Hölder continuous subsolutions imply Hölder continuous solutions on domains of plurisubharmonic type m

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Abstract

In this paper, we prove the existence of Hölder continuous solutions for an arbitrary non-negative Borel measure μ if there exists a Hölder continuous subsolution on a domain Ω of plurisubharmonic type m in $\mathbb{C}^n$.

1 Introduction

Let $0 < \alpha \leq 1$. Through the paper by $C^{0,\alpha}(A), A \subset \mathbb{C}^n$ we denote the set of real-valued functions which are α-Hölder continuous on A. Hence, $\varphi \in C^{0,\alpha}(A)$ if and only if there exists $C > 0$ such that for $x, y \in A$ we have

$$|\varphi(x) - \varphi(y)| \leq C\|x - y\|^\alpha,$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{C}^n$. Let Ω be a bounded domain in $\mathbb{C}^n$ and $\psi \in C^{0,\alpha}(\partial\Omega)$. Assume that μ is a non-negative Borel measure on Ω. The Dirichlet problem with Hölder continuous solutions to the complex Monge-Ampère equation on Ω is the following

$$MA(\Omega, \mu, \psi) : \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}^{0,\gamma}(\overline{\Omega}), 0 < \gamma \leq 1; \\ (dd^c u)^n = \mu \\ \lim\limits_{z \to x} u(z) = \psi(x), \quad \text{for} \quad x \in \partial\Omega. \end{cases} \tag{1.1}$$

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where $PSH(\Omega)$ is the set of plurisubharmonic (psh) functions in Ω and $d = \partial + \overline{\partial}$, $d^c = (\frac{i}{4})(\overline{\partial} - \partial)$. Then $dd^c = (\frac{i}{2})\partial\overline{\partial}$ and $(dd^c u)^n$ stands for the complex Monge-Ampère operator of u.
In the case $\mu = f dV_{2n}$ where f is a function defined on Ω and dV_{2n} denotes the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ the regularity of the equation (1.1) was studied extensively by many authors. In [3], Bedford and Taylor proved that if $\psi \in C^{0,2\alpha}(\partial\Omega)$, $f^{\frac{1}{n}} \in C^{0,\alpha}(\overline{\Omega})$ then the equation (1.1) has a unique solution $u = u(\Omega, f, \psi)$ and $u(\Omega, f, \psi) \in C^{0,\alpha}(\overline{\Omega})$. Higher regularity of solutions of (1.1) has been investigated by Caffarelli, Kohn, Nirenberg and Spruck in [5]. In [5] under assuming smoothness of the data ψ, f and nondegeneracy of the density $f > 0$, they showed that $u(\Omega, f, \psi) \in C^{\infty}(\Omega)$. Krylov in [16] proved that if $\psi \in C^{3,1}(\partial\Omega)$ and $f \geq 0, f^{\frac{1}{n}} \in C^{1,1}(\overline{\Omega})$ then $u(\Omega, f, \psi) \in C^{1,1}(\overline{\Omega})$. When Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, in [9], Guedj, Kołodziej and Zeriahi investigated the Hölder continuity of solutions of (1.1). They in [9] proved that if $\psi \in C^{1,1}(\partial\Omega)$, $0 \leq f \in L^p(\Omega)$, for some $p > 1$, is bounded near the boundary then $u = u(\Omega, f, \psi)$ is the α-Hölder continuity with $0 < \alpha < \frac{2}{nq+1}$, $\frac{1}{p} + \frac{1}{q} = 1$. Next, Charabati extended the above result of Guedj, Kołodziej and Zeriahi to bounded strongly hyperconvex Lipschitz domains in $\mathbb{C}^n$ (see [7]). Recently, one are interested in a new direction of study for Hölder continuous solutions of the equation (1.1). Namely, let μ be a non-negative Borel measure on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and ψ is a Hölder continuous function on $\partial\Omega$. Assume that there exists $\varphi \in PSH(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ such that $\mu \leq (dd^c\varphi)^n$. Is there a $0 < \gamma \leq 1$ and a real-valued function u such that

$$MA(\Omega, \mu, \psi) : \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}^{0,\gamma}(\overline{\Omega}), 0 < \gamma \leq 1; \\ (dd^c u)^n = \mu \\ \lim\limits_{z \to x} u(z) = \psi(x), \quad \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1.2}$$

