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New Results on Robust Finite-Time Passivity for Fractional-Order Neural Networks with Uncertainties

Mai Viet Thuan[1] · Dinh Cong Huong[2,3] · Duong Thi Hong[1]

Abstract
In this paper, the robust finite-time passivity for a class of fractional-order neural networks with uncertainties is considered. Firstly, the definition of finite-time passivity of fractional-order neural networks is introduced. Then, by using finite-time stability theory and linear matrix inequality approach, new sufficient conditions that ensure the finite-time passivity of the fractional-order neural network systems are derived via linear matrix inequalities which can be effectively solved by various computational tools. Finally, three numerical examples with simulation results are given to illustrate the effectiveness of the proposed method.

Keywords Fractional order neural networks · Finite-time stability · Finite-time passivity · Linear matrix inequalities

1 Introduction

Fractional order calculus is a natural generalization of classical integer order calculus. In recent years, this field of science has gained many interests due to the fact that the mathematical methods for fractional calculus have great development and fractional order models have come to play an important role in many applications and real-world physical phenomena [1–4]. In particularly, fractional-order neural networks is one of an important applications of fractional calculus. Compared with integer order neural networks, fractional order neural networks can describe the real dynamic characteristics of actual network systems accurately.

✉ Mai Viet Thuan
thuanmv@tnus.edu.vn

Dinh Cong Huong
dinhconghuong@tdtu.edu.vn

Duong Thi Hong
duonghong42@gmail.com

1 Department of Mathematics and Informatics, Thainguyen University of Science, Thai Nguyen, Vietnam

2 Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam

3 Faculty of Applied Sciences, Ton Duc Thang University, Ho Chi Minh City, Vietnam

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Therefore, fractional-order neural networks has received considerable attention [5]. The problem of stability analysis of fractional-order neural network systems was considered in [6–11]. Finite-time stability for fractional-order neural network systems was studied in [12–14]. The authors in [15] investigated the problem of mixed H_∞/passive projective synchronization for fractional-order neural networks with uncertain parameters by using the adaptive sliding mode control approach. The problem of adaptive synchronization of fractional-order memristor-based neural networks with time delay was considered in [16] by combining the adaptive control, linear delay feedback control, and a fractional-order inequality. As an extension, synchronization problem of Caputo fractional-order complex-valued neural networks with time delay was investigated in [17].

The passivity theory originated from circuit theory plays an important role in the stability analysis of dynamical systems [18–20]. The main idea of passivity theory is that the passive properties of the system can keep the system internally stable. Based on Lyapunov–Krasovskii functional method, a lot of results have been obtained for the problem of passivity analysis of integer order neural networks (see [21–26] and the references therein). Recently, the problem of mixed H_∞ and passivity based synchronization criteria for memristor-based recurrent neural networks with time-varying delays was considered in [27] based on the master-slave concept, differential inclusions theory and Lyapunov–Krasovskii stability theory. It should be noticed that all the results mentioned above were developed in the context of Lyapunov stability. However, in some practical process, the main attention may be related to the behavior of the dynamical systems over a fixed finite-time interval. To discuss this transient performance of control dynamics, literatures on finite-time stability (FTS) and finite-time boundedness (FTB) have attracted particular interests of researchers. Comparing with Lyapunov asymptotic stability (LAS), which deals with the asymptotic behavior of a system over a sufficiently long-time interval, FTS and FTB concern the domain of the state trajectory over a specified finite time interval. FTS and FTB means that once we fix a time interval, the state of a system does not exceed a certain bound during this specified time interval. It should be pointed out that FTS and LAS are independent concepts; indeed a system can be FTS but not LAS, and vice versa. From the view of engineering, it is interesting and worthy to investigate the problem of finite-time passivity of dynamical systems. The dynamical systems is said to be finite-time passive if the systems is not only finite-time stable but can also satisfies the given passive index. Based on this idea, the problem of optimal finite-time passive control problem for a class of uncertain nonlinear Markovian jumping systems was considered in [28] by using a fuzzy Lyapunov–Krasovskii functional approach. Recently, by employing an appropriate mode-dependent Lyapunov function and some appropriate free-weighting matrices, the problem of finite-time passivity and passification for stochastic time-delayed Markovian switching systems were studied in [29]. Based on passive control theory, some sufficients for the existence of finite-time robust passive controller for a class of uncertain Lipschitz nonlinear systems with time-delays was proposed in [30]. For neural network systems, some novel results have been derived [31,32]. Particularly, in [31], the authors investigated the problem of finite-time boundedness and finite-time passivity of discrete-time delayed neural network systems with time-varying delays by using the Lyapunov theory together with the zero inequalities, convex combination and reciprocally convex combination approaches. Recently, the problem of finite-time non-fragile passivity control for neural network systems with time-varying delay was considered in [32] by using Lyapunov–Krasovskii functional method. However, all the above results are limited to integer order systems. To the best of authors knowledge, so far, no result on the finite-time passivity for fractional-order neural network systems with uncertainties has been reported. This motivates our present research.

