# The existence of solutions for a new class of differential inclusions involving proximal normal cone mappings

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**Abstract**

In this paper, a new class of differential inclusions involving proximal normal cone mappings and positive semi-definite linear mappings will be introduced and studied for the existence of solutions. The considered differential inclusions arise from the reformulation of finite-dimensional differential variational inequalities and it also can be seen as a new variant of sweeping processes. Our contributions are establishing the existence of absolutely continuous solutions to the systems.

*Keywords:* Differential inclusion, proximal normal cone, existence of solutions, differential variational inequality.

*2010 MSC:* 34A60, 34A12, 49K24

## 1. Introduction

Differential inclusions in the form of convex sweeping processes were introduced by Moreau [1] in the seventies. These are constrained differential inclusions involving normal cone mappings which appears naturally in several applications such as electrical circuits, hysteresis, crowd motion, etc. Among major subjects for convex sweeping processes, the well-posedness, in the sense of the existence and uniqueness of solutions, has been studied in numerous papers. Ever since conceived by Moreau [1], the well-posedness of variants of sweeping processes has been extensively studied and developed in various contexts including sweeping processes with perturbations and with prox-regular state dependent sets, second-order sweeping processes [2, 3, 4], sweeping processes associated with maximal monotone operators and with perturbations [5, 6], higher order Moreau's sweeping process in the finite-dimensional spaces [7], etc. In recent developments, differential inclusions involving normal cones has been known as an effective tool to study differential variational inequalities in [8, 9, 10] as well as evolution variational inequalities in [11, 12]. Regarding to evolution variational inequalities, a new variant of Moreau's convex sweeping process with velocity constraint has been studied in the papers [13, 14].

In this paper, we consider a new class of differential inclusions involving proximal normal cone mappings with positive semi-definite linear mappings and study for the existence and uniqueness

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of solutions. Such kind of differential inclusions arises from the reformulation of finite-dimensional differential variational inequalities studied in [8, 11, 12] and it also can be seen as a new variant of non-convex sweeping processes with perturbations. In such differential inclusions, since a proximal normal cone mapping to prox-regular non-convex closed sets is not maximal monotone, the existence and uniqueness of solutions can not be derived from the theory of differential inclusions governed by the maximal monotone operators studied in [15] or recently developed in [16]. Also, the existing results on existence and uniqueness of solutions of the classes of sweeping processes [5, 13, 14, 17, 18] do not apply to the class of differential inclusions in the generality as treated in this paper.

In the present paper, our contributions are deriving results on the existence of solutions. To do so, the approach we take in this paper includes rewritting the differential inclusions as sweeping processes with state-velocity constraints and employing the matrix analysis technique of the involved positive semi-definite linear mappings. For the positive definite case, the result sweeping process is reformulated as a coupled system of ordinary differential equations and a sweeping process whose evolution interconnects each other. We employ the catching-up like method to establish the existence of solutions to the coupled system. Note that the general results on the existence and uniqueness of solutions of implicit sweeping processes with velocity constraints is studied in [13] in Hilbert spaces. However, it works only for convex closed sets and it can not be applied to the case of this paper. For the case positive seni-definite of linear mappings, we employ the singular value decomposition and state coordinate tranformation to decompose the system into two subsystems: The first one is of the considered differential inclusion forms and the second one is a standard non-convex sweeping process. The existence of solutions is then followed.

The rest of this paper is organized as follows. In Section 2, we will recall notations and preliminaries used later in the paper. The class of differential inclusions involving proximal normal cones mappings that the paper studies will be introduced in Section 3 with the motivations of the study. The main results of this paper will be presented in Section 4. Finally, the paper closes with the conclusions in Section 5.

## 2. Notations and preliminaries

Throughout the paper, we will denote the set of all real numbers by $\mathbb{R}$, non-negative real numbers by $\mathbb{R}_+$, $n$-tuple real numbers by $\mathbb{R}^n$, $n \times m$ real-valued matrices by $\mathbb{R}^{n\times m}$, $n$-tuple real numbers with non-negative components by $\mathbb{R}^n_+$. For $A \in \mathbb{R}^{n\times n}$, its transpose and inverse will be denoted by $A^T$ and $A^{-1}$, respectively. A matrix $J \in \mathbb{R}^{n\times n}$ is called positive definite if $x^T Jx > 0$ for all $x \in \mathbb{R}^n, x \neq 0$. If $x^T Jx \geqslant 0$ for all $x \in \mathbb{R}^n$, $J$ is called positive semi-definite. It is well-known that for every symmetric positive definite matrix $J \in \mathbb{R}^{n\times n}$, there exists a unique symmetric positive definite matrix $J^{\frac{1}{2}}$ such that $J = J^{\frac{1}{2}} J^{\frac{1}{2}}$. Such $J^{\frac{1}{2}}$ is called the square root of the matrix $J$.

For a subset $X$ of $\mathbb{R}^n$, we denote by $\mathrm{d}(x, X) := \inf \{\|x - a\| \mid a \in X\}$ the distance from $x$ to $X$. Also, we denote by $\mathrm{proj}_X(x)$ the set of all $z \in X$ such that $\|z-x\| = \mathrm{d}(x, X)$. The Hausdorff distance between two sets $X$ and $Y$ of $\mathbb{R}^n$ is defined as

$$\mathrm{d}_H(X,Y) := \max\left\{\sup_{x\in X}\mathrm{d}(x,Y), \sup_{y\in Y}\mathrm{d}(y,X)\right\}. \tag{1}$$

For $T>0$, we denote by $\mathrm{C}([0,T],\mathbb{R}^n)$, $\mathrm{AC}([0,T],\mathbb{R}^n)$ the linear space of continuous functions and absolutely continuous functions from $[0,T]$ to $\mathbb{R}^n$, respectively, and endow them with the norms

$$\|x\|_{\mathrm{C}([0,T],\mathbb{R}^n)} = \max_{t\in[0,T]}\|x(t)\|, \quad \|z\|_{\mathrm{AC}([0,T],\mathbb{R}^n)} = \|z(0)\| + \int_0^T \|\dot z(s)\| ds \tag{2}$$

for $x\in \mathrm{C}([0,T],\mathbb{R}^n), z\in \mathrm{AC}([0,T],\mathbb{R}^n)$. Also, we denote by $\mathrm{L}_p([0,T],\mathbb{R}^n), p\geqslant 1$, the linear space of Lebesgue $p$-integrable functions from $[0,T]$ to $\mathbb{R}^n$ and endow it with the norm

$$\|x\|_{\mathrm{L}_p([0,T],\mathbb{R}^n)} := \left(\int_0^T \|x(s)\|^p ds\right)^{\frac{1}{p}}, x\in \mathrm{L}_p([0,T],\mathbb{R}^n). \tag{3}$$

Let $\mathrm{L}_\infty([0,T],\mathbb{R}^n)$ be the space of essentially bounded measurable functions form $[0,T]$ to $\mathbb{R}^n$ with the norm

$$\|x\|_{\mathrm{L}_\infty([0,T],\mathbb{R}^n)} = \inf\{\lambda\geqslant 0 \mid \|x(t)\|\leqslant\lambda \text{ for almost all } t\in[0,T]\}.$$

**Definition 2.1.** Let $\{x_\nu \mid \nu\in\mathbb{N}\}$ be a sequence of Lebesgue integrable functions in $\mathrm{L}_1([0,T],\mathbb{R}^n)$ and let $x\in \mathrm{L}_1([0,T],\mathbb{R}^n)$. The sequence $\{x_\nu \mid \nu\in\mathbb{N}\}$ is said to be

(a) strongly convergent to $x$ if $\lim\limits_{\nu\to\infty}\|x_\nu - x\|_{\mathrm{L}_1([0,T],\mathbb{R}^n)} = 0$;

(b) weakly convergent to $x$ if

$$\lim_{\nu\to\infty}\int_0^T\langle x_\nu(s), w(s)\rangle ds = \int_0^T \langle x(s), w(s)\rangle ds, \forall w\in \mathrm{L}_\infty([0,T],\mathbb{R}^n).$$

Every strongly convergent sequence is weakly convergent with the same limit, but the inverse does not hold. However, it has been known as Mazur's lemma that the following statement holds; see for instance [19].

