REGULARITY AND KOSZUL PROPERTY OF SYMBOLIC POWERS OF MONOMIAL IDEALS

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Abstract. Let I be a homogeneous ideal in a polynomial ring over a field. Let $I^{(n)}$ be the n-th symbolic power of I. Motivated by results about ordinary powers of I, we study the asymptotic behavior of the regularity function $\operatorname{reg}(I^{(n)})$ and the maximal generating degree function $d(I^{(n)})$, when I is a monomial ideal. It is known that both functions are eventually quasi-linear. We show that, in addition, the sequences $(\operatorname{reg} I^{(n)}/n)_n$ and $(d(I^{(n)})/n)_n$ converge to the same limit, which can be described combinatorially. We construct an example of an equidimensional, height two squarefree monomial ideal I for which $d(I^{(n)})$ and $\operatorname{reg}(I^{(n)})$ are not eventually linear functions. For the last goal, we introduce a new method for establishing the componentwise linearity of ideals. This method allows us to identify a new class of monomial ideals whose symbolic powers are componentwise linear.

1. Introduction

Let $R = k[x_1, \dots, x_r]$ be a polynomial ring over a field k. In this paper we investigate the maximal generating degree and the regularity of symbolic powers of monomial ideals in R. Let I be a homogeneous ideal of R. Then the n-th *symbolic power* of I is defined by

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} I^n R_{\mathfrak{p}} \cap R,$$

where $\operatorname{Min}(I)$ is as usual the set of minimal associated prime ideals of I.

Symbolic powers were studied by many authors. While sharing some similar features with ordinary powers, the symbolic powers are usually much harder to deal with. One difficulty lies in the fact that the symbolic Rees algebra, defined as

$$\mathcal{R}_s(I) = R \oplus I^{(1)} \oplus I^{(2)} \oplus \cdots ,$$

is not noetherian in general. Examples of non-noetherian symbolic Rees algebras were discovered by Roberts [31] and simpler examples were provided by Goto-Nishida-Watanabe [11].

Denote by $\operatorname{reg}(I)$ and $d(I)$ to be the regularity of I and the maximal degree of the homogeneous generators I, respectively. By celebrated results by Cutkosky-Herzog-Trung [6] and Kodiyalam [23], we know that $\operatorname{reg} I^n$ and $d(I^n)$ are eventually linear functions with the same leading coefficients. In particular, there exist the limits

$$\lim_{n\to\infty} \frac{\operatorname{reg} I^n}{n} = \lim_{n\to\infty} \frac{d(I^n)}{n}$$

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and the common limit is integral. On the other hand, by [3, Proposition 7], when I defines $2r+1$ points on a rational normal curve in $\mathbb{P}^r$, where $r \geq 2$, then for all $n \geq 1$,

$$\operatorname{reg} I^{(n)} = 2n + 1 + \left\lfloor \frac{n-2}{r} \right\rfloor.$$

Hence the function $\operatorname{reg} I^{(n)}$ is not eventually linear in general. Cutkosky [5] could even construct a smooth curve in $\mathbb{P}^3$ whose homogeneous defining ideal I has the property that $\lim_{n\to\infty} \operatorname{reg} I^{(n)}/n$ is an irrational number. Another peculiar example is given in [6, Example 4.4]: given any prime number $p \equiv 2$ modulo 3, there exist some field k of characteristic p, and some collection of 17 fat points in $\mathbb{P}^2_k$ whose defining ideal I has the property that $\operatorname{reg} I^{(n)}$ is *not eventually quasi-linear.*

While the question about eventual quasi-linear behavior of $\operatorname{reg} I^{(n)}$ has a negative answer in general, various basic questions remain tantalizing. For example:

(1) There was no known example of a homogeneous ideal I in a polynomial ring for which the limit $\lim_{n\to\infty} \operatorname{reg}(I^{(n)})/n$ does not exist (Herzog-Hoa-Trung [18, Question 2]);
(2) It remains an open question whether for every such homogeneous ideal I, the function $\operatorname{reg} I^{(n)}$ is bounded by a linear function;
(3) Even an answer for the analogue of the last question for $d(I^{(n)})$ remains unknown.

In [21, Theorem 4.9], it is shown that $\lim_{n\to\infty} \operatorname{reg}(I^{(n)})/n$ exists if I is a squarefree monomial ideal (but a description of the limit was not provided). By [18, Section 2], Question (2) (and hence of course (3)) has a positive answer if either I is a monomial ideal, or $\dim(R/I) \leq 2$, or the singular locus of R/I has dimension at most 1. The general case remains open for all of these questions.

In the present paper, we address the following related questions for a *monomial* ideal I of R.

Question 1.1. Does the limit $\displaystyle\lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n}$ exists? If it does, describe the limit in terms of I. The same questions for $\displaystyle\lim_{n\to\infty} \frac{d(I^{(n)})}{n}$.

Question 1.2 (Minh-T.N. Trung [25, Question A, part (i)])**.** Is the function $\operatorname{reg} I^{(n)}$ eventually linear if I is squarefree?

A motivation for Question 1.2 is a result of Herzog, Hibi, Trung [17], that $\operatorname{reg} I^{(n)}$ is eventually quasi-linear. Another motivation is a recent result of Hoa et al. [20] on the existence of $\lim\limits_{n\to\infty} \operatorname{depth} I^{(n)}$ when I is a squarefree monomial ideal. It is worth pointing out that Question 1.2 has a negative answer for non-squarefree monomial ideals; see Remark 5.16.

Extending previous result of Hoa and T.N. Trung, our first main result answers Question 1.1 in the positive for *both* limits (they are actually the same). We also describe explicitly the limits in terms of certain polyhedron associated to I. Our second main result answers the other question in the negative. In fact, a counterexample is given using equidimensional height 2 squarefree monomial ideals, in other words, *cover ideals* of graphs. Interestingly, at the same time, our counterexample also gives a negative answer for the analogue of Question 1.2 for the function $d(I^{(n)})$.

In detail, the main tool for Question 1.1 comes from the theory of convex polyhedra. Assume that I admits a minimal primary decomposition

$$I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_t$$

where $Q_1, \ldots, Q_s$ are are the primary monomial ideals associated to the minimal prime ideals of I. We define certain polyhedron associated to I as follows:

$$\mathcal{SP}(I) = NP(Q_1) \cap \cdots \cap NP(Q_s) \subset \mathbb{R}^r,$$

where $NP(Q_i)$ is the Newton polyhedron of Q_i. Then $\mathcal{SP}(I)$ is a convex polyhedron in $\mathbb{R}^r$. For a vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{R}^r$, denote $|\mathbf{v}| = v_1 + \cdots + v_r$. Let

$$\delta(I) = \max\{|\mathbf{v}| \mid \mathbf{v} \text{ is a vertex of } \mathcal{SP}(I)\}.$$

Answering Question 1.1, our first two main results are:

Theorem 1.3 (Theorems 3.3 and 3.6)**.** *For all monomial ideal I, there are equalities*

$$\lim_{n\to\infty} \frac{d(I^{(n)})}{n} = \lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n} = \delta(I).$$

We next study componentwise linear ideals in the sense of Herzog and Hibi [14] which are also known as *Koszul* ideals [19]. Our main tool is the following new result on Koszul ideals, and we prove it by using the theory of linearity defect.

Proposition 1.4 (See Theorem 5.1)**.** *Let R be a polynomial ring over k with the graded maximal ideal $\mathfrak{m}$. Let x be a non-zero linear form, I', T non-zero homogeneous ideals of R such that the following conditions are simultaneously satisfied:*

(i) *I' is Koszul;*
(ii) *$T \subseteq \mathfrak{m}I'$;*
(iii) *x is a regular element with respect to R/T and $\operatorname{gr}_{\mathfrak{m}} T$, the associated graded module of T with respect to the $\mathfrak{m}$-adic filtration.*

Denote $I = xI' + T$. Then I is Koszul if and only if so is T.

