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Versions of the subgradient extragradient method for pseudomonotone variational inequalities with non-Lipschitz mappings

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Abstract We propose two algorithms of the subgradient extragradient type for variational inequalities in Hilbert spaces. For the first algorithm, a sufficient condition for the weak convergence is established under pseudomonotonicity and uniform continuity assumptions. The strong convergence is also proved even with Q-linear rate, under strong pseudomonotonicity and Lipschitz continuity hypothese. To avoid these restrictive hypothese, the second algorithm is designed by modifying the first one with the use of an idea of the Mann algorithm in adding one step with new parameters to each iteration. These two algorithms improve related results in the literature. Finally, some numerical experiments are presented to show the efficiency and advantages of the proposed algorithms.

Keywords Extragradient method · Subgradient extragradient method · Mann type algorithm · variational inequality · pseudomonotonicity

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1 Introduction

Variational inequalities (VI) introduced by Stampacchia [36] have been proved to be a simple, natural and unified framework for various (even seemmingly unrelated) problems in mathematics, physics, engineering, social sciences and other fields. Besides qualitative studies, much attention has been given to develop effective and implementable numerical methods, including projection-type methods, see, e.g., [2,11,20,23,24]. Among all the iterative

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methods for VI, the simplest one is the gradient projection method in which only one projection on the feasible set is performed. The convergence of this method can be proved under a strong condition that the underlying mapping is strongly monotone or inverse strongly monotone.

To relax this assumption. Korpelevich [25] (also Antipin [1]) introduced the extragradient method in an Euclidean space for VI with monotone and Lipschitz continuous mappings. For finite-dimensional spaces, it is known now that the extragradient method can be successfully applied for solving pseudomonotone variational inequalities, see [11, Theorem 12.2.11]. Note that a monotone (pseudomonotone) variational inequality expresses an optimality condition for a convex (pseudoconvex, respectively) minimization problem. So, the above reducing monotonicity to pseudomonotonicity allows us to consider the remarkably larger class of the pseudoconvex optimization problems. In recent years, the extragradient method has been further developed and extended in both finite and infinite dimensional spaces, see, e.g., [3,6,7,8,21,22,28,29,35,37,38,39,41] and the references therein. The extragradient method requires two projections onto the feasible set per iteration. Finding this projection is minimizing the distance function to the feasible set, which is relatively easy only for simple sets. Therefore, its computation is expensive if the feasible sets have complicated structures. Hence, many efforts have been made to reduce the overall number of projections or to use projections on simpler sets such as closed half-spaces. The subgradient extragradient method proposed by Censor et al. [6,7] is an important modification of the extragradient method in this direction (see also [14,40] for single projection methods). On the other hand, Lipchitz continuity is also a restrictive condition which can prevent the use of the method when this condition is violated or the Lipschitz constant (of the mapping defining the variational inequality) is difficult to compute. In this case, an Armijo-type procedure is used to avoid Lipschitz assumptions. The interested reader is referred to [16,19,31] for more details.

Recently, the subgradient extragradient method [6] has received significant attention from researchers. A number of results on weak and strong convergence have been established, but under relatively restrictive assumptions such as those on monotonicity and Lipschitz continuity in [7,26,34,38] and on strong pseudomonotonicity in [15,39].

Inspired by the above observations, in this paper, we develop new versions (Algorithms 3.1 and 3.2) of the subgradient extragradient method for variational inequalities in Hilbert spaces to obtain sufficient conditions for both weak and strong convergence under sufficiently relaxed assumptions. Algorithms 3.1 and 3.2 are relatively easy to implement and their number of iterations and cpu time are smaller than some recently considered algorithms of similar types. These facts are verified by numerical examples in both finite and infinite dimensional cases.

The structure of the paper is as follows. In Section 2, we recall some definitions and preliminary results for the use in what follows. Section 3 is devoted the main results. Here we first propose Algorithm 3.1 and establish a sufficient conditions for its weak convergence under pseudomonotonicity and uniform continuity assumptions (Theorem 3.1). This algorithm is also proved to be strongly convergent with a Q-linear rate, but under more restirctive assumptions of κ-strong pseudomonotonivity and Lipschitz continuity. To weaken these hypothesese, we modify Algorithm 3.1 by adding one step with two new sequences of parameters to each iteration following an idea of the Mann algorithm [30] in choosing parameters for linesearches to get Algorithm 3.2. Then, we can come back to using the assumptions of Theorem 3.1 with an additional hypothesis on the new parameters to receive a strong convergence (Theorem 3.3). Section 4 contains numerical examples to illustrate our two algorithms and their advantages over some recent considered methods. Final remarks and conclusions are given in Section 5.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ and its strong convergence is written as $x_n \to x$. For $x,y,z \in H$ and $\alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2. \quad (1)$$

Definition 2.1 ([33, Chapter 9]). Suppose that a sequence $\{x_n\}$ in H converges strongly to $p \in H$. We say that $\{x_n\}$ converges to p with a Q-linear rate if there exists $\delta \in (0,1)$ such that, for large n,

$$\|x_{n+1}-p\| \le \delta\|x_n-p\|.$$

Definition 2.2 ([18]) Let $A : H \to H$ be a mapping. Then,

1. A is called L-`Lipschitz continuous` (with Lipschitz constant $L > 0$) if

$$\|Ax-Ay\| \le L\|x-y\| \quad \forall x,y \in H.$$

   Particularly, if $L = 1$ then A is called `nonexpansive` and if $L \in (0,1)$, A is called a `contraction`.
2. A is said to be `monotone` if

$$\langle Ax-Ay, x-y\rangle \ge 0 \quad \forall x,y \in H.$$

3. A is termed `pseudomonotone` if

$$\langle Ay, x-y\rangle \ge 0 \Longrightarrow \langle Ax, x-y\rangle \ge 0 \quad \forall x,y \in H.$$

4. A is called κ-`strongly pseudomonotone` if there exists a constant $\kappa > 0$ such that

$$\langle Ay, x-y\rangle \ge 0 \Longrightarrow \langle Ax, x-y\rangle \ge \kappa\|x-y\|^2 \quad \forall x,y \in H.$$

5. A is called `sequentially weakly continuous` if, for each sequence $\{x_n\}$, we have: $\{x_n\}$ converging weakly to x implies Ax_n converging weakly to Ax.

It is easy to see that every monotone mapping is pseudomonotone but the converse is not true. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x-P_C x\| \le \|x-y\|\ \forall y \in C$. P_C is called the (metric) *projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 ([17]) *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.2 ([12]) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then, $z = P_C x$ if and only if $\langle x-z, z-y\rangle \ge 0\ \forall y \in C$.*

Lemma 2.3 ([12]) *Let C be a nonempty, closed and convex subset in a real Hilbert space H and $x \in H$. Then,*
(i) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \ \forall y \in C$;
(ii) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$;
(iii) $\langle (I - P_C)x - (I - P_C)y, x - y\rangle \ge \|(I - P_C)x - (I - P_C)y\|^2 \ \forall y \in C$.

