FINITE-TIME STABILITY OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTERVAL TIME-VARYING DELAY

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Abstract. In this paper, we propose a novel approach to study finite-time stability of fractional differential equations (FDEs) with delays via Laplace transforms and LMI techniques. The advantage of our proposed method is that we can construct a simple Lyapunov functional to derive delay-dependent stability conditions for the systems with interval time-varying delay. The conditions are presented in terms of the Mittag-Leffler function and linear matrix inequalities (LMIs), which are less conservative and more easier to verify than the existing ones. The proposed method is also applicable for finite-time stability of linear uncertain time-varying delay FDEs. A numerical example is given to show the validity and effectiveness of the proposed results.

Key words. fractional derivatives, finite-time stability, Laplace transforms, time-varying delay, linear matrix inequality.

AMS subject classifications. 34D20, 93D05, 93D20

1. Introduction. Over the last decades, considerable attention has been paid to stability theory of fractional differential equations (FDES)(see [1-3] and the refrences therein). Stability analysis of FDEs is more complicated than that of ordinary differential equations, because fractional derivatives are nonlocal and have weakly singular kernels. In recent years, various effective methods have been employed to derive stability criteria for the FDEs. The most well-known one is the Lyapunov function method, which was used in [4,5] by applying the Lyapunov stability theorem extended to FDEs. In addition, Laplace transform and Lambert functions approach ([6,7]), Gronwall's approach ([8-10]) and Razumikhin approach ([5, 11, 12]) are also used to investigate the stability of FDEs. On the other hand, there has been a considerable research interest in study of FDEs with time-varying delays. The stability analysis of FDEs with time-varying delays are typical required. It is no doubt that the Lyapunov function method provides a very effective tool to analyze stability of nonlinear systems. However, this method is not effectively applied for FDEs with time-varying delays. In fact, it is very difficult to construct a Lyapunov-Krasovskiii functional and calculate its fractional-order derivative for FDEs with delays. This is the main reason that there are very few delay-dependent criteria for asymptotic stability of FDEs with delays. In [13, 14], to overcome the difficulty of calculating the fractional-order derivative the authors attempt to construct an appropriate Lyapunov-Krasovskii functional associated with the RiemannLiouville fractional integral. However, the proof of the main theorems in these papers contains a gap, that is, the positive definiteness of the constructed Lyapunov-Krasovskii functional can not be guaranteed by the positivity of the RiemannLiouville fractional-order integral and the definite integral functionals. Very recently, the authors of [15] tried to overcome this difficulty by proving a Lyapuov stability theorem for FDEs with delays (Theorem 1 and Theorem 2 in [15]), unfortunately, the obtained result is also incorrect, since their proof is based on a wrong derivation, that is, the definite positiveness of the Lyapunov functional for systems with delays $V(x_t)$ defined on infinite-dimensional space

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$C([-h,0]$ and the Lyapunov functional for systems without delays $V(x(t))$ defined on finite-dimensional space R^n are different.

Motivated by the above discussion, we study finite-time stability problem of a class of nonlinear FDEs with delay. A central analysis technique is enabled by proposing Laplace transform approach combining with Lyapunov function method to study finite-time stability of FDEs with interval time-varying delay. The approach is used to derive new delay-dependent sufficient conditions for finite-time stability in terms of the Mittag-Leffler function and LMIs, which are less conservative and more easier to verify than the existing ones. The obtained results are applied to finite-time stability of linear uncertain time-varying delay systems. A numerical example is given to show the effectiveness of the obtained result.

The paper is organized as follows. In Section 2, we provide some preliminaries on the fractional-order derivative, the Laplace transform, the finite-time stability problem and some auxiliary lemmas needed in next section. In Section 3, delay-dependent sufficient conditions for finite-time stability of FDEs with time-varying delays are established. The validity and effectiveness of the theoretical result is illustrated by a numerical example.