**Lemma 2.2** ([19])**.** *Let $\mathcal{X}$ be a Banach space and let $\{x_n\}$ be convergent weakly to $x$ in $\mathcal{X}$. Then, $\{z_n\}$ converges strongly to $x$ for some sequence $\{z_n\}$ of the form*

$$z_n = \sum_{k=n}^{\tau(n)} a_k^n x_k \text{ where } a_k^n\geqslant 0 \text{ and } \sum_{k=1}^{\tau(n)} a_k^n = 1. \tag{4}$$

For a set-valued mapping $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, we denote by $\mathrm{dom}(F)$, $\mathrm{im}(F)$ or $\mathrm{rge}(F)$ and $\mathrm{gr}(F)$ its domain, image and graph, i.e. $\mathrm{dom}(F) := \{x\in\mathbb{R}^n \mid F(x)\neq\emptyset\}$, $\mathrm{gr}(F) := \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m \mid y\in F(x)\}$, $\mathrm{rge}(F) = \mathrm{im}(F) := \{y\in\mathbb{R}^m \mid \text{ there exists } x\in\mathbb{R}^n \text{ such that } y\in F(x)\}$. For $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, we say that $F$ has closed (convex) values if $F(x)$ is a closed (convex) subset of $\mathbb{R}^m$ for every $x\in\mathrm{dom}(F)$. The mapping $F$ is called closed if $\mathrm{gr}(F)$ is closed in $\mathbb{R}^n\times\mathbb{R}^m$. The inverse mapping of $F$ is the set-valued mapping $F^{-1}:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ defined by its graph $\mathrm{gr}(F^{-1}) = \{(y,x)\in\mathbb{R}^m\times\mathbb{R}^n \mid (x,y)\in\mathrm{gr}(F)\}$.

**Definition 2.3.** Let $\mathcal{C}\subset\mathbb{R}^n$ be a non-empty closed set and $x\in\mathcal{C}$. A vector $v\in\mathbb{R}^n$ is called proximal normal vector to $\mathcal{C}$ at $x$ if either of them is 0 or there exists $r>0$ such that $x\in\mathrm{proj}_{\mathcal{C}}(x+rv/\|v\|)$.

It is not difficult to see that $v$ is a proximal normal vector to $\mathcal{C}$ at $x$ if and only if there exists $M \geqslant 0$ such that $\langle v, y-x\rangle \leqslant M\|y-x\|^2, \forall y \in \mathcal{C}$. The set of all proximal normal vectors to $\mathcal{C}$ at $x$ is a convex cone, which is denoted by $\mathcal{N}_{\mathcal{C}}^P(x)$,

$$\mathcal{N}_{\mathcal{C}}^P(x) = \{v \in \mathbb{R}^n \mid \exists M \geqslant 0 \text{ such that } \langle v, y-x\rangle \leqslant M\|y-x\|^2, \forall y \in \mathcal{C}\}. \tag{5}$$

For $x \notin \mathcal{C}$, we define $\mathcal{N}_{\mathcal{C}}^P(x) = \emptyset$. If $\mathcal{C}$ is a closed convex set then the proximal normal cone mapping $\mathcal{N}_{\mathcal{C}}^P(\cdot)$ coincides the normal cone mapping $\mathcal{N}_{\mathcal{C}}(\cdot)$ in the convex analysis literature

$$\mathcal{N}_{\mathcal{C}}(x) = \begin{cases} \{\zeta \in \mathbb{R}^n \mid \langle \zeta, z-x\rangle \leqslant 0, \forall z \in \mathcal{C}\} & \text{if } x \in \mathcal{C}, \\ \emptyset & \text{if } x \notin \mathcal{C}. \end{cases}$$

**Definition 2.4.** Let $\mathcal{C}$ be a non-empty closed subset of $\mathbb{R}^n$ and $\rho > 0$. The set $\mathcal{C}$ is said to be *$\rho$-prox-regular* if each $z$ in the $\rho$-enlargement of $\mathcal{C}$, i.e. $z \in \mathcal{E}_\rho(\mathcal{C}) := \{z \in \mathbb{R}^n \mid \mathrm{d}(z, \mathcal{C}) < \rho\}$, has a unique nearest point $\mathrm{proj}_{\mathcal{C}}(z)$ and the projected mapping $\mathrm{proj}_{\mathcal{C}}(\bullet)$ is continuous on $\mathcal{E}_\rho(\mathcal{C})$.

**Example 2.5.** In $\mathbb{R}^2$, the set

$$\mathcal{C} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{matrix} 0 \leqslant y \leqslant 1 \\ -1+\sqrt{1-y^2} \leqslant x \leqslant 1-\sqrt{1-y^2} \end{matrix} \right\} \text{ is } \frac{1}{2} - \text{prox-regular.}$$

**Lemma 2.6** ([5]). *Let $\mathcal{C}$ be an $\rho$-prox-regular closed subset of $\mathbb{R}^n$. Then, the following statements hold.*

(a) *For all $x \in \mathcal{C}$ and $\zeta \in \mathcal{N}_{\mathcal{C}}^P(x)$ such that $\|\zeta\| < \rho$, one has $x = \mathrm{proj}_{\mathcal{C}}(x+\zeta)$.*

(b) *For all $x \in \mathcal{C}$ and $\zeta \in \mathcal{N}_{\mathcal{C}}^P(x)$, one has*

$$\langle \zeta, y-x\rangle \leqslant \frac{\|\zeta\|}{2\rho}\|y-x\|^2, \forall y \in \mathcal{C}.$$

(c) *For all $x, \tilde{x} \in \mathcal{C}, \zeta \in \mathcal{N}_{\mathcal{C}}^P(x), \tilde{\zeta} \in \mathcal{N}_{\mathcal{C}}^P(\tilde{x})$ and $\|\zeta\| \leqslant \rho, \|\tilde{\zeta}\| \leqslant \rho$, one has*

$$\langle \zeta - \tilde{\zeta}, x - \tilde{x}\rangle \geqslant -\frac{1}{\rho}\|x-\tilde{x}\|^2.$$

(d) *The square distance function $d^2(\bullet, \mathcal{C})$ is continuously differentiable on $\mathcal{E}_\rho(\mathcal{C})$ and*

$$\nabla\left(\frac{1}{2}d^2(\bullet, \mathcal{C})\right)(x) = x - \mathrm{proj}_{\mathcal{C}}(x), \forall x \in \mathcal{E}_\rho(\mathcal{C}).$$

*Moreover, for any positive number $\delta < \rho$ and $x, \tilde{x} \in \mathcal{E}_\delta(\mathcal{C})$, the following inequality holds*

$$\|\mathrm{proj}_{\mathcal{C}}(x) - \mathrm{proj}_{\mathcal{C}}(\tilde{x})\| \leqslant \frac{\rho}{\rho-\delta}\|x-\tilde{x}\|.$$

Finally, we recall some versions of Gronwall's inequality that will be used later in this paper.

**Lemma 2.7** ([20], Lemma 4.1)**.** *Let $T > 0$ be given and $a, b \in \mathrm{L}_1([0,T],\mathbb{R})$ with $b(t) \geqslant 0$ for almost all $t \in [0,T]$. Let $w : [0,T] \to \mathbb{R}_+$ be an absolutely continuous function satisfying*

$$(1-\alpha)\dot{w}(t) \leqslant a(t)w(t) + b(t)w^{\alpha}(t) \text{ for almost all } t \in [0,T], \tag{6}$$

*where $0 \leqslant \alpha < 1$. Then, one has*

$$w^{1-\alpha}(t) \leqslant w^{1-\alpha}(0)\exp\left(\int_0^t a(s)ds\right) + \int_0^t \exp\left(\int_s^t a(\tau)d\tau\right) b(s)ds \tag{7}$$

*for almost all $t \in [0,T]$.*

**Lemma 2.8** ([20], Lemma 4.2')**.** *Let $T > 0$ be given and let $a \in \mathrm{L}_1([0,T],\mathbb{R})$ and $v \in \mathrm{L}_\infty([0,T],\mathbb{R})$. Assume that the bounded measurable function $u$ satisfies almost everywhere the inequality*

$$u(t) \leqslant \int_0^t a(s)u(s)ds + v(t), t \in [0,T]. \tag{8}$$

*Then, we have*

$$u(t) \leqslant v(t) + \int_0^t \exp\left(\int_s^t a(\tau)d\tau\right) v(s)ds, \forall t \in [0,T]. \tag{9}$$

**Lemma 2.9.** *Let $\{x_\nu\}_{\nu=1}^{\infty}$ be a sequence of absolutely continuous functions from $[0,T]$ to $\mathbb{R}^n$. Suppose that $\lim\limits_{\nu\to\infty} x_\nu(0) = 0$, $\lim\limits_{\nu\to\infty} \int_0^T \alpha_\nu(t)dt = 0$ and*

$$\frac{d}{dt}(\|x_\nu(t))\|^2) \leqslant \beta_\nu(t)\|x_\nu(t)\|^2 + \alpha_\nu(t)\|x_\nu(t)\| + \gamma\|x_\nu(t)\| \int_0^T \|x_\nu(s)\|ds, \forall \nu \in \mathbb{N}, \tag{10}$$

*for almost all $t \in [0,T]$, where $\alpha_\nu, \beta_\nu \in \mathrm{L}_1([0,T],\mathbb{R}_+)$ and $\gamma \in \mathbb{R}_+$. Suppose that the sequence $\{\beta_\nu\}_{\nu=1}^{\infty}$ is bounded in $\mathrm{L}_1([0,T],\mathbb{R})$ and*

$$\int_0^T \exp\left(\int_s^T \beta_\nu(\tau)d\tau\right) ds < \frac{1}{\gamma T}. \tag{11}$$