Lemma 2.4 ([10]) *For $x \in H$ and $\alpha \ge \beta > 0$ the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \le \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$

$$\|x - P_C(x - \beta Ax)\| \le \|x - P_C(x - \alpha Ax)\|.$$

Lemma 2.5 ([5]) *Given $x \in H$ and $v \in H$, $v \neq 0$, let $T = \{z \in H : \langle v, z - x\rangle \le 0\}$. Then, for all $u \in H$, the projection $P_T(u)$ is defined by*

$$P_T(u) = u - \max\left\{0, \frac{\langle v, u - x\rangle}{||v||^2}\right\} v.$$

In particular, if $u \notin T$ then

$$P_T(u) = u - \frac{\langle v, u - x\rangle}{||v||^2} v.$$

For more properties of the metric projection, the interested reader is referred to Chapter 4 in [5] and Section 3 in [12].

For $A : H \to H$ and a nonempty, closed, and convex subset C of H, the (Stampacchia) variational inequality (for A on C) is

(VI) find $x^* \in C$ such that $\langle Ax^*, x - x^*\rangle \ge 0 \ \forall x \in C$.

The Minty variational inequality (known also as the dual problem of (VI)) is

(MVI) find $x^* \in C$ such that $\langle Ax, x^* - x\rangle \le 0 \ \forall x \in C$.

Let Sol(VI) and Sol(MVI) denote the solution set of (VI) and (MVI), respectivey. Note that $x^* \in$ Sol(VI) if and only if $x^* = P_C(x^* - \gamma Ax^*)$ for $\gamma > 0$.

The following lemmas are useful for the convergence study in this paper.

Lemma 2.6 ([9]) *If A is pseudomonotone and continuous, then* Sol(VI)) = Sol(MVI).

Lemma 2.7 ([32]) *Let C be a nonempty subset of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold*
(i) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_n\}$ is in C.*
Then, $\{x_n\}$ converges weakly to a point in C.

Lemma 2.8 ([27]) *For a sequence of nonnegative real numbers $\{a_n\}$, assume that there exists a subsequence $\{a_{n_j}\}$ satisfying $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ tending to ∞ such that, for large k,*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.9 ([42]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} b_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

3 The Main Results

In this section, we propose two algorithms of the subgradient extragradient type for solving (VI). In the first algorithm, we impose the following three conditions to get its weak convergence.

Condition 1 *The feasible set C is nonempty, closed, and convex.*

Condition 2 *The mapping $A : H \to H$ is pseudomonotone on H, and sequentially weakly continuous on C and uniformly continuous on bounded subsets of H.*

Condition 3 *The solution set* Sol(VI) *is nonempty.*

Algorithm 3.1 *Let the parameters be $\gamma > 0, l \in (0,1), \mu \in (0,1)$.*

Initialization: *Choose arbitrarily $x_1 \in C$.*

Iterative Steps: *Given a current iterate x_n, calculate x_{n+1} as follows:*

Step 1. *Compute*

$$y_n := P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{2}$$

If $y_n = x_n$ or $Ay_n = 0$, then stop and y_n is a solution of (VI). *Otherwise,*
Step 2. *Compute*

$$x_{n+1} := P_{T_n}(x_n - \lambda_n A y_n),$$

where

$$T_n = \{x \in H : \langle x_n - \lambda_n A x_n - y_n, x - y_n \rangle \leq 0\}.$$

*Set $n := n+1$ and go to **Step 1**.*

We start the analysis of the algorithm convergence by proving the following lemmas.

Lemma 3.10 *Assume that Conditions 1–3 hold. Then, the Armijo linesearch rule (2) is well-defined and $\lambda_n \leq \gamma$ for all n.*

Proof If $x_n \in \Omega$ then $x_n = P_C(x_n - \gamma A x_n)$, therefore (2) holds with $m = 0$. For $x_n \notin \Omega$, suppose to the contrary that, for all m, we have

$$\gamma l^m \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \|x_n - P_C(x_n - \gamma l^m Ax_n)\|.$$

Then,

$$\|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \frac{\|x_n - P_C(x_n - \gamma l^m Ax_n)\|}{\gamma l^m}. \tag{3}$$

If $x_n \in C$, then since P_C ans A are continuous, we have $\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m Ax_n)\| = 0$. From the uniform continuity of A on bounded subsets of H, we have

$$\lim_{m\to\infty} \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| = 0.$$

Combining this and (3), we get

$$\lim_{m\to\infty} \frac{\|x_n - P_C(x_n - \gamma l^m Ax_n)\|}{\gamma l^m} = 0. \tag{4}$$

For $z_m := P_C(x_n - \gamma l^m Ax_n)$, we have

$$\langle z_m - x_n + \gamma l^m Ax_n, x - z_m\rangle \geq 0 \ \ \forall x \in C.$$

Hence,

$$\langle \frac{z_m - x_n}{\gamma l^m}, x - z_m\rangle + \langle Ax_n, x - z_m\rangle \geq 0 \ \ \forall x \in C.$$

Taking the limit $m \to \infty$ in this inequality and using (4), we obtain $\langle Ax_n, x - x_n\rangle \geq 0 \ \ \forall x \in C$, which implies that $x_n \in \mathrm{Sol(VI)}$. This is a contradiction.

For $x_n \notin C$, we have

$$\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m Ax_n)\| = \|x_n - P_C x_n\| > 0. \tag{5}$$

and

$$\lim_{m\to\infty} \gamma l^m \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| = 0.$$

Combining this, (3), and (5), we get another contradiction. □

Remark 3.1 Note that if $y_n = x_n$ or $Ay_n = 0$, then y_n is a solution of (VI), and Algorithm 3.1 stops. Indeed, $y_n = x_n$ means $P_C(x_n - \gamma Ax_n) = x_n$, i.e., x_n satisfies the basic chracterization of a solution of (VI) metioned just after the statement of the problem.

If $Ay_n = 0$, substituting $y_n - \lambda_n Ay_n$ into the place of y and y_n into x in $\|x - P_C y\|^2 \leq \langle x - y, x - P_C y\rangle$ (which holds for all $x \in C$ and $y \in H$), we obtain

$$0 = \langle Ay_n, y_n - P_C(y_n - \lambda_n Ay_n)\rangle \geq \frac{1}{\lambda_n}\|y_n - P_C(y_n - \lambda_n Ay_n)\|^2 \geq \frac{1}{\gamma}\|y_n - P_C(y_n - \lambda_n Ay_n)\|^2.$$

Hence, $\|y_n - P_C(y_n - \gamma Ay_n)\| = 0$ and so y_n is a solution of (VI).

Lemma 3.11 *Assume that Conditions 1–3 hold and $\{x_n\}$ is a sequence generated by Algorithm 3.1. If $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, then any weak cluster point of $\{x_n\}$ is a solution of* (VI).