(Question 17 in [10]). This problem makes readers to connect to the earlier result proved by Kołodziej in [13] (also see Problem C in [14]). This is if Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $\psi \in C(\partial\Omega)$. Let μ be a non-negative Borel measure on Ω. If there exists a subsolution for μ in the sense that there exists $v \in PSH(\Omega) \cap L^{\infty}(\Omega)$, $\mu \leq (dd^c v)^n$, $\lim\limits_{z \to x} v(z) = \psi(x)$ for all $x \in \partial\Omega$ then we can find $u \in PSH(\Omega) \cap L^{\infty}(\Omega), \lim\limits_{z \to x} u(z) = \psi(x), x \in \partial\Omega$ and $(dd^c u)^n = \mu$ on Ω. The equation (1.2) was solved recently by Ngoc Cuong Nguyen in [18]. In [18], under the assumption that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, $\psi = 0$ on $\partial\Omega$ and $\int\limits_{\Omega} (dd^c\varphi)^n < +\infty$, Ngoc Cuong Nguyen showed that there exists $0 < \gamma \leq 1$ and $u \in C^{0,\gamma}(\Omega)$ such that u satisfies the equation (1.2). Recently, in the most new preprint (see [19]), Ngoc Cuong Nguyen removed the hypothesis $\int\limits_{\Omega} (dd^c\varphi)^n < +\infty$. However, in question 17 in [10], Zeriahi asked that to look for a Hölder continuous solution of equation (1.2) under assumption that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$. In this paper we try to replace the hypothesis Ω is a strictly pseudoconvex domain by a more weak hypothesis. This is Ω is a bounded domain of plurisubharmonic type m (see the precise definition in Section 2 below). Then we get the following.

Theorem 1.1. *Let Ω be a bounded domain of plurisubharmonic type m and μ be a non-negative Borel measure on Ω. Assme that there exists $\varphi \in \mathcal{E}_0(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $\mu \leq (dd^c\varphi)^n$.*
Then
i) *there exists a unique $w \in \mathcal{E}_0(\Omega) \cap C^0(\overline{\Omega})$ such that $(dd^c w)^n = \mu$ on Ω.*
ii) *if the function ρ in the definition of this domain is in the class $\mathcal{E}_0(\Omega)$ then there is a $0 < \gamma \leq \alpha$ such that the Dirichlet problem*

$$MA(\Omega, \mu, \gamma, 0) : \begin{cases} u \in PSH(\Omega) \cap \mathcal{C}^{0,\gamma}(\overline{\Omega}), \\ (dd^c u)^n = \mu, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

is solvable on Ω.

Note that techniques we use in the proof of the paper come from results in [18] and [11]. We also give an example in Section 2 which shows that the function ρ in the definition of domains of plurisubharmonic type m in [11], may be, is not in the class $\mathcal{E}_0(\Omega)$ introduced in [6]. Now we say something about the organization of the paper. In Section 2 we recall some elements of pluripotential theory and the classes $\mathcal{E}_0(\Omega)$ and $\mathcal{E}'_0(\Omega)$ introduced and invesigated by Cegrell in [6] and Ngoc Cuong Nguyen in [18] recently. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminairies

First, some elements of pluripotential theory that will be used throughout the paper can be found in [1], [2], [3], [4], [6], [7], [9], [11], [12], [14], [15], [20].
2.1. First, we recall the definition of a domain of plurisubharmonic type m in $\mathbb{C}^n$ introduced in [11] (also see [20] and [2]).
Let $m > 0$ and let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Ω is said to be of plurisubharmonic type m if there exists a bounded negative function $\rho \in C^{0,\frac{2}{m}}(\overline{\Omega})$ such that $\{\rho < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\rho(z) - \|z\|^2$ is plurisubharmonic in Ω.
From the above definition we note that every smooth bounded strictly pseudoconvex domain in $\mathbb{C}^n$ is of plurisubharmonic type 1. Moreover, every domain of plurisubharmonic type m is a hyperconvex domain. Here a domain Ω in $\mathbb{C}^n$ is called to be hyperconvex if there exists a plurisubharmonic function $\varphi : \Omega \longrightarrow (-\infty, 0)$ such that for every $c < 0$ the set $\Omega_c = \{z \in \Omega : \varphi(z) < c\} \Subset \Omega$. However, from the above definition, in

general, ρ is not a defining function for Ω. Moreover, under the hypotheses for ρ, may be, $\int\limits_{\Omega}(dd^c\rho)^n = +\infty$. We consider the following example.