2. Prilimeries. The following are some notations and definitions used in this paper. $\mathbb{N}$ denotes the set of all non-negative integers, $\mathbb{C}$ denotes the complex space; $R^{n\times r}$ denotes the space of all $(n\times r)-$ matrices; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{max}(A) = max\{Re(\lambda) : \lambda\in\lambda(A)\}$; $\lambda_{min}(A) = min\{Re(\lambda) : \lambda\in\lambda(A)\}$; $\|A\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(A^\top A)}$; $C([h,0],R^n)$ denotes the set of all R^n-valued continuously functions on $[h,0]$; $[a]$ denotes the integer part of number a.

We first give some basic concepts of fractional calculus used in the paper.

Definition 2.1. *([1]) The Caputo fractional derivative is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, \quad t\ge 0,\ n-1<\alpha\le n.$$

where $n\in\mathbb{N},\Gamma(.)$ *is the gamma function,* $\Gamma(s) = \int\limits_0^\infty e^{-t}t^{s-1}dt, s\in\mathbb{C}, Re(s)>0$. *In particular, for* $0<\alpha<1$, *we have*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(s)}{(t-s)^{\alpha}}ds, \quad t\ge 0.$$

Lemma 2.2. [1]. *For* $\alpha\in(0,1)$, *and* $z\in\mathbb{C}$, $Re(z)>0$, *we have*
(i) The Gamma function converges in the right haft of the complex plane $Re(z)>0$.
(ii) $\Gamma(z+1) = z\Gamma(z)$. *In particular,*

$$\Gamma(n+1) = n!,\ n=1,2,\dots,\quad \Gamma(1)=1.$$

The Mittag-Lefller function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)},$$

where $\alpha > 0,\ \beta > 0$, and $z \in C$. For $\beta = 1$, we denote $E_\alpha(z) := E_{\alpha,1}(z)$.

LEMMA 2.3. [1]. *Given $\alpha > 0$, we have*
(i) $E_\alpha(z) \geq 1,\ \forall z \in R^+$,
(ii) $E_\alpha(z)$ is a increasing function on R^+.

The Laplace transform $\mathbb{L}[f(t)](s)$ of the integrable function $f(.)$, which is defined by

$$F(s) = \mathbb{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt.$$

LEMMA 2.4. [1]. *Let $f(.) : R^+ \to R$ be a integrable function, $\alpha \in (0,1), \beta > 0$, we have*
(i)

$$\mathbb{L}[D^\alpha f(t)](s) = s^\alpha \mathbb{L}[f(t)](s) - s^{\alpha-1} f(0),$$

(ii)

$$\mathbb{L}[t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(ht^\alpha)](s) = \frac{k! s^{\alpha-\beta}}{s^\alpha - h)^{k+1}}, \quad k = 0, 1, 2, ... \textit{ provided} \quad Re(s) > h^{1/\alpha}.$$

(iii)

$$\mathbb{L}[f * g(t)](s) = \mathbb{L}[f(t)](s) \cdot \mathbb{L}[g(t)](s),$$

where $f(t),\ g(t)$ are integrable functions on R^+, the convolution of $f(t)$ and $g(t)$ is defined by $f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

Consider a fractional differential equation with delay of the form

$$\begin{cases} D^\alpha x(t) &= Ax(t) + Dx(t-h(t)) + f(t, x(t), x(t-h(t))), \\ x(\theta) &= \varphi(\theta),\ \theta \in [-h_2, 0], \end{cases} \tag{2.1}$$

where $\alpha \in (0,\ 1), x(t) \in R^n$, the delay function $h(t)$ is continuous and satisfies the following condition:

$$0 < h_1 \leq h(t) \leq h_2, \quad t \geq 0,$$

the constant matrices $A, D \in R^{n\times n}$, the initial function $\varphi \in C([-h_2, 0], R^n)$ with the norm

$$\|\varphi\| = \sup_{t\in[-h_2,0]} \|\varphi(t)\|.$$

The nonlinear function $f : R^+ \times R^n \times R^n \to R^n$ satisfies the following condition

$$\exists E_1,\ E_2 \in R^{n\times n} : f(t, x, x_h)^T f(t, x, x_h) \leq x^T E_1^T E_2 x(t) + x_h^T E_2^T E_2 x_h(t), \tag{2.2}$$

for all $(t, x, x_h) \in R^+ \times R^n \times R^n$.