Proof We have

$$\langle x_{n_k} - \lambda_{n_k} A x_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0 \;\; \forall x \in C.$$

or equivalently

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \langle A x_{n_k}, x - y_{n_k} \rangle \;\; \forall x \in C.$$

Consequently,

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle A x_{n_k}, y_{n_k} - x_{n_k} \rangle \leq \langle A x_{n_k}, x - x_{n_k} \rangle \;\; \forall x \in C. \tag{6}$$

Now, we claim that

$$\liminf_{k\to\infty} \langle A x_{n_k}, x - x_{n_k} \rangle \geq 0. \tag{7}$$

Indeed, suppose first that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. By Lemma 2.1, $\{A x_{n_k}\}$ is bounded. Taking $k \to \infty$ in (6), since $\|x_{n_k} - y_{n_k}\| \to 0$, we get (7). Next, we assume that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Setting $z_{n_k} := P_C(x_{n_k} - \lambda_{n_k}.l^{-1} A x_{n_k})$, as $\lambda_{n_k} l^{-1} > \lambda_{n_k}$, Lemma 2.4 yields

$$\|x_{n_k} - z_{n_k}\| \leq \frac{1}{l} \|x_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Hence, z_{n_k} weakly converges to $z \in C$. Because A is (uniformly) continuous on the bounded set $\{x_n\} \cup \{z_n\}$, we obtain

$$\|A x_{n_k} - A z_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{8}$$

As $\lambda_{n_k} l^{-1} = \gamma l^{m_{n_k}} l^{-1} = \gamma l^{m_{n_k} - 1}$, by the Armijo linesearch rule (2), we have

$$\lambda_{n_k} l^{-1} \|A x_{n_k} - A P_C(x_{n_k} - \lambda_{n_k} l^{-1} A x_{n_k})\| > \mu \|x_{n_k} - P_C(x_{n_k} - \lambda_{n_k} l^{-1} A x_{n_k})\|.$$

That is,

$$\frac{1}{\mu} \|A x_{n_k} - A z_{n_k}\| > \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}}.$$

Combining this and (8), we obtain

$$\lim_{k\to\infty} \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0.$$

Furthermore, we have from the definition of z_{n_k} that

$$\langle x_{n_k} - \lambda_{n_k} l^{-1} A x_{n_k} - z_{n_k}, x - z_{n_k} \rangle \leq 0 \;\; \forall x \in C.$$

Hence,

$$\frac{1}{\lambda_{n_k} l^{-1}} \langle x_{n_k} - z_{n_k}, x - z_{n_k} \rangle + \langle A x_{n_k}, z_{n_k} - x_{n_k} \rangle \leq \langle A x_{n_k}, x - x_{n_k} \rangle \;\; \forall x \in C.$$

Taking the limit as $k \to \infty$, we get

$$\liminf_{k\to\infty} \langle A x_{n_k}, x - x_{n_k} \rangle \geq 0.$$

Therefore, the claim (7) is proved.

Furthermore, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Ax_{n_k}, x - x_{n_k}\rangle + \langle Ax_{n_k}, x - x_{n_k}\rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle. \qquad (9)$$

As $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$, by the uniform continuity of A on bounded subsets, we get $\lim_{k\to\infty}\|Ax_{n_k} - Ay_{n_k}\| = 0$, which together with (7) and (9) implies that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0. \qquad (10)$$

Finally, we show that $z \in \mathrm{Sol(VI)}$. Take a sequence $\{\varepsilon_k\}$ of positive numbers, decreasing and tending to 0. Choose an increasing sequence $\{N_k\}$ such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \varepsilon_k \geq 0 \;\; \forall j \geq N_k, \qquad (11)$$

where the existence of N_k follows from (10). Moreover, for each k setting $v_{N_k} = Ay_{N_k}\|Ay_{N_k}\|^{-2}$, we have $\langle Ay_{N_k}, v_{N_k}\rangle = 1$. We deduce from (11) that, for each k,

$$\langle Ay_{N_k}, x + \varepsilon_k v_{N_k} - y_{N_k}\rangle \geq 0.$$

In view of the pseudomonotonicity of A on H, we get

$$\langle A(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - y_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Ax, x - y_{N_k}\rangle \geq \langle Ax - A(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - y_{N_k}\rangle - \varepsilon_k\langle Ax, v_{N_k}\rangle. \qquad (12)$$

We show that $\lim_{k\to\infty}\varepsilon_k v_{N_k} = 0$. Indeed, since $x_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$, we obtain $y_{N_k} \rightharpoonup z$. Since A is sequentially weakly continuous on C, $\{Ay_{n_k}\}$ converges weakly to Az. We have $Az \neq 0$ (otherwise, z is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|.$$

Since $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \to 0$, we obtain

$$0 \leq \limsup_{k\to\infty}\|\varepsilon_k v_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\varepsilon_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty}\varepsilon_k v_{N_k} = 0$.

Letting $k \to \infty$, the right-hand side of (12) tends to zero due to the uniform continuity of A. Thus, $\liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0$. Hence, we have, for all $x \in C$,

$$\langle Ax, x - z\rangle = \lim_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0.$$

By Lemma 2.6, $z \in \mathrm{Sol(VI)}$ and the proof is complete. □

Remark 3.2 When A is monotone, the imposed sequential weak continuity of A can be omitted.

Lemma 3.12 *Assume that Conditions 1–3 hold and $\{x_n\}$ is a sequence generated by Algorithm 3.1. Then,*

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1-\mu)\|y_n - x_n\|^2 - (1-\mu)\|x_{n+1} - y_n\|^2, \qquad (13)$$

for all $p \in$ Sol(VI).

Proof Since $p \in \text{Sol}(\text{VI}) \subset C \subset T_n$, by Lemma 2.3, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|P_{T_n}(x_n-\lambda_n Ay_n)-P_{T_n}p\|^2 \leq \langle x_{n+1}-p, x_n-\lambda_n Ay_n-p\rangle \\ &= \frac{1}{2}\|x_{n+1}-p\|^2+\frac{1}{2}\|x_n-\lambda_n Ay_n-p\|^2-\frac{1}{2}\|x_{n+1}-x_n+\lambda_n Ay_n\|^2 \\ &= \frac{1}{2}\|x_{n+1}-p\|^2+\frac{1}{2}\|x_n-p\|^2+\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle x_n-p,\lambda_n Ay_n\rangle \\ &\quad -\frac{1}{2}\|x_{n+1}-x_n\|^2-\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle x_{n+1}-x_n,\lambda_n Ay_n\rangle \\ &= \frac{1}{2}\|x_{n+1}-p\|^2+\frac{1}{2}\|x_n-p\|^2-\frac{1}{2}\|x_{n+1}-x_n\|^2-\langle x_{n+1}-p,\lambda_n Ay_n\rangle.\end{aligned}$$

Then,

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-x_n\|^2+2\lambda_n\langle Ay_n, p-x_{n+1}\rangle.$$

As $\langle Ap, y_n-p\rangle \geq 0$, by the pseudomonotonicity of A on H, we get $\langle Ay_n, p-y_n\rangle \leq 0$. Hence,

