex and is defined by a smooth function

$$\rho(z', z_n) = \mathrm{Re}(z_n) + h\big(\mathrm{Im}(z_n), z'\big),$$

where h is a function of class C^∞. We may also assume that there exists a real positive number ϵ_0 such that for every $-\epsilon_0 < \epsilon < \epsilon_0$, the level sets $\{\rho(z) = \epsilon\}$ are linearly convex.

For each $\epsilon \in (0, \epsilon_0/2)$, $\eta \in \Omega \cap U$ with $|\rho(\eta)| < \epsilon_0/2$ and each unit vector $v \in \mathbb{S}^{n-1} := \{v \in \mathbb{C}^n \colon |v| = 1\}$, we set

$$\tau(\eta, v, \epsilon) := \sup\{r > 0 : \rho(\eta + \lambda v) - \rho(\eta) < \epsilon \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < r\}.$$

Then, it is easy to see that $\tau(\eta, v, \epsilon)$ is the distance from η to $S_{\eta,\epsilon} := \{\rho(z) = \rho(\eta) + \epsilon\}$ along the complex line $\{\eta + \lambda v \colon \lambda \in \mathbb{C}\}$.

To every point $\eta \in \Omega \cap U$ and every sufficiently small positive constant ϵ we associate

(1) A holomorphic coordinate system $(z_1, z_2, \ldots, z_n)$ centered at η and preserving orthogonality,
(2) Points $p_1, p_2, \ldots, p_n$ on the hypersurface $S_{\eta,\epsilon}$ and,
(2) Positive real numbers $\tau_1(\eta, \epsilon), \tau_2(\eta, \epsilon), \ldots, \tau_n(\eta, \epsilon)$.

The construction proceeds as follows. We first set

$$e_n := \frac{\nabla\rho(\eta)}{|\nabla\rho(\eta)|} \text{ and } \tau_n(\eta, \epsilon) := \tau(\eta, e_n, \epsilon).$$

Working with sufficiently small ϵ, there exists a unique point p_n in $S_{\eta,\epsilon}$ where this distance is achieved. Choose a parameterization of the complex line from η to p_n such that $z_n(0) = \eta$ and p_n lies on the positive $\mathrm{Re}(z_n)$ axis. By the choice of real axis for z_n, we have $\frac{\partial r}{\partial x_n}(\eta) = 1$ and thus, if U is small enough,

$$\frac{\partial r}{\partial x_n}(z) \approx 1 \text{ for all } z \in U.$$

We also have

$$\tau_n(\eta, \epsilon) \approx \epsilon, \tag{4}$$

where the constant is independent of η and ϵ. Now consider the orthogonal complement H_n of the span of the coordinate z_n in $\mathbb{C}^n$. For any $\gamma \in H_n \cap \mathbb{S}^{n-1}$, compute $\tau(\eta, \gamma, \epsilon)$. Because of the assumption of finite type, the largest such distance is finite and is achieved at a vector $e_{n-1} \in H_n \cap \mathbb{S}^{n-1}$. Set $\tau_{n-1}(\eta, \epsilon) := \tau(\eta, e_{n-1}, \epsilon)$. Let $p_{n-1} \in S_{\eta,\epsilon}$ be a point such that $p_{n-1} = \eta + \tau_{n-1}(\eta, \epsilon)e_{n-1}$. The coordinate z_{n-1} is defined by parameterizing the complex line from η to p_{n-1} in such a way that $z_{n-1}(0) = \eta$ and p_{n-1} lies on the positive $\mathrm{Re}(z_{n-1})$ axis. For the next step, define H_{n-1} as the orthogonal complement of the span of z_{n-1} and z_n and repeat the above construction. Continuing this process, we obtain n coordinate functions z_k, vectors e_k, the numbers $\tau_k(\eta, \epsilon)$ and the distinguished points p_k $(1 \leqslant k \leqslant n)$. Let $z_k = x_k + iy_k$ $(1 \leqslant k \leqslant n)$ denote the underlying real coordinates.