A common method (among a dozen of others), to establish the Koszul property of an ideal is to show that it has linear quotients. Compared with this method, the criterion of Proposition 1.4 has the advantage that it does not require the knowledge of a system of generators of the ideal. It just asks for the knowledge of a decomposition which is in many cases not hard to obtain, the more so if we work with monomial ideals. Indeed, let I be a monomial ideal of R, and x one of its variables. Then we always have a decomposition $I = xI' + T$, where I', T are monomial ideals, and x does not divide any minimal generator of T. For such a decomposition, condition (iii) in Proposition 1.4 is automatic. Hence given conditions (i) and (ii), we can prove the Koszulness of I by passing to T, which lives in a smaller polynomial ring.

Proposition 1.4 is interesting in its own and has further applications, which we hope to pursue in future work. The main application of this proposition in our paper is to study the Koszulness of symbolic powers of cover ideals of graphs. Let G be an arbitrary simple graph with the vertex set $V(G) = \{1, \ldots, r\}$ and the edge set $E(G)$. Recall that the cover ideal of G is defined by

$$J(G) = \bigcap_{\{i,j\}\in E(G)} (x_i, x_j).$$

Herzog, Hibi, and Trung showed that the symbolic Rees algebra of $J(G)$ is by elements of degree either 1 or 2 [17, Thereom 5.1]. Using this, we have a fairly useful description of the function $d(J(G)^{(n)})$ (see Theorem 4.9).

By using Proposition 1.4, we prove:

Theorem 1.5 (Theorem 5.7)**.** *Let G be the graph obtained by adding to each vertex of a graph H at least one pendant. Then all the symbolic powers of $J(G)$ are Koszul.*

It is worth mentioning that, via Alexander duality, this can be seen as a generalization of previous work of Villarreal [37] and Francisco-Hà [8] on the Cohen-Macaulay property of graphs.

In order to give a counter-example to Question 1.2, we apply Theorem 1.5 for corona graphs. Namely,

Theorem 1.6 (Theorem 5.15)**.** *For $m \geqslant 3$ and $s \geqslant 2$, let $G = \operatorname{cor}(K_m, s)$ be the graph obtained from the complete graph on m vertices K_m by adding exactly s pendants to each of its vertex. Let $J = J(G)$. Then for all $n \geqslant 0$,*

(1) $\operatorname{reg}(J^{(2n)}) = d(J^{(2n)}) = m(s+1)n$;
(2) $\operatorname{reg}(J^{(2n+1)}) = d(J^{(2n+1)}) = m(s+1)n + m + s - 1$.

In particular, for all n,

$$\operatorname{reg}(J^{(n)}) = d(J^{(n)}) = (m+s-1)n + (m-2)(s-1)\left\lfloor \frac{n}{2} \right\rfloor,$$

which is not an eventually linear function of n.

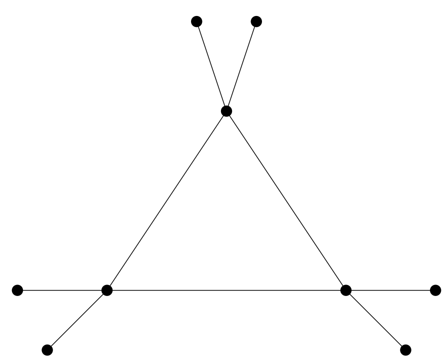

Figure 1. The graph $\operatorname{cor}(K_3, 2)$

Let us summarize the structure of this article. In Section 2, we recall some necessary background. In Section 3, we prove show that for any monomial ideal I, the limits $\lim_{n\to\infty} \operatorname{reg} I^{(n)}/n$ and $\lim_{n\to\infty} d(I^{(n)})/n$ exist and equal to each other. We identify them in terms of the afore-mentioned polyhedron associated to I. In Section 4, we describe structural properties of the symbolic powers of a cover ideal $J(G)$, and compute the function $d(J(G)^{(n)})$ in terms of the graph G in certain situations. In Section 5, we first prove the Koszulness criterion of Proposition 1.4. The main result of this section is the Koszul property of the symbolic powers for certain class of cover ideals, stated in Theorem 5.7. Combining this with results in Section 4, Theorem 1.6 is deduced at the end of this section.

2. Preliminaries

For standard terminology and results in commutative algebra, we refer to the book of Eisenbud [7]. Good references for algebraic aspects of monomial ideals and simplicial complexes are the books of Herzog and Hibi [15], Miller and Sturmfels [24], and Villarreal [38].

2.1. **Regularity.** Let R be a standard graded algebra over a field k. Let M be a finitely generated graded nonzero R-module. Let

$$F : \cdots \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be the minimal graded free resolution of M over R. For each $i \geq 0$, $j \in \mathbb{Z}$, denote $\beta_i^R(M) = \operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^R(k, M)$ and $\beta_{i,j}^R(M) = \dim_k \operatorname{Tor}_i^R(k, M)_j$. We usually omit the superscript R and write simply $\beta_i(M)$ and $\beta_{i,j}(M)$ whenever this is possible. Let

$$t_i(M) = \sup\{j \mid \beta_{i,j}(M) \neq 0\}$$

where, by convention, $t_i(M) = -\infty$ if $F_i = 0$. The *CastelnuovoMumford regularity* of M measures the growth of the generating degrees of the F_i, $i \geq 0$. Concretely, it is defined by

$$\operatorname{reg}_R(M) = \sup\{t_i(M) - i \mid i \geqslant 0\}.$$

In the remaining of this paper, we denote by $d(M)$ the number $t_0(M)$. Hence $d(M)$ is the maximal degree of a minimal homogeneous generator of M. The definition of the regularity implies

$$d(M) \leqslant \operatorname{reg}_R(M).$$

If M is generated by elements of the same degree d, and $\operatorname{reg}_R M = d$, we say that M has a *linear resolution* over R. We also say M has a d-linear resolution in this case.

If R is a standard graded polynomial ring over k, it is customary to denote $\operatorname{reg}_R M$ simply by $\operatorname{reg} M$.

2.2. **Linearity defect, Koszul modules, Betti splittings.** We use the notion of linearity defect, formally introduced by Herzog and Iyengar [19]. Let R be a standard graded k-algebra, and M a finitely generated graded R-module. The linearity defect of M over R, denoted by $\operatorname{ld}_R M$, is defined via certain filtration of the minimal graded free resolution of M. For details of this construction, we refer to [19, Section 1]. We say M is called a *Koszul module* if $\operatorname{ld}_R M = 0$. We say that R is a *Koszul algebra* if $\operatorname{reg}_R k = 0$. As a matter of fact, R is a Koszul algebra if and only if k is a Koszul R-module [19, Remark 1.10].

For each $d \in \mathbb{Z}$, denote by $M_{\langle d \rangle}$ the submodule of M generated by homogeneous elements of degree d. Following Herzog and Hibi [14], M is called *componentwise linear* if for all $d \in \mathbb{Z}$, $M_{\langle d \rangle}$ has a d-linear resolution. By results of Römer [32, Theorem 3.2.8] and Yanagawa [39, Proposition 4.9], if R is a Koszul algebra, then M is Koszul if and only if M is componentwise linear.

Because of the last result and for unity of treatment, we use the terms *Koszul* modules throughout, instead of *componentwise linear* modules.

The following result is folklore; see for example [1, Proposition 3.4].

Lemma 2.1. *Let R be a standard graded k-algebra, and M be a Koszul R-module. Then* $\operatorname{reg}_R M = d(M)$.

We also recall the following base change result for the linearity defect.

Lemma 2.2 (Nguyen and Vu [28, Corollary 3.2]). *Let $R \to S$ be a flat extension of standard graded k-algebras. Let I be a homogeneous ideal of R. Then*

$$\operatorname{ld}_R I = \operatorname{ld}_S IS.$$

Let $(R, \mathfrak{m})$ be a noetherian local ring (or a standard graded k-algebra) and $P, I, J \neq (0)$ be proper (homogeneous) ideals of R such that $P = I + J$.