LEMMA 2.5. [16]. *Assume that the initial function $\varphi(t) \in C([-h,0],R^n)$, then the fractional differential delay equation (2.1) under the assumption (2.2) has a unique solution.*

DEFINITION 2.6. *For given positive numbers c_1, c_2, T, the system (2.1) is finite-time stable w.r.t. (c_1, c_2, T), if*

$$\sup_{s\in[-h_2,0]} \varphi(s)^T\varphi(s) \le c_1 \Rightarrow x(t)^T x(t) \le c_2,\ \forall t \in [0,T].$$

LEMMA 2.7. [17]. *Assume that $\alpha \in (0,\ 1)$, let $x(t) \in R^n$ be a continuously differentiable vector function, $P = P^T > 0,\ P \in R^{n\times n}$, then the following inequality holds:*

$$D^\alpha\Big(x(t)^T Px(t)\Big) \le 2x(t)^T PD^\alpha(x(t)), \qquad t \ge 0.$$

LEMMA 2.8. *Let $T, h_1 > 0,\ a \ge 1,\ b \ge 0$ and $G(t) : [-h_1,\ T] \to \mathbb{R}^+$ be a non-decreasing function satisfying*

$$G(t) \le aG(0) + bG(t-h_1),\ \forall t \in [0,T],$$

then, we have

$$G(t) \le G(0)a \sum_{j=0}^{[T/h_1]+1} b^j, \quad \forall t \in [0,\ T].$$

Proof. For each $t \in [0,T]$, there is $m \in \mathbb{N}$ such that

$$mh_1 \le t < (m+1)h_1.$$

By induction we obtain that

$$G(t) \le \begin{cases} \Big[a + ba + \cdots + b^m a\Big]G(0) + b^{m+1}G(t-(m+1)h_1) & m \ge 1, \\ aG(0) + bG(t-(m+1)h_1) & m = 0. \end{cases}$$

since $G(t)$ is nondecreasing on $-h_1 \le t-(m+1)h_1 < 0$, we get

$$G(t-(m+1)h_1) \le G(0).$$

Hence,

$$\begin{aligned} G(t) \le & \begin{cases} \Big[a + ba + \cdots + b^m a + b^{m+1}a\Big]G(0) & m \ge 1, \\ (a+ba)G(0) & m = 0, \end{cases} \\ =& a\sum_{j=0}^{m+1} b^j G(0). \end{aligned}$$

Besides, $t \le T$ leads to $m \le [T/h_1]$ and $G(t) \le a \sum_{j=0}^{[T/h_1]+1} b^j G(0)$. This completes the proof. □

3. Main result. In this section, we present delay-dependent conditions for finite-time stability of system (2.1). The conditions are presented in terms the Mittag-Leffler function and LMIs.

THEOREM 3.1. *The system (2.1) is finite-time stable w.r.t.* (c_1, c_2, T) *if there exist a positive scalar* γ*, symmetric positive definite matrices* $P, Q \in R^{n\times n}$ *satisfying the following conditions*

$$\begin{bmatrix} PA + A^TP - h_2P + \gamma E_1^TE_1 + \gamma E_2^TE_2 & PD & P \\ D^TP & -Q & 0 \\ P & 0 & -\gamma I \end{bmatrix} < 0, \tag{3.1}$$

$$Q + \gamma E_2^TE_2 < h_2P, \tag{3.2}$$

$$Cond(P) \sum_{j=0}^{[\frac{T}{h_1}]+1} (E_\alpha(h_2T^\alpha) - 1)^j E_\alpha(h_2T^\alpha) \le \frac{c_2}{c_1}, \tag{3.3}$$

where $Cond(P) = \dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}$.