Motivated by the above discussions, the problem of finite-time passivity of fractional-order neural network systems is considered. Firstly, definition of finite-time passivity for the fractional-order neural network systems is given. The definition can be regarded as an extension of definition for integer order neural network systems to fractional-order ones. Then, by using finite-time stable theory and linear matrix inequality approach, sufficient conditions are derived to guarantee that the fractional-order neural network systems is not only finite-time stable but can also satisfies the given passive index. Finally, the feasibility and the effectiveness of our obtained result is illustrated by some numerical examples.

Notation The following notations will be used in this paper: $\mathbb{R}^n$ denotes the $n-$dimensional linear vector space over the reals with the Euclidean norm $\|.\|$ given by $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$; $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices. For a real matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. A matrix P is positive definite ($P > 0$) if $x^T Px > 0, \forall x \neq 0$; $P > Q$ means $P - Q > 0$. The symmetric term in a matrix is denoted by $*$.

2 Problem Statement and Preliminaries

In order to describe the model, we firstly give some useful definitions and lemmas on Riemann–Liouville fractional integral and Caputo fractional derivative of order $\alpha > 0$.

Definition 1 [1] The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $f(t)$ is defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, $$

where $\Gamma(.)$ is the gamma function, $\Gamma(s) = \int\limits_0^\infty e^{-t} t^{s-1} dt, s > 0$.

Definition 2 [1] The Caputo fractional-order derivative of order $\alpha > 0$ for a function $f(t)$ is defined as

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t \geq 0,\ n-1 < \alpha \leq n, $$

where n is a positive integer. In particular, when $0 < \alpha < 1$, we have

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(s)}{(t-s)^{\alpha}} ds, \quad t \geq 0. $$

Lemma 1 ([33]) *If* $x(t) \in C^n([0, +\infty), \mathbb{R})$ *and* $n - 1 < \alpha < n$, $(n \geq 1, n \in \mathbb{Z}^+)$, *then*

$$ {}_0I_t^\alpha \left({}_0^C D_t^\alpha x(t) \right) = x(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} x^{(k)}(0). $$

In particular, when $0 < \alpha < 1$, *we have*

$$ {}_0I_t^\alpha \left({}_0^C D_t^\alpha x(t) \right) = x(t) - x(0). $$

Now, we consider a class of fractional-order neural network systems with parameter uncertainties described by

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = -[A + \Delta A(t)]x(t) + [D + \Delta D(t)]f(x(t)) + W\omega(t), \quad t \geq 0, \\ y(t) = Mf(x(t)) + N\omega(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $0 < \alpha < 1$ is the fractional commensurate order of the system, $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the pseudo state vector, $y(t) \in \mathbb{R}^p$ is the output vector, $\omega(t) \in \mathbb{R}^m$ is the disturbance input, n is the number of neurals, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in \mathbb{R}^n$ denotes the activation function, $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $D \in \mathbb{R}^{n\times n}$ is the interconnection weight matrix, $W \in \mathbb{R}^{n\times m}$, $M \in \mathbb{R}^{p\times n}$, $N \in \mathbb{R}^{p\times m}$ are known real matrices, $\Delta A(t) = G_a F_a(t) H_a$, $\Delta D(t) = G_d F_d(t) H_d$, G_a, G_d, H_a, H_d are known real constant matrices of appropriate dimensions; $F_a(t), F_d(t)$ are unknown real time-varying matrices satisfying $F_a^T(t)F_a(t) \leq I$, $F_d^T(t)F_d(t) \leq I, \forall t \geq 0$.

In order to obtain the main results about the finite-time passivity of the system (1), the following conditions are needed in a later study.

Assumption 1 The activation functions $f_i(.)$ are continuous, $f_i(0) = 0\,(i = 1, \ldots, n)$, and satisfies Lipschitz condition on $\mathbb{R}$ with Lipschitz constant $l_i > 0$:

$$|f_i(x) - f_i(y)| \leq l_i|x - y|, \forall x, y \in \mathbb{R}. \tag{2}$$

Especially, when $y = 0$, we have

$$|f_i(x)| \leq l_i|x|, \quad \forall x \in \mathbb{R}. \tag{3}$$

Assumption 2 The disturbance input $\omega(t) \in \mathbb{R}^m$ is time-varying satisfying the following condition:

$$\exists d > 0 : \omega^T(t)\omega(t) < d, \forall t \in [0, T_f]. \tag{4}$$

Definition 3 [34,35] (*Finite-time boundedness*) Given positive numbers $T_f, c_1, c_2 (c_1 < c_2), d$, and a symmetric positive definite matrix $R \in \mathbb{R}^{n\times n}$. The system (1) with the output $y(t) = 0$ is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d) if $x_0^T R x_0 \leq c_1 \Longrightarrow x^T(t)Rx(t) < c_2, \forall t \in [0, T_f]$, for all the disturbance input $\omega(t) \in \mathbb{R}^m$ satisfying the Assumption 2.

