Weak solutions to equations of complex Monge-Ampère type on open subsets of $\mathbb{C}^n$

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Abstract

In this paper, we prove the existence of weak solutions to equations of complex Monge-Ampère type in the class $\mathcal{D}(\Omega)$ on an open subset Ω of $\mathbb{C}^n$.

1 Introduction

Let Ω be an open subset in $\mathbb{C}^n$ and μ a positive Borel measure on Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow [0, +\infty)$ is a $dt \times d\mu$-measurable function. The equation of the form

$$(dd^c u)^n = F(u, .)d\mu, \tag{1.1}$$

where u is a plurisubharmonic function on Ω is called to be the equation of complex Monge-Ampère type. The proof of the existence of weak solutions of this equation has been investigated by many authors; see for example [3], [5], [12], [14], [15], [16], [17], [26], [28], [29], [30] and references therein for futher results about complex Monge-Ampère equations. In the case, Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and μ vanishes on all pluripolar sets and $\mu(\Omega) < \infty$, F is bounded by an integrable function for μ which is independent of the first variable then for all $f \in MPSH(\Omega) \cap \mathcal{E}(\Omega)$, in [15] Cegrell and Kołodziej proved that (1.1) has a solution $u \in \mathcal{F}^a(\Omega, f)$ where $MPSH(\Omega)$ denotes the set of maximal plurisubharmonic functions and $\mathcal{F}^a(\Omega, f)$ is the set of plurisubharmonic functions introduced and investigated by Cegrell in [11] and [12]. Next, in [17] Czyż investigated the equation (1.1) in the class $\mathcal{N}(\Omega, f)$. He proved that if μ vanishes on pluripolar sets of Ω, F is a continuous function of the first variable and above bounded by a function $g \in L^1((-\varphi)\mu)$ which is independent of the first variable then the equation (1.1) is solvable in the class $\mathcal{N}(\Omega, f)$ (see [17] for more details). Recently, under the same assumption that μ vanishes on all pluripolar

2010 *Mathematics Subject Classification*: 32U05. 32U10,32W20.

Key words and phrases: Plurisubharmonic functions, weak solutions to equations of complex Monge-Ampère type, the class $\mathcal{D}(\Omega)$.

sets of Ω and there exists a subsolution $v_0 \in \mathcal{N}^a(\Omega)$, i.e there exists a function $v_0 \in \mathcal{N}^a(\Omega)$ such that

$$(dd^c v_0)^n \geq F(v_0, .)d\mu, \tag{1.2}$$

Benelkourchi in [5] proved that (1.1) has a solution $u \in \mathcal{N}^a(\Omega, f)$.
In this note we study weak solutions of the equation (1.1) on an arbitrary open subset of $\mathbb{C}^n$ in the class $\mathcal{D}(\Omega)$ introduced and investigated by Błocki in [8]. The main result is the following.

Theorem 1.1. *Let Ω be an open subset in $\mathbb{C}^n$ and μ be a nonnegative Borel measure in Ω which vanishes on pluripolar sets of Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow [0, +\infty)$ is a $dt \times d\mu$-measurable function such that:*

(1) For all $z \in \Omega$, the function $t \longmapsto F(t, z)$ is continuous and nondecreasing.

(2) For all $t \in \mathbb{R}$, the function $z \longmapsto F(t, z)$ belongs to $L^1(d\mu)$.

(3) $\exists\ u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ such that

$$F(u, .)d\mu \leq (dd^c u)^n.$$

Then there exists $\psi \in \mathcal{D}(\Omega), u \leq \psi \leq 0$ such that

$$(dd^c \psi)^n = F(\psi, .)d\mu.$$

The organization of the paper is as follows. In Section 2 we recall some notions of pluripotential theory which is necessary for the next results of the paper. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminairies

In this section, we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [1]-[30]. Let n be a positive integer and let Ω be an open set in $\mathbb{C}^n$. We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on Ω and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. We first recall the definition of the complex Monge-Ampère measure on an open set in $\mathbb{C}^n$ (see [8]).

Definition 2.1. A plurisubharmonic function u defined on Ω belongs to $\mathcal{D}(\Omega)$ if there exists a nonnegative Radon measure μ on Ω such that if $\Omega' \subset \Omega$ is an open subset and $\{u_j\} \subset PSH(\Omega') \cap \mathcal{C}^\infty(\Omega')$ is a sequence which decreases to u in Ω' then $(dd^c u_j)^n$ tends weakly to μ in Ω'. The measure μ we then denote by $(dd^c u)^n$.

Note that $PSH(\Omega) \cap L^\infty_{loc}(\Omega) \subset \mathcal{D}(\Omega)$. Moreover, if $n = 1$ then $SH(\Omega) = \mathcal{D}(\Omega)$.