Proof. Consider the following quadratic non-negative function

$$V(t, x(t)) = x(t)^TPx(t), \quad t \ge 0.$$

Taking the fractional-order of α derivative of $V(\cdot)$ in t along the solution of the system and using Lemma 2.7 and condition (2.2), we have

$$\begin{aligned} D^\alpha(V(t, x(t))) \le & 2x(t)^TPD^\alpha(x(t)) = 2x(t)^TP\Big(Ax(t) + Dx(t - h(t)) + f(\cdot)\Big) \\ \le & 2x(t)^TP\Big(Ax(t) + Dx(t - h(t)) + f(\cdot)\Big) \\ & - x(t - h(t))^TQx(t - h(t)) + x(t - h(t))^TQx(t - h(t)) \\ & - \gamma f(\cdot)^Tf(\cdot) + \gamma x^TE_1^TE_1x(t) + \gamma x(t - h(t))^TE_2^TE_2x(t - h(t)) \\ = & \xi(t)^T \begin{bmatrix} PA + A^TP - h_2P + \gamma E_1^TE_1 + \gamma E_2^TE_2 & PD & P \\ D^TP & -Q & 0 \\ P & 0 & -\gamma I \end{bmatrix} \xi(t) \\ & + h_2x(t)^TPx(t) + x(t - h(t))^T[Q + \gamma E_2^TE_2]x(t - h(t)). \end{aligned}$$

where $\xi(t) = \Big[x(t)^T \ x(t - h(t))^T \ f(\cdot)\Big]^T$.

Using the conditions (3.1), (3.2) gives

$$D^\alpha(V(t, x(t))) \le h_2V(t, x(t)) + h_2x(t - h(t))^TPx(t - h(t)).$$

Let us set

$$M(t) = D^\alpha(V(t, x(t))) - h_2V(t, x(t)), \tag{3.4}$$

we have

$$M(t) \le h_2x(t - h(t))^TPx(t - h(t)), \quad t \ge 0.$$

Applying the Laplace transform to both side of (3.4), by Lemma 2.4 (ii), we have

$$s^{\alpha}\mathbb{V}(s) - V(0,x(0))s^{\alpha-1} = h_2\mathbb{V}(s) + \mathbb{M}(s),$$

where $\mathbb{V}(s) = \mathbb{L}[V(t,x(t))](s),\ \mathbb{M}(s) = \mathbb{L}[M(t)](s)$, and hence

$$\mathbb{V}(s) = (s^{\alpha} - h_2)^{-1}(V(0,x(0))s^{\alpha-1} + \mathbb{M}(s)). \tag{3.5}$$

On the other hand, we can verify the correctness of following relations

$$(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha}) \geq 0 \quad \forall t \geq 0,\ \tau \in [0,t],\ h_2 > 0,$$

$$\sup_{0\leq\tau\leq t} M(\tau) \leq h_2 \sup_{0\leq\tau\leq t} x(\tau-h(\tau))^T Px(\tau-h(\tau)),$$

$$\begin{aligned}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau =& \int_0^t u^{\alpha-1}E_{\alpha,\alpha}(h_2u^{\alpha})du \\ =&\frac{1}{\alpha}\int_0^{t^{\alpha}} E_{\alpha,\alpha}(h_2v)dv = \frac{1}{\alpha}\int_0^{t^{\alpha}}\sum_{n=0}^{\infty}\frac{(h_2v)^n}{\Gamma(\alpha n+\alpha)}dv \\ =&\frac{1}{h_2}\sum_{n=0}^{\infty}\frac{(h_2v)^{n+1}}{\Gamma(\alpha n+\alpha)}\frac{1}{(n+1)\alpha}\Big|_{v=0}^{v=t^{\alpha}} = \frac{E_{\alpha}(h_2t^{\alpha})-1}{h_2}.\end{aligned}$$

Taking the inverse Laplace transform to both sides of equation (3.5), by Lemma 2 (ii)-(iii), we have

$$\begin{aligned}V(t,x(t)) =&V(0,x(0))E_{\alpha}(h_2t^{\alpha}) + \int_0^t M(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau \\ \leq&V(0,x(0))E_{\alpha}(h_2t^{\alpha}) + \sup_{0\leq\tau\leq t} M(\tau)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau \\ \leq&V(0,x(0))E_{\alpha}(h_2t^{\alpha}) + \Big(E_{\alpha}(h_2t^{\alpha})-1\Big)\sup_{0\leq\tau\leq t} x(\tau-h(\tau))^TPx(\tau-h(\tau)) \\ \leq&V(0,x(0))E_{\alpha}(h_2t^{\alpha}) + \Big(E_{\alpha}(h_2t^{\alpha})-1\Big)\sup_{-h_2\leq\theta\leq t-h_1} x(\theta)^TPx(\theta),\end{aligned}$$

and we obtain that

$$\begin{aligned}x(t)^TPx(t) \leq&\varphi(0)^TP\varphi(0)E_{\alpha}(h_2t^{\alpha}) \\ &+\Big(E_{\alpha}(h_2t^{\alpha})-1\Big)\sup_{-h_2\leq\theta\leq t-h_1} x(\theta)^TPx(\theta),\ t\geq 0.\end{aligned} \tag{3.6}$$