2.2. Example. Let $\mathbb{B}(0,1) = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$ be the unit ball in $\mathbb{C}^2$. Set

$$\rho(z,w) = -(1 - |z|^2 - |w|^2)^{\frac{1}{2}}.$$

Is is clear that $-1 \leq \rho(z,w) < 0$ on $\mathbb{B}(0,1)$, $\lim\limits_{(z,w)\to\partial\mathbb{B}(0,1)} \rho(z,w) = 0$, $\rho(z,w) \in C^2(\mathbb{B}(0,1))$ and is a radial symmetric function. By an elementary computation we note that the function $\rho(z,w)$ is a Hölder continuous function on $\overline{\mathbb{B}}(0,1)$ with exponent $0 < \alpha \leq \frac{1}{4}$. Next, we prove that $\int\limits_{\mathbb{B}(0,1)} (dd^c\rho)^2 = +\infty$. Indeed, set $r = \sqrt{|z|^2 + |w|^2}$. Then we can write $\rho(r) = -\sqrt{(1-r^2)}$. By Proposition 2.3 in [17] we have

$$(dd^c\rho)^2 = \frac{1}{8}\frac{(2 - |z|^2 - |w|^2)}{(1 - |z|^2 - |w|^2)^2} dV_4,$$

where dV_4 is the Lebesgue measure in $\mathbb{C}^2 \cong \mathbb{R}^4$.
Then

$$\begin{aligned}\int\limits_{\mathbb{B}(0,1)} (dd^c\rho)^2 &= \frac{1}{8}\int\limits_0^1 \Big(\int\limits_{\partial\mathbb{B}(0,r)} \frac{(2-r^2)}{(1-r^2)^2} d\sigma(x)\Big) dr \\ &= \frac{1}{8}\sigma(\partial\mathbb{B}(0,1)) \int\limits_0^1 \frac{(2-r^2)r^3 dr}{(1-r^2)^2} \\ &= \frac{1}{16}\sigma(\partial\mathbb{B}(0,1)) \int\limits_0^1 \frac{(2-t)t dt}{(1-t)^2} \\ &= \frac{1}{16}\sigma(\partial\mathbb{B}(0,1)) \int\limits_0^1 \Big(-1 + \frac{1}{(1-t)^2}\Big) dt = +\infty,\end{aligned}$$

where $\sigma(\partial\mathbb{B}(0,1))$ denotes the Lebesgue measure of the sphere $\partial\mathbb{B}(0,1)$. The desired conclusion follows.

2.3. Remark. A similar result as the above example can be found in Demailly's paper (see [8], p. 542).

2.4. In the note, by $PSH(\Omega)$ we denote the set of plurisubharmonic functions on Ω while by $PSH^-(\Omega)$ we denote the set of negative plurisubharmonic functions on Ω.

Now we recall some classes of plurisubharmonic functions which are due to Cegrell (see [6]) and Ngoc Cuong Nguyen (see [18]). Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. As in [6] we define the following subclass of $\mathrm{PSH}^-(\Omega)$.

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) \cap \mathrm{L}^\infty(\Omega) : \lim_{z\to\partial\Omega}\varphi(z) = 0, \ \int\limits_{\Omega}(dd^c\varphi)^n < \infty\}.$$

The following subclass $\mathcal{E}_0'(\Omega)$ of $\mathcal{E}_0(\Omega)$ introduced in [18],

$$\mathcal{E}_0'(\Omega) = \{\varphi \in \mathcal{E}_0(\Omega) : \int\limits_{\Omega} (dd^c\varphi)^n \leq 1\}.$$

2.5. Through the paper we will use the following notation. We will write "$A \lesssim B$" if there exists a constant C such that $A \leq CB$. Moreover, we write $u \in C^{0,\alpha}(\overline{\Omega})$ if u is α-Hölder continuous on $\overline{\Omega}$.