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-x_n\|^2+2\lambda_n\langle Ay_n, y_n-x_{n+1}\rangle.$$

Consequently,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2-\|(x_n-y_n)+(y_n-x_{n+1})\|^2+2\lambda_n\langle Ay_n, y_n-x_{n+1}\rangle \\ &= \|x_n-p\|^2-\|x_n-y_n\|^2-\|y_n-x_{n+1}\|^2+2\langle x_n-y_n, y_n-x_{n+1}\rangle+2\lambda_n\langle Ay_n, y_n-x_{n+1}\rangle \\ &= \|x_n-p\|^2-\|x_n-y_n\|^2-\|y_n-x_{n+1}\|^2+2\langle x_n-\lambda_n Ay_n-y_n, x_{n+1}-y_n\rangle.\end{aligned} \tag{14}$$

Now, we estimate $\langle x_n-\lambda_n Ay_n-y_n, x_{n+1}-y_n\rangle$. As $x_{n+1} \in T_n$, we have

$$2\langle x_n-\lambda_n Ax_n-y_n, x_{n+1}-y_n\rangle \leq 0.$$

Therefore,

$$\begin{aligned}2\langle x_n-\lambda_n Ay_n-y_n, x_{n+1}-y_n\rangle &= 2\langle x_n-\lambda_n Ax_n-y_n, x_{n+1}-y_n\rangle+2\lambda_n\langle Ax_n-Ay_n, x_{n+1}-y_n\rangle \\ &\leq 2\lambda_n\langle Ax_n-Ay_n, x_{n+1}-y_n\rangle \\ &\leq 2\lambda_n\|Ax_n-Ay_n\|.\|x_{n+1}-y_n\| \\ &\leq 2\mu\|x_n-y_n\|.\|x_{n+1}-y_n\| \\ &\leq \mu\|x_n-y_n\|^2+\mu\|x_{n+1}-y_n\|^2.\end{aligned}$$

This together with (14) implies that

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2-(1-\mu)\|x_n-y_n\|^2-(1-\mu)\|y_n-x_{n+1}\|^2.$$

□

Theorem 3.1 *Assume that Conditions 1–3 hold. Then, any sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges weakly to a solution of* (VI).

Proof By virtue of Lemma 3.12, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| \ \ \forall p \in \mathrm{Sol(VI)}.$$

Hence, for all $p \in$ Sol(VI), $\lim_{n\to\infty} \|x_n - p\|$ exists. Also from Lemma 3.12, we have

$$(1-\mu)\|x_n - y_n\|^2 + (1-\mu)\|y_n - x_{n+1}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

which implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{15}$$

Since $\{x_n\}$ is a bounded sequence, there exists the subsequence $\{x_{n_k}\}$ convergent weakly to $z \in C$. It follows from Lemma 3.11 and (15) that $z \in$ Sol(VI). Therefore, according to Lemma 2.7, the sequence $\{x_n\}$ converges weakly to a solution of (VI). □

Remark 3.3 Some main assumptions in Theorem 3.1 are weaker than the corresponding ones in Theorem 5.1 in [6] as follows:

(i) the monotonicity of A on H in Theorem 5.1 in [6] is replaced by the pseudomonotonicity on H and sequential weak continuity on C;

(ii) the Lipschits continuity of A is replaced by the uniform continuity on bounded subsets of H.

It is evedent that Lipschitz continuity (on C) implies uniform continuity (on C) (simply take $\delta = \varepsilon/L$ to see that $\|Ax - Ay\| \leq \varepsilon$ if $\|x-y\| \leq \delta$ in the definition of uniform continuity). The following example shows that the converse is false.

Example 1 Let $A : [0,1] \to [0,+\infty)$ be defined by $Ax = \sqrt{x}$. We claim that A is uniformly continuous and even monotone on $[0,1]$, but not Lipschitz continuous. Indeed, A is monotone because

$$\langle Ax - Ay, x - y\rangle = (\sqrt{x} - \sqrt{y})(x-y) = (\sqrt{x} - \sqrt{y})^2(\sqrt{x} + \sqrt{y}) \geq 0 \ \ \forall x, y \in [0,1].$$

To see the uniform continuity, for any $\varepsilon > 0$, we choose $\delta = \varepsilon^2$ to have, from $|x-y| \leq \delta$,

$$|Ax - Ay| = |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} \leq \varepsilon.$$

Now suppose that, for some $L > 0$,

$$|Ax - Ay| \leq L|x-y| \ \ \forall x, y \in [0,1].$$

Taking $x = 0$ and $y = (2(L+1))^{-2}$, we arrive at that contradiction $|Ax - Ay| > L|x-y|$ because

$$|Ax - Ay| = (2(L+1))^{-1} \text{ and } L|x-y| = L(2(L+1))^{-2}.$$

Additionally, notice that in this case, Sol(VI) is nonempty (equal to $\{0\}$).

To get a sufficient condition for the strong convergence of Algorithm 3.1, we modify the assumptions, back to a Lipschitz continuity condition, as follows. Here, we assert furthermore that the iterative sequence generated by Algorithm 3.1 converges at a Q-linear rate to the unique solution of (VI). This result extends and improves the corresponding one in [15], where the strong pseudomonotonicity of A was imposed.

Theorem 3.2 *If we retain Conditions 1 and 3 and replace Condition 2 by*

Condition $2'$ *A is κ-strongly pseudomonotone on C and L-Lipschitz continuous on H,*

then, any sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution p of (VI) *with a Q-linear rate.*

Proof First, we claim that $\eta \le \lambda_n$, where $\eta := \min\{\gamma, \frac{\mu l}{L}\}$.

Indeed, as $\lambda_n \le \gamma$ by Lemma 3.10, the claim is checked if $\lambda_n = \gamma$. Consider the case $\lambda_n < \gamma$. Then, $\lambda_n l^{-1}$ must violate inequality (2), i.e.

$$\|Ax_n - A(P_C(x_n - \frac{\lambda_n}{l}Ax_n))\| > \frac{\mu}{\frac{\lambda_n}{l}}\|x_n - P_C(x_n - \frac{\lambda_n}{l}Ax_n)\|,$$

combining this with the Lipschitz condition we obtain the claim: $\lambda_n > \mu l L^{-1}$.