Definition 2.3. The decomposition of P as $I + J$ is called a *Betti splitting* if for all $i \geq 0$, the following equality of Betti numbers holds: $\beta_i(P) = \beta_i(I) + \beta_i(J) + \beta_{i-1}(I \cap J)$.

We have the following reformulations of Betti splittings.

Lemma 2.4 ([28, Lemma 3.5]). *The following are equivalent:*

(i) *The decomposition $P = I + J$ is a Betti splitting;*
(ii) *The natural morphisms $\operatorname{Tor}^R(k, I \cap J) \to \operatorname{Tor}^R(k, I)$ and $\operatorname{Tor}^R(k, I \cap J) \to \operatorname{Tor}^R(k, J)$ are both zero;*
(iii) *The mapping cone construction for the map $I \cap J \to I \oplus J$ yields a minimal free resolution of P.*

2.3. **Symbolic powers of monomial ideals.** Let $R = k[x_1, \ldots, x_r]$ be a standard graded polynomial ring, and I a monomial ideal of R. Let $\mathcal{G}(I)$ denotes the set of minimal monomial generators of I. In the present paper, when saying about minimal generators of a monomial ideal we mean *minimal monomial generators* of it. Let

$$I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_t$$

be a minimal primary decomposition of I, where Q_i is a primary monomial ideal for $i = 1, \ldots, t$, and $P_j = \sqrt{Q_j}$ is a minimal prime of I for $j = 1, \ldots, s$. For each $i = 1, \ldots, s$, the monomial ideal Q_j is obtained from minimal generators of I by setting $x_i = 1$ for all i for which $x_i \notin P_j$, thus

$$d(Q_j) \leqslant d(I), \quad \text{for } j = 1, \ldots, s. \tag{2.1}$$

In the case of monomial ideals, we have a simple formula for the symbolic powers in terms of the minimal primary components (see [17, Lemma 3.1]).

Lemma 2.5. *With notations as above, for all $n \geq 1$, there is an equality*

$$I^{(n)} = Q_1^n \cap Q_2^n \cap \cdots \cap Q_s^n.$$

A function $f \colon \mathbb{N} \to \mathbb{N} \cup \{-\infty\}$ is called *quasi-linear* if there exist a positive integer N and rational numbers $a_i \in \mathbb{Q}$ and $b_i \in \mathbb{Q} \cup \{-\infty\}$, for $i = 0, \ldots, N-1$, such that

$$f(n) = a_i n + b_i, \text{ for all } n \in \mathbb{N} \text{ with } n \equiv i \pmod N.$$

In this case, the smallest such number N is called the period of f.

Assume that f is not identically $-\infty$. Then $\lim\limits_{n \to \infty} \dfrac{f(n)}{n}$ exists if and only if $a_0 = \cdots = a_{N-1}$. In this case we say that f has a constant leading coefficient.

Lemma 2.6. *With notations as above, for every $i \geqslant 0$, $t_i(I^{(n)})$ is quasi-linear in n for $n \gg 0$. In particular, $d(I^{(n)})$ and $\operatorname{reg}(I^{(n)})$ are quasi-linear in n for $n \gg 0$.*

Proof. By [17, Theorem 3.2], the symbolic Rees ring $\mathcal{R}_s(I) = \bigoplus_{n=0}^{\infty} I^{(n)}$ is finitely generated. By the very same way as the proof of [6, Theorem 4.3], we obtain $t_i(I^{(n)})$ is quasi-linear in n for $n \gg 0$. $\square$

If I is a monomial ideal of R, the minimal graded free resolution of I is $\mathbb{Z}^r$-graded. For each $\boldsymbol{\alpha} \in \mathbb{Z}^r$, we denote by $\beta_{i,\boldsymbol{\alpha}}(I)$ the number $\dim_k \operatorname{Tor}_k^R(k, I)_{\boldsymbol{\alpha}}$. Clearly $\beta_{i,\boldsymbol{\alpha}}(I) = 0$ if $\boldsymbol{\alpha} \notin \mathbb{N}^r$.

When we talk about a monomial $x^{\boldsymbol{\alpha}}$ of R, we always mean $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ and $x^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. A vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ is called squarefree if for all $i = 1, \ldots, r$, α_i is either 0 or 1. Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ be the canonical basis of the $\mathbb{Z}$-module space $\mathbb{Z}^r$. For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ the *upper Koszul simplicial complex* associated with I at degree $\boldsymbol{\alpha}$ is defined by

$$K^{\boldsymbol{\alpha}}(I) = \{\text{squarefree vector } \tau \mid x^{\boldsymbol{\alpha} - \tau} \in I\},$$

where we use the convention $\boldsymbol{\alpha} - \tau = \boldsymbol{\alpha} - \sum_{i \in \tau} \mathbf{e}_i$. The multigraded Betti numbers of I can be computed as follows.

Lemma 2.7. ([24, Theorem 1.34]) *For all $i \geqslant 0$ and all $\alpha \in \mathbb{N}^r$, there is an equality $\beta_{i,\boldsymbol{\alpha}}(I) = \dim_k \widetilde{H}_{i-1}(K^{\boldsymbol{\alpha}}(I); k)$.*

Let $\overline{I}$ be the integral closure of the monomial ideal I. To describe $\overline{I}$ geometrically, let $E(I) = \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in \mathbb{N}^r \text{ and } x^{\boldsymbol{\alpha}} \in I\}$. The Newton polyhedron of I is the convex polyhedron in $\mathbb{R}^r$ defined by $NP(I) = \operatorname{conv}\{E(I)\}$. Then $\overline{I}$ is a monomial ideal determined by (see [7, See Exercises 4.22 and 4.23]):

$$E(\overline{I}) = NP(I) \cap \mathbb{N}^r. \tag{2.2}$$

For each $n \geqslant 1$, let

$$\mathcal{SP}_n(I) = \bigcap_{i=1}^{s} NP(Q_i^n),$$

and

$$J_n = \overline{Q_1^n} \cap \cdots \cap \overline{Q_s^n}.$$

Then from Equation (2.2) we have $E(J_n) = \mathcal{SP}_n(I) \cap \mathbb{N}^r$.

We will denote $\mathcal{SP}_1(I)$ simply by $\mathcal{SP}(I)$.

For subsets X and Y of $\mathbb{R}^r$ and a positive integer n, we denote

$$nX = \{ny \mid y \in X\},$$
$$X + Y = \{x + y : x \in X, y \in Y\}.$$

Denote by $\mathbb{R}_+$ the set of non-negative real numbers. The following lemma gives the structure of the convex polyhedron $\mathcal{SP}_n(I)$.

Lemma 2.8. *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ be the set of vertices of $\mathcal{SP}(I)$. Then*

$$\mathcal{SP}_n(I) = n\,\mathcal{SP}(I) = n \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\} + \mathbb{R}_+^r.$$

Proof. For each $i = 1, \ldots, s$, we have $NP(Q_j^n) = nNP(Q_j)$ by [30, Lemma 2.5]. It follows that $\mathcal{SP}_n(I) = n\,\mathcal{SP}(I)$.

For $\mathbf{v} \in \mathcal{SP}(I)$ and $\mathbf{u} \in \mathbb{R}_+^r$, one has $\mathbf{v} + \mathbf{u} \in \mathcal{SP}(I)$ again by [30, Lemma 2.5]. Combining this with [33, Formula (28), Page 106] we have

$$\mathcal{SP}(I) = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\} + \mathbb{R}_+^r.$$

Thus, $\mathcal{SP}_n(I) = n\,\mathcal{SP}(I) = n \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\} + \mathbb{R}_+^r$, as required. $\square$

The following result was proved in [35, Lemma 6].