Definition 4 (*Finite-time passivity*) The system (1) is said to be finite-time passive with respect to (c_1, c_2, T_f, R, d) if the following conditions are satisfied:

(i) When the input $y(t) \equiv 0$, the system (1) is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d).
(ii) Under the zero initial condition, there exist a scalar $\gamma > 0$ such that the following inequality holds

$$2\, {}_0I_t^{\alpha}\left(y^T(t)\omega(t)\right) \geq -\gamma\, {}_0I_t^{\alpha}\left(\omega^T(t)\omega(t)\right), \forall t \in [0, T_f].$$

Remark 1 It should be noted that when $\alpha = 1$ the Definition 4 is turned into the definition of finite-time passivity of integer-order systems which have been considered in [30]. Therefore, this definition generalize those given in the literature.

Now, we recalled the following auxiliary lemmas which are essential in order to derive our main results in this paper.

Lemma 2 ([36]) *Let $x(t) \in \mathbb{R}^n$ be a vector of diffrentiable function. Then, for any time instant $t \geq t_0$, the following relationship holds*

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}\left(x^T(t)Px(t)\right) \leq x^T(t)P\,{}_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall\alpha \in (0,1), \forall t \geq t_0 \geq 0,$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 3 ([37]) *Given constant matrices X, Y, Z with appropriate dimensions satisfying $Y = Y^T > 0$, $X = X^T$, then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0.$$

3 Main Results

Firstly, we derive a sufficient condition for the finite-time boundedness of the fractional-order neural network systems with parameter uncertainties (1).

Theorem 1 *For given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R. Assume that the Assumptions 1, 2 are satisfied. The system (1) with the output $y(t) \equiv 0$ is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d) if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive numbers $\theta, \epsilon_1, \epsilon_2, \epsilon_3$ satisfying the following conditions*

$$\begin{bmatrix} \Xi_{11} & PD & PG_a & PG_d & PW \\ * & \Xi_{22} & 0 & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & -\epsilon_3 I \end{bmatrix} < 0, \tag{5a}$$

$$\lambda_2 c_1 + \frac{d\epsilon_3}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_1 c_2, \tag{5b}$$

where

$$\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \lambda_1 = \lambda_{\min}(\overline{P}),\ \lambda_2 = \lambda_{\max}(\overline{P}),\ L = \operatorname{diag}\{l_1, l_2, \ldots, l_n\},$$
$$\Xi_{11} = -PA - A^T P + \epsilon_1 H_a^T H_a + \theta L^T L,$$
$$\Xi_{22} = \epsilon_2 H_d^T H_d - \theta I.$$

Proof Consider the following non-negative quadratic function

$$V(x(t)) = x^T(t)Px(t).$$

It follows from Lemma 2 that we obtain the $\alpha-$order $(0 < \alpha < 1)$ Caputo derivative of $V(x(t))$ along the trajectories of the system (1) as follows:

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha}V(x(t)) &\leq 2x^T(t)P\,{}_{0}^{C}D_t^{\alpha}x(t) \\ &= x^T(t)\left[-PA - A^T P\right]x(t) - 2x^T(t)PG_a F_a(t)H_a x(t) \\ &\quad + 2x^T(t)PDf(x(t)) + 2x^T(t)PG_d F_d(t)H_d f(x(t)) + 2x^T(t)PW\omega(t). \end{aligned} \tag{6}$$

By using the Cauchy matrix inequality, we have the following estimates

$$-2x^T(t)PG_aF_a(t)H_ax(t) \le \epsilon_1^{-1}x^T(t)PG_aG_a^TPx(t) + \epsilon_1 x^T(t)H_a^TH_ax(t), \tag{7}$$

$$2x^T(t)PG_dF_d(t)H_df(x(t)) \le \epsilon_2^{-1}x^T(t)PG_dG_d^TPx(t) + \epsilon_2 f^T(x(t))H_d^TH_df(x(t)), \tag{8}$$

$$2x^T(t)PW\omega(t) \le \epsilon_3^{-1}x^T(t)PWW^TPx(t) + \epsilon_3\omega^T(t)\omega(t). \tag{9}$$