As $\langle Ap, y_n - p\rangle \ge 0$, the strong pseudomonotonicity of A gives $\langle Ay_n, y_n - p\rangle \ge \kappa\|y_n - p\|^2$. Hence,

$$\begin{aligned}\langle Ax_n, p - y_n\rangle &= \langle Ax_n - Ay_n, p - y_n\rangle - \langle Ay_n, y_n - p\rangle\\ &\le \|Ax_n - Ay_n\|\|y_n - p\| - \kappa\|y_n - p\|^2\\ &\le L\|x_n - y_n\|\|y_n - p\| - \kappa\|y_n - p\|^2.\end{aligned}$$

Furthermore, as $y_n \in C$,

$$\langle x_n - \lambda_n Ax_n - y_n, y_n - p\rangle \ge 0,$$

and so

$$\begin{aligned}\langle x_n - y_n, p - y_n\rangle &\le \lambda_n\langle Ax_n, p - y_n\rangle\\ &\le \lambda_n L\|x_n - y_n\|\|y_n - p\| - \lambda_n\kappa\|y_n - p\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\lambda_n\kappa\|y_n - p\|^2 &\le \lambda_n L\|x_n - y_n\|\|y_n - p\| + \langle x_n - y_n, y_n - p\rangle\\ &\le \lambda_n L\|x_n - y_n\|\|y_n - p\| + \|x_n - y_n\|\|y_n - p\|\\ &= (1 + \lambda_n L)\|x_n - y_n\|\|y_n - p\|.\end{aligned}$$

Therefore, we get

$$\|y_n - p\| \le \frac{1 + \lambda_n L}{\lambda_n\kappa}\|x_n - y_n\| \le \frac{1 + \gamma L}{\eta\kappa}\|x_n - y_n\|.$$

Moreover,

$$\|x_n - p\| \le \|x_n - y_n\| + \|y_n - p\| \le \frac{1 + \gamma L + \eta\kappa}{\eta\kappa}\|x_n - y_n\|.$$

This implies that

$$\|x_n - y_n\| \ge \frac{\eta\kappa}{1 + \gamma L + \eta\kappa}\|x_n - p\|$$

Substituting this into (13), we get

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 - (1-\mu)\left(\frac{\eta\kappa}{1+\gamma L+\eta\kappa}\right)^2 \|x_n-p\|^2$$
$$= \left(1-(1-\mu)\left(\frac{\eta\kappa}{1+\gamma L+\eta\kappa}\right)^2\right)\|x_n-p\|^2.$$

As $\sqrt{1-(1-\mu)\left(\frac{\eta\kappa}{1+\gamma L+\eta\kappa}\right)^2} \in (0,1)$, this inequality shows that $\{x_n\}$ converges strongly to p with a Q-linear rate. □

Although Algorithm 3.1, under Conditions 1, 3 and 2′, strongly converges with a Q-linear rate, the restrictive Condition 2′ prevents its applications. Hence, we modify it to get Algorithm 3.2 with additional parameter sequences $\{\alpha_n\}$ and $\{\beta_n\}$, and use again Conditions 1-3 and add Condition 4 below to keep the strong convergence. Furthermore, we have an important additional property of the solution being the limit of the generated sequence $\{x_n\}$.

Condition 4 *$\{\alpha_n\}$ is in $(0,1)$ and tends to 0 such that $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\{\beta_n\}$ is in $[c,1-\alpha_n)$ for some $c>0$.*

Algorithm 3.2 *Let the parameters be $\gamma>0, l\in(0,1), \mu\in(0,1)$, and $\{\alpha_n\}$, $\{\beta_n\}$ given in Condition 4.*

Initialization: *Choose arbitrarily $x_1 \in C$.*

Iterative Steps: *Given a current iterate x_n, calculate x_{n+1} as follows:*

Step 1. *Compute*

$$y_n := P_C(x_n-\lambda_n Ax_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m\|Ax_n-Ay_n\| \le \mu\|x_n-y_n\|.$$

If $y_n=x_n$ or $Ay_n=0$, then stop; x_n is a solution of (VI). *Otherwise,*
Step 2. *Compute*

$$z_n := P_{T_n}(x_n-\lambda_n Ay_n),$$

where

$$T_n=\{x\in H : \langle x_n-\lambda_n Ax_n-y_n, x-y_n\rangle \le 0\}.$$

Step 3. *Compute*

$$x_{n+1}=(1-\alpha_n-\beta_n)x_n+\beta_n z_n.$$

Set $n:=n+1$ and go to ***Step 1****.*

Observe that in comparison with Algorithm 3.1, in Algorithm 3.2 the only addition is Step 3.

Theorem 3.3 *Assume that Conditions 1–4 hold. Then, any sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $p \in \mathrm{Sol(VI)}$*, where* $p = \operatorname*{argmin}_{z\in \mathrm{Sol(VI)}} \|z\|$.

Proof **Claim 1.** The sequence $\{x_n\}$ is bounded.

Indeed, applying Lemma 3.12 to Algorothm 3.2, in (13) in the place of x_{n+1} we write z_n and for p we take $p = \operatorname*{argmin}_{z\in \mathrm{Sol(VI)}} \|z\|$ (which is $P_{\mathrm{Sol(VI)}}0$) to obtain

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - (1-\mu)\|x_n - y_n\|^2 - (1-\mu)\|z_n - y_n\|^2. \tag{16}$$

Then,

$$\|z_n - p\| \le \|x_n - p\|. \tag{17}$$

We have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\| \\ &= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p) - \alpha_n p\| \\ &\le \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p)\| + \alpha_n\|p\|.\end{aligned} \tag{18}$$

Furthermore, using (17), we get

$$\begin{aligned}&\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p)\|^2 \\ &\quad=(1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\langle x_n - p, z_n - p\rangle + \beta_n^2\|z_n - p\|^2 \\ &\quad\le(1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\|z_n - p\|\|x_n - p\| + \beta_n^2\|z_n - p\|^2 \\ &\quad\le(1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\|x_n - p\|^2 + \beta_n^2\|x_n - p\|^2 \\ &\quad=(1-\alpha_n)^2\|x_n - p\|^2.\end{aligned}$$

Hence, from (18), we obtain

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| \\ &\le \max\{\|x_n - p\|, \|p\|\} \\ &\le \dots \le \max\{\|x_1 - p\|, \|p\|\}.\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded and hence so is $\{z_n\}$ by (17).

Claim 2

$$\beta_n(1-\mu)\|x_n - y_n\|^2 + \beta_n(1-\mu)\|z_n - y_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|p\|^2.$$

Indeed, using (1), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 =&\|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\|^2 \\ =&\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p) + \alpha_n(-p)\|^2 \\ =&(1-\alpha_n-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + \alpha_n\|p\|^2 - \beta_n(1-\alpha_n-\beta_n)\|x_n - z_n\|^2 \\ &- \alpha_n(1-\alpha_n-\beta_n)\|x_n\|^2 - \alpha_n\beta_n\|z_n\|^2 \\ \le&(1-\alpha_n-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + \alpha_n\|p\|^2,\end{aligned}$$

which together with (16) implies that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le &(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|x_n-p\|^2-\beta_n(1-\mu)\|x_n-y_n\|^2\\ &-\beta_n(1-\mu)\|z_n-y_n\|^2+\alpha_n\|p\|^2\\ =&(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\mu)\|x_n-y_n\|^2-\beta_n(1-\mu)\|z_n-y_n\|^2+\alpha_n\|p\|^2\\ \le&\|x_n-p\|^2-\beta_n(1-\mu)\|x_n-y_n\|^2-\beta_n(1-\mu)\|z_n-y_n\|^2+\alpha_n\|p\|^2.\end{aligned}$$

This inequality implies Claim 2.