We assume that ξ_0 is an accumulating point for a sequence of automorphisms of Ω. Let $\{q_j\} \subset \Omega$ be a sequence converging to ξ_0. Moreover, we may assume that $q_j \in \Omega \cap U$ for all j. Let us set $\epsilon_j := -\rho(q_j)$ for all j. Then, by argument as above, we construct the new coordinates $(z_1^j, \ldots, z_n^j)$, the positive numbers $\tau_{j,1}, \ldots, \tau_{j,n}$, and the points $p_1^j, \ldots, p_n^j$ associated with q_j and ϵ_j.

The change of coordinates from the canonical system to the system $(z_1^j, \ldots, z_n^j)$ is the composition of a translation T_j and of a unitary transform A_j. In addition, we may assume that $(A_j \circ T_j)^{-1}$ is defined in a fixed neighborhood of the origin and thus the corresponding defining function ρ_j is defined by

$$\rho_j := \rho \circ (A_j \circ T_j)^{-1},$$

which is given in a fixed neighborhood of 0 by

$$\rho_j(z) = -\epsilon_j + \mathrm{Re}(\sum_{k=1}^{n} a_k^j z_k) + \sum_{2 \leqslant |\alpha|+|\beta| \leqslant 2m} C_{\alpha\beta}^j z'^{\alpha} z'^{\beta} + O(|z|^{2m+1}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n-1}$ and $z'^{\alpha} = z_1^{\alpha_1} \cdot \ldots \cdot z_{n-1}^{\alpha_{n-1}}$. We note that $O(|z|^{2m+1})$ is independent of j.

Let $\rho \circ A$ be the limit of ρ_j when j goes to infinity, where A is a unitary transform and this convergence is $\mathcal{C}^\infty$ on a fixed compact neighborhood of ξ_0. Then, for every j less than or equal to n and for every multi-index α and β satisfying $2 \leqslant |\alpha| + |\beta| \leqslant 2m$, there exist two complex numbers a_j and $C_{\alpha\beta}$ such that

$$\lim_{j \to \infty} a_k^j = a_k \text{ and } \lim_{j \to \infty} C_{\alpha\beta}^j = C_{\alpha\beta}.$$

Now let us consider the dilation

$$\Lambda_j(z) := (\tau_{j,1} z_1, \ldots, \tau_{j,n} z_n)$$

and the function

$$\tilde{\rho}_j = \frac{1}{\epsilon_j} \rho_j \circ \Lambda_j.$$

Therefore, the defining function $\tilde{\rho}_j$ has the following form

$$\tilde{\rho}_j(z) = -1 + \frac{1}{\epsilon_j} \mathrm{Re}\big(\sum_{j=1}^{n} a_j^j \tau_{j,j} z_j\big) + \frac{1}{\epsilon_j} \sum_{2 \leqslant |\alpha|+|\beta| \leqslant 2m} C_{\alpha\beta}^j \tau_j^{\alpha+\beta} z'^{\alpha} z'^{\beta} + O((\epsilon_j)^{1/2m} |z|^{2m+1}),$$

where $\tau_j^{\alpha+\beta} = \tau_{j,1}^{\alpha_1+\beta_1} \cdot \ldots \cdot \tau_{j,n-1}^{\alpha_{n-1}+\beta_{n-1}}$. Furthermore, it follows from [Ni09, Prop.3.1] that the functions $\tilde{\rho}_j$ are smooth and plurisubharmonic, and after taking a subsequence, we may assume that $\{\tilde{\rho}_j\}$ that converges uniformly on compacta of $\mathbb{C}^n$ to a smooth plurisubharmonic function $\tilde{\rho}$ of the form

$$\tilde{\rho}(z) = -1 + \mathrm{Re}\Big(\sum_{k=1}^{n} b_k z_k\Big) + P(z'),$$

where P is a plurisubharmonic polynomial of degree less than or equal to $2m$.