We now estimate the value $x(t)^TPx(t)$ on $\tau \in [-h_2,t]$. Firstly, note that by Lemma 2.3 the function $E_{\alpha}(h_2t^{\alpha}) \geq 1$, we have

$$x(\tau)^TPx(\tau) \leq \sup_{\theta\in[-h_2,0]} \varphi(\theta)^TP\varphi(\theta)E_{\alpha}(h_2t^{\alpha}), \quad \tau \in [-h_2,0].$$

Since $E_{\alpha}(.)$ is non-decreasing, applying the derived condition (3.6) for $0 \leq \tau \leq t$, we get

$$\begin{aligned} x(\tau)^T Px(\tau) &\le \varphi(0)^T P\varphi(0)E_\alpha(h_2\tau^\alpha) + \Big(E_\alpha(h_2\tau^\alpha) - 1\Big) \sup_{-h_2\le\theta\le\tau-h_1} x(\theta)^T Px(\theta) \\ &\le \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T P\varphi(\theta)E_\alpha(h_2t^\alpha) + \Big(E_\alpha(h_2t^\alpha) - 1\Big) \sup_{-h_2\le\theta\le t-h_1} x(\theta)^T Px(\theta), \end{aligned}$$

which implies

$$\begin{aligned} \sup_{-h_2\le\theta\le t} x(\theta)^T Px(\theta) \le & \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T P\varphi(\theta)E_\alpha(h_2t^\alpha) \\ & + \Big(E_\alpha(h_2t^\alpha) - 1\Big) \sup_{-h_2\le\theta\le t-h_1} x(\theta)^T Px(\theta) \\ \le & \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T P\varphi(\theta)E_\alpha(h_2T^\alpha) \\ & + \Big(E_\alpha(h_2T^\alpha) - 1\Big) \sup_{-h_2\le\theta\le t-h_1} x(\theta)^T Px(\theta). \end{aligned}$$

Let us denote $G(t) = \sup_{-h_2\le\theta\le t} x(\theta)^T Px(\theta)$, we have

$$G(t) \le E_\alpha(h_2T^\alpha)G(0) + \Big(E_\alpha(h_2T^\alpha) - 1\Big)G(t-h_1),\ \forall t\in[0,T],$$

Applying Lemma 2.8 with $a = E_\alpha(h_2T^\alpha),\ b = \Big(E_\alpha(h_2T^\alpha) - 1\Big)$, we obtain that

$$x(t)^T Px(t) \le G(t) \le G(0)q \le \sup_{\theta\in[-h,0]} \varphi(\theta)^T P\varphi(\theta)q, \qquad \forall t\in[0,T],$$

where $q = E_\alpha(h_2T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2T^\alpha) - 1)^j$. On the other hand, we can see that

$$x(t)^T x(t) \le \frac{1}{\lambda_{min}(P)} x(t)^T Px(t), \quad \varphi(\theta)^T P\varphi(\theta) \le \lambda_{max}(P)\varphi(\theta)^T\varphi(\theta).$$

Therefore, using condition (3.3), we obtain for all $t\in[0,\ T]$ that

$$\begin{aligned} x(t)^T x(t) \le & \frac{1}{\lambda_{min}(P)} x(t)^T Px(t) \le \frac{1}{\lambda_{min}(P)} q \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T P\varphi(\theta) \\ \le & \frac{1}{\lambda_{min}(P)} q \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T\varphi(\theta)\lambda_{max}(P) \\ = & Cond\,(P) q \sup_{\theta\in[-h_2,0]} \varphi(\theta)^T\varphi(\theta) \\ \le & Cond\,(P) q c_1 \le c_2. \end{aligned}$$

This completes the proof of the theorem. □

REMARK 2.1. In the proof of Theorem 3.1, we choose a simple Lyapunov functional as for systems without delay, which does not involve any integral-delay function. By doing so, no definite positiveness of the functional is required to obtain $V(t,x_t) \ge \alpha\|x(t)\|^2$. Meanwhile, in [13, 14] the constructed Lyapunov-Krasovskii functional can not be positive definite, and it shows that our method is applicable and more effective.