Claim 3

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|x_n-z_n\|\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle].$$

Indeed, setting $t_n=(1-\beta_n)x_n+\beta_n z_n$, we have

$$\begin{aligned}\|t_n-p\| =&\|(1-\beta_n)(x_n-p)+\beta_n(z_n-p)\|\\ \le&(1-\beta_n)\|x_n-p\|+\beta_n\|x_n-p\|\\ =&\|x_n-p\|,\end{aligned} \tag{19}$$

and

$$\|t_n-x_n\|=\beta_n\|x_n-z_n\|.$$

Then, by the formula of x_{n+1} in Step 3,

$$\begin{aligned}\|x_{n+1}-p\|^2 =&\|(1-\beta_n)x_n+\beta_n z_n-\alpha_n x_n-p\|^2\\ =&\|(1-\alpha_n)(t_n-p)-\alpha_n(x_n-t_n)-\alpha_n p\|^2.\end{aligned}$$

Substituting, in the write-hand side of the simple inequality for any Hilbert space $\|x+y\|^2 \le \|x\|^2+2\langle y,x+y\rangle$, $x=(1-\alpha_n)(t_n-p)$, $y=-\alpha_n(x_n-t_n)-\alpha_n p$, and $x+y=x_{n+1}-p$, we get

$$\begin{aligned}\|x_{n+1}-p\|^2 =&\|(1-\alpha_n)(t_n-p)-\alpha_n(x_n-t_n)-\alpha_n p\|^2\\ \le&(1-\alpha_n)^2\|t_n-p\|^2-2\langle\alpha_n(x_n-t_n)+\alpha_n p,x_{n+1}-p\rangle\\ =&(1-\alpha_n)^2\|t_n-p\|^2+2\alpha_n\langle x_n-t_n,p-x_{n+1}\rangle+2\alpha_n\langle p,p-x_{n+1}\rangle\\ \le&(1-\alpha_n)\|t_n-p\|^2+2\alpha_n\|x_n-t_n\|\|x_{n+1}-p\|+2\alpha_n\langle p,p-x_{n+1}\rangle\\ \le&(1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|x_n-z_n\|\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle].\end{aligned}$$

Claim 4. The sequence $\{\|x_n-p\|\}$ converges to zero.

We have two cases.

Case 1: There exists an $N\in\mathbb{N}$ such that $\|x_{n+1}-p\|\le\|x_n-p\|$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. It follows from Claim 2 that

$$\lim_{n\to\infty}\|x_n-y_n\|=0 \text{ and } \lim_{n\to\infty}\|z_n-y_n\|=0.$$

So, $\lim_{n\to\infty}\|z_n-x_n\|=0$. Hence,

$$\|x_{n+1}-x_n\|\le\alpha_n\|x_n\|+\beta_n\|x_n-z_n\|\to 0.$$

As $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_j}\}$ converges weakly to a point q such that

$$\limsup_{n\to\infty}\langle p,p-x_n\rangle=\lim_{j\to\infty}\langle p,p-x_{n_j}\rangle=\langle p,p-q\rangle.$$

Observe that Lemma 3.11 is valid also for $\{x_n\}$ generated by Algorithm 3.2. Indeed, examining its proof we see that only Conditions 1-3 and the formula of y_n in terms of x_n are used, not depending on x_{n+1} defined in the additional Step 3 of Algorithm 3.2. Applying now this lemma, we have $q \in \mathrm{Sol(VI)}$.

Since $q \in \mathrm{Sol(VI)}$ and $p = P_{\mathrm{Sol(VI)}}0$, we have

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \langle p, p - q\rangle \le 0.$$

As $\|x_{n+1} - x_n\| \to 0$, we get

$$\limsup_{n\to\infty}\langle p, p - x_{n+1}\rangle \le 0.$$

Therefore, by Claim 3 and Lemma 2.9, we get $\lim_{n\to\infty}\|x_n - p\| = 0$, that is $x_n \to p$.

Case 2: There exists a subsequence $\{\|x_{n_j} - p\|\}$ of $\{\|x_n - p\|\}$ such that $\|x_{n_j} - p\| < \|x_{n_j+1} - p\|$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.8 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\| \le \|x_{m_k+1} - p\| \text{ and } \|x_k - p\| \le \|x_{m_k+1} - p\|.$$

As $\beta_n \ge c\ \forall n \in \mathbb{N}$, by Claim 2, we have

$$\begin{aligned} c(1-\mu)\|x_{m_k} - y_{m_k}\|^2 + c(1-\mu)\|z_{m_k} - y_{m_k}\|^2 \le& \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}\|p\|^2 \\ \le& \alpha_{m_k}\|p\|^2. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\|x_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k\to\infty}\|z_{m_k} - y_{m_k}\| = 0.$$

As proved in the first case, we obtain

$$\|x_{m_k+1} - x_{m_k}\| \to 0,\ \limsup_{k\to\infty}\langle p, p - x_{m_k+1}\rangle \le 0.$$

In view of Claim 3, we have

$$\begin{aligned} \|x_{m_k+1} - p\|^2 \le& (1-\alpha_{m_k})\|x_{m_k} - p\|^2 \\ &+ \alpha_{m_k}[2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle] \\ \le& (1-\alpha_{m_k})\|x_{m_k+1} - p\|^2 \\ &+ \alpha_{m_k}[2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle]. \end{aligned}$$

This leads to

$$\|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 \le 2\beta_{m_k}\|x_{m_k} - z_{m_k}\|\|x_{m_k+1} - p\| + 2\langle p, p - x_{m_k+1}\rangle.$$

Therefore, $\limsup_{k\to\infty}\|x_k - p\| \le 0$, that is $x_k \to p$. The proof is complete. □

Remark 3.4 In comparsion with Theorem 5.1 in [7] and Theorem 3.3 in [34], our Theorem 3.3 has the following improvements.

1. The Lipschitz continuity (of A) is replaced by the uniform continuity on bounded subsets.
2. The monotonicity (of A) is replaced by the pseudomonotonicity on H and the sequential weak continuity on C.
   Moreover, in our Theorem 3.3, we do not need to add a mapping $f : H \to H$ is contraction as required in Theorem 3.3 in [34].

The following example illustrates that the replacement 2 is indeed an improvement.

Example 2 Let $H = l_2$ be a real Hilbert space whose elements are square-summable sequences of real scalars, i.e.,

$$H = l_2 := \{u = (u_1, u_2, ..., u_n, ...) : \sum_{n=1}^{\infty} |u_n|^2 < +\infty\}$$

(with the usual inner product). Let $\beta > 1$ and

$$C = \{u = (u_1, u_2, ..., u_n, ...) \in H : |u_n| \leq \frac{1}{n} \text{ for all } n\},$$
$$Au = \left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right)u.$$

We show that A is pseudomonotone on H, uniformly continuous and sequentially weakly continuous on C, but not Lipschitz continuous on H. Moreover, Sol(VI)$\neq \emptyset$.