*Then, one has*

$$\lim_{\nu\to\infty} \|x_\nu\|_{\mathrm{C}([0,T],\mathbb{R}^n)} = 0. \tag{12}$$

*Proof.* It follows from (10) that

$$\frac{d}{dt}(\|x_\nu(t))\|^2) \leqslant \beta_\nu(t)\|x_\nu(t)\|^2 + (\alpha_\nu(t) + \gamma T\|x_\nu\|_{\mathrm{C}([0,T],\mathbb{R}^n)})\|x_\nu(t)\|, \tag{13}$$

for all $\nu \in \mathbb{N}$ and for almost all $t \in [0,T]$. By Lemma 2.7, one has

$$\begin{aligned}
\|x_\nu(t)\| &\leqslant \|x_\nu(0)\|\exp\left(\int_0^t \beta_\nu(\tau)d\tau\right) + \int_0^t \exp\left(\int_s^t \beta_\nu(\tau)d\tau\right)(\alpha_\nu(s) + \gamma T\|x_\nu\|_{\mathrm{C}([0,T],\mathbb{R}^n)})ds \\
&\leqslant \|x_\nu(0)\|\exp\left(\int_0^t \beta_\nu(\tau)d\tau\right) + \int_0^t \exp\left(\int_s^t \beta_\nu(\tau)d\tau\right)\alpha_\nu(s)ds \\
&\qquad + \gamma T\|x_\nu\|_{\mathrm{C}([0,T],\mathbb{R}^n)}) \int_0^t \exp\left(\int_s^t \beta_\nu(\tau)d\tau\right) ds, \forall t \in [0,T].
\end{aligned}$$

It follows that

$$p\|x_\nu\|_{\mathrm{C}([0,T],\mathbb{R}^n)}) \leqslant \|x_\nu(0)\|\exp\left(\int_0^T \beta_\nu(\tau)d\tau\right) + \int_0^T \exp\left(\int_s^T \beta_\nu(\tau)d\tau\right)\alpha_\nu(s)ds$$

where $p := 1 - \gamma T \int_0^T \exp\left(\int_s^T \beta_\nu(\tau)d\tau\right) ds > 0$. Thus, one has

$$\|x_\nu\|_{C([0,T],\mathbb{R}^n)}) \leqslant \frac{1}{p}\|x_\nu(0)\| \exp\left(\int_0^T \beta_\nu(\tau)d\tau\right) + \frac{1}{p}\int_0^T \exp\left(\int_s^T \beta_\nu(\tau)d\tau\right) \alpha_\nu(s)ds \overset{\nu\to\infty}{\longrightarrow} 0$$

due to boundedness of the sequence $\{\beta_\nu\}_{\nu=1}^\infty$, $\lim\limits_{\nu\to\infty} x_\nu(0) = 0$, and $\lim\limits_{\nu\to\infty} \int_0^T \alpha_\nu(t)dt = 0$. The equality (12) is proved. □

## 3. Differential inclusions involving proximal normal cone mappings

In this paper, we are interested in studying a class of finite-dimensional differential inclusions involving proximal normal cone mappings of the form

$$\begin{cases} \dot{x}(t) \in f(x(t)) - \left(J + \left(\mathcal{N}^P_{\mathcal{C}(t)}\right)^{-1}\right)^{-1}(x(t)) \\ x(0) = x_0 \end{cases} \tag{14}$$

where $x \in \mathbb{R}^n$, $J \in \mathbb{R}^{n\times n}$ and $\mathcal{N}^P_{\mathcal{C}}$ is the proximal normal cone mapping to the $\rho$-prox-regular closed set $\mathcal{C} \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

**Definition 3.1.** An absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ is said to be a solution of the differential inclusion (14) for the initial state $x_0$ if $x(0) = x_0$ and $x(\cdot)$ satifies (14) for almost all $t \in [0, T]$.

For each $J$, we denote by $\mathcal{D}_J$ the domain of the set-valued mapping $\left(J + (\mathcal{N}^P_{\mathcal{C}})^{-1}\right)^{-1}$. It is not hard to verify the following lemma that presents the characterizations of $\mathcal{D}_f$.

**Lemma 3.2.** *For any $\rho$-prox-regular closed set $\mathcal{C}$, one has*

$$\mathcal{D}_J = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^n, y \in \mathcal{N}^P_{\mathcal{C}}(x - Jy)\} \tag{15}$$

$$= \{x \in \mathbb{R}^n \mid \exists z \in \mathcal{C}, x \in z + J(\mathcal{N}^P_{\mathcal{C}}(z))\}. \tag{16}$$

*Proof.* For the claim (15), it is obvious that $x \in \mathcal{D}_J$ iff there exists $y \in \mathbb{R}^n$ such that $y \in \left(J + (\mathcal{N}^P_{\mathcal{C}})^{-1}\right)^{-1}(x)$ or equivalently $x \in Jy + (\mathcal{N}^P_{\mathcal{C}})^{-1}(y)$. The latter is equivalent to $y \in \mathcal{N}^P_{\mathcal{C}}(x - Jy)$. Thus, the equality (15) is proved.

In order to prove the second claim (16), let $x \in \mathcal{D}_J$. Due to (15), there exists $y \in \mathbb{R}^n$ such that $y \in \mathcal{N}^P_{\mathcal{C}}(x - Jy)$. Defining $z := x - Jy \in \mathcal{C}$, we have $y \in \mathcal{N}^P_{\mathcal{C}}(z)$ and hence

$$x = x - Jy + Jy = z + Jy \in z + J(\mathcal{N}^P_{\mathcal{C}}(z)).$$

Conversely, let $x \in \mathbb{R}^n$ such that $x \in z + J(\mathcal{N}^P_{\mathcal{C}}(z))$ for some $z \in \mathcal{C}$. Then, $x = z + Jy$ for some $y \in \mathcal{N}^P_{\mathcal{C}}(z)$. Moreover, $y \in \mathcal{N}^P_{\mathcal{C}}(z)$ is equivalent to $z \in (\mathcal{N}^P_{\mathcal{C}})^{-1}(y)$. Therefore, $x \in Jy + (\mathcal{N}^P_{\mathcal{C}})^{-1}(y)$ and hence $y \in \mathcal{N}^P_{\mathcal{C}}(x - Jy)$. By (15), the latter yields $x \in \mathcal{D}_J$. □

The study of differential inclusions of the form (14) is motivated from the recent study of differential variational inequalities in the papers [8, 21, 22, 23] setting as follows. Let $\mathcal{C} : [0, \infty) \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with non-empty closed convex values and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a given function. We consider the finite-dimensional differential variational inequalities (DVI) of the form

$$\dot{x}(t) = f(x(t)) + Gz(t) \tag{17a}$$

$$\mathcal{C}(t) \ni w(t) = Hx(t) + Jz(t) \tag{17b}$$

$$0 \leqslant \langle \zeta - w(t), z(t) \rangle, \forall \zeta \in \mathcal{C}(t) \tag{17c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $z(t), w(t) \in \mathbb{R}^n$, the matrices $H \in \mathbb{R}^{n\times n}, J \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times n}$ are given matrices, and the notation $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathbb{R}^n$.

In order to study the DVI (17), one often reformulations it in term of the differential inclusions. Indeed, the conditions (17b) and (17c) can be rewritten in the form of the generalized equations

$$- z(t) \in \mathcal{N}_{\mathcal{C}(t)}(Hx(t) + Jz(t)) \tag{18}$$

where $\mathcal{N}_C(\cdot)$ is the set-valued normal cone mapping in the convex analysis literature. In views of (18), the DVI (17) can be rewritten as

$$\begin{cases} \dot{x}(t) & = f(x(t)) + Gz(t) \\ -z(t) & \in \mathcal{N}_{\mathcal{C}(t)}(Hx(t) + Jz(t)). \end{cases} \tag{19}$$

Moreover, one can see that

$$-z(t) \in \mathcal{N}_{\mathcal{C}(t)}(Hx(t) + Jz(t)) \Longleftrightarrow -z(t) \in \left(J + \mathcal{N}^{-1}_{\mathcal{C}(t)}\right)^{-1} (Hx(t)).$$

Therefore, the system (19) can be further written in the form of differential inclusions as

$$\dot{x}(t) \in f(x(t)) - G\left(J + \mathcal{N}^{-1}_{\mathcal{C}(t)}\right)^{-1} (Hx(t)). \tag{20}$$

Now, a question arises is how to deal with the system in the case that $\mathcal{C}(t)$ is not convex. In that case, the normal cone mapping $\mathcal{N}_{\mathcal{C}(t)}$ would be replaced by the proximal normal cone mapping $\mathcal{N}^P_{\mathcal{C}(t)}$ defined as (5). For such case, the differential inclusion (20) must read as

$$\dot{x}(t) \in f(x(t)) - G\left(J + (\mathcal{N}^P_{\mathcal{C}(t)})^{-1}\right)^{-1} (Hx(t)). \tag{21}$$

Note that the differential inclusion (14) is a particular class of the system (21) as taking $H = \mathrm{I}_n = G$. It is also interesting that the differential inclusions of the forms (21) and (14) have not been studied elsewhere in the literature, except the case $J = 0$ and $H = \mathrm{I}_n = G$, i.e. the system

$$\dot{x}(t) \in f(x(t)) - \mathcal{N}^P_{\mathcal{C}(t)}(x(t)). \tag{22}$$

The differential inclusions of the form (22) are well-known as sweeping processes with perturbations which have been extensively studied in the series of papers [1, 6, 5] and the references therein on Hilbert spaces and even more general moving sets $\mathcal{C}(t)$ of uniformly prox-regular properties.