From the Assumption 1, we have

$$0 \le -\theta f^T(x(t))f(x(t)) + \theta x^T(t)L^TLx(t). \tag{10}$$

From (6)–(10), we obtain

$${}_0^C D_t^\alpha V(x(t)) \le \eta^T(t)\Omega\eta(t) + \epsilon_3\omega^T(t)\omega(t), \tag{11}$$

where

$$\eta(t) = \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}, \ \Omega = \begin{bmatrix} \Omega_{11} & PD \\ D^TP & \Omega_{22} \end{bmatrix},$$

with

$$\Omega_{11} = -PA - A^TP + \theta L^TL + \epsilon_1 H_a^TH_a + \epsilon_1^{-1}PG_aG_a^TP + \epsilon_2^{-1}PG_dG_d^TP + \epsilon_3^{-1}PWW^TP,$$
$$\Omega_{22} = \epsilon_2 H_d^TH_d - \theta I.$$

Using the Schur Complement Lemma (Lemma 3), we can see that $\Omega < 0$ is equivalent to (5a). Therefore, from the condition (5a), we have the following estimate

$${}_0^C D_t^\alpha V(x(t)) \le \epsilon_3\omega^T(t)\omega(t), \quad \forall t \in [0, T_f]. \tag{12}$$

Integrating with order α both sides of (12) from 0 to t $(0 < t < T_f)$ and using Lemma 1, we have

$$\begin{aligned} x^T(t)Px(t) &\le x^T(0)Px(0) + {}_0I_t^\alpha\left(\epsilon_3\omega^T(t)\omega(t)\right) \\ &= x^T(0)Px(0) + \frac{\epsilon_3}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega^T(s)\omega(s)ds \\ &\le x^T(0)Px(0) + \frac{d\epsilon_3}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &\le x^T(0)Px(0) + \frac{d\epsilon_3}{\Gamma(\alpha+1)}T_f^\alpha. \end{aligned} \tag{13}$$

On the other hand, the following condition holds

$$x^T(t)Px(t) = x^T(t)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(t) \ge \lambda_{\min}(\overline{P})x^T(t)Rx(t) = \lambda_1 x^T(t)Rx(t), \tag{14}$$

and

$$x^T(0)Px(0) = x^T(0)R^{\frac{1}{2}}\overline{P}R^{\frac{1}{2}}x(0) \le \lambda_{\max}(\overline{P})x^T(0)Rx(0) = \lambda_2 x^T(0)Rx(0) \le \lambda_2 c_1. \tag{15}$$

Combining (13), (14) and (15), we get

$$\lambda_1 x^T(t)Rx(t) \le V(x(t)) = x^T(t)Px(t) \le \lambda_2 c_1 + \frac{d\epsilon_3}{\Gamma(\alpha+1)}T_f^\alpha.$$

Condition (5b) implies that $x^T(t)Rx(t) < c_2$. Thus, the system (1) with the output $y(t) = 0$ is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d). This completes the proof of the theorem. □

Remark 2 The problem of finite-time stability for fractional-order neural networks have been attracted a lot of research attention by many authors in the literatures. For example, by using Hölder inequality, Gronwall inequalities and inequality scaling skills, the authors in [38] studied the problem of finite-time stability of fractional delayed neural networks of retarded-type. The result of [38] was later improved in [39]. As an extension, the problem of finite-time stability for fractional-order complex-valued neural networks was considered in [40]. We note that all the above results considered the problem of finite-time stability for fractional-order neural networks. To the best of authors knowledge, so far, no result on the finite-time boundedness for Caputo fractional neural networks with uncertainties has been reported. By using finite-time stable theory and linear matrix inequality approach, Theorem 1 solves the problem of finite-time boundedness for fractional-order neural networks with uncertainties in the form of linear matrix inequalities. Compared with the existing results [38,40] based on matrix norm computation, our linear matrix inequality approach has the advantage that the linear matrix inequalities can be solved numerically and effectively by using the interior-point method [37].

Next, we present a sufficient condition for finite-time passivity of the fractional-order neural network systems with parameter uncertainties (1).