As $\beta > 1$, we have $\left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right) > 0 \ \ \forall u \in H$. Let $u, v \in H$ be such that $\langle Au, v-u\rangle \geq 0$. Then, $\langle u, v-u\rangle \geq 0$. Consequently,

$$\begin{aligned}\langle Av, v-u\rangle &= \left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right)\langle v, v-u\rangle \\ &\geq \left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right)(\langle v, v-u\rangle - \langle u, v-u\rangle) \\ &= \left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right)\|v-u\|^2 \geq 0,\end{aligned}$$

which implies that A is pseudomonotone on H.

Because A is continuous on C and C is compact, A is uniformly continuous and sequentially weakly continuous on C.

Now, suppose that A is Lipschitz continuous on H with constant $L > 0$. For $u = (L, 0, ..., 0, ...)$ and $v = (0, 0, ..., 0, ...)$,

$$\|Au - Av\| = \|Au\| = \left((\|u\|+1)\beta - \frac{1}{\|u\|+1}\right)\|u\| = \left((L+1)\beta - \frac{1}{L+1}\right)L.$$

Thus, $\|Au - Av\| \leq L\|u - v\|$ is equivalent to

$$\left((L+1)\beta - \frac{1}{L+1}\right)L \leq L^2,$$

which implies the contradiction that $\beta < 1$. Finally, it is easy to see that Sol(VI)$= \{0\}$.

Remark 3.5 To avoid possible repeated mistakes, we note that the following example in Section 4 of [41] does not illustrate the main result in that paper (Section 3 there), because the mapping A_β used there defined as below is not weakly sequentially continuous as mistakenly asserted there.

Let $H = l_2$, Let α and β be positive numbers such that $\beta > \alpha > \frac{\beta}{2}$, and

$$C_\alpha = \{u \in H : \|u\| \leq \alpha\}, \ \ A_\beta u = (\beta - \|u\|)\, u.$$

To show that A_β is not weakly sequentially continuous, take $e_n = (0, ..., 0, 1, 0, ...)$ with 1 at the n-th position ($\{e_n\}$ is the standard basis of H) and $\beta = 2\sqrt{2}$. Then, $e_n + e_1 \rightharpoonup e_1$, $A_\beta(e_n + e_1) = \sqrt{2}(e_n + e_1) \rightharpoonup \sqrt{2}e_1$ and $A_\beta(e_1) = (2\sqrt{2} - 1)e_1$. Thus, $A_\beta(e_n + e_1)$ does not converge weakly to $A_\beta e_1$. □

4 Numerical Illustrations

In this section, we provide two numerical examples to test the proposed algorithms. All the codes were written in Matlab (R2015a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz. We apply Algorithms 3.1 and 3.2 to solve the variational inequality problem (VI) and compare our algorithms with other ones. In the numerical results reported in tables, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 3 Assume that $A : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $A(x) = Mx + q$, which is taken from [13] and has been considered by many authors for numerical experiments (see, for example, [38] with $M = BB^T + S + D$), B is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, D is a $m \times m$ diagonal matrix, whose diagonal entries are positive (so M is positive definite), q is a vector in $\mathbb{R}^m$, and

$$C := \left\{x \in \mathbb{R}^m : Qx \leq b\right\},$$

where Q is an $l \times m$ matrix and b is a nonnegative vector.

It is clear that A is monotone and Lipschitz continuous with the Lipschitz constant $L = ||M||$. For $q = 0$, the unique solution of the corresponding variational inequality is $\{0\}$. We will compare Algorithm 3.1 (SEGM) with the Tseng's algorithm in [37] (Tseng algorithm). All entries of the matrices B, S, and D are generated randomly. We use

(i) the same stopping rule $\|y_n - x_n\|^2 < 10^{-6}$;
(ii) the same starting point $x_0 = (1, 1, ..., 1) \in \mathbb{R}^m$.

The results are described in Table 4.1 and Figures 1-3.

	$m = 10$		$m = 50$		$m = 80$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
EGM	3.4944	97	10.0465	76	33.2126	124
SEGM	1.4976	44	3.2916	23	19.2349	69

Table 4.1: Comparison of the two algorithms with different m

The convergence behavior of the algorithms with different starting points is given in Figures 1-3, where the values of error $\|x_n - 0\|$ (for both algorithms) are presented on the y-axis and the numbers of iterations are presented on the x-axis.

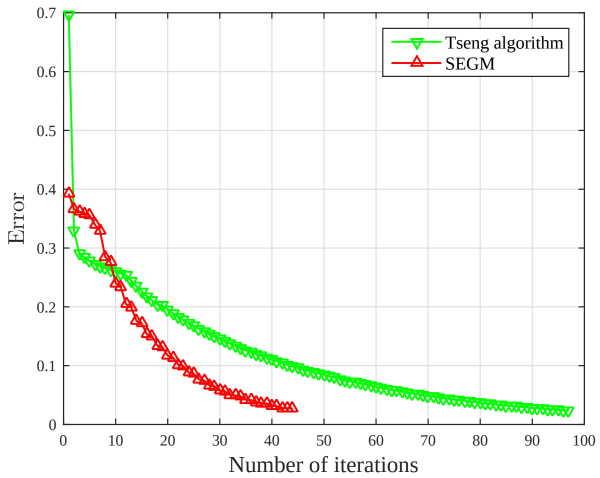

Fig. 1: Comparison of the two algorithms in Example 3 with $m = 10$.

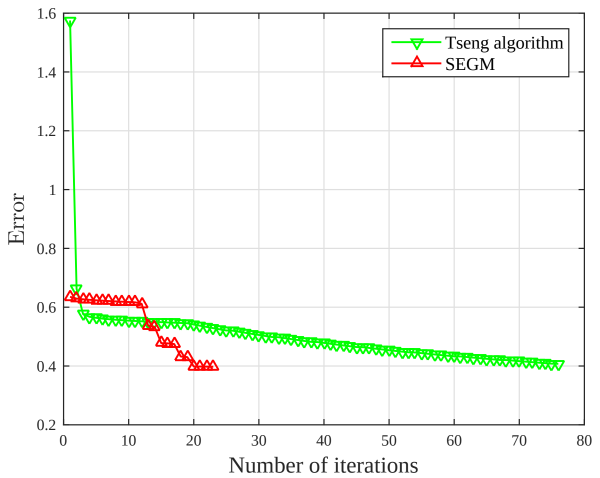

Fig. 2: Comparison of the two algorithms in Example 3 with $m = 50$.