## 4. Main results

In this section, we consider the existence and uniqueness of absolutely continuous solutions of the differential inclusions involving proximal normal cone mappings of the form (14). In our point of view, uniqueness of solutions of the differential inclusion (14) is hard to guarantee even in very simple form. The following is an example.

**Example 4.1.** In $\mathbb{R}^2$, we consider the 1/4-prox-regular set

$$\mathcal{C} = \mathbb{B}\left(\mathrm{col}(1/2,0),1/2\right) \cup \mathbb{B}\left(\mathrm{col}(2,0),1/2\right)$$

and consider the differential inclusion

$$\begin{cases} -\dot{x}(t) & \in \left(I_2 + (\mathcal{N}_{\mathcal{C}}^P)^{-1}\right)^{-1}(x(t)),\ t \in [0,2] \\ x(0) & = \mathrm{col}(1,0) \in \mathcal{C}. \end{cases} \tag{23}$$

It can be verified that $x_1(t) = \mathrm{col}(1,0), \forall t \in [0,1]$ and

$$x_2(t) = e^{-t}\mathrm{col}(1,0) + \left(1 - e^{-t}\right)\mathrm{col}(3/2,0), t \in [0,2]$$

are two absolutely continuous solutions of the differential inclusion (23).

In the sequel, we only study the existence of absolutely continuous solutions of the differential inclusion (14). We will make the following assumption on the set-valued function $\mathcal{C}(\cdot)$:

**Assumption 1.** For each $t \in [0,T]$, suppose that $\mathcal{C}(t)$ is a $\rho$-prox-regular closed subset of $\mathbb{R}^n$ and there exists an absolutely continuous function $v$ such that

$$|\mathrm{d}(x,\mathcal{C}(t)) - \mathrm{d}(x,\mathcal{C}(s))| \leqslant |v(t) - v(s)| \tag{24}$$

for all $t, s \in [0,T]$ and for all $x \in \mathbb{R}^n$.

The first main result is the following theorem that establishes the existence of absolutely continuous soluttions in the case that the matrix $J$ is positive definite and with zero perturbation.

**Theorem 4.2.** *Suppose that the set-valued mapping $\mathcal{C}(\cdot)$ satisfies Assumption 1 and $J \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Then, the absolutely continuous solutions of the differential inclusion*

$$\begin{cases} -\dot{x}(t) & \in \left(J + \left(\mathcal{N}_{\mathcal{C}(t)}^P\right)^{-1}\right)^{-1}(x(t)),\ t \in [0,T], \\ x(0) & = x_0 \end{cases} \tag{25}$$

*locally exist for any initial state* $x_0 \in \mathcal{D}_J(0) := \mathrm{dom}\left(J + \left(\mathcal{N}_{\mathcal{C}(0)}^P\right)^{-1}\right)^{-1}$.

*Proof.* First, observe that the differential inclusion (25) can be rewritten in the following form

$$\begin{cases} -\dot{x}(t) & \in \mathcal{N}_{\mathcal{C}(t)}^P(x(t) + J\dot{x}(t)),\ t \in [0,T], \\ x(0) & = x_0. \end{cases} \tag{26}$$

Associated to (26), we consider the dynamical system involving an ordinary differential equation and a sweeping process with perturbation of the form

$$\begin{cases} \dot{y}(t) & = -J^{-1}y(t) + J^{-1}\dot{z}(t), y(0) = y_0 \\ \dot{z}(t) & \in \dfrac{y(t)}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)), z(0) = z_0, \end{cases} \tag{27}$$

where $x_0 + Jy_0 = z_0$. Regarding to this system, we claim that if $(y, z) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{AC}([0, T], \mathbb{R}^n)$ satisfies the system (27) on $[0, T]$, then $x(t) := z(t) - Jy(t), t \in [0, T]$, is an absolutely continuous solution of the differential inclusion (26) or equivalently (25). Indeed, one has that $\dot{x}(t) = \dot{z}(t) - J\dot{y}(t)$ and

$$x(t) + J\dot{x}(t) = z(t) - Jy(t) + J\dot{z}(t) - J^2\dot{y}(t) = z(t)$$

for almost all $t \in [0, T]$. Moreover, we have

$$\begin{aligned} \dot{z}(t) \in \frac{y(t)}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) &\iff -2\dot{z}(t) + y(t) \in 2\mathcal{N}^P_{\mathcal{C}(t)}(z(t)) = \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \\ &\iff -\left[2\dot{z}(t) - y(t)\right] \in \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \\ &\iff -\left[2\dot{z}(t) - (\dot{z}(t) - J\dot{y}(t))\right] \in \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \\ &\iff -\dot{x}(t) \in \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) = \mathcal{N}^P_{\mathcal{C}(t)}(x(t) + J\dot{x}(t)). \end{aligned}$$

Thus, the claim is proved. Due to the claim, in order to prove Theorem 4.2, it suffices to show that for each $z_0 \in \mathcal{C}(0)$ and $y_0 \in \mathbb{R}^n$, the system (27) has at least one solution $(y, z) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{AC}([0, T], \mathbb{R}^n)$. The proof of the latter would be done in three steps as follows.

*Step 1. Construct a sequence of approximation solutions.* For each $\kappa \in \mathbb{N}^*$, we partition the interval $[0, T]$ into $\kappa$ sub-intervals by the points $t_i^\kappa = \dfrac{iT}{\kappa}, 0 \leqslant i \leqslant \kappa$. On the interval $[t_0^\kappa, t_1^\kappa]$, observe that the sweeping process with constant perturbation

$$\begin{cases} \dot{z}(t) & \in \dfrac{y_0}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)), t \in [t_0^\kappa, t_1^\kappa], \\ z(0) & = z_0, \end{cases} \tag{28}$$

has a unique absolutely continuous solution due to [5, Lem. 3.1] and [16, Prop. 1], which is denoted by $z_0(t)$, and moreover, the following estimation holds

$$\left\| \dot{z}_0(t) - \frac{y_0}{2} \right\| \leqslant \frac{1}{2}\|y_0\| + |\dot{v}(t)| \tag{29}$$

for almost all $t \in [t_0^\kappa, t_1^\kappa]$. By employing $z_0(t)$, we consider the ordinary differential equation

$$\begin{cases} \dot{y}(t) & = -J^{-1}y(t) + J^{-1}\dot{z}_0(t), t \in [0, t_1^\kappa], \\ y(0) & = y_0. \end{cases} \tag{30}$$

This linear differential equation admits a unique absolutely continuous solution, denoted by $y_0(t)$. By the similar arguments for the interval $[t_1^\kappa, t_2^\kappa]$, the sweeping process with constant perturbation

$$\begin{cases} \dot{z}(t) & \in \dfrac{y_0(t_1^\kappa)}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)), t \in [t_1^\kappa, t_2^\kappa] \\ z(t_1^\kappa) & = z_0(t_1^\kappa) \end{cases} \tag{31}$$

has a unique absolutely continuous solution, which is denoted by $z_1(t)$. Moreover, the following estimation holds

$$\left\| \dot{z}_1(t) - \frac{y_0(t_1^\kappa)}{2} \right\| \leqslant \frac{1}{2}\|y_0(t_1^\kappa)\| + |\dot{v}(t)| \tag{32}$$

for almost all $t \in [t_1^\kappa, t_2^\kappa]$. Then, by employing $z_1(t)$, we consider the ordinary differential equation

$$\begin{cases} \dot{y}(t) &= -J^{-1}y(t) + J^{-1}\dot{z}_1(t), t \in [t_1^\kappa, t_2^\kappa], \\ y(t_1^\kappa) &= y_0(t_1^\kappa). \end{cases} \tag{33}$$

This differential equation has one and only one absolutely continuous solution, denoted by $y_1(t)$. Continuing on $[t_2^\kappa, t_3^\kappa]$ and so on, one finally comes up with a finite sequence of functions $z_i(t)$ and $y_i(t), i \in \{0, 1, \ldots, \kappa - 1\}$, such that

$$\begin{cases} \dot{z}_i(t) &\in \frac{y_{i-1}(t_i^\kappa)}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z_i(t)) \text{ for almost all } t \in [t_i^\kappa, t_{i+1}^\kappa] \\ z_i(t_i^\kappa) &= z_{i-1}(t_i^\kappa) \end{cases}$$

and

$$\begin{cases} \dot{y}_i(t) &= -J^{-1}y_i(t) + J^{-1}\dot{z}_i(t) \;\text{ for almost all } t \in [t_i^\kappa, t_{i+1}^\kappa] \\ y_i(t_i^\kappa) &= y_{i-1}(t_i^\kappa) \end{cases}$$

where $y_{-1}(t_0^\kappa) = y_0$ and $z_{-1}(t_0^\kappa) = z_0$. Moreover, one has

$$\left\| \dot{z}_i(t) - \frac{y_i(t_i^\kappa)}{2} \right\| \leqslant \frac{1}{2}\|y_i(t_i^\kappa)\| + |\dot{v}(t)| \text{ for almost all } t \in [t_i^\kappa, t_{i+1}^\kappa].$$