Theorem 2 *For given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R. Assume that the Assumptions 1, 2 are satisfied. The system (1) is robustly finite-time passive with respect to (c_1, c_2, T_f, R, d) if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive numbers $\gamma, \theta, \epsilon_1, \epsilon_2, \epsilon_3$ satisfying the following conditions*

$$\begin{bmatrix} \varXi_{11} & PD & 0 & PG_a & PG_d & PW \\ * & \varXi_{22} & -M^T & 0 & 0 & 0 \\ * & * & \varXi_{33} & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & -\epsilon_3 I \end{bmatrix} < 0, \tag{16a}$$

$$\lambda_2 c_1 + \frac{d\epsilon_3}{\Gamma(\alpha+1)} T_f^{\alpha} < \lambda_1 c_2, \tag{16b}$$

where

$$\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \lambda_1 = \lambda_{\min}(\overline{P}),\ \lambda_2 = \lambda_{\max}(\overline{P}),\ L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\},$$
$$\varXi_{11} = -PA - A^T P + \epsilon_1 H_a^T H_a + \theta L^T L,$$
$$\varXi_{22} = \epsilon_2 H_d^T H_d - \theta I,$$
$$\varXi_{33} = \epsilon_3 I - (N + N^T + \gamma I).$$

Proof When $y(t) = 0$, (16a) and (16b) imply (5a) and (5b). Therefore, from Theorem 1, the system is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d). To show the finite-time passive analysis of the system (1), we choose the non-negative quadratic function as given in Theorem 1. We have the following estimate:

$$ {}_0^C D_t^{\alpha} V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t) \le \xi^T(t)\hat{\varOmega}\xi(t), \tag{17}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ f(x(t)) \\ \omega(t) \end{bmatrix},\ \hat{\varOmega} = \begin{bmatrix} \varOmega_{11} & PD & 0 \\ * & \varOmega_{22} & -M^T \\ * & * & \varOmega_{33} \end{bmatrix},$$

with

$$\Omega_{11} = -PA - A^T P + \theta L^T L + \epsilon_1 H_a^T H_a + \epsilon_1^{-1} P G_a G_a^T P + \epsilon_2^{-1} P G_d G_d^T P + \epsilon_3^{-1} P W W^T P,$$
$$\Omega_{22} = \epsilon_2 H_d^T H_d - \theta I,$$
$$\Omega_{33} = \epsilon_3 I - (N + N^T + \gamma I).$$

By using the Schur Complement Lemma (Lemma 3), we have that $\hat{\Omega} < 0$ is equivalent to (16a). Therefore, from the Condition (16a), we obtain

$$ {}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t) \le 0. \tag{18}$$

Now, we set

$$\begin{aligned} J &= {}_0 I_t^\alpha \left(-2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(-2y^T(s)\omega(s) - \gamma\omega^T(s)\omega(s)\right) ds,\ t \in [0, T_f]. \end{aligned}$$

Noting the zero initial condition and using Lemma 1, we have

$$\begin{aligned} J &= {}_0 I_t^\alpha \left({}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)\right) - V(x(t)) \\ &\le {}_0 I_t^\alpha \left({}_0^C D_t^\alpha V(x(t)) - 2y^T(t)\omega(t) - \gamma\omega^T(t)\omega(t)\right) \end{aligned} \tag{19}$$

due to $V(x(t)) \ge 0$.

Combining (18) and (19), we get $J < 0$. Hence

$$2\,{}_0 I_t^\alpha \left(y^T(t)\omega(t)\right) \ge -\gamma\ {}_0 I_t^\alpha \left(\omega^T(t)\omega(t)\right), \forall t \in [0, T_f].$$

Hence it can be concluded that the system (1) is finite-time passive. This completes the proof of the theorem. □

Remark 3 Since the Condition (16a) are linear matrix inequalities, we can solve the condition by using Matlab's LMI Control Toolbox in [37]. Therefore, from Theorem 2, we have the following Algorithm to solve the problem of robust finite-time passivity for the fractional-order neural network systems with parameter uncertainties (1):

Step 1 Solve the linear matrix inequalities (16a) and obtain symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive numbers $\gamma, \theta, \epsilon_1, \epsilon_2, \epsilon_3$.
Step 2 Compute $\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \lambda_1 = \lambda_{\min}(\overline{P}),\ \lambda_2 = \lambda_{\max}(\overline{P})$.
Step 3 Check Condition (16b) in Theorem 2. If they hold, enter Step 4; else return to Step 1.
Step 4 The system (1) is finite-time passive with respect to (c_1, c_2, T_f, R, d).

Remark 4 The present method used in Theorem 2 can be regarded as an extension of passivity-based methods for integer order neural networks systems (see, e.g., [21,22,24]) to fractional order ones. To the best of our knowledge, this is the first time that finite-time passivity of fractional-order neural networks with uncertainties is investigated. Thanks to Lemmas 1 and 2, the investigation is readily achievable.