3 The proof of Theorem 1.1.

Now we prove **i)** of Theorem 1.1.
Indeed, under the hypotheses of Theorem 1.1 and by using Theorem 4.4 in [1] there exists $w \in \mathcal{E}_0(\Omega)$, $\lim\limits_{z\to\partial\Omega} w(z) = 0$ with $(dd^cw)^n = \mu$ on Ω. Take a sequence $\varepsilon_j \searrow 0$. Next, choose a sequence of strictly psedoconvex domains $\Omega_j, j \geq 1$ such that

$$\Omega_j \Subset U_j = \{w < -\varepsilon_j\} \Subset \Omega_{j+1} \Subset U_{j+1} = \{w < -\varepsilon_{j+1}\} \Subset \Omega,$$

and $\Omega = \bigcup\limits_{j=1}^{\infty} \Omega_j$. Let $\rho_j \in C^2$ in a neighbourhood of $\overline{\Omega_j}$ be a defining function of Ω_j, i.e a function such that

$$\rho_j < 0 \ \text{ on } \Omega_j, \quad \rho_j = 0 \ \text{ and } \ d\rho_j \neq 0 \ \text{ on } \ \partial\Omega_j,$$

and $\Omega_j = \{\rho_j < 0\}$. By the hypothesis and using Lemma 2.7 in [18] we note that μ is Hölder continuous on $\mathcal{E}_0'$. On the other hand, for all $K \Subset \Omega_j \subset \Omega$ we have $C_n(K,\Omega) \leq C_n(K,\Omega_j)$ for $j \geq 1$. By Proposition 2.9 in [18] there exist uniform constants $\alpha_1 > 0, C > 0$ such that for all $K \Subset \Omega_j$ we have

$$\mu(K) \leq Cexp\Big(\frac{-\alpha_1}{[C_n(K,\Omega_j)]^{\frac{1}{n}}}\Big). \tag{3.1}$$

Theorem 5.9 in [15] implies that there exists a continuous solution u_j of the following Dirichlet problem

$$MA(\Omega,\mu,\rho_j) : \begin{cases} u_j \in PSH(\Omega_j) \cap \mathcal{C}(\overline{\Omega_j}); \\ (dd^cu_j)^n = \mu \\ \lim\limits_{z\to x} u_j(z) = \rho_j(x) = 0, \quad \forall \ \ x \in \partial\Omega. \end{cases}$$

Then it follows that $u_j \in \mathcal{E}_0(\Omega_j) \cap C(\overline{\Omega_j})$. From the definition of u_j, by using the comparison principle we get that $u_{j-1} \geq u_j \geq w$ on Ω_{j-1}. On the other hand, note that $w + \varepsilon_j = 0$ on ∂U_j and $u_{j+1} \leq 0$ on U_j. We have

$$(dd^cu_{j+1})^n = (dd^cw)^n = (dd^c(w+\varepsilon_j))^n. \tag{3.2}$$

Again using the comparison principle we obtain that

$$w \leq u_{j+1} \leq w + \varepsilon_j, \tag{3.3}$$

on U_j. From (3.3) it follows that

$$1_{U_j}|u_{j+1} - w| < \varepsilon_j.$$

Thus, we get that the sequence $\{u_j\}$ is uniformly convergent to w on compact subsets $K \Subset \Omega$. Hence, w is in $\mathcal{E}_0(\Omega) \cap C(\Omega)$. If we set $w = 0$ on $\partial\Omega$ then we get that $w \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ with $(dd^c w)^n = \mu$ on Ω and the proof of **i)** is complete.