Now, define $\theta_\kappa(t_0^\kappa) = 0$ and $\theta_\kappa(t) = t_i^\kappa$ if $t \in (t_i^\kappa, t_{i+1}^\kappa], \forall i \in \{0, 1, \ldots, \kappa - 1\}$, and define the functions $z^\kappa : [0, T] \to \mathbb{R}^n$, $y^\kappa : [0, T] \to \mathbb{R}^n$ as the concatenations of the functions $z_i$ and $y_i$, respectively, i.e.

$$z^\kappa(t) := z_i(t) \text{ and } y^\kappa(t) := y_i(t) \text{ if } t \in [t_i^\kappa, t_{i+1}^\kappa], i \in \{0, 1, \ldots, \kappa - 1\}.$$

By the construction, one then has

$$\begin{cases} \dot{z}^\kappa(t) &\in \frac{y^\kappa(\theta_\kappa(t))}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z^\kappa(t)) \text{ for almost all } t \in [0, T] \\ z^\kappa(0) &= z_0 \end{cases} \tag{34}$$

and

$$\begin{cases} \dot{y}^\kappa(t) &= -J^{-1}y^\kappa(t) + J^{-1}\dot{z}^\kappa(t) \;\text{ for almost all } t \in [0, T] \\ y^\kappa(0) &= y_0. \end{cases} \tag{35}$$

Moreover, one has the estimation

$$\left\| \dot{z}^\kappa(t) - \frac{y^\kappa(\theta_\kappa(t))}{2} \right\| \leqslant \frac{1}{2}\|y^\kappa(\theta_\kappa(t))\| + |\dot{v}(t)| \text{ for almost all } t \in [0, T]. \tag{36}$$

It follows from (35) and (36) that

$$\|\dot{y}^\kappa(t)\| \leqslant \delta\|y^\kappa(t)\| + \delta\|y^\kappa(\theta_\kappa(t))\| + \delta|\dot{v}(t)|$$

for almost all $t \in [0,T]$, where $\delta := \|J^{-1}\|$. Thus, one has

$$\begin{aligned}
\|y^\kappa(t)\| &\leqslant \|y^\kappa(t_i^\kappa)\| + \delta \int_{t_i^\kappa}^t \|y^\kappa(s)\| ds + \delta \int_{t_i^\kappa}^t \|y^\kappa(t_i^\kappa)\| ds + \delta \int_{t_i^\kappa}^t |\dot v(s)| ds \\
&\leqslant (1 + \delta(t - t_i^\kappa)) \|y^\kappa(t_i^\kappa)\| + \delta \int_{t_i^\kappa}^t \|y^\kappa(s)\| ds + \delta \int_{t_i^\kappa}^t |\dot v(s)| ds \\
&\leqslant \{1 + \delta(t_{i+1}^\kappa - t_i^\kappa)\} \|y^\kappa(t_i^\kappa)\| + \delta \int_{t_i^\kappa}^t \|y^\kappa(s)\| ds + \delta \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds
\end{aligned}$$

for all $t \in [t_i^\kappa, t_{i+1}^\kappa]$. By Gronwall's inequality, the latter inequality yields

$$\|y^\kappa(t)\| \leqslant \left( \left(1 + \delta(t_{i+1}^\kappa - t_i^\kappa)\right) \|y^\kappa(t_i^\kappa)\| + \delta \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds \right) e^{\delta(t - t_i^\kappa)} \tag{37}$$

for all $t \in [t_i^\kappa, t_{i+1}^\kappa]$ and in particular

$$\begin{aligned}
\|y^\kappa(t_{i+1}^\kappa)\| &\leqslant \left(1 + \delta(t_{i+1}^\kappa - t_i^\kappa)\right) \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} + \delta \left( \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds \right) e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} \\
&\leqslant \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} + \delta(t_{i+1}^\kappa - t_i^\kappa) \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} + \delta \left( \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds \right) e^{\delta(t_{i+1}^\kappa - t_i^\kappa)}.
\end{aligned}$$

Since the above argument is valid for any $i \in \{0, 1, \ldots, \kappa - 1\}$, one gets

$$\begin{aligned}
\|y^\kappa(t_{i+1}^\kappa)\| &\leqslant \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} + \delta(t_{i+1}^\kappa - t_i^\kappa) \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} + \delta \left( \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds \right) e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} \\
&\leqslant \|y^\kappa(t_{i-1}^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_{i-1}^\kappa)} + \delta(t_i^\kappa - t_{i-1}^\kappa) \|y^\kappa(t_{i-1}^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_{i-1}^\kappa)} + \delta(t_{i+1}^\kappa - t_i^\kappa) \|y^\kappa(t_i^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} \\
&\qquad + \delta \left( \int_{t_{i-1}^\kappa}^{t_i^\kappa} |\dot v(s)| ds \right) e^{\delta(t_{i+1}^\kappa - t_{i-1}^\kappa)} + \delta \left( \int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)| ds \right) e^{\delta(t_{i+1}^\kappa - t_i^\kappa)} \leqslant \\
&\ \ \vdots \\
&\leqslant \|y_0\| e^{\delta(t_{i+1}^\kappa - t_0^\kappa)} + \delta \sum_{\ell = i+1}^{1} (t_\ell^\kappa - t_{\ell-1}^\kappa) \|y^\kappa(t_{\ell-1}^\kappa)\| e^{\delta(t_{i+1}^\kappa - t_{\ell-1}^\kappa)} + \delta \sum_{\ell = i+1}^{1} \int_{t_{\ell-1}^\kappa}^{t_\ell^\kappa} |\dot v(s)| ds e^{\delta(t_{i+1}^\kappa - t_{\ell-1}^\kappa)} \\
&\leqslant e^{\delta T} \|y_0\| + \delta T e^{\delta T} \max\{\|y^\kappa(t_i^\kappa)\|\} + \delta e^{\delta T} \int_0^T |\dot v(s)| ds,
\end{aligned}$$

for all $i \in \{0, 1, \ldots, \kappa - 1\}$ by backward induction. Therefore, one has that

$$\max\{\|y^\kappa(t_i^\kappa)\|\} \leqslant e^{\delta T} \|y_0\| + \delta T e^{\delta T} \max\{\|y^\kappa(t_i^\kappa)\|\} + \delta e^{\delta T} \int_0^T |\dot v(s)| ds$$

or equivalently

$$(1 - \delta T e^{\delta T}) \max\{\|y^\kappa(t_i^\kappa)\|\} \leqslant e^{\delta T} \|y_0\| + \delta e^{\delta T} \int_0^T |\dot v(s)| ds.$$

In view of this, if $1 - \delta T e^{\delta T} > 0$, the quantity $\max\limits_{0 \leqslant i \leqslant \kappa} \|y^\kappa(t_i^\kappa)\|$ is upper bounded by a constant that does not depend on the partition of the interval $[0,T]$,

$$\max_{0 \leqslant i \leqslant \kappa} \|y^\kappa(t_i^\kappa)\| \leqslant \frac{e^{\delta T} \|y_0\| + \delta e^{\delta T} \int_0^T |\dot v(s)| ds}{1 - \delta T e^{\delta T}} =: \gamma. \tag{38}$$

Due to (37), (38) and Bernoulli's inequality, one gets

$$\begin{aligned}\|y^\kappa(t)\| &\leqslant \left(\left(1+\delta(t_{i+1}^\kappa - t_i^\kappa)\right)\|y^\kappa(t_i^\kappa)\| + \delta\int_{t_i^\kappa}^{t_{i+1}^\kappa} |\dot v(s)|ds\right)e^{\delta(t-t_i^\kappa)}\\ &\leqslant \left((1+\delta T)\gamma + \delta\int_0^T |\dot v(s)|ds\right)e^{\delta T} =: \Gamma_1, \forall t\in[0,T].\end{aligned}\tag{39}$$

It follows from (36) and (38) that

$$\|\dot z^\kappa(t)\| \leqslant \|y^\kappa(\theta_\kappa(t))\| + |\dot v(t)| \leqslant \max_{0\leqslant i\leqslant \kappa}\|y^\kappa(t_i^\kappa)\| + |\dot v(t)| \leqslant \gamma + |\dot v(t)|\tag{40}$$

and

$$\left\|\dot z^\kappa(t) - \frac{y^\kappa(\theta_\kappa(t))}{2}\right\| \leqslant \frac{1}{2}\|y^\kappa(\theta_\kappa(t))\| + |\dot v(t)| \leqslant \frac{\gamma}{2} + |\dot v(t)| =: \Delta(t)\tag{41}$$

for all $t\in[0,T]$.