REMARK 2.2. We may notice that the conditions (3.1), (3.2) in Theorem 3.1 are LMIs, so we first find solutions P, Q, γ, by using LMI Toolbox in Matlab. Then, we check the condition (3.3) for given c_1, c_2, T.

REMARK 2.3. Theorem 3.1 gives sufficient conditions for the finite-time stability. It is of interest to minimize the trajectory bound c_2(or maximize the time bound T), that is, the smaller the c_2 is (or the bigger the time T is), the better performance the system has. This problem can be viewed as an optimization parameter, we can give, as [18], the following optimization algorithm to get the minimal value of c_2 :

$$\begin{aligned} min \quad & c_2 \\ & P, Q, \gamma \\ & \text{s.}t.(3.1)-(3.3) \end{aligned}$$

In the sequel, we give an application to a class of linear uncertain FDEs with delay. Consider the following system with interval time-varying delay:

$$\begin{cases} D^{\alpha}x(t) & = [A+\Delta A]x(t) + [D+\Delta D]x(t-h(t)), \\ x(\theta) & = \varphi(\theta), \ \theta \in [-h_2, 0], \end{cases} \tag{3.7}$$

where the time-varying uncertainties $\Delta A, \Delta D$ satisfy

$$\Delta A = K^A H^A(t) M^A, \quad \Delta D = K^D H^D(t) M^D,$$

and $\{K^A, \ M^A, \ K^D, \ M^D\}$ are known real constant matrices of appropriate dimensions, and $H^A(t), H^D(t)$ are unknown matrices uncertainty satisfying

$$H^A(t)^T H^A(t) \leq I, \ H^D(t)^T H^D(t) \leq I, \ t \geq 0.$$

To apply Theorem 3.1, we denote $f(\cdot) = \Delta A x(t) + \Delta D x(t-h(t))$. Observe that

$$\begin{aligned} \|f(\cdot)\|^2 \leq & 2x(t)^T \Delta A^T \Delta A x(t) + 2x(t-h(t))^T \Delta D^T \Delta D x(t-h(t)) \\ \leq & x(t)^T E_1^T E_2 x(t) + x(t-h(t))^T E_2^T E_2 x(t-h(t)), \end{aligned}$$

where

$$E_1 = \sqrt{2\lambda_{max}(M^{AT}M^A)\lambda_{max}(K^{AT}K^A)}\ I,$$

$$E_2 = \sqrt{2\lambda_{max}(M^{DT}M^D)\lambda_{max}(K^{DT}K^D)}\ I.$$

With the same notation stated in Theorem 1, we have the following result for the system (3.7).

COROLLARY 3.2. *The system (3.7) is finite-time stable w.r.t.* (c_1, c_2, T) *if there exist a positive scalar* γ, *symmetric positive definite matrices* P, Q *satisfying (3.1)-(3.3).*

The next numerical example demonstrates the validity and effectiveness of our results.

Consider system (2.1), where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix},$$

$$\alpha = 1/2,\ h(t) = 2,\ h_1 = 2,\ h_2 = 4.5,$$

$$E_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}.$$

By using LMI Toolbox in Matlab, the conditions (3.1)-(3.3) are feasible with

$$P = \begin{bmatrix} 19.1802 & -2.3808 \\ -2.3808 & 3.4955 \end{bmatrix},\ Q = \begin{bmatrix} 40.3127 & 0.2724 \\ 0.2724 & 12.9275 \end{bmatrix},\ \gamma = 22.6771.$$