In what follows, let us denote by $\Gamma_j := \Lambda_j^{-1} \circ A_j \circ T_j$ for all j. Then, one can deduce that $\{\Gamma_j(\Omega \cap U)\}$ converges to the following model

$$\widetilde{M}_P := \left\{ z \in \mathbb{C}^n \colon \tilde{\rho}(z) = -1 + \mathrm{Re}\Big(\sum_{k=1}^{n} b_k z_k\Big) + P(z') < 0 \right\},$$

which is clearly biholomorphically equivalent to

$$M_P := \{z \in \mathbb{C}^n \colon \hat{\rho}(z) := \mathrm{Re}(z_n) + P(z') < 0\}.$$

Let us consider a sequence of the biholomorphisms $F_j \colon f_j(\Omega \cap U) \to \Gamma_j(\Omega \cap U)$ defined by $F_j = \Gamma_j \circ f_j^{-1}$. Since $F_j(0) = 0 \in \widetilde{M}_P$, it follows that our sequence $\{F_j\}$ is not compactly divergence. Moreover, the normality of $\{F_j\}$ is ensured by [Ni09, Lemma 4.1].

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{q_j\} \subset \Omega$ be a sequence given in Theorem 1.1, that is, $\lim_{j\to\infty} q_j = \xi_0$ and $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$. Firstly, let us set $\delta_j = 2(1 - \sigma_\Omega(q_j))$ for all j. Then by our assumption, for each j, there exists an injective holomorphic map $f_j : \Omega \to \mathbb{B}^n$ such that $f_j(q_j) = (0', 0)$ and $\mathbb{B}(0; 1 - \delta_j) \subset f_j(\Omega)$. Then by [DN09, Proposition 2.2] and the hypothesis of Theorem 1.1, without loss of generality we may assume that for each compact subset $K \Subset \mathbb{B}^n$ and each neighborhood U of ξ_0, there exists an integer j_0 such that $f_j^{-1}(K) \subset \Omega \cap U$ for all $j \geqslant j_0$, i.e. $f_j(\Omega \cap U)$ converges to $\mathbb{B}^n$.

Next, it follows from [Ni09, Lemma 4.1] that the sequence $\Gamma_j \circ f_j^{-1} \colon f_j(\Omega \cap U) \to \Gamma_j(\Omega \cap U)$ is normal and its limit is a holomorphic mapping from $\mathbb{B}^n$ to $\widetilde{M}_P$. Moreover, by Montel's theorem the sequence $f_j \circ \Gamma_j^{-1} \colon \Gamma_j(\Omega \cap U) \to f_j(\Omega \cap U) \subset \mathbb{B}^n$ is also normal. In addition, our the sequence $\{\Gamma_j \circ f_j^{-1}\}$ is not compactly divergent since $\Gamma_j \circ f_j^{-1}(0, 0') = (0, 0')$. Then by [DN09, Proposition 2.1], after taking some subsequence of $\{\Gamma_j \circ f_j^{-1}\}$, we may assume that such a subsequence converges uniformly on every compact subset of $\mathbb{B}^n$ to a biholomorphism F from $\mathbb{B}^n$ onto $\widetilde{M}_P$, which is clearly equivalent to M_P.

On the other hand, by [Ni09, Theorem 1.1] Ω is also biholomorphically equivalent to M_P, and hence Ω is biholomorphically equivalent to $\mathbb{B}^n$. Therefore, $\partial\Omega$ is strongly pseudoconvex at ξ_0 (ξ_0 is of the D'Angelo type 2), which ends our proof. □

4. Proofs of Theorem 1.2 and Theorem 1.4

This section is devoted to proofs of Theorem 1.2 and Theorem 1.4.