Definition 2.2. *A bounded domain $\Omega \subset \mathbb{C}^n$ is said to be hyperconvex if there exists $\varphi \in PSH^-(\Omega)$ such that $\{\varphi < -\varepsilon\} \Subset \Omega$ for every $\varepsilon > 0$.*

For a bounded hyperconvex domain, Cegrell [11] introduced the following classes of plurisubharmonic functions.

Definition 2.3. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function φ on Ω belongs to $\mathcal{E}_0(\Omega)$ if $\{\varphi < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\int_\Omega (dd^c\varphi)^n < +\infty$.

Let $\mathcal{F}(\Omega)$ be the family of plurisubharmonic functions φ defined on Ω, such that there exists a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to φ on Ω as $j \to \infty$ and

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty.$$

We denote by $\mathcal{E}(\Omega)$ the family of plurisubharmonic functions φ defined on Ω such that for every open set $G \Subset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ satisfying $\psi = \varphi$ in G.

Definition 2.4. Let $\mathcal{K} \in \{\mathcal{F}, \mathcal{E}, \mathcal{D}\}$. We denote by $\mathcal{K}^a(\Omega)$ the subclass of $\mathcal{K}(\Omega)$ such that the Monge-Ampère measure $(dd^c.)^n$ vanishes on all pluripolar sets of Ω.

Proposition 2.5. *If Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ then $\mathcal{E}(\Omega) = PSH^-(\Omega) \cap \mathcal{D}(\Omega)$.*

Proof. See Theorem 2.4 in [8]. □

Remark 2.6. If Ω is an open subset of $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega)$ then $u|_{\mathbb{B}} - \sup_{\mathbb{B}} u \in \mathcal{E}(\mathbb{B})$ for all open ball $\mathbb{B} \Subset \Omega$.

Now we recall the definition of the important class of plurisubharmonic functions. This is the class of maximal plurisubharmonic functions. Let Ω be an open subset of $\mathbb{C}^n$.

Definition 2.7. *A plurisubharmonic function u on Ω is said to be maximal plurisubharmonic (briefly, $u \in MPSH(\Omega)$) if for every $v \in PSH(\Omega)$, $v \leq u$ outside a compact subset of Ω implies $v \leq u$ on Ω.*

It is well known (see, e.g.,[25]), locally bounded plurisubharmonic functions are maximal if and only if they satisfy the homogeneous Monge-Ampère equation $(dd^cu)^n = 0$. In [9], Błocki extended the above result for the class $\mathcal{E}(\Omega)$ in the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$.
Next, we recall the class $\mathcal{N}(\Omega)$ introduced and investigated in [12]. Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $\{\Omega_j\}_{j\geq 1}$ a fundamental sequence of Ω. This is an increasing sequence of strictly pseudoconvex subsets Ω_j of Ω such that $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Let $u \in \mathrm{PSH}^-(\Omega)$. For each $j \geq 1$, put

$$u^j = \sup\{\varphi : \varphi \in \mathrm{PSH}^-(\Omega), \quad \varphi \leq u \ \text{ on } \ \Omega \setminus \overline{\Omega}_j\}.$$

As in [11], the function $\widetilde{u} = \left(\lim_{j\to\infty} u^j\right)^* \in \mathrm{PSH}(\Omega)$ and $\widetilde{u} \in MPSH(\Omega)$. Set

$$\mathcal{N} = \mathcal{N}(\Omega) = \{u \in \mathcal{E} : \widetilde{u} = 0\}$$

or equivalently,

$$\mathcal{N} = \mathcal{N}(\Omega) = \{u \in \mathrm{PSH}^-(\Omega) : u^j \uparrow 0\}.$$

By using the comparison principle in [12], we can prove that $\mathcal{F}(\Omega) \subset \mathcal{N}(\Omega)$.

3 Weak solutions to equations of complex Monge-Ampère type.

In this section we prove Theorem 1.1. We need the following lemma.

Lemma 3.1. *Let Ω be an open set in $\mathbb{C}^n$ and μ be a non-negative Radon measure on Ω which vanishes on pluripolar sets of Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow [0, +\infty)$ is a $dt \times d\mu$-measurable function such that:*
(i) For all $z \in \Omega$, the function $t \longmapsto F(t, z)$ is continuous and nondecreasing.
(ii) For all $t \in \mathbb{R}$, the function $z \longmapsto F(t, z)$ belongs to $L^1(d\mu)$.
(iii) $\exists\ u \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ such that

$$F(u, .)d\mu \leq 1_{\{u>-\infty\}}(dd^c u)^n.$$

Then, for every open ball $\mathbb{B} \Subset \Omega$, there exists $\psi \in \mathcal{D}(\Omega)$ satisfying
(i) $u \leq \psi \leq 0$ on Ω;
(ii) $(dd^c\psi)^n \geq 1_{\{u>-\infty\}}F(\psi, .)d\mu$ in Ω;
(iii) $(dd^c\psi)^n = 1_{\{u>-\infty\}}F(\psi, .)d\mu$ in $\mathbb{B}$.