ii). As in [9] , for $\delta > 0$ by Ω_δ we denote

$$\Omega_\delta = \{z \in \Omega : dist(z, \partial\Omega) > \delta\},$$

and set

$$w_\delta(z) = \sup_{\|\zeta\| \leq \delta} w(z + \zeta), \quad \text{for} \quad z \in \Omega_\delta,$$

$$\hat{w}_\delta(z) = \frac{1}{\tau_{2n}\delta^{2n}} \int\limits_{\|\zeta - z\| \leq \delta} w(\zeta) dV_{2n}(\zeta), \quad z \in \Omega_\delta,$$

where τ_{2n} denotes the volume of the unit ball in $\mathbb{C}^n$ and dV_{2n} is the Lebesgue measure of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Since $(dd^c w)^n \leq (dd^c \varphi)^n$ by the comparison principle in [4] it follows that

$$\varphi \leq w \leq 0 \quad \text{on} \quad \Omega. \tag{3.4}$$

Repeating arguments as in [18] we infer that there exist constants $c_0 = c_0(\varphi), 1 > \delta_0 > 0$ small such that for all $0 < \delta < \delta_0$, $z \in \partial\Omega_\delta$ we have

$$w_\delta(z) \leq w(z) + c_0\delta^\alpha, \tag{3.5}$$

where α is the exponent of φ.

Now set

$$\widetilde{w} = \begin{cases} \max\{\hat{w}_\delta - c_0\delta^\alpha, w\} & \text{on} \ \ \Omega_\delta, \\ w & \text{on} \ \ \Omega \setminus \Omega_\delta. \end{cases}$$

Note that $\widetilde{w} \in PSH(\Omega) \cap C(\overline{\Omega})$, $0 \geq \widetilde{w} \geq w$ then $\widetilde{w} \in \mathcal{E}_0(\Omega)$. If put $C_1 = \int\limits_\Omega (dd^c\widetilde{w})^n$, $C_2 = \int\limits_\Omega (dd^c w)^n$ and $C_3 = \max\{C_1, C_2\}$ then $\frac{\widetilde{w}}{\sqrt[n]{C_3}}, \frac{w}{\sqrt[n]{C_3}} \in \mathcal{E}'_0(\Omega)$. On the other hand, from the hypothesis and using Lemma 2.7 and 2.4 in [18] it follows that the measure $\nu = (dd^c w)^n$ is α-Hölder continuous on $\mathcal{E}'_0(\Omega)$. Hence, Definition 2.3 in [18] implies that there exists $0 < \alpha_1 \leq 1$ such that

$$\int\limits_\Omega \Big| \frac{\widetilde{w}}{\sqrt[n]{C_3}} - \frac{w}{\sqrt[n]{C_3}} \Big| d\nu \leq C \| \frac{\widetilde{w}}{\sqrt[n]{C_3}} - \frac{w}{\sqrt[n]{C_3}} \|_1^{\alpha_1}. \tag{3.6}$$

From (3.6) we infer that

$$\int\limits_\Omega \Big| \widetilde{w} - w \Big| d\nu \lesssim \Big(\int\limits_\Omega |\widetilde{w} - w| dV_{2n} \Big)^{\alpha_1}. \tag{3.7}$$

We need the following.

Lemma 3.1. *For $1 > \delta > 0$ small enough there exists $C > 0$ independent of δ such that*

$$\int_{\Omega_\delta} |\hat{w}_\delta - w| dV_{2n} \leq C\delta.$$

Proof. Indeed, by Jensen's formula and using polar coordinates as in [9] we have

$$\hat{w}_\delta(z) - w(z) = \frac{1}{\sigma_{2n-1}\delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} \Big(\int_{|\zeta - z| \leq t} dd^c w \wedge \beta_{n-1} \Big) dt,$$

for every $z \in \Omega_\delta$, where σ_{2n-1} denotes the surface measure of the unit sphere. Hence, we get

$$\int_{\Omega_{2\delta}} |\hat{w}_\delta - w| dV_{2n}$$
$$\leq \frac{1}{\sigma_{2n-1}\delta^{2n}} \int\limits_{\Omega_{2\delta}} \Big(\int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} \Big(\int_{|\zeta - z| \leq t} dd^c w \wedge \beta_{n-1} \Big) \Big) dt dV_{2n}(z).$$