Proof of Theorem 1.2. Let $\{q_j\} \subset D^s_{P,r}$ be a sequence converging to $(0', 1)$ for some fixed $r \in (0, 1)$. For simplicity, let us denote by $a_j = q_{jn}$ for $j \geqslant 1$. Let us denote by $x_j := 1 - \mathrm{Re}(a_j), y_j := \mathrm{Im}(a_j)$ for convenience. Then we have $x_j \to 0^+$, $y_j \to 0$, and $y_j^2/x_j \to \alpha$ as $j \to \infty$.

We now consider the sequence of automorphisms $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ given in (3). Then, Lemma 2.1 yields

$$\lim_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D^s_{P,r}(\alpha);\ \lim_{j\to\infty} \psi_j^{-1}(D^s_P) = D^s_P(\alpha). \tag{5}$$

Moreover, we have that $\psi_j^{-1}(q_j) = \Big(\dfrac{q_{j1}}{\lambda_j^{1/2m_1}}, \dots, \dfrac{q_{jn-1}}{\lambda_j^{1/2m_{n-1}}}, 0\Big) \in D^s_{P,r}(\alpha) \cap \{z_n = 0\}$, where $\lambda_j = 1 - |a_j|^2$ and $D^s_{P,r}(\alpha) \cap \{z_n = 0\} \Subset D^s_P(\alpha)$. Therefore, by (5) and by Lemma 2.1 in [NNC21] there exists $j_0 \in \mathbb{N}^*$ such that

$$\sigma_\Omega(q_j) = \sigma_{\psi_j^{-1}(\Omega)}(\psi_j^{-1}(q_j)) > \delta/d > 0,\ \forall j \geqslant j_0,$$

where d denotes the diameter of D_P and $\delta := \mathrm{dist}(Z_{r,\alpha}(P), Z_{1,\alpha}(P))/2$ with $Z_{\rho,\alpha}(P) = \Big\{z' \in \mathbb{C}^{n-1} \colon P(z') = \rho\dfrac{2s-\alpha}{s(2-\alpha)}\Big\}$ for $0 < \rho \leqslant 1$. This finishes the proof with $\gamma_1 = \delta/d$. □

Proof of Corollary 1.3. We first consider an arbitrary sequence $\{q_j\} \subset D^s_{P,r} \cap \Gamma_c$ converging to $p = (0', 1)$. Let us write $a_j = q_{jn} = 1 - x_j + iy_j$. Then we have

$$\frac{y_j^2}{x_j} = \frac{|y_j|}{|x_j|} \cdot |y_j| \leqslant c \cdot |y_j|,\ j \geqslant 1.$$

This implies that $\alpha := \lim_{j\to\infty} \dfrac{y_j^2}{x_j} = 0$, and hence by Remark 2.1, we obtain $\lim_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D_{P,r}$ and $\lim_{j\to\infty} \psi_j^{-1}(D^s_P) = D_P$, where $\psi_j \in \mathrm{Aut}(D_P)$ given in (3).

Next, the above argument shows that

$$\lim_{D^s_P \cap \Gamma_c \ni q \to (0',1)} \psi_a^{-1}(D^s_P) = D_P;\ \lim_{D^s_P \cap \Gamma_c \ni q \to (0',1)} \psi_a^{-1}(D^s_{P,r}) = D_{P,r}, \tag{6}$$

where $\psi_a \in \mathrm{Aut}(D_P)$ given by

$$\psi_a(z) = \left(\frac{(1-|a|^2)^{1/2m_1}}{(1+\bar{a}z_n)^{1/m_1}} z_1, \dots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1+\bar{a}z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1 + \bar{a}z_n}\right),\ j \geqslant 1,$$

where $a := q_n$. In addition, for $q \in D^s_{P,r} \cap \Gamma_c$ one has

$$\psi_a^{-1}(q) = \Big(\frac{q_1}{\lambda^{1/2m_1}}, \dots, \frac{q_{n-1}}{\lambda^{1/2m_{n-1}}}, 0\Big) \in D_{P,r} \cap \{z_n = 0\} \Subset D_P \cap \{z_n = 0\},$$