Lemma 2.9. *Let Q be a monomial ideal of R. Then the Newton polyhedron $NP(Q)$ is the set of solutions of a system of inequalities of the form*

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x} \rangle \geq b_j, j = 1, \ldots, q\},$$

such that the following conditions are simultaneously satisfied:

(i) *Each hyperplane with the equation $\langle \mathbf{a}_j, \mathbf{x} \rangle = b_j$ defines a facet of $NP(Q)$, which contains s_j affinely independent points of $E(G(Q))$ and is parallel to $r - s_j$ vectors of the canonical basis. In this case s_j is the number of non-zero coordinates of $\mathbf{a}_j$.*
(ii) *$\mathbf{0} \neq \mathbf{a}_j \in \mathbb{N}^r, b_j \in \mathbb{N}$ for all $j = 1, \ldots, q$.*
(iii) *If we write $\mathbf{a}_j = (a_{j,1}, \ldots, a_{j,r})$, then $a_{j,i} \leq s_j d(Q)^{s_j - 1}$ for all $i = 1, \ldots, r$.*

Using this, we can give information about facets of $\mathcal{SP}_n(I)$, which will be useful to bound from below the maximal generating degree of $I^{(n)}$ by some linear function of n.

Lemma 2.10. *The polyhedron $\mathcal{SP}(I)$ is the solutions in $\mathbb{R}^r$ of a system of linear inequalities of the form*

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x} \rangle \geqslant b_j, \ j = 1, 2, \ldots, q\},$$

where for each j, the following conditions are fulfilled:

(i) *$\mathbf{0} \neq \mathbf{a}_j \in \mathbb{N}^r$, $b_j \in \mathbb{N}$;*
(ii) *$|\mathbf{a}_j| \leqslant r^2 d(I)^{r-1}$;*
(iii) *The equation $\langle \mathbf{a}_j, \mathbf{x} \rangle = b_j$ defines a facet of $\mathcal{SP}(I)$.*

Proof. Note that $\mathcal{SP}(I)$ is the solution in $\mathbb{R}^r$ of the system of all linear inequalities that arise from those inequalities defining $NP(Q_j)$ where $j = 1, \ldots, s$. Now combining Lemma 2.9 with the fact that $d(Q_j) \leqslant d(I)$ (Inequality (2.1)), the lemma follows. □

Let Δ be a simplicial complex on $\{1, \ldots, r\}$. For a subset $F = \{i_1, \ldots, i_j\}$ of $\{1, \ldots, r\}$, set $x^F = x_{i_1} \cdots x_{i_j}$ and $P_F = (x_i : i \notin F)$. Then the *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal

$$I_\Delta = (x^G \mid G \notin \Delta) \subseteq R.$$

Let $\mathcal{F}(\Delta)$ denote the set of all facets of Δ. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. Then I_Δ admits the primary decomposition

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F.$$

Thanks to Lemma 2.5, for every integer $n \geqslant 1$, the n-th symbolic power of I_Δ is given by

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^n.$$

2.4. **Graph theory.** Let G be a finite simple graph. We use the symbols $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. When there is no confusion, the edge $\{u,v\}$ of G is written simply as uv. Two vertices u and v are *adjacent* if $\{u,v\} \in E(G)$.

For a subset S of $V(G)$, we define

$$N_G(S) = \{v \in V(G) \setminus S \mid uv \in E(G) \text{ for some } u \in S\}$$

and $N_G[S] = S \cup N_G(S)$. When there is no confusion, we shall omit G and write $N(S)$ and $N[S]$. If S consists of a single vertex u, denote $N_G(u) = N_G(S)$ and $N_G[u] = N_G[S]$. Define $G[S]$ to be the induced subgraph of G on S, and $G \setminus S$ to be the subgraph of G with the vertices in S and their incident edges deleted.

The *degree* of a vertex $u \in V(G)$, denoted by $\deg_G(u)$, is the number of edges incident to u. If $\deg_G(u) = 0$, then u is called an *isolated vertex*; if $\deg_G(u) = 1$, then u is a *leave*. An edge emanating from a leaf is called a *pendant*.

A *vertex cover* of G is a subset of $V(G)$ which meets every edge of G; a vertex cover is *minimal* if none of its proper subsets is itself a cover. The *cover ideal* of G is defined by $J(G) := (x^\tau \mid \tau \text{ is a minimal vertex cover of } G)$. Note that $J(G)$ has the primary decomposition

$$J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).$$

An *independent set* in G is a set of vertices no two of which are adjacent to each other. An independent set in G is *maximal* (with respect to set inclusion) if the set cannot be extended to a larger independent set. The set of all independent sets of G, denoted by $\Delta(G)$, is a simplicial complex, called the *independence complex* of G.

3. Asymptotic maximal generating degree and regularity

Let I be a monomial ideal of $R = k[x_1, \ldots, x_r]$ and let

$$I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_t$$

be a minimal primary decomposition of I, where $Q_1, \ldots, Q_s$ are the components associated to the minimal primes of I. By Lemma 2.5 we have

$$I^{(n)} = Q_1^n \cap Q_2^n \cap \cdots \cap Q_s^n.$$

Recall that

$$\mathcal{SP}_n(I) = NP(Q_1^n) \cap NP(Q_2^n) \cap \cdots \cap NP(Q_s^n) = n\,\mathcal{SP}(I),$$

and

$$J_n = \overline{Q_1^n} \cap \overline{Q_2^n} \cap \cdots \cap \overline{Q_s^n}.$$

Observe that $x^{\boldsymbol{\alpha}} \in J_n$ if and only if $\boldsymbol{\alpha} \in \mathcal{SP}_n(I) \cap \mathbb{N}^r$. We note two simple facts.

Remark 3.1. Let J be a monomial ideal and $x^{\boldsymbol{\alpha}} \in J$, with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$. Then $x^{\boldsymbol{\alpha}} \in \mathcal{G}(J)$ if and only if for every i with $\alpha_i \geq 1$, we have $x^{\boldsymbol{\alpha} - \mathbf{e}_i} \notin J$.

Lemma 3.2. *Let J be a monomial ideal and $x^{\boldsymbol{\alpha}} \in J$. For $i = 1, \ldots, r$, let $m_i \in \mathbb{N}$ be an integer such that $x^{\boldsymbol{\alpha} - m_i \mathbf{e}_i} \notin J$ if $\alpha_i \geqslant m_i$. Then there are integers $0 \leqslant n_i \leqslant m_i - 1$ such that $x^{\boldsymbol{\alpha} - (n_1 \mathbf{e}_1 + \cdots + n_r \mathbf{e}_r)} \in \mathcal{G}(J)$.*

Proof. Just choose $0 \leqslant n_i < m_i$ for $i = 1, \dots, r$ such that
$$x^{\boldsymbol{\alpha}-(n_1\mathbf{e}_1+\cdots+n_r\mathbf{e}_r)} \in J$$
and $n_1 + \cdots + n_r$ is as large as possible. $\square$

The first main result of this paper is

Theorem 3.3. *There is an equality* $\lim_{n\to\infty} \dfrac{d(I^{(n)})}{n} = \delta(I)$.

In fact, setting $\rho = r^2 d(I)^{r-1}$, we will prove that for all $n \geqslant 1$, the followings bounds for $d(I^{(n)})$ hold:

$$\delta(I)n - r\rho(1 + s(r-1)d(I)) \leqslant d(I^{(n)}) \leqslant \delta(I)n + r + r(r-1)d(I). \tag{3.1}$$

This clearly implies the conclusion of Theorem 3.3.

For the upper bound, we need the following auxiliary statements.

Lemma 3.4. *Let $x^{\boldsymbol{\alpha}} \in I^{(n)}$ be a monomial. Assume that for some $1 \leq i \leq r$, we have $x^{\alpha-\mathbf{e}_i} \notin I^{(n)}$. Denote $m = (r-1)d(I) + 1$. If $\alpha_i \geqslant m$, then $x^{\boldsymbol{\alpha}-m\mathbf{e}_i} \notin J_n$.*

Proof. Since $x^{\alpha-\mathbf{e}_i} \notin I^{(n)}$, $x^{\alpha-\mathbf{e}_i} \notin Q_j^n$ for some $1 \leqslant j \leqslant s$. By [36, Theorem 7.58], we have $\overline{Q_j^n} = Q_j^{n-p}\overline{Q_j^p}$ for some $0 \leqslant p \leqslant r-1$.