*Step 2. Prove that the sequences $\{z^\kappa(\cdot)\mid \kappa\in\mathbb{N}\}$ and $\{y^\kappa(\cdot)\mid\kappa\in\mathbb{N}\}$ are Cauchy in* $\mathrm{C}([0,T],\mathbb{R}^n)$. For any $\kappa,\ell\in\mathbb{N}$, it first follows from (35) that

$$\|y^\kappa(t) - y^\ell(t)\| \leqslant \delta\int_0^t \|y^\kappa(s) - y^\ell(s)\|ds + \delta\|z^\kappa(t) - z^\ell(t)\|$$

for all $t\in[0,T]$ with recalling that $\delta := \|J^{-1}\|$. By Lemma 2.8, this implies

$$\|y^\kappa(t) - y^\ell(t)\| \leqslant \delta\|z^\kappa(t) - z^\ell(t)\| + \delta\int_0^t e^{\delta(t-s)}\|z^\kappa(s) - z^\ell(s)\|ds, \forall t\in[0,T].\tag{42}$$

Also, by the construction and (41), one has on the one hand that

$$\frac{\rho}{\Delta(t)}\left(-\dot z^\kappa(t) + \frac{1}{2}y^\kappa(\theta_\kappa(t))\right)\in\mathcal{N}^P_{\mathcal{C}(t)}(z^\kappa(t)),\quad \frac{\rho}{\Delta(t)}\left(-\dot z^\ell(t) + \frac{1}{2}y^\ell(\theta_\ell(t))\right)\in\mathcal{N}^P_{\mathcal{C}(t)}(z^\ell(t))$$

and on the other hand

$$\left\|\frac{\rho}{\Delta(t)}\left(-\dot z^\kappa(t) + \frac{1}{2}y^\kappa(\theta_\kappa(t))\right)\right\|\leqslant\rho,\ \left\|\frac{\rho}{\Delta(t)}\left(-\dot z^\ell(t) + \frac{1}{2}y^\ell(\theta_\ell(t))\right)\right\|\leqslant\rho.$$

Thus, by Lemma 2.6, we have

$$\left\langle \dot z^\kappa(t) - \frac{1}{2}y^\kappa(\theta_\kappa(t)) - \dot z^\ell(t) + \frac{1}{2}y^\ell(\theta_\ell(t)), z^\kappa(t) - z^\ell(t)\right\rangle \leqslant \frac{\Delta(t)}{\rho^2}\|z^\kappa(t) - z^\ell(t)\|^2$$

for almost all $t\in[0,T]$. It follows that

$$\begin{aligned}\left\langle \dot z^\kappa(t) - \dot z^\ell(t), z^\kappa(t) - z^\ell(t)\right\rangle &\leqslant \frac{\Delta(t)}{\rho^2}\|z^\kappa(t) - z^\ell(t)\|^2 + \frac{1}{2}\left\langle y^\kappa(\theta_\kappa(t)) - y^\ell(\theta_\ell(t)), z^\kappa(t) - z^\ell(t)\right\rangle\\ &\leqslant \frac{\Delta(t)}{\rho^2}\|z^\kappa(t) - z^\ell(t)\|^2 + \frac{1}{2}\|y^\kappa(t) - y^\ell(t)\|\|z^\kappa(t) - z^\ell(t)\|\\ &\quad + \frac{1}{2}\left(\|y^\kappa(t) - y^\kappa(\theta_\kappa(t))\| + \|y^\ell(t) - y^\ell(\theta_\ell(t))\right)\|z^\kappa(t) - z^\ell(t)\|.\end{aligned}$$

Due to (42), one further gets

$$\begin{aligned}\langle \dot z^\kappa(t) - \dot z^\ell(t), z^\kappa(t) - z^\ell(t)\rangle &\leqslant \left(\frac{\Delta(t)}{\rho^2} + \frac{\delta}{2}\right)\|z^\kappa(t) - z^\ell(t)\|^2 + \frac{\delta}{2}\|z^\kappa(t) - z^\ell(t)\|\int_0^t e^{\delta(t-s)}\|z^\kappa(s) - z^\ell(s)\|ds\\ &\quad + \frac{1}{2}\left(\|y^\kappa(t) - y^\kappa(\theta_\kappa(t))\| + \|y^\ell(t) - y^\ell(\theta_\ell(t))\right)\|z^\kappa(t) - z^\ell(t)\|\end{aligned}$$

and hence

$$\frac{d}{dt}\|z^\kappa(t)-z^\ell(t)\|^2 \leqslant 2\left(\frac{\Delta(t)}{\rho^2}+\frac{\delta}{2}\right)\|z^\kappa(t)-z^\ell(t)\|^2+\delta e^{\delta T}\|z^\kappa(t)-z^\ell(t)\|\int_0^t\|z^\kappa(s)-z^\ell(s)\|ds$$
$$+\left(\|y^\kappa(t)-y^\kappa(\theta_\kappa(t))\|+\|y^\ell(t)-y^\ell(\theta_\ell(t))\right)\|z^\kappa(t)-z^\ell(t)\|.$$

Note that $\Delta(\cdot)\in \mathrm{L}_1([0,T],\mathbb{R}_+)$ and one has

$$\begin{aligned}
\|y^\kappa(t)-y^\kappa(\theta_\kappa(t))\| &\leqslant \int_{\theta_\kappa(t)}^t\|\dot{y}^\kappa(s)\|ds\leqslant \delta\int_{\theta_\kappa(t)}^t(\|y^\kappa(s)\|+\|\dot{z}^\kappa(s)\|)ds\\
&\leqslant \delta\int_{\theta_\kappa(t)}^t(\Gamma_1+\gamma+|\dot{v}(s)|)ds\overset{\kappa\to\infty}{\longrightarrow}0,\\
\|y^\ell(t)-y^\ell(\theta_\ell(t))\| &\leqslant \int_{\theta_\ell(t)}^t\|\dot{y}^\ell(s)\|ds\leqslant \delta\int_{\theta_\ell(t)}^t(\|y^\ell(s)\|+\|\dot{z}^\ell(s)\|)ds\\
&\leqslant \delta\int_{\theta_\ell(t)}^t(\Gamma_1+\gamma+|\dot{v}(s)|)ds\overset{\ell\to\infty}{\longrightarrow}0.
\end{aligned}$$

By denoting $G_{\kappa,\ell}(t)=\|y^\kappa(t)-y^\kappa(\theta_\kappa(t))\|+\|y^\ell(t)-y^\ell(\theta_\ell(t))\|$, one has on the one hand that $\lim\limits_{\kappa,\ell\to\infty}G_{\kappa,\ell}(t)=0$ for all $t\in[0,T]$, and onther other hand

$$G_{\kappa,\ell}(t)=\|y^\kappa(t)-y^\kappa(\theta_\kappa(t))\|+\|y^\ell(t)-y^\ell(\theta_\ell(t))\|\leqslant 4\Gamma_1,\forall t\in[0,T].$$

Therefore, by Lebesgue's dominated convergence theorem, one has

$$\lim_{\kappa,\ell\to\infty}\int_0^T\left(\|y^\kappa(t)-y^\kappa(\theta_\kappa(t))\|+\|y^\ell(t)-y^\ell(\theta_\ell(t))\right)dt=\lim_{\kappa,\ell\to\infty}\int_0^T G_{\kappa,\ell}(t)dt=0.$$

Moreover, $z^\kappa(0)=z^\ell(0)=z_0$ for all $\kappa,\ell\in\mathbb{N}$. In view of these achivements and if

$$\int_0^T\exp\left(\int_s^T 2\left(\frac{\frac{\gamma}{2}+|\dot{v}(\tau)|}{\rho^2}+\frac{\delta}{2}\right)d\tau\right)ds<\frac{1}{\delta e^{\delta T}T}, \tag{43}$$

Lemma 2.9 yields

$$\lim_{\kappa,\ell\to\infty}\|z^\kappa-z^\ell\|_{\mathrm{C}([0,T],\mathbb{R}^n)}=0 \text{ and hence } \lim_{\kappa,\ell\to\infty}\|y^\kappa-y^\ell\|_{\mathrm{C}([0,T],\mathbb{R}^n)}=0.$$

Therefore, the sequence $\{z^\kappa\mid\kappa\in\mathbb{N}\}$ and $\{y^\kappa\mid\kappa\in\mathbb{N}\}$ are Cauchy in $\mathrm{C}([0,T],\mathbb{R}^n)$ and hence they converge to the functions $z,y\in\mathrm{C}([0,T],\mathbb{R}^n)$, respectively.