where $\lambda = 1 - |a|^2$. Therefore, by (6) and by Lemma 2.1 in [NNC21] we finally conclude that there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \delta_r/d > 0,\ \forall q \in D_{P,r_0} \cap \Gamma_c \cap B(p, \epsilon_0),$$

where d denotes the diameter of D_P and $\delta_r := \mathrm{dist}(Z_r(P), Z_1(P))/2$ with $Z_r(P) = \Big\{z' \in \mathbb{C}^{n-1} \colon P(z') = r\Big\}$. Hence, the proof is complete with $\gamma_2 = \delta_r/d$. □

Proof of Theorem 1.4. Suppose that $\{q_j\}$ converges Λ-tangentially to $(0', 1)$ in D_P. For simplicity, let us denote by $a_j = \eta_{jn}$. Then we consider the sequence of automorphisms $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ given in (3).

Let us set $b_j = (b'_j, 0) := \psi_j^{-1}(q_j)$ for all $j \geqslant 1$. Then, a straightforward computation shows that

$$b_j = \psi_j^{-1}(q_j) = \Big(\frac{\eta_{j1}}{\lambda_j^{1/2m_1}}, \ldots, \frac{\eta_{j(n-1)}}{\lambda_j^{1/2m_{n-1}}}, 0\Big) \in D_P \cap \{z_n = 0\},$$

where $\lambda_j = 1 - |a_j|^2$ for all $j \geqslant 1$.

Since $\{q_j\}$ converges Λ-tangentially to $(0', 1)$ in D_P, it follows that there exists a sequence $\{r_j\} \subset (0, 1)$ with $r_j \to 1$ as $j \to \infty$ such that

$$|a_j|^2 + \frac{1}{r_j} P(q'_j) = |\eta_{jn}|^2 + \frac{1}{r_j} P(q'_j) \geqslant 1, \ \forall j \geqslant 1,$$

which implies that

$$1 > P(b'_j) = \frac{1}{\lambda_j} P(q'_j) = \frac{1}{1 - |a_j|^2} P(q'_j) \geqslant r_j$$

for all $j \geqslant 1$. Therefore, we obtain that $P(b'_j) \to 1$ as $j \to \infty$, and hence by passing to a subsequence if necessary, we may assume that $\psi_j^{-1}(q_j)$ converges to some strongly pseudoconvex boundary point $p \in \partial D_P \cap \{z_n = 0\}$.

Since $\psi_j(0', 0) = (0', a_j) \to (0', 1)$ as $j \to \infty$ and the boundary point $(0', 1)$ is of D'Angelo finite type, by [Ber94, Proposition 2.1] it follows that

$$\lim_{j \to \infty} \psi_j^{-1}(\Omega_j) = \lim_{j \to \infty} \psi_j^{-1}(\Omega_j \cap U) = \lim_{j \to \infty} \psi_j^{-1}(D_P \cap U) = D_P.$$

In addition, for any $\epsilon > 0$ sufficiently small there exists $j_0 \geqslant 1$ such that

$$\psi_j^{-1}(\overline{\Omega_j}) \backslash B((0', -1), \epsilon) = \overline{D_P} \backslash B((0', -1), \epsilon)$$

for any $j \geqslant j_0$. Hence, since $\sigma_{D_P}(b_j) \to 1$ as $j \to \infty$ and by Theorem 3.1 in [KZ16], one concludes that $\sigma_{\Omega_j}(q_j) = \sigma_{\psi_j^{-1}(\Omega_j)}(b_j) \to 1$ as $j \to \infty$. □

The following proposition provides a uniform lower bound for the squeezing function near $(0', 1) \in \partial D_P$.