We now discuss a special case of system (1). Particularly, when $\Delta A(t) = 0$, $\Delta D(t) = 0$, then system (1) is reduced to the Caputo fractional order neural networks which has been considered in [9]

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -Ax(t) + Df(x(t)) + W\omega(t), \quad t \ge 0, \\ y(t) = Mf(x(t)) + N\omega(t), \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{20}$$

The authors in [9] considered the problem of Lyapunov asymptotic stability of the system (20) when output $y(t) \equiv 0$ and disturbance input $\omega(t) \equiv 0$. In contrast, in this paper, we study the problems of finite-time boundedness and finite-time passivity of the Caputo fractional-order neural network (20). According to Theorems 1 and 2, we immediately have the following results.

Corollary 1 *For given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R. Assume that the Assumptions 1, 2 are satisfied. The system (20) with the output $y(t) \equiv 0$ is robustly finite-time stable with respect to (c_1, c_2, T_f, R, d) if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive numbers θ, ϵ satisfying the following conditions*

$$\begin{bmatrix} \left(-PA - A^T P + \theta L^T L\right) & PD & PW \\ * & -\theta I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0, \tag{21a}$$

$$\lambda_2 c_1 + \frac{d\epsilon}{\Gamma(\alpha+1)} T_f^\alpha < \lambda_1 c_2, \tag{21b}$$

where

$$\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \lambda_1 = \lambda_{\min}(\overline{P}),\ \lambda_2 = \lambda_{\max}(\overline{P}),\ L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}.$$

Corollary 2 *For given positive numbers c_1, c_2, T_f and a symmetric positive definite matrix R. Assume that the Assumptions 1, 2 are satisfied. The system (20) is robustly finite-time passive with respect to (c_1, c_2, T_f, R, d) if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive numbers γ, θ, ϵ satisfying the following conditions*

$$\begin{bmatrix} \left(-PA - A^T P + \theta L^T L\right) & PD & 0 & PW \\ * & -\theta I & -M^T & 0 \\ * & * & \left(\epsilon I - (N + N^T + \gamma I)\right) & 0 \\ * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{22a}$$

$$\lambda_2 c_1 + \frac{d\epsilon}{\Gamma(\alpha+1)} T_f^\alpha < \lambda_1 c_2, \tag{22b}$$

where

$$\overline{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},\ \lambda_1 = \lambda_{\min}(\overline{P}),\ \lambda_2 = \lambda_{\max}(\overline{P}),\ L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}.$$

Remark 5 In this paper, we derive some sufficient conditions that ensure the finite-time passivity of the fractional-order neural networks by constructing a simple Lyapunov functional and using linear matrix inequality approach. To get less conservative criteria by constructing a more complex Lyapunov function is not easy because it is very difficult to calculate the fractional order derivative of a complex Lyapunov function [9,41]. Therefore, constructing a more suitable and effective Lyapunov functional is essential and require further investigation.

4 Numerical Examples

In this section, we provide three examples to demonstrate the effectiveness of the proposed method.

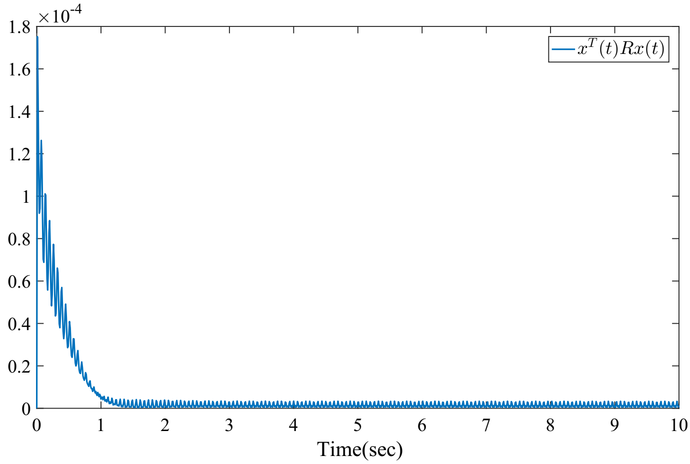

Fig. 1 $x^T(t)Rx(t)$ of the system for $\alpha = 0.96$

Example 1 In system (20) with output $y(t) \equiv 0$, we consider the following neural networks of $n = 3$ neurons with hub structure [42], $\alpha = 0.96$, $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ is pseudo state, $f(x(t)) = (\sin(x_1(t)), \tanh(x_2(t)), \tanh(x_3(t)))^T \in \mathbb{R}^3$ is activation function, $\omega(t) = 0.1 \cos t \in \mathbb{R}$ is disturbance input, $A = \text{diag}\{6, 2, 2\}$, and

$$D = \begin{bmatrix} 3 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; W = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.9 \end{bmatrix}.$$