Proof. Fix a ball $\mathbb{B} \Subset \Omega$. From the hypothesis we have

$$1_{\{u>-\infty\}}F(u, .)d\mu \leq 1_{\{u>-\infty\}}(dd^c u)^n \leq (dd^c u)^n \tag{3.1}$$

on Ω. Set

$$w = \sup\{\varphi \in \mathcal{D}(\Omega) \cap PSH^-(\Omega) : (dd^c\varphi)^n \geq 1_{\{u>-\infty\}}F(\varphi, .)d\mu \ \text{ on } \Omega\}.$$

By the hypothesis we infer that $u \leq w \leq 0$. Hence, by [8] it follows that $w \in \mathcal{D}(\Omega)$. By using Choquet's Lemma in [6] and Proposition 2.3 in [**?**] we can choose an increasing sequence $\{\varphi_j\}_{j\geq 1} \subset \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ such that $\varphi_j \nearrow w$ a.e on Ω and

$$(dd^c\varphi_j)^n \geq 1_{\{u>-\infty\}}F(\varphi_j, .)d\mu. \tag{3.2}$$

The main result in [13] implies that $(dd^c\varphi_j)^n$ is weakly convergent to $(dd^c w)^n$ as $j \to \infty$. On the other hand, by the hypothesis we have $F(\varphi_j, .)d\mu$ is also weakly convergent to $F(w, .)d\mu$ as $j \to \infty$. Thus, from (3.2) we get that

$$(dd^c w)^n \geq 1_{\{u>-\infty\}}F(w, .)d\mu. \tag{3.3}$$

Now, since the measure $1_{\{u>-\infty\}}F(w, .)d\mu$ vanishes on all pluripolar sets of Ω, by [11] we can find $h \in \mathcal{F}^a(\mathbb{B})$ such that

$$(dd^c h)^n = 1_{\{u>-\infty\}}F(w, .)d\mu \text{ in } \mathbb{B}.$$

Put $g = \max(h, w)$. Then $g \in \mathcal{F}^a(\mathbb{B})$. We have

$$(dd^c h)^n = 1_{\{u>-\infty\}}F(w, .)d\mu. \tag{3.4}$$

$$(dd^c w)^n \geq 1_{\{u>-\infty\}}F(w, .)d\mu. \tag{3.5}$$

Coupling (3.4) and (3.5) and using Proposition 2.3 in [**?**] we infer that

$$(dd^c g)^n \geq 1_{\{u>-\infty\}}F(g, .)d\mu. \tag{3.6}$$

By [5] there exists $g_1 \in \mathcal{N}^a(\mathbb{B})$, $g_1 \geq g$ on $\mathbb{B}$ with

$$(dd^c g_1)^n = 1_{\{u>-\infty\}} F(g_1, .)d\mu. \tag{3.7}$$

on $\mathbb{B}$. However, by the hypothesis (ii) we have

$$\int\limits_{\mathbb{B}} (dd^c g_1)^n = \int\limits_{\mathbb{B}} 1_{\{u>-\infty\}} F(g_1, .)d\mu \leq \int\limits_{\mathbb{B}} F(0, .)d\mu < +\infty.$$

Proposition 2.2 in [18] implies that $g_1 \in \mathcal{F}^a(\mathbb{B})$. Let ψ be the smallest plurisubharmonic majorant of the function

$$\eta = \begin{cases} g_1 & \text{in } \mathbb{B}, \\ w & \text{in } \Omega \backslash \mathbb{B}. \end{cases}$$

Since $w \leq g \leq g_1$ on $\mathbb{B}$, then we have $w \leq \psi$ on Ω. By [8] it follows that $\psi \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ and $u \leq \psi$ on Ω. We prove that

$$(dd^c \psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu, \tag{3.8}$$

on Ω and

$$(dd^c \psi)^n = 1_{\{u>-\infty\}} F(\psi, .)d\mu, \tag{3.9}$$

on $\mathbb{B}$.