Since $x^{\boldsymbol{\alpha}} \in Q_j^n$ and $x^{\boldsymbol{\alpha}-\mathbf{e}_i} \notin Q_j^n$, it follows that x_i divides some generator of Q_j^n. As the monomial ideal Q_j is primary, $x_i^{d(Q_j)} \in Q_j$. In particular, $x_i^{d(I)} \in Q_j$ because $d(Q_j) \leqslant d(I)$.

We now assume on the contrary that $x^{\boldsymbol{\alpha}-m\mathbf{e}_i} \in J_n$. Then $x^{\boldsymbol{\alpha}-m\mathbf{e}_i} \in \overline{Q_j^n}$. Since $\overline{Q_j^n} = Q_j^{n-p}\overline{Q_j^p}$, there are two monomials $m_1 \in Q_j^{n-p}$ and $m_2 \in \overline{Q_j^p}$ such that $x^{\boldsymbol{\alpha}-m\mathbf{e}_i} = m_1 m_2$. It follows that $x^{\boldsymbol{\alpha}-\mathbf{e}_i} = (m_1 x_i^{m-1})m_2$. Observe that $x_i^{m-1} \in Q_j^{r-1}$ as $m-1 = (r-1)d(I)$. Thus $x^{\boldsymbol{\alpha}-\mathbf{e}_i} = (m_1 x_i^{m-1})m_2 \in Q_j^{n-p}Q_j^{r-1} \subseteq Q_j^n$, a contradiction. The lemma follows. $\square$

Lemma 3.5. *There is an inequality $d(J_n) < \delta(I)n + r$.*

Proof. Let $x^{\boldsymbol{\alpha}} \in \mathcal{G}(J_n)$, $\mathbf{v}_1, \dots, \mathbf{v}_d$ be all the vertices of $\mathcal{SP}(I)$. By Lemma 2.8, we can represent $\boldsymbol{\alpha}$ as
$$\boldsymbol{\alpha} = n(\lambda_1\mathbf{v}_1 + \cdots + \lambda_d\mathbf{v}_d) + \mathbf{u}$$
where $\lambda_i \geqslant 0$, $\lambda_1 + \cdots + \lambda_d = 1$, and $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{R}_+^r$.

Since $x^{\boldsymbol{\alpha}}$ is a minimal generator of J_n, necessarily $u_i < 1$ for every i. Therefore,
$$|\boldsymbol{\alpha}| \leqslant \delta(I)n + (u_1 + \cdots + u_r) < \delta(I)n + r.$$
It follows that $d(J_n) < \delta(I)n + r$, as required. $\square$

Now we are ready for the

Proof of the inequality on the right of (3.1). Let $x^{\boldsymbol{\alpha}}$ be a minimal generator of $I^{(n)}$. By Remark 3.1 we have $x^{\boldsymbol{\alpha}-\mathbf{e}_i} \notin I^{(n)}$ for each $i = 1, \dots, r$, whenever $\alpha_i \geqslant 1$. For $1 \leq i \leq r$, set $m_i = (r-1)d(I) + 1$. By Lemma 3.4, $x^{\boldsymbol{\alpha}-m_i\mathbf{e}_i} \notin J_n$ if $\alpha_i \geqslant (r-1)d(I) + 1$.

By Lemma 3.2, there are integers $0 \leqslant n_i \leqslant (r-1)d(I)$ such that the monomial $x^{\boldsymbol{\alpha}-(n_1\mathbf{e}_1+\cdots+n_r\mathbf{e}_r)}$ is a minimal generator of J_n. Thus
$$d(J_n) \geqslant |\boldsymbol{\alpha}| - (n_1 + \cdots + n_r) \geqslant |\boldsymbol{\alpha}| - r(r-1)d(I),$$

and hence $|\boldsymbol{\alpha}| \leqslant d(J_n)+r(r-1)d(I)$. It follows that $d(I^{(n)}) \leqslant d(J_n)+r(r-1)d(I)$. Together with Lemma 3.5, we obtain

$$d(I^{(n)}) \leqslant \delta(I)n + r + r(r-1)d(I).$$

This is the desired inequality. □

For the remaining inequality in (3.1), we will make some use of Lemma 2.10.

Proof of the inequality on the left of (3.1). Let $\mathbf{v} = (v_1, \ldots, v_r)$ be a vertex of the polyhedron $\mathcal{SP}(I)$ such that $\delta(I) = |\mathbf{v}|$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ where $\alpha_i = \lceil nv_i \rceil$. Because $n\mathbf{v}$ is a vertex of $\mathcal{SP}_n(I)$, $x^{\boldsymbol{\alpha}} \in J_n$.

For each $i = 1, \ldots, s$, we have $x^{\boldsymbol{\alpha}} \in \overline{Q_i^n} = Q_i^{n-(r-1)}\overline{Q_i^{r-1}}$ by [36, Theorem 7.58], so we can write

$$x^{\boldsymbol{\alpha}} = m_1 m_2 m_3$$

where $m_1 \in Q_i$, $m_2 \in Q_i^{n-r}$ and $m_3 \in \overline{Q_i^{r-1}}$. Let $f_i = m_1^{r-1}$ so that $\deg(f_i) \leqslant (r-1)d(Q_i) \leqslant (r-1)d(I)$. We have $x^{\boldsymbol{\alpha}} f_i = (m_1^r m_2)m_3 \in Q_i^n$.

Let $x^{\boldsymbol{\beta}} = f_1 \cdots f_s$ and $x^{\boldsymbol{\gamma}} = x^{\boldsymbol{\alpha}} x^{\boldsymbol{\beta}}$. Then $x^{\boldsymbol{\gamma}} \in Q_i^n$ for all i, consequently $x^{\boldsymbol{\alpha}} x^{\boldsymbol{\beta}} \in I^{(n)}$. Moreover, $\gamma_i = 0$ if and only if $\alpha_i = 0$, if and only if $v_i = 0$. Note that $|\boldsymbol{\beta}| = \deg(f_1) + \cdots + \deg(f_s) \leqslant s(r-1)d(I)$.

By Lemma 2.10, the convex polyhedron $\mathcal{SP}(I)$ is the solutions in $\mathbb{R}^r$ of a system of linear inequalities of the form

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle \geqslant b_j,\ j = 1, 2, \ldots, q\},$$

such that:

(1) each equation $\langle \mathbf{a}_j, \mathbf{x}\rangle = b_j$ defines a facets of $\mathcal{SP}(I)$,
(2) $\mathbf{a}_j \in \mathbb{N}^r$, $b_j \in \mathbb{N}$, and,
(3) $|\mathbf{a}_j| \leqslant r^2 d(I)^{r-1}$ for any j.

Let $\rho = r^2 d(I)^{r-1}$ so that $|\mathbf{a}_j| \leqslant \rho$ for every $j = 1, \ldots, q$.

Since $\mathbf{v}$ is a vertex of $\mathcal{SP}(I)$, by [33, Formula 23 in Page 104], we may assume that $\mathbf{v}$ is the unique solution of the following system

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_i, \mathbf{x}\rangle = b_i, i = 1, \ldots, r\}.$$

For an index i with $\gamma_i \geqslant 1$, since the last system has a unique solution, we deduce that $\mathbf{a}_{j,i} \neq 0$ for some $1 \leqslant j \leqslant r$. For simplicity, we denote $\mathbf{a} = \mathbf{a}_j = (a_1, \ldots, a_r)$ so that $a_i \geqslant 1$.