Then system (20) with output $y(t) \equiv 0$ can be rewritten as

$$\begin{cases} {}_0^C D_t^{0.96} x_1(t) = -6x_1(t) + 3\sin(x_1(t)) - 2\tanh(x_2(t)) - 2\tanh(x_3(t)) + 0.02\cos t, & t \geq 0, \\ {}_0^C D_t^{0.96} x_2(t) = -2x_2(t) + \sin(x_1(t)) + \tanh(x_2(t)) + 0.05\cos t, & t \geq 0, \\ {}_0^C D_t^{0.96} x_3(t) = -2x_3(t) + \sin(x_1(t)) + \tanh(x_3(t)) + 0.09\cos t, & t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3. \end{cases} \tag{23}$$

We consider problem of finite-time boundedness of the Caputo fractional-order neural networks (23). We see that the Assumptions 1 and 2 hold with $L = \text{diag}\{1, 1, 1\}, d = 0.01$. Let $c_1 = 1, c_2 = 3.7, T_f = 10$ and matrix $R = I$. By using Corollary 1, we found that the Conditions (21a), (21b) are satisfied with $\theta = 0.8763$, $\epsilon = 1.0818$ and

$$P = \begin{bmatrix} 0.1692 & -0.0033 & -0.0035 \\ -0.0033 & 0.4879 & -0.0249 \\ -0.0035 & -0.0249 & 0.4790 \end{bmatrix}.$$

Thus the conditions of Corollary 2 are satisfied. Therefore, system (23) is robust finite-time boundedness with respect to $(1, 3.7, 10, I, 0.01)$.

The trajectory of $x^T(t)Rx(t)$ for the system is shown in Fig. 1, which clearly demonstrate that the system (23) is robustly finite-time boundedness.

Example 2 In system (20), consider the following neural networks of $n = 3$ neurons with ring structure [42], $\alpha = 0.98$, $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ is pseudo state, $f(x(t)) =$

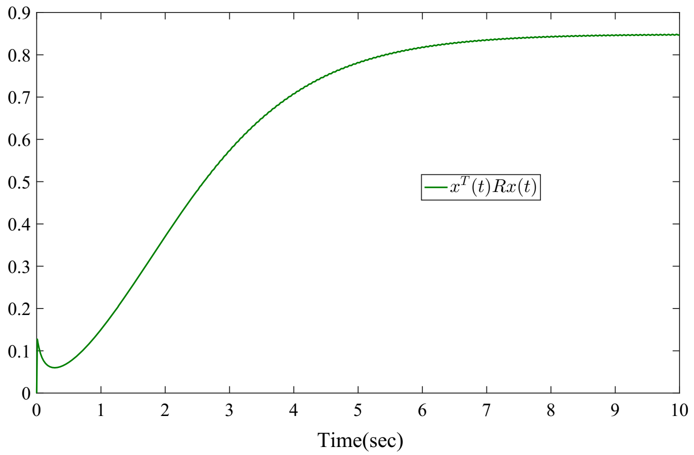

Fig. 2 $x^T(t)Rx(t)$ of the system for $\alpha = 0.98$

$(\sin(x_1(t)), \sin(x_2(t)), \sin(x_3(t)))^T \in \mathbb{R}^3$ is activation function, $\omega(t) = 0.1\sin t \in \mathbb{R}$ is disturbance input, $y(t) \in \mathbb{R}$ is output vector, $A = \text{diag}\{5, 6, 5.5\}$, and

$$D = \begin{bmatrix} 3 & 1 & -2.5 \\ -2.5 & 3 & 1 \\ 1 & -2.5 & 3 \end{bmatrix}, \ W = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}, \ M = \begin{bmatrix} 0.1 \ 0.3 \ 0.1 \end{bmatrix}, \ N = \begin{bmatrix} 1 \end{bmatrix}.$$

Then system (20) can be rewritten as

$$\begin{cases} {}_0^C D_t^{0.98} x_1(t) = -5x_1(t) + 3\sin(x_1(t)) + \sin(x_2(t)) - 2.5\sin(x_3(t)) + 0.02\sin t, t \geq 0, \\ {}_0^C D_t^{0.98} x_2(t) = -6x_2(t) - 2.5\sin(x_1(t)) + 3\sin(x_2(t)) + \sin(x_3(t)) + 0.03\sin t, t \geq 0, \\ {}_0^C D_t^{0.98} x_3(t) = -5.5x_3(t) + \sin(x_1(t)) - 2.5\sin(x_2(t)) + 3\sin(x_3(t)) + 0.05\sin t, t \geq 0, \\ y(t) = 0.1\sin(x_1(t)) + 0.3\sin(x_2(t)) + 0.1\sin(x_3(t)) + 0.1\sin t, t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^3. \end{cases} \tag{24}$$