*Step 3. Prove that the pair $z(\cdot),y(\cdot)$ is an absolutely continuous solution of the system* (27). Due to (40), one can assume that the sequence $\{\dot{z}^\kappa\}$ converges weakly to a function $w\in\mathrm{L}_1([0,T],\mathbb{R}^n)$ and one then has

$$\lim_{\kappa\to\infty}z_i^\kappa(t)-z_{i,0}=\lim_{\kappa\to\infty}\int_0^t\dot{z}_i^\kappa(s)ds=\lim_{\kappa\to\infty}\int_0^t\langle\dot{z}^\kappa(s),e_i\rangle ds=\int_0^t\langle w(s),e_i\rangle ds=\int_0^t w_i(s)ds \tag{44}$$

for any $t\in[0,T]$ where $e_i$ stands for the $i$-th unit vector of $\mathbb{R}^n$. Moreover, since $z^\kappa(t)$ is pointwisely convergent to $z(t)$ in $\mathbb{R}^n$, it implies from (44) that

$$z(t)=z_0+\int_0^t w(s)ds,\ \text{for any } t\in[0,T].$$

This means that $z$ is absolutely continuous, $z(0) = z_0$ and $\dot{z}(t) = w(t)$ for almost all $t \in [0, T]$. Next, note that the ordinary differential equation

$$\dot{y}^\kappa(t) = -J^{-1}y^\kappa(t) + J^{-1}\dot{z}^\kappa(t), y^\kappa(0) = y_0, z^\kappa(0) = z_0 \tag{45}$$

can be rewritten in the equivalent integral equation form

$$y^\kappa(t) = y_0 - J^{-1}\int_0^t y^\kappa(s)ds + J^{-1}[z^\kappa(t) - z_0], z^\kappa(0) = z_0. \tag{46}$$

Since $\lim_{\kappa\to\infty} z^\kappa(t) = z(t)$, $\lim_{\kappa\to\infty} y^\kappa(t) = y(t)$ and $\|y^\kappa(t)\| \leqslant \Gamma_1, \forall t \in [0, T]$, letting $\kappa \to \infty$ in (46) we obtain

$$y(t) = y_0 - J^{-1}\int_0^t y(s)ds + J^{-1}(z(t) - z_0).$$

By taking derivative both sides, the latter yields

$$\dot{y}(t) = -J^{-1}y(t) + J^{-1}\dot{z}(t), y(0) = y_0, z(0) = z_0 \tag{47}$$

for almost all $t \in [0, T]$. The remain is proving that

$$\dot{z}(t) \in \frac{y(t)}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \text{ for almost all } t \in [0, T]. \tag{48}$$

Since $\theta_\kappa(t) \to t$ and $y^\kappa(t) \to y(t)$ as $\kappa \to \infty$ for any $t \in [0, T]$, one has $y^\kappa(\theta_\kappa(t)) \to y(t)$ as $\kappa \to \infty$. Also, note that $\dot{z}^\kappa$ converges weakly to $\dot{z}$ in $\mathrm{L}_1([0, T], \mathbb{R}^n)$, Mazur's lemma yields the existence of a sequence $\{\zeta^\ell\}$ that converges strongly to the function $\dot{z} - \frac{1}{2}y$ in $\mathrm{L}_1([0, T], \mathbb{R}^n)$, where

$$\zeta^\ell(t) = \sum_{k=\ell}^{\tau(\ell)} \delta^\ell_k \left(\dot{z}^k(t) - \frac{1}{2}y^k(\theta_k(t))\right), \text{ for almost all } t \in [0, T],$$

for some $\delta^\ell_k \geqslant 0, \sum_{k=\ell}^{\tau(\ell)} \delta^\ell_k = 1$. By extracting a subsequence if it is necessary, without loss of generality, we can assume that $\zeta^\ell(t) \longrightarrow \dot{z}(t) - \frac{1}{2}y(t)$ as $\ell \to \infty$ for almost all $t \in [0, T]$. By the construction, we have

$$\dot{z}^\kappa(t) \in \frac{y^\kappa(\theta_\kappa(t))}{2} - \mathcal{N}^P_{\mathcal{C}(t)}(z^\kappa(t)) \text{ for almost all } t \in [0, T] \text{ and for all } \kappa \in \mathbb{N}^*. \tag{49}$$

By Lemma 2.6, this implies that

$$\begin{aligned}\left\langle -\dot{z}^\kappa(t) + \frac{y^\kappa(\theta_\kappa(t))}{2}, \xi - z^\kappa(t)\right\rangle &\leqslant \frac{\left\|\dot{z}^\kappa(t) - \frac{y^\kappa(\theta_\kappa(t))}{2}\right\|}{2\rho}\|\xi - z^\kappa(t)\|^2 \\ &\leqslant \frac{\Delta(t)}{2\rho}\|\xi - z^\kappa(t)\|^2, \forall \xi \in \mathcal{C}(t)\end{aligned} \tag{50}$$

for almost all $t \in [0, T]$ and for all $\kappa \in \mathbb{N}^*$. Thus, one has

$$\sum_{k=\ell}^{\tau(\ell)} \delta^\ell_k \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), \xi - z^k(t)\right\rangle \leqslant \frac{\Delta(t)}{2\rho}\sum_{k=\ell}^{\tau(\ell)} \delta^\ell_k\|\xi - z^\kappa(t)\|^2, \forall \xi \in \mathcal{C}(t) \tag{51}$$

for almost all $t \in [0,T]$. On the other hand, we have

$$\begin{aligned}
&\sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), \xi - z^k(t) \right\rangle \\
&\quad = \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), \xi - z(t) \right\rangle + \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), z(t) - z^k(t) \right\rangle \\
&\quad = \left\langle -\zeta^\ell(t), \xi - z(t) \right\rangle + \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), z(t) - z^k(t) \right\rangle . \qquad (52)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left| \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \left\langle -\dot{z}^k(t) + \frac{1}{2}y^k(\theta_k(t)), z(t) - z^k(t) \right\rangle \right| \\
&\quad \leqslant \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \|\dot{z}^k(t) - \frac{1}{2}y^k(\theta_k(t))\| \|z(t) - z^k(t)\| \leqslant \Delta(t) \sum_{k=\ell}^{\tau(\ell)} \delta_k^\ell \|z(t) - z^k(t)\| \overset{\ell \to \infty}{\longrightarrow} 0. \qquad (53)
\end{aligned}$$

For each $t \in [0,T]$, due to (51), (52) and (53), by letting $\ell \to \infty$, we obtain

$$\left\langle -\dot{z}(t) + \frac{1}{2}y(t), \xi - z(t) \right\rangle \leqslant \frac{\Delta(t)}{2\rho}\|\xi - z(t)\|^2, \forall \xi \in \mathcal{C}(t).$$

Due to (5), this turns out that

$$-\dot{z}(t) + \frac{1}{2}y(t) \in \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \text{ or } \dot{z}(t) \in \frac{1}{2}y(t) - \mathcal{N}^P_{\mathcal{C}(t)}(z(t)) \text{ for almost all } t \in [0,T]. \qquad (54)$$

The proof is done. □

In the remain of this paper, we aim at providing a generalization of Theorem 4.2 to the case that $J$ is a symmetric positive semi-definite matrix. Of course, we only need to deal without the positive definite property of $J$. In this case, $\operatorname{rank}(J) = r$ for some $r < n$. Moreover, there exists an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $J$ has the singular valued decomposition as

$$J = U\Lambda U^T = U \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} U^T \qquad (55)$$

where $U^T = U^{-1}$ and $\Lambda \in \mathbb{R}^{r\times r}$ is a diagonal matrix with positive entries on the principal diagonal. To proceed, we need to introduce some nomenclatures and auxiliary results. Let $r \in \{1, 2, \ldots, n\}$ and let $\mathcal{C}$ be a closed subset of $\mathbb{R}^n$. We define

$$\mathcal{C}_1^r := \left\{ \zeta_1 \in \mathbb{R}^r \mid \exists \zeta_2 \in \mathbb{R}^{n-r}, \operatorname{col}(\zeta_1, \zeta_2) \in \mathcal{C} \right\}, \ \mathcal{C}_2^r := \left\{ \zeta_2 \in \mathbb{R}^{n-r} \mid \exists \zeta_1 \in \mathbb{R}^{r}, \operatorname{col}(\zeta_1, \zeta_2) \in \mathcal{C} \right\}.$$

Note that these sets can not be closed if $\mathcal{C}$ is not compact. In the remain, we suppose that these sets are closed under the suitable choices of the set $\mathcal{C}$.