Let $m = \rho(1 + s(r-1)d(I)) + 1$. If $\gamma_i \geqslant m$, we have

$$\begin{aligned}\langle \mathbf{a}, \boldsymbol{\gamma} - m\mathbf{e}_i\rangle &= \langle \mathbf{a}, \boldsymbol{\alpha}\rangle + \langle \mathbf{a}, \boldsymbol{\beta}\rangle - a_i m \leqslant \langle \mathbf{a}, n\mathbf{v} + \mathbf{e}_1 + \cdots + \mathbf{e}_r\rangle + \langle \mathbf{a}, \boldsymbol{\beta}\rangle - a_i m \\ &= \langle \mathbf{a}, n\mathbf{v}\rangle + |\mathbf{a}| + \langle \mathbf{a}, \boldsymbol{\beta}\rangle - a_i m = nb_j + |\mathbf{a}| + \langle \mathbf{a}, \boldsymbol{\beta}\rangle - a_i m \\ &\leqslant nb_j + |\mathbf{a}| + |\mathbf{a}||\boldsymbol{\beta}| - m < nb_j\end{aligned}$$

since $m = \rho(1 + s(r-1)d(I)) + 1 > |\mathbf{a}| + |\mathbf{a}||\boldsymbol{\beta}|$. Consequently, $x^{\boldsymbol{\gamma} - m\mathbf{e}_i} \notin J_n$, and hence $x^{\boldsymbol{\gamma} - m\mathbf{e}_i} \notin I^{(n)}$.

By Lemma 3.2, there are non-negative integers $n_i \leqslant \rho(1 + s(r-1)d(I))$ for $i = 1, \ldots, r$ such that $x^{\boldsymbol{\gamma} - (n_1\mathbf{e}_1 + \cdots + n_r\mathbf{e}_r)}$ is a minimal generator of $I^{(n)}$. Therefore

$$\begin{aligned}d(I^{(n)}) &\geqslant |\boldsymbol{\gamma}| - (n_1 + \cdots + n_r) \geqslant |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| - r\rho(1 + s(r-1)d(I)) \\ &\geqslant |\boldsymbol{\alpha}| - r\rho(1 + s(r-1)d(I)) \geqslant |n\mathbf{v}| - r\rho(1 + s(r-1)d(I)) \\ &= \delta(I)n - r\rho(1 + s(r-1)d(I)).\end{aligned}$$

This finishes the proof of (3.1) and hence that of Theorem 3.3. □

The second main result of this paper is

Theorem 3.6. *There is an equality* $\lim_{n\to\infty} \dfrac{\operatorname{reg}(I^{(n)})}{n} = \delta(I)$.

Recall that for any finitely generated graded R-module M, and for any $i \geq 0$, we have the notation

$$t_i(M) = \sup\{j : \operatorname{Tor}_i^R(k, M)_j \neq 0\}.$$

From Theorem 3.3 and the fact that $d(M) \leq \operatorname{reg} M$, we see that Theorem 3.6 will follow from a suitable linear upper bound for $\operatorname{reg} I^{(n)}$. This is accomplished by

Lemma 3.7. *For all $i \geqslant 0$, there is an inequality*

$$t_i(I^{(n)}) \leqslant \delta(I)n + 2r + r(r^2 d(I)^{r-1} + (r-1)d(I)).$$

Proof. By Lemma 2.10, the convex polyhedron $\mathcal{SP}(I)$ is the solutions in $\mathbb{R}^r$ of a system of linear inequalities of the form

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle \geqslant b_j,\ j = 1, 2, \dots, q\},$$

where for each j, the equation $\langle \mathbf{a}_j, \mathbf{x}\rangle = b_j$ defines a facets of $\mathcal{SP}(I)$, $\mathbf{a}_j \in \mathbb{N}^r$, $b_j \in \mathbb{N}$, and $|\mathbf{a}_j| \leqslant r^2 d(I)^{r-1}$.

Let $\rho = r^2 d(I)^{r-1}$ so that $|\mathbf{a}_j| \leqslant \rho$ for every j.

Take $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ such that $\beta_{i,\boldsymbol{\alpha}}(I_\Delta^{(n)}) \neq 0$. Since $\beta_{i,\boldsymbol{\alpha}}(I_\Delta^{(n)}) = \dim_k \widetilde{H}_{i-1}(K^{\boldsymbol{\alpha}}(I_\Delta^{(n)}); k) \neq 0$ by Lemma 2.7, we have $K^{\boldsymbol{\alpha}}(I_\Delta^{(n)})$ is not a cone. Hence, for each $j = 1, \dots, r$, we have $j \notin \tau$ for some $\tau \in \mathcal{F}(K^{\boldsymbol{\alpha}}(I_\Delta^{(n)}))$.

Since $\tau \cup \{j\} \notin K^{\boldsymbol{\alpha}}(I_\Delta^{(n)})$, we have $x^{\boldsymbol{\alpha}-\tau-\mathbf{e}_j} \notin I^{(n)}$. Let $m = (r-1)d(I) + 1$.

Claim: If $\alpha_j \geqslant \rho + m$, then $x^{\boldsymbol{\alpha}-(\rho+m)\mathbf{e}_j} \notin J_n$.

Indeed, by Lemma 3.4, $x^{\alpha-\tau-m\mathbf{e}_j} \notin J_n$. Therefore, $\langle \mathbf{a}_i, \boldsymbol{\alpha} - \tau - m\mathbf{e}_j\rangle < nb_i$ for some $1 \leqslant i \leqslant q$. Since $x^{\boldsymbol{\alpha}-\tau} \in I^{(n)} \subseteq J_n$, we have $\langle \mathbf{a}_i, \boldsymbol{\alpha} - \tau\rangle \geqslant nb_i$. It follows that $\mathbf{a}_{i,j} \geqslant 1$. Thus

$$\begin{aligned}
\langle \mathbf{a}_i, \boldsymbol{\alpha} - (\rho+m)\mathbf{e}_j\rangle &= \langle \mathbf{a}_i, \boldsymbol{\alpha} - \tau - m\mathbf{e}_j\rangle + \langle \mathbf{a}_i, \tau - \rho\mathbf{e}_j\rangle < nb_j + \langle \mathbf{a}_i, \tau - \rho\mathbf{e}_j\rangle \\
&= nb_j + \langle \mathbf{a}_i, \tau\rangle - \langle \mathbf{a}_i, \rho\mathbf{e}_j\rangle = nb_j + \langle \mathbf{a}_i, \tau\rangle - \mathbf{a}_{i,j}\rho \\
&\leqslant nb_j + \langle \mathbf{a}_i, \tau\rangle - \rho \leqslant nb_j.
\end{aligned}$$

The last inequality holds since $\rho \geqslant |\mathbf{a}_i| \geqslant \langle \mathbf{a}_i, \tau\rangle$. Consequently, $x^{\boldsymbol{\alpha}-(\rho+m)\mathbf{e}_j} \notin J_n$, as desired.

By Lemma 3.4, there are integers $0 \leqslant n_i \leqslant \rho + m - 1$ for $i = 1, \dots, r$, for which

$$x^{\boldsymbol{\alpha}-\tau-n_1\mathbf{e}_1-\cdots-n_r\mathbf{e}_r}$$

is a minimal generator of J_n. It follows that

$$d(J_n) \geqslant |\boldsymbol{\alpha}| - |\tau| - (n_1 + \cdots + n_r) \geqslant |\boldsymbol{\alpha}| - r - r(\rho + m - 1),$$

and hence

$$|\boldsymbol{\alpha}| \leqslant d(J_n) + r + r(\rho + m - 1) = d(J_n) + r + r(r^2 d(I)^{r-1} + (r-1)d(I)).$$

Together with Lemma 3.5, this yields

$$\begin{aligned}
t_i(I^{(n)}) &\leqslant d(J_n) + r + r\left(r^2 d(I)^{r-1} + (r-1)d(I)\right) \\
&\leqslant \delta(I)n + 2r + r\left(r^2 d(I)^{r-1} + (r-1)d(I)\right),
\end{aligned}$$

and the proof is complete. $\square$

Proof of Theorem 3.6. By Lemma 3.7 we have

$$\operatorname{reg} I^{(n)} = \max\{t_i(I^{(n)}) - i \mid i \geqslant 0\} \leqslant \delta(I)n + 2r + r(r^2 d(I)^{r-1} + (r-1)d(I)).$$

On the other hand, by the proof of Theorem 3.3 (more precisely (3.1)), there exists $c \in \mathbb{R}$ such that

$$d(I^{(n)}) \geqslant \delta(I)n + c \text{ for all } n \geqslant 1.$$

In particular, $\operatorname{reg} I^{(n)} \geqslant d(I^{(n)}) \geqslant \delta(I)n + c$ for all $n \geqslant 1$. Thus,

$$\delta(I)n + c \leqslant \operatorname{reg} I^{(n)} \leqslant \delta(I)n + 2r + r(r^2 d(I)^{r-1} + (r-1)d(I))$$

for all $n \geqslant 1$. It follows that

$$\lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n} = \delta(I),$$

as required. □

Remark 3.8. Although the limits $\lim_{n\to\infty} \dfrac{d(I^{(n)})}{n}$ and $\lim_{n\to\infty} \dfrac{\operatorname{reg}(I^{(n)})}{n}$ do exist, it is not true that the limit

$$\lim_{n\to\infty} \frac{t_i(I^{(n)})}{n}$$

exists for all $i \geq 0$.