We consider problem of finite-time passivity of the Caputo fractional-order neural networks (24). We see that the Assumptions 1 and 2 hold with $L = \text{diag}\{1, 1, 1\}, d = 0.01$. Let $c_1 = 1, c_2 = 2.3, T_f = 10$ and matrix $R = I$. By using Corollary 2, we found that the Conditions (22a), (22b) are satisfied with $\theta = 1.0692, \epsilon = 1.5909, \gamma = 1.2841$ and

$$P = \begin{bmatrix} 0.2068 \ 0.0192 \ 0.0210 \\ 0.0192 \ 0.1840 \ 0.0173 \\ 0.0210 \ 0.0173 \ 0.1945 \end{bmatrix}.$$

Thus the conditions of Corollary 2 are satisfied. Therefore, system (24) is robustly finite-time passive with respect to $(1, 2.3, 10, I, 0.01)$.

The trajectory of $x^T(t)Rx(t)$ for the system is shown in Fig. 2, which clearly demonstrate that the system (24) is robustly finite-time passive.

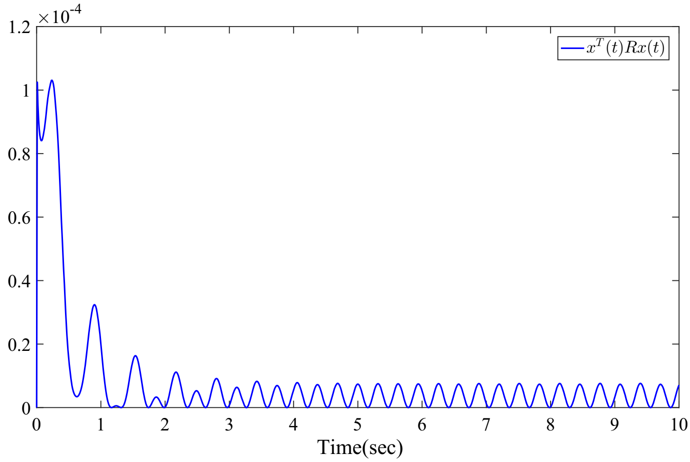

Fig. 3 $x^T(t)Rx(t)$ of the system for $\alpha = 0.95$

Example 3 We consider the fractional-order neural network systems with parameter uncertainties:

$$\begin{cases} {}_0^C D_t^{0.95} x(t) = -[A+G_a F_a(t) H_a] x(t) + [D+G_d F_d(t) H_d] f(x(t)) + W\omega(t), \quad \forall t \geq 0, \\ y(t) = Mx(t) + N\omega(t), \\ x(0) = x_0 \in \mathbb{R}^2, \end{cases} \tag{25}$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \ G_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ H_a = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}, \ F_a(t) = \sin t,$$
$$D = \begin{bmatrix} 0.1 & 0.5 \\ 0.9 & 1 \end{bmatrix}, \ G_d = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \ H_d = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}, \ F_d(t) = \sin t,$$
$$W = \begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix}, \ M = \begin{bmatrix} 1 & 2 \end{bmatrix}, \ N = \begin{bmatrix} 1 \end{bmatrix}.$$

The disturbance is choose as $\omega(t) = \sqrt{0.1} \sin t \in \mathbb{R}$. Hence the disturbance satisfying Condition 4 with $d = 0.1$. Take the activation function as $f(x(t)) = (\tanh x_1(t), \tanh x_2(t))^T \in \mathbb{R}^2$. We have the activation function $f(x(t))$ satisfies the Condition 2 with $L = \text{diag}\{1, 1\}$. Let $c_1 = 1, c_2 = 3.3, T_f = 10$ and matrix $R = I$. By using Theorem 2 and Remark 2, we found that the Conditions (16a), (16b) are satisfied with $\theta = 5.0421, \gamma = 4.6726, \epsilon_1 = 3.5343, \epsilon_2 = 1.9506, \epsilon_3 = 2.8204$, and

$$P = \begin{bmatrix} 2.2328 & -0.0080 \\ -0.0080 & 1.4915 \end{bmatrix}.$$

Thus the conditions of Theorem 2 are satisfied. Therefore, system (25) is robustly finite-time passive with respect to $(1, 3.3, 10, I, 0.1)$.

The trajectory of $x^T(t)Rx(t)$ for the system is shown in Fig. 3, which clearly demonstrate that the system (25) is robustly finite-time passive.

5 Conclusion

This paper has dealt with the problem of finite-time passivity analysis of fractional-order neural network systems with uncertainties. By extending the concept of passivity-based methods for integer-order neural network systems to fractional-order neural network systems and utilizing a recently established lemma for the Caputo fractional derivative of a quadratic function and using linear matrix inequalities approach, LMI-based conditions that ensure the passivity of the fractional-order neural network systems have been derived. The effectiveness of proposed method has been demonstrated by three numerical examples.

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