**Lemma 4.3.** *Let $1 \leqslant r \leqslant n$ and let $\Lambda \in \mathbb{R}^{r\times r}$ be a symmetric positive definite matrix. Let $\mathcal{C}$ be a closed subset of $\mathbb{R}^n$ such that $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ are closed. Then, if*

$$(x_1 - \Lambda y_1, y_1) \in \operatorname{gr}\left(\mathcal{N}^P_{\mathcal{C}_1^r}\right) \text{ and } (x_2, y_2) \in \operatorname{gr}\left(\mathcal{N}^P_{\mathcal{C}_2^r}\right) \qquad (56)$$

*then*

$$\left(\begin{pmatrix} x_1 - \Lambda y_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in \operatorname{gr}\left(\mathcal{N}_{\mathcal{C}}^P\right). \tag{57}$$

*As a result, one has*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \left(\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} + \left(\mathcal{N}_{\mathcal{C}}^P\right)^{-1}\right)^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{58}$$

*Proof.* Due to (56) and the property (5) of proximal normal cones, there exist $M_1 \geqslant 0$ and $M_2 \geqslant 0$ such that

$$\langle y_1, c_1^r - (x_1 - \Lambda y_1)\rangle \leqslant M_1 \|c_1^r - (x_1 - \Lambda y_1)\|^2 \text{ and } \langle y_2, c_2^r - x_2\rangle \leqslant M_2\|c_2^r - x_2\|^2$$

for all $c_1^r \in \mathcal{C}_1^r$ and all $c_2^r \in \mathcal{C}_2^r$. In view of these inequalities, one then has

$$\begin{aligned}\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} x_1 - \Lambda y_1 \\ x_2 \end{pmatrix}\right\rangle &= \langle y_1, c_1 - (x_1 - \Lambda y_1)\rangle + \langle y_2, c_2 - x_2\rangle \\ &\leqslant M_1\|c_1 - (x_1 - \Lambda y_1)\|^2 + M_2\|c_2 - x_2\|^2 \leqslant \max\{M_1, M_2\} \left\| \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} x_1 - \Lambda y_1 \\ x_2 \end{pmatrix}\right\|^2\end{aligned}$$

for all $c = \operatorname{col}(c_1, c_2) \in \mathcal{C}$ where $c_1 \in \mathbb{R}^r$ and $c_2 \in \mathbb{R}^{n-r}$. Due to (5), the latter inequality yields that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{N}_{\mathcal{C}}^P \begin{pmatrix} x_1 - \Lambda y_1 \\ x_2 \end{pmatrix}$$

and the claim (57) is proved. Finally, one can easily verify (58) once (57) is valid. $\square$

**Lemma 4.4.** *Suppose that $\mathcal{C}(\cdot)$ satisfies Assumption 1 such that $\mathcal{C}_1^r(t)$ and $\mathcal{C}_2^r(t)$ are closed for all $t \in [0, T]$. Let $\Lambda$ be a symmetric positive definite matrix of $\mathbb{R}^{r\times r}, r < n$. Then, the absolutely continuous solutions of the differential inclusion*

$$\begin{cases} -\dot{x}(t) & \in \left(\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} + \left(\mathcal{N}_{\mathcal{C}(t)}^P\right)^{-1}\right)^{-1} (x(t)), \ t \in [0, T], \\ x(0) & = x_0 \end{cases} \tag{59}$$

*locally exist for each $x_0 \in \mathcal{D}$, where*

$$\mathcal{D} = \operatorname{dom}\left(\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} + \left(\mathcal{N}_{\mathcal{C}(0)}^P\right)^{-1}\right)^{-1}.$$

*Proof.* For $x_0 \in \mathcal{D}$, we partition it as $x_0 = \operatorname{col}(x_1^0, x_2^0)$ with $x_1^0 \in \mathbb{R}^r$ and $x_2^0 \in \mathbb{R}^{n-r}$. One can verify that $x_2^0 \in \mathcal{C}_2^r(0)$ and $x_1^0 \in \operatorname{dom}(\Lambda + (\mathcal{N}_{\mathcal{C}_1^r(0)}^P)^{-1})^{-1}$. Due to Lemma 4.3, we observe that if $x_1(t)$ is an absolutely continuous solution of the differential inclusion

$$\begin{cases} -\dot{x}_1(t) & \in \left(\Lambda + \left(\mathcal{N}_{\mathcal{C}_1^r(t)}^P\right)^{-1}\right)^{-1} (x_1(t)), t \in [0, T] \\ x_1(0) & = x_1^0 \end{cases} \tag{60}$$

and $x_2(t)$ is an absolutely continuous solution of the sweeping process

$$\begin{cases} -\dot{x}_2(t) & \in \mathcal{N}^P_{\mathcal{C}_2^r(t)}(x_2(t)), t \in [0, T] \\ x_2(0) & = x_2^0 \end{cases} \tag{61}$$

then

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, t \in [0, T]$$

is an absolutely continuous solution of the differential inclusion (59). Therefore, to prove the solutions existence of (59), it suffices to show the existence of solutions of the differential inclusions (60) and (61). However, the existence of solutions of the system (60) is followed from Theorem 4.2, while the existence of solutions of (61) is a well-known result in sweeping process literature, refered to the papers [5, 16]. ☐

**Theorem 4.5.** *Let $\mathcal{C}(\cdot)$ satisfy Assumption 1 and be such that $\mathcal{C}_1^r(t)$ and $\mathcal{C}_2^r(t)$ are closed for all $t \in [0, T]$. Let $J \in \mathbb{R}^{n \times n}$ be a symmetric positive semi-definite matrix. Then, the absolutely continuous solutions of the differential inclusion*

$$\begin{cases} -\dot{x}(t) & \in \left(J + \left(\mathcal{N}^P_{\mathcal{C}(t)}\right)^{-1}\right)^{-1}(x(t)), \ t \in [0, T], \\ x(0) & = x_0 \end{cases} \tag{62}$$

*locally exist for every $x_0 \in \mathcal{D}_J(0) := \operatorname{dom}\left(J + \left(\mathcal{N}^P_{\mathcal{C}(0)}\right)^{-1}\right)^{-1}$.*

*Proof.* Let $x_0 \in \mathcal{D}_J(0) := \operatorname{dom}\left(J + (\mathcal{N}^P_{\mathcal{C}(0)})^{-1}\right)^{-1}$. Suppose that $J$ has the singular value decomposition in the form of (55). Then, the system (62) can be written in the form

$$\begin{cases} -U^T\dot{x}(t) & \in \left(\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} + U^T \left(\mathcal{N}^P_{\mathcal{C}(t)}\right)^{-1} U\right)^{-1} (U^T x(t)), \ t \in [0, T], \\ x(0) & = x_0. \end{cases} \tag{63}$$

On the other hand, we claim that

$$U^T(\mathcal{N}^P_{\mathcal{C}(t)})^{-1}U = (\mathcal{N}^P_{U^T\mathcal{C}(t)})^{-1}. \tag{64}$$

Indeed, on the one hand, one has

$$v \in U^T(\mathcal{N}^P_{\mathcal{C}(t)})^{-1}U(x) \Longleftrightarrow Uv \in (\mathcal{N}^P_{\mathcal{C}(t)})^{-1}Ux \Longleftrightarrow Ux \in \mathcal{N}^P_{\mathcal{C}(t)}(Uv).$$

On the other hand, the fact $Ux \in \mathcal{N}^P_{\mathcal{C}(t)}(Uv)$ is equivalent to the existence of a nonnegative number $M$ such that

$$\begin{aligned} \langle x, U^T c - v\rangle = \langle Ux, c - Uv\rangle \leqslant M\|c - Uv\|^2 = M\langle c - Uv, c - Uv\rangle \\ = M\langle U^T c - v, U^T c - v\rangle = M\|U^T c - v\|^2 \end{aligned}$$

for all $c \in U^T\mathcal{C}(t)$. Therefore, we have

$$\langle x, U^T c - v\rangle \leqslant M\|U^T c - v\|^2 \text{ for all } c \in U^T\mathcal{C}(t). \tag{65}$$

This is equivalent to the fact $x \in \mathcal{N}^P_{U^T\mathcal{C}(t)}(v)$ or $v \in (\mathcal{N}^P_{U^T\mathcal{C}(t)})^{-1}(x)$. The claim (64) is proved.

In views of the equality (64) and by letting $z(t) := U^T x(t)$, the system (63) transforms into the following one

$$\begin{cases} -\dot{z}(t) & \in \left(\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} + \left(\mathcal{N}^P_{U^T\mathcal{C}(t)}\right)^{-1}\right)^{-1} (z(t)),\ t \in [0,T], \\ z(0) & = U^T x_0. \end{cases} \tag{66}$$

For all $t \in [0,T]$, note that $U^T\mathcal{C}(t)$ is $\tilde{\rho}$-prox-regular with $\tilde{\rho} = \|U\|\rho$. Thus, Lemma 4.4 ensures the existence of absolutely continuous solutions of the differential inclusion (66). Therefore, we conclude that the system (62) has at least one absolutely continuous solution. $\square$

## 5. Conclusions

In this paper, we studied the existence of absolutely continuous solutions of a class of differential inclusions involving proximal normal cone mappings in $\mathbb{R}^n$. The motivation of the study of this class of dynamical systems stems from the investigation of finite-dimensional differential variational inequalities in the recent years. Our contributions are that we derived serveral sufficient conditions for existence of absolutely continuous solutions.

**Acknowledgements.** This research is funded by Vietnam National Foundation for Science and Technology Developemnt (NAFOSTED) under grant number 101.02-2017.310 and by Vietnam Ministry of Education and Training under the grant TN. 18. Individually, L.Q. Thuan would like to thank Vietnam Institute for Advanced Study in Mathematics (VIASM) for supporting and providing a fruitful research environment and hospitality.

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