Proposition 4.1. *Let Ω be a subdomain of D_P and $\Omega \cap U = D_P \cap U$ for a fixed neighborhood U of $p = (0', 1)$ in $\mathbb{C}^n$. Then, there exist $\epsilon_0, \gamma_0 > 0$ depending only on D_P such that*

$$\sigma_\Omega(z) > \gamma_0, \ \forall z \in D_P \cap B(p; \epsilon_0).$$

Proof. By Theorem 3.1 in [KZ16], for any $p \in \{(z', 0) \in D_P \colon P(z') = 1\}$ we have $\lim_{z \to p} \sigma_{D_P}(z) = 1$. Then, there exists $r_0 \in (0, 1)$ such that

$$\sigma_{D_P}(z', 0) > 3/4, \ \forall z' \in \mathbb{C}^{n-1} \text{ with } P(z') \geqslant r_0. \tag{7}$$

For $q \in D_P$, we consider the automorphism $\psi_a \in \operatorname{Aut}(D_P)$, given by

$$\psi_a(z) = \left(\frac{(1 - |a|^2)^{1/2m_1}}{(1 + \bar{a} z_n)^{1/m_1}} z_1, \ldots, \frac{(1 - |a|^2)^{1/2m_{n-1}}}{(1 + \bar{a} z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1 + \bar{a} z_n}\right),$$

where $a := q_n$. In addition, let us set $b := \psi_a^{-1}(q)$. Then, a straightforward computation shows that

$$b = (b', 0) = \psi_a^{-1}(q) = \Big(\frac{q_1}{\lambda^{1/2m_1}}, \ldots, \frac{q_{n-1}}{\lambda^{1/2m_{n-1}}}, 0\Big) \in D_P \cap \{z_n = 0\},$$

where $\lambda = 1 - |a|^2$.

Now we consider the following three cases:

Case 1. $q \in D_{P,r_0}$. In this case, we have

$$|a|^2 + \frac{1}{r_0}P(q') = |q_n|^2 + \frac{1}{r_0}P(q') < 1,$$

which implies that

$$P(b') = \frac{1}{\lambda}P(q') = \frac{1}{1-|a|^2}P(q') < r_0.$$

Since $\psi_a(0', 0) = (0', a) \to (0', 1)$ as $a \to 1$ and the boundary point $(0', 1)$ is of D'Angelo finite type, again by [Ber94, Proposition 2.1] it follows that

$$\lim_{a\to 1} \psi_a^{-1}(\Omega) = \lim_{a\to 1} \psi_a^{-1}(\Omega \cap U) = \lim_{a\to 1} \psi_a^{-1}(D_P \cap U) = D_P.$$

Therefore, by Lemma 2.1 in [NNC21] there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \frac{\delta_{r_0}}{d} > 0, \ \forall q \in D_{P,r_0} \cap B(p, \epsilon_0),$$

where d denotes the diameter of D_P and $\delta_{r_0} := \mathrm{dist}(Z_{r_0}(P), Z_1(P))/2$ with $Z_{r_0}(P) = \Big\{z' \in \mathbb{C}^{n-1} \colon P(z') = r_0\Big\}$.

Case 2. $q \in D_P \backslash D_{P,r_0}$. Then we have

$$|a|^2 + \frac{1}{r_0}P(q') = |q_n|^2 + \frac{1}{r_0}P(q') \geqslant 1,$$

which implies that

$$P(b') = \frac{1}{\lambda}P(q') = \frac{1}{1-|a|^2}P(q') \geqslant r_0.$$

As in Case 1 and by (7), there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \frac{1}{2}, \ \forall q \in (D_P \backslash D_{P,r_0}) \cap B(p, \epsilon_0),$$

Hence, altogether, the proof is complete with $\gamma_0 = \min\{\frac{\delta_{r_0}}{d}, \frac{1}{2}\}$.

□

We close this section with an example, which is a generalization of Example 1.1.