Example 3.9. In the polynomial ring $R = \mathbb{Q}[x, y, z, u, v]$, consider the ideal I with the primary decomposition

$$I = (x^2, y^2, z^2)^2 \cap (x^3, y^3, u) \cap (z, v).$$

By [27, Lemma 4.2] we have

$$\operatorname{depth}(R/I^{(n)}) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

From the Auslander-Buchsbaum formula, we get

$$\operatorname{pd} I^{(n)} = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

In particular, $t_3(I^{(n)}) = -\infty$ if n is even, and $t_3(I^{(n)}) > 0$ if n is odd. Since $t_3(I^{(n)})$ is a quasi-linear function in n for $n \gg 0$, we deduce that

$$\liminf_{s\to\infty} \frac{t_3(I^{(2s+1)})}{2s+1} \geqslant 0.$$

So the limit $\lim_{n\to\infty} \dfrac{t_3(I^{(n)})}{n}$ does not exit.

4. Cover ideals

In this section we investigate the symbolic powers of cover ideals of graphs. Our main results in this section are:

(1) Theorem 4.6, which determines explicitly the invariant $\delta(J(G))$ in terms of the combinatorial data of G;
(2) Theorem 4.9, which computes the maximal generating degrees of the symbolic powers of $J(G)$.

Combining these results with a result on the Koszul properties of the symbolic powers of some cover ideals, we construct in Theorem 5.15 a family of graphs G for which both $\operatorname{reg} J(G)^{(n)}$ and $d(J(G)^{(n)})$ are not eventually linear function of n.

Let Δ be a simplicial complex on the vertex set $\{1,\dots,r\}$ and $n \geqslant 1$. We first describe $\mathcal{SP}_n(I_\Delta)$ in a more specific way. For $F \in \mathcal{F}(\Delta)$, $NP(P_F^n)$ is defined by the system

$$\sum_{i\notin F} x_i \geqslant n, x_1 \geqslant 0, \dots, x_r \geqslant 0,$$

so that $\mathcal{SP}_n(I_\Delta)$ is determined by the following system of inequalities:

$$\begin{cases} \sum_{i\notin F} x_i \geqslant n, & \text{for } F \in \mathcal{F}(\Delta), \\ x_1 \geqslant 0, \dots, x_r \geqslant 0. \end{cases} \tag{4.1}$$

From this, one has

Remark 4.1. Let $x^{\boldsymbol{\alpha}} \in I_\Delta^{(n)}$ be a monomial. The following are equivalent:

(1) $x^{\boldsymbol{\alpha}} \in \mathcal{G}(I_\Delta^{(n)})$;
(2) for every i such that $\alpha_i \geqslant 1$, we have $x^{\boldsymbol{\alpha}-\mathbf{e}_i} \notin I_\Delta^{(n)}$,
(3) for every i such that $\langle \boldsymbol{\alpha}, \mathbf{e}_i\rangle \geqslant 1$, there exists $F \in \mathcal{F}(\Delta)$ such that $i \notin F$ and $\left\langle \boldsymbol{\alpha}, \sum_{j\notin F} \mathbf{e}_j \right\rangle = n$.

The following lemma is a consequence of the last remark.

Lemma 4.2. *Let $p \geqslant 1$, $m_1,\dots,m_p \geqslant 0$ be integers and $x^{\boldsymbol{\alpha}_j} \in I_\Delta^{(m_j)}$ be monomials for $j = 1,\dots,p$. Assume that $x^{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_p} \in I_\Delta^{(m_1)} \cdots I_\Delta^{(m_p)} \subseteq I_\Delta^{(m_1+\cdots+m_p)}$ is a minimal generator of $I_\Delta^{(m_1+\cdots+m_p)}$. Then for all $n_1,\dots,n_p \geqslant 0$, $x^{n_1\boldsymbol{\alpha}_1+\cdots+n_p\boldsymbol{\alpha}_p}$ is a minimal generator of $I_\Delta^{(m_1n_1+\cdots+m_pn_p)}$.*

In particular:

(i) *For every subset $W \subseteq [p]$, $x^{\sum_{i\in W}\boldsymbol{\alpha}_i}$ is a minimal generator of $I_\Delta^{(\sum_{i\in W} m_i)}$.*
(ii) *If $x^{\boldsymbol{\alpha}} \in \mathcal{G}(I_\Delta)$ then $x^{n\boldsymbol{\alpha}} \in \mathcal{G}(I_\Delta^{(n)})$ for all $n \geqslant 1$.*

Proof. We claim that for every $1 \leqslant i \leqslant p$, if $m_i = 0$ then $\boldsymbol{\alpha}_i = \mathbf{0}$. Indeed, for example, assume $m_p = 0$ and $\boldsymbol{\alpha}_p \neq \mathbf{0}$. Then

$$x^{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_p} = x^{\boldsymbol{\alpha}_p} x^{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_{p-1}} \notin \mathcal{G}(I_\Delta^{(m_1+\cdots+m_{p-1})}) = \mathcal{G}(I_\Delta^{(m_1+\cdots+m_p)}),$$

a contradiction. Hence the claim is true. In view of the desired conclusion, we can assume that $m_i \geqslant 1$ for all $i = 1,\dots,p$.

Take arbitrary i such that $\langle n_1\boldsymbol{\alpha}_1+\cdots+n_p\boldsymbol{\alpha}_p, \mathbf{e}_i\rangle \geqslant 1$. Then $\langle \boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_p, \mathbf{e}_i\rangle \geqslant 1$. Since $x^{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_p} \in \mathcal{G}(I_\Delta^{(m_1+\cdots+m_p)})$, by Remark 4.1, there exists $F \in \mathcal{F}(\Delta)$ such that $i \notin F$ and

$$\left\langle \boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_p, \sum_{j\notin F} \mathbf{e}_j \right\rangle = m_1+\cdots+m_p.$$

For all $u = 1,\dots,p$, since $x^{\boldsymbol{\alpha}_u} \in I_\Delta^{(m_u)}$,

$$\left\langle \boldsymbol{\alpha}_u, \sum_{j\notin F} \mathbf{e}_j \right\rangle \geqslant m_u.$$

Thus the equality actually happens for all $u = 1, \dots, p$. This implies that

$$\left\langle n_1\boldsymbol{\alpha}_1 + \cdots + n_p\boldsymbol{\alpha}_p, \sum_{j\notin F} \mathbf{e}_j \right\rangle = m_1n_1 + \cdots + m_pn_p.$$

Hence by Remark 4.1, $x^{n_1\boldsymbol{\alpha}_1+\cdots+n_p\boldsymbol{\alpha}_p}$ is a minimal generator of $I_\Delta^{(m_1n_1+\cdots+m_pn_p)}$. The proof is concluded. □

Lemma 4.3. *For all $n \geqslant 1$, there is an inequality $d(I_\Delta^{(n)}) \leqslant \delta(I_\Delta)n$.*

Proof. For simplicity, denote $\delta = \delta(I_\Delta)$. Let $x^{\boldsymbol{\alpha}}$ be a minimal generator of $I_\Delta^{(n)}$. We may assume that $\alpha_i \geqslant 1$ for $i = 1, \dots, p$ and $\alpha_i = 0$ for $i = p+1, \dots, r$ for some $1 \leqslant p \leqslant r$.