Indeed, by the definition of ψ we note that $\psi = w$ in the interior of $\Omega \setminus \mathbb{B}$ and $g_1 = \psi$ in $\mathbb{B}$. Hence, it follows that

$$(dd^c \psi)^n = (dd^c g_1)^n = 1_{\{u>-\infty\}} F(g_1, .)d\mu = 1_{\{u>-\infty\}} F(\psi, .)d\mu$$

on $\mathbb{B}$ and we have (3.9). It remains to prove (3.8) holds. By the definition of ψ it is clear that $\psi = w$ on the interior of $\Omega \setminus \mathbb{B}$. Thus, $(dd^c\psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu$ on the interior of $\Omega \setminus \mathbb{B}$. We will to prove $(dd^c\psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu$ on $\Omega \setminus \mathbb{B}$. It suffices to prove $(dd^c\psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu$ on $\partial\mathbb{B}$. By the definition of ψ it follows that $w = \psi$ on $\partial\mathbb{B} \setminus E$, where E is a pluripolar subset of $\partial\mathbb{B}$ containing $\{w = -\infty\}$. Let $K \subset \partial\mathbb{B}\backslash E$ be a compact set. Since $K \subset \{\psi + \frac{1}{j} > w\}$, by Theorem 4.1 in [24] we have

$$\begin{aligned} 1_{\{w>-\infty\}} F(\psi, .)d\mu(K) &= 1_{\{w>-\infty\}} F(w, .)d\mu(K) \\ &\leq \int_K (dd^c w)^n = \lim_{j \to +\infty} \int_K (dd^c \max(\psi + \frac{1}{j}, w))^n \\ &\leq \int_K (dd^c \max(\psi, w))^n = \int_K (dd^c \psi)^n. \end{aligned}$$

Hence, it follows that

$$(dd^c \psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu \text{ on } \partial\mathbb{B}\backslash E.$$

Because μ vanishes on pluripolar sets then

$$(dd^c \psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu \text{ on } \partial\mathbb{B}.$$

Combining this with (3.8) and (3.9) we obtain

$$(dd^c \psi)^n \geq 1_{\{w>-\infty\}} F(\psi, .)d\mu \geq 1_{\{u>-\infty\}} F(\psi, .)d\mu,$$

on Ω because $u \leq w$ on Ω. The proof of lemma is finished. □

Proof of Theorem 1.1. Similarly as in the proof of Lemma 3.1, we now set

$$w := \sup\{\varphi \in \mathcal{D}(\Omega) \cap PSH^-(\Omega) : (dd^c\varphi)^n \geq 1_{\{u>-\infty\}}F(\varphi, .)d\mu \text{ on } \Omega\}.$$

Then $u \leq w \leq 0$ on Ω. Hence, as above, $w \in \mathcal{D}(\Omega)$. Moreover, we have

$$(dd^c w)^n \geq 1_{\{u>-\infty\}}F(w, .)d\mu$$

on Ω. On the other hand, from $u \leq w$ on Ω then $\{u > -\infty\} \subset \{w > -\infty\}$. Therefore,

$$\begin{aligned} 1_{\{w>-\infty\}}(dd^c w)^n &\geq 1_{\{w>-\infty\}}1_{\{u>-\infty\}}F(w, .)d\mu \\ &= 1_{\{u>-\infty\}}F(w, .)d\mu = F(w, .)d\mu, \end{aligned}$$

because μ vanishes on pluripolar sets of Ω. Let $\mathbb{B} \Subset \Omega$ be an arbitrary ball of Ω. By Lemma 3.1 there exists $\psi \in \mathcal{D}(\Omega) \cap PSH^-(\Omega)$ such that
(a) $w \leq \psi \leq 0$ on Ω.
(b) $(dd^c\psi)^n \geq 1_{\{w>-\infty\}}F(\psi, .)d\mu$ on Ω.
(c) $(dd^c\psi)^n = 1_{\{w>-\infty\}}F(\psi, .)d\mu$ on $\mathbb{B}$.
By (a) we get that $u \leq \psi \leq 0$ on Ω. (b) implies that

$$(dd^c\psi)^n \geq 1_{\{w>-\infty\}}F(\psi, .)d\mu \geq 1_{\{u>-\infty\}}F(\psi, .)d\mu$$

on Ω. By the definition of w it follows that $\psi = w$ on Ω. Hence, $(dd^c w)^n = (dd^c\psi)^n$ on Ω. By (c) we get that

$$(dd^c\psi)^n = 1_{\{w>-\infty\}}F(\psi, .)d\mu = F(\psi, .)d\mu,$$

on $\mathbb{B}$. Because $\mathbb{B}$ is arbitrary then we obtain that $(dd^c\psi)^n = F(\psi, .)d\mu$ on Ω and the proof of Theorem 1.1 is complete. □

Acknowlegements. This work was done during the stay of the first-named author at Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to express his gratitude to the Institute for the support.

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