Example 4.1. Fix positive integers $m_1, \ldots, m_{n-1}$ and denote by $\Lambda := (1/m_1, \ldots, 1/m_{n-1})$. Let us consider a general ellipsoid D_P in $\mathbb{C}^n$ ($n \geqslant 2$), defined by

$$D_P := \{(z', z_n) \in \mathbb{C}^n \colon |z_n|^2 + P(z') < 1\},$$

where $P(z')$ is a $(1/m_1, \ldots, 1/m_{n-1})$-homogeneous polynomial given by

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} {z'}^K \bar{z}'^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \neq 0$. Suppose that the domain D_P is a WB-domain, i.e., ∂D_P is strongly pseudoconvex at every boundary point outside the set $\{(0', e^{i\theta}) \colon \theta \in \mathbb{R}\}$ (cf. [AGK16]).

Now let us denote by $\rho(z) := |z_n|^2 - 1 + P(z')$ a local defining function for D_P and consider a sequence $\{a_j = (a_j', a_{jn})\} \subset D_P$ which converges Λ-tangentially to $p := (0', 1)$. Since D_P is invariant under the map $z' \mapsto z'; z_n \mapsto e^{i\theta} z_n$ and σ_{D_P} is invariant under biholomorphisms, we may assume that $\mathrm{Im}(a_{jn}) = 0$ for all j. Since $\mathrm{dist}(a_j, \partial D_P) \approx$

$-\rho(a_j) \approx 1 - |a_{jn}|^2 - P(a'_j)$ and $\{a_j\}$ converges Λ-tangentially to p, it follows that $P(a'_j) \geqslant c_j \operatorname{dist}(a_j, \partial D_P)$ for some sequence $\{c_j\} \subset \mathbb{R}$ with $0 < c_j \to +\infty$. This implies that $P(a'_j) \geqslant c'_j(1 - |a_{jn}|^2 - P(a'_j))$ for some sequence $\{c'_j\} \subset \mathbb{R}$ with $0 < c'_j \to +\infty$ and hence

$$P(a'_j) \geqslant \frac{c'_j}{1 + c'_j}(1 - |a_{jn}|^2), \forall j \geqslant 1.$$

Let us denote by $\tilde{\psi}_j$ the automorphism of D_P, given by

$$\tilde{\psi}_j(z) = \left(\frac{(1 - |a_{jn}|^2)^{1/2m_1}}{(1 - \bar{a}_{jn} z_n)^{1/m_1}} z_1, \ldots, \frac{(1 - |a_{jn}|^2)^{1/2m_{n-1}}}{(1 - \bar{a}_{jn} z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n - a_{jn}}{1 - \bar{a}_{jn} z_n} \right),$$

and hence $\tilde{\psi}_j(a_j) = (b'_j, 0)$, where

$$b'_j = \left(\frac{a_{j1}}{(1 - |a_{jn}|^2)^{1/2m_1}}, \ldots, \frac{a_{j(n-1)}}{(1 - |a_{jn}|^2)^{1/2m_{n-1}}} \right).$$

Thanks to the boundedness of $\{b'_j\}$, without loss of generality we may assume that $b'_j \to b' \in \mathbb{C}^{n-1}$ as $j \to \infty$. In addition, we have that $P(b'_j) = \dfrac{1}{1 - |a_{jn}|^2} P(a'_j) \geqslant \dfrac{c'_j}{1 + c'_j}, \forall j \geqslant 1$. Therefore, we arrive at the situation $b'_j \to b'$ with $P(b') = 1$ and thus $\tilde{\psi}_j(a_j)$ converges to the strongly pseudoconvex boundary point $(b', 0)$ of ∂D_P, which implies by [KZ16, Theorrem 3.1] that $\sigma_{D_P}(a_j) = \sigma_{D_P}(\tilde{\psi}_j(a_j)) \to 1$ as $j \to \infty$ even the boundary point p is weakly pseudoconvex.

Acknowledgement. Part of this work was done while the first author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for financial support and hospitality. The first author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2021.42. The third author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2019.304.

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