Applying Fubini's theorem and by the hypothesis $dd^c\rho \geq \beta$ we infer that

$$\int_{\Omega_{2\delta}} |\hat{w}_\delta - w| dV_{2n} \lesssim \delta^2 \int_{\Omega_\delta} dd^c w \wedge \beta_{n-1} \lesssim \delta^2 \int_{\Omega_\delta} dd^c w \wedge (dd^c\rho)^{n-1}$$
$$\lesssim \delta^2 \int_\Omega dd^c w \wedge (dd^c\rho)^{n-1} \leq C\delta, \tag{3.8}$$

where by the hypothesis $\rho \in \mathcal{E}_0(\Omega)$ and using the inequality

$$\int\limits_\Omega (dd^c w) \wedge (dd^c\rho)^{n-1} \leq \Big(\int\limits_\Omega (dd^c w)^n \Big)^{\frac{1}{n}} \Big(\int\limits_\Omega (dd^c\rho)^n \Big)^{\frac{n-1}{n}} < +\infty,$$

in [6]. Hence the desired conclusion follows. The proof is complete. □

We continue to prove **ii)**. Note that the proof of Theorem 1.1 in [9] and, hence, the proof of Proposition 2.10 in [18] are valid under the hypotheses of Theorem 1.1. By applying Proposition 2.10 in [18] to $\nu = (dd^c w)^n$ and $\widetilde{w}$ it follows that there exists $0 < \alpha_2 < 1$ such that

$$\sup_\Omega \Big(\widetilde{w} - w \Big) \leq C \Big(\int\limits_\Omega \max\{\widetilde{w} - w, 0\} d\nu \Big)^{\alpha_2}$$
$$\leq C \Big(\int\limits_\Omega |\widetilde{w} - w| d\nu \Big)^{\alpha_2}. \tag{3.9}$$

By (3.7) the righ-hand side of (3.9) is less than $C\Big(\int\limits_\Omega |\widetilde{w} - w| dV_{2n} \Big)^{\alpha_1\alpha_2}$. Hence,

$$\sup_\Omega \Big(\widetilde{w} - w \Big) \leq C \Big(\int\limits_\Omega |\widetilde{w} - w| dV_{2n} \Big)^{\alpha_1\alpha_2}. \tag{3.10}$$

On the other hand, by the definition of $\widetilde{w}$, on Ω_δ we have the following:

$$\begin{aligned} 0 \le \widetilde{w} - w &= \max\Big\{\hat{w} - w - c_0\delta^\alpha, 0\Big\} \\ &\le |\hat{w} - w| + c_0\delta^\alpha. \end{aligned} \tag{3.11}$$

Lemma 3.1 and (3.11) imply that

$$\begin{aligned} \int_{\Omega_\delta} |\widetilde{w} - w| dV_{2n} &\le \int_{\Omega_\delta} |\hat{w} - w| dV_{2n} + c_0\delta^\alpha \int_{\Omega_\delta} dV_{2n} \\ &\le \int_{\Omega_\delta} |\hat{w} - w| dV_{2n} + c_0\delta^\alpha \int_{\Omega} dV_{2n} \\ &\le \int_{\Omega_\delta} |\hat{w} - w| dV_{2n} + c_0\delta^\alpha . Vol(\Omega), \\ &\le C\delta + Vol(\Omega) c_0 \delta^\alpha, \\ &\lesssim \delta^\xi, \end{aligned} \tag{3.12}$$

where $\xi = \min\{1, \alpha\}$. Thus, from (3.10) we deduce the following.

$$\begin{aligned} \sup_\Omega\Big(\widetilde{w} - w\Big) &\le C\Big(\int_\Omega |\widetilde{w} - w| dV_{2n}\Big)^{\alpha_1\alpha_2}, \\ &= C\Big(\int_{\Omega_\delta} |\widetilde{w} - w| dV_{2n}\Big)^{\alpha_1\alpha_2} \\ &\le C\delta^{\xi\alpha_1\alpha_2}. \end{aligned} \tag{3.13}$$

Using (3.13) and the definition of $\widetilde{w}$ we infer that

$$\begin{aligned} \sup_{\Omega_\delta}(\hat{w}_\delta - w) &\le \sup_\Omega(\widetilde{w} - w) + c_0\delta^\alpha \\ &\le C\delta^\gamma, \end{aligned}$$

where $\gamma = \min\{\alpha, \xi\alpha_1\alpha_2\}$. By Lemma 4.2 in [9] it follows that

$$\sup_{\Omega_\delta}(w_\delta - w) \le C\delta^\gamma,$$

and the desired conclusion follows.

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