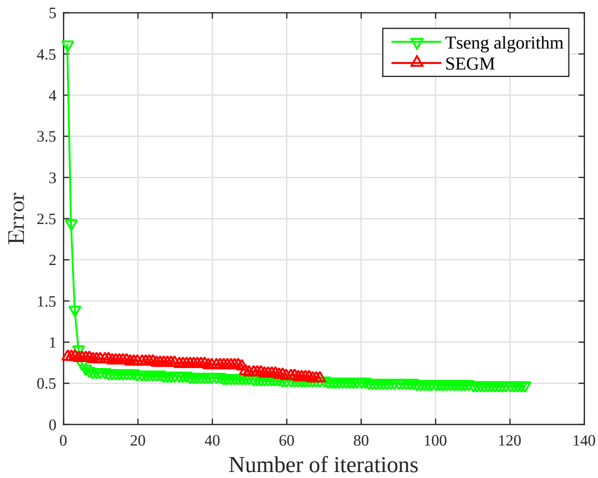

Fig. 3: Comparison of the two algorithms in Example 3 with $m = 80$.

Example 4 Suppose that $H = L^2([0,1])$. It is known that

$$P_C(x) = \begin{cases} \dfrac{b - \langle a, x\rangle}{\|a\|} a + x, & \text{if } \langle a, x\rangle > b, \\ x, & \text{if } \langle a, x\rangle \le b, \end{cases}$$

where $C := \left\{x \in L^2([0,1]) : \langle a, x\rangle \le b\right\}$, $0 \neq a \in L^2([0,1])$, and $b \in \mathbb{R}$.

Let

$$C := \left\{x \in L^2([0,1]) : \int_0^1 (t^2+1)x(t)dt \le 1\right\}$$

and $A : C \to H$ be defined by $(Ax)(t) = \max\{0, x(t)\}$.

It is easy to see that A is 1-Lipschitz continuous and monotone on C and the set of the solutions of the variational inequality (VI) is Sol(VI)$= \{0\}$.

We will apply Algorithm 3.2 (MSEGM), Shehu's algorithm [34] (Shehu algorithm), and Tseng's Algorithm [37] (Tseng algorithm) to solve (VI). We use

(ii) the same stopping criterion $\|y_n - x_n\| < 10^{-3}$;
(iii) the same initial point x_0.

Moreover, for Algorithm 3.2, we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5 * (1 - \alpha_n)$. We also choose the same $\alpha_n = \frac{1}{n+1}$ for Tseng's algorithm and Shehu's algorithm. We now make comparison of the three algorithms with different x^0 and report the results in Table 4.2 and Figures 4-6.

	$x_0 = sin(-3t) + cos(-10t)$		$x_0 = \frac{1}{85}(t^3+1)e^{3t}$		$x_0 = 2(t^4 - e^{-t})$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Shehu algorithm	0.093601	10	0.4056	49	0.1404	13
Tseng algorithm	0.093601	10	0.4056	49	0.1248	13
MSEGM	0.0468	7	0.2652	29	0.078	9

Table 4.2: Comparison of the three algorithms in Example 4

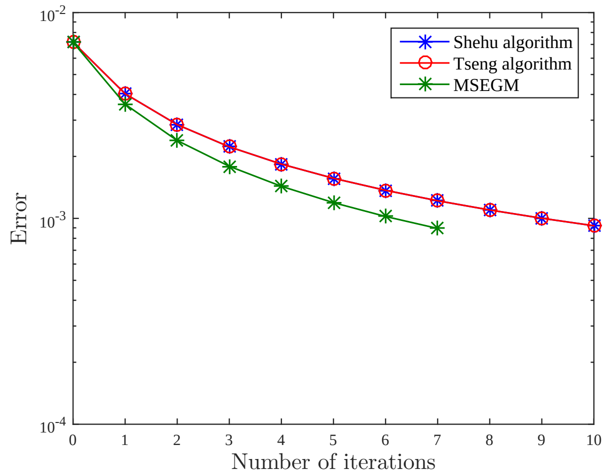

Fig. 4: Comparison of the three algorithms in Example 4 with $x_0 = sin(-3t) + cos(-10t)$.

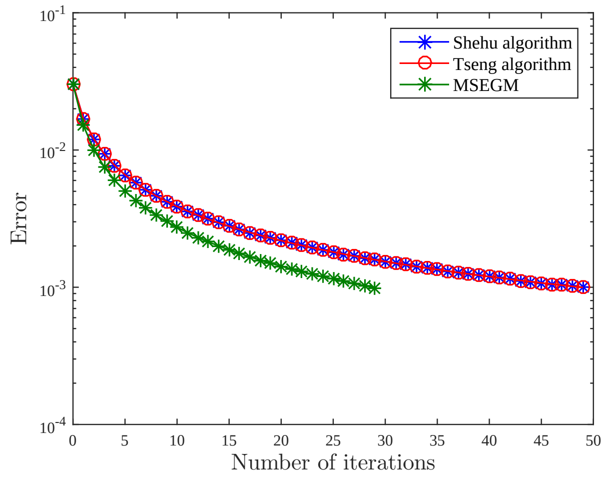

Fig. 5: Comparison of the three algorithms in Example 4 with $x_0 = \frac{1}{85}(t^3+1)e^{3t}$.

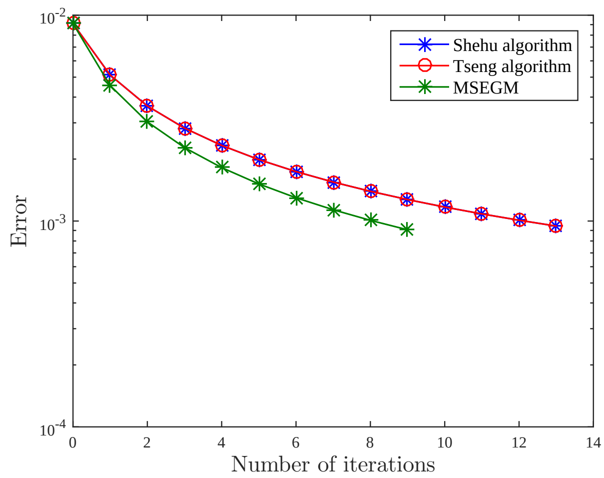

Fig. 6: Comparison of the three algorithms in Example 4 with $x_0 = 2(t^4 - e^{-t})$.

In Figures 4-6 the values of error $\|x_n - y_n\|$ (for the three algorithms) are presented on the y-axis and the numbers of iterations are presented on the x-axis. From Tables 4.1 and 4.2, we observe that the number of iterations and CPU time required for our algorithms are less than that of Tseng's algorithm and Shehu's algorithm.

5 Conclusions

In this paper, we introduce two algorithms of the type of subgradient extragradient method for solving variational inequalities in real Hilbert spaces. The first algorithm converges weakly under pseudomonotonicity and uniform continuity assumptions. The strong convergence with a Q-linear rate is also guaranteed under κ-strong pseudomonotonicity and Lipschitz continuity. The second one converges strongly even under the same assumptions as for the above weak convergence together with an additional condition on the new parameters in the last step added to each iteration. The obtained results extend some recent ones in the literature. The implementation and also some advantages of the proposed algorithms are illustrated by numerical experiments for variational inequalities in both cases of finite and infinite dimensional spaces.

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