For $c_1 = 0.01;\ c_2 = 2e + 176,\ T = 5$, we can calculate

$$Cond(P) = 6.2169,\ E_\alpha(h_2 T^\alpha) = 1.8765e + 44,\ [\frac{T}{h_1}] + 1 = 3,$$

and

$$c_1 Cond(P) \sum_{j=0}^{[\frac{T}{h_1}]+1} (E_\alpha(h_2T^\alpha) - 1)^j E_\alpha(h_2T^\alpha) = 1.5417e + 176 < c_2.$$

Therefore, the system, by Theorem 3.1, is finite-time stable w.r.t. (0.01, 2e+176, 5). We now show that our conditions are less conservative than the existing ones. For example, for the linear systems, i.e, $f(\cdot) \equiv 0, E_1 = E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we show that this system is finite-time stable w.r.t. (0.01, e+176, 5) , but finite-time unstable by using the conditions obtained in [8]. In fact, by using LMI Toolbox in Matlab, LMIs (3.1)-(3.2) are feasible with

$$P = \begin{bmatrix} 3.8446 & -0.1918 \\ -0.1918 & 1.0090 \end{bmatrix},\ Q = \begin{bmatrix} 9.6274 & 0.8815 \\ 0.8815 & 3.9804 \end{bmatrix},\ \gamma = 9.4453.$$

on the other hand, it is easy to see that

$$c_1 Cond(P) \sum_{j=0}^{[\frac{T}{h_1}]+1} (E_\alpha(h_2T^\alpha) - 1)^j E_\alpha(h_2T^\alpha) = 9.6032e + 175 < e + 176,$$

which implies that the system is finite-time stable w.r.t. (0.01, e+176, 5) due to Theorem 3.1. Besides, the system is FTS w.r.t (c_1, c_2, T) by the conditions obtained in [8] if

$$c_1 \Big(1 + \frac{(\|A\| + \|D\|)T^\alpha}{\Gamma(\alpha+1)}\Big)^2 \Big[E_\alpha((\|A\| + \|D\|)T^\alpha)\Big]^2 < c_2.$$

However, we see that $\Gamma(3/2) = 0.886$, and

$$0.01 \times \Big(1 + \frac{7\sqrt{5}}{0.886}\Big)^2 \times \Big[E_{1/2}(7\sqrt{5})\Big]^2 = 8.8814e + 213 > e + 176,$$

which implies that the system is finite-time unstable w.r.t (0.01, e+176, 5) by the conditions of [8]. Moreover, considering special linear case of system (2.1), the authors of [Theorem 4.2, 15] provided sufficient conditions on finite-time stability of linear system

$$D^{\alpha}x(t) = Dx(t-h),$$

in the form:

$$c_1\Big(E_\alpha(\|D\|T^\alpha) + E_\alpha(\|D\|h^\alpha)\Big)^2 < c_2.$$

We can show that our results is less conservative than this result. Indeed, using LMI Toolbox in Matlab, LMIs (3.1)-(3.2) are feasible with

$$P = \begin{bmatrix} 0.2851 & 0.0404 \\ 0.0404 & 0.1142 \end{bmatrix},\ Q = \begin{bmatrix} 0.7660 & 0.2246 \\ 0.2246 & 0.4463 \end{bmatrix},\ \gamma = 9.4453.$$

Hence, the system is finite-time stable w.r.t. (0.01,7e+175, 5) due to Theorem 3.1 from

$$c_1 Cond(P) \sum_{j=0}^{[\frac{T}{h_1}]+1} (E_\alpha(h_2T^\alpha)-1)^j E_\alpha(h_2T^\alpha) = 6.9392e+175 < 7e+175.$$

Besides,

$$c_1\Big(E_\alpha(\|D\|T^\alpha) + E_\alpha(\|D\|h^\alpha)\Big)^2 = 5.6144e+215 > 7e+175,$$

the system is finite-time unstable by using the conditions obtained in [15].

4. Conclusion. We have studied the finite-time stability of nonlinear FDEs with interval time-varying delay. By using Laplace transform and LMI technique, we have presented delay-dependent sufficient conditions for finite-time stability in terms of the Mittag-Leffler function and LMIs. The result obtained is useful in the stability analysis of FDEs with time-varying delay. Extending the results of this paper to singular FDEs with time-varying delay is a future work.

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