For each $i = 1, \dots, p$, there is a facet $F_i \in \mathcal{F}(\Delta)$ which does not contain i such that $\boldsymbol{\alpha}$ lies in the hyperplane $\sum_{j\notin F_i} x_j = n$. From the system (4.1) we imply that the intersection of $\mathcal{SP}_n(I_\Delta)$ with the set

$$\begin{cases} \sum_{j\notin F_i} x_j = n & \text{for } i = 1, \dots, p, \\ x_s = 0 & \text{for } s = p+1, \dots, r, \end{cases}$$

is a compact face of $\mathcal{SP}_n(I_\Delta)$.

Since $\boldsymbol{\alpha}$ belongs to this face, there is a vertex $\boldsymbol{\gamma}$ of $\mathcal{SP}_n(I_\Delta)$ lying on this face such that $|\boldsymbol{\alpha}| \leqslant |\boldsymbol{\gamma}|$. As $\boldsymbol{\gamma}/n$ is a vertex of $\mathcal{SP}(I_\Delta)$, $|\boldsymbol{\alpha}| \leqslant |\boldsymbol{\gamma}| = |\boldsymbol{\gamma}/n| \cdot n \leqslant \delta n$. The conclusion follows. □

Example 4.4. Let G be a graph on the vertex set $\{1, \dots, r\}$. Let

$$I(G) = (x_ix_j \mid \{i,j\} \in E(G)) \subseteq k[x_1, \dots, x_r]$$

be the edge ideal of G. Then $d(I(G)^{(n)}) = 2n$ for all $n \geqslant 1$.

Indeed, for any $n \geqslant 1$ we have $d(I(G)^{(n)}) \leqslant 2n$ by [2, Corollary 2.11]. On the other hand, if x_ix_j is a minimal generator of $I(G)$, then $(x_ix_j)^n$ is a minimal generator of $I(G)^{(n)}$, and so $d(I(G)^{(n)}) \geqslant 2n$. Hence, $d(I(G)^{(n)}) = 2n$.

Of course, $I(G)^{(n)}$ need not be generated in degree $2n$. For example, if $I(G) = (xy, xz, yz)$ then

$$I(G)^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 = (x^2y^2, x^2z^2, y^2z^2, xyz).$$

We do not know whether for any graph G, $\operatorname{reg} I(G)^{(n)}$ is asymptotically linear in n. This is the case when G is a cycle (see [13, Corollary 5.4]).

Let G be a graph on $[r] = \{1, \dots, r\}$. Then the polyhedron $\mathcal{SP}(J(G))$ is defined by the following system of inequalities:

$$\begin{cases} x_i + x_j \geqslant 1, \text{ for } \{i,j\} \in E(G), \\ x_1 \geqslant 0, \dots, x_r \geqslant 0. \end{cases} \tag{4.2}$$

The following lemma is quite useful to identify the vertices of $\mathcal{SP}(J(G))$.

Lemma 4.5. *Let G be a graph on $[r]$ with no isolated vertex, and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$. Assume that $\boldsymbol{\alpha}$ is a vertex of $\mathcal{SP}(J(G))$. Then $\alpha_i \in \{0, 1/2, 1\}$ for every $i = 1, \dots, r$. Denote $S_0 = \{i : \alpha_i = 0\}$, $S_1 = \{i : \alpha_i = 1\}$ and $S_{1/2} = \{i : \alpha_i = 1/2\}$. Then the following statements hold:*

(i) *S_0 is an independent set of G.*
(ii) *$S_1 = N(S_0)$.*

(iii) *The induced subgraph of G on $S_{1/2}$ has no bipartite component.*
(iv) *If v is a leaf not lying in S_0 and $N(v) = \{u\}$ then $u \notin S_1$.*

Proof. Since $\boldsymbol{\alpha}$ is a vertex of $\mathcal{SP}(J(G))$, by [33, Formula (23), Page 104], $\boldsymbol{\alpha}$ is the unique solution of a system

$$\begin{cases} x_i + x_j = 1, \text{ for } \{i,j\} \in E_1, \\ x_i = 0, \text{ for } i \in V_1, \end{cases} \tag{4.3}$$

of exactly r linearly independent equations, where $E_1 \subseteq E(G)$ and $V_1 \subseteq \{1, \dots, r\}$ with $|E_1| + |V_1| = r$.

Step 1: Let H be the subgraph of G with the same vertex set and $E(H) = E_1$. Let $H_1, \dots, H_s$ be connected components of H. Assume that $V(H_i) \cap V_1 \neq \emptyset$ for $i = 1, \dots, t$; and $V(H_i) \cap V_1 = \emptyset$ for $i = t+1, \dots, s$ for some $0 \leqslant t \leqslant s$. We show that $\alpha_j \in \{0, 1\}$ if $j \in \bigcup_{i=1}^t V(H_i)$ and $\alpha_j = 1/2$ if $j \in \bigcup_{i=t+1}^s V(H_i)$.

For each $i \in \{1, \dots, t\}$ and each $j \in H_i$, we take $p \in V(H_i) \cap S$. Then $\alpha_p = 0$ by the assumption. Since H_i is connected, there is a path from p to j in H_i, say

$$p = j_0, j_1, \dots, j_m = j.$$

Since $\alpha_{j_u} + \alpha_{j_{u+1}} = 1$ for $u = 0, \dots, m-1$, we deduce that $\alpha_{j_m} = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$

For each $u = t+1, \dots, s$, from the above discussion, the system

$$\begin{cases} x_i + x_j = 1, \\ \{i,j\} \in E(H_u), \end{cases} \tag{4.4}$$

also has a unique solution. As $V(H_u) \cap V_1 = \emptyset$, H_u cannot be an isolated vertex, so $E(H_u) \neq \emptyset$. Since $x_i = 1/2$ for all $i \in V(H_u)$ is a solution of the last system, it is the unique one. Hence we see that $\alpha_i \in \{0, 1, 1/2\}$ for all i.

Step 2: If there are adjacent vertices $i, j \in S_0$ then as $\boldsymbol{\alpha} \in \mathcal{SP}(J(G))$, we get $0 = \alpha_i + \alpha_j \geq 1$. This is a contradiction. Hence S_0 is an independent set, proving (i).

Step 3: Similarly there can be no edge connecting any $i \in S_0$ with some $j \in S_{1/2}$. Hence $N(S_0) \subseteq S_1$.

Now assume that S_1 has a vertex, say i, that is not adjacent to any vertex in S_0. Then $\boldsymbol{\gamma} = \boldsymbol{\alpha} - \frac{1}{2}\mathbf{e}_i$ is a point of $\mathcal{SP}(J(G))$. On the other hand, $\boldsymbol{\alpha} + \frac{1}{2}\mathbf{e}_i$ is obviously a point of $\mathcal{SP}(J(G))$. Hence we have a convex decomposition

$$\boldsymbol{\alpha} = \frac{1}{2}(\boldsymbol{\alpha} - \mathbf{e}_i/2) + \frac{1}{2}(\boldsymbol{\alpha} + \mathbf{e}_i/2),$$

contradicting the fact that $\boldsymbol{\alpha}$ is a vertex of $\mathcal{SP}(J(G))$. Thus, as G has no isolated vertex, every vertex in S_1 is adjacent to one in S_0, and thus $S_1 \subseteq N(S_0)$. In particular, $S_1 = N(S_0)$, proving (ii).

Step 4: Next we show (iii). Assume the contrary, the induced subgraph of G on $S_{1/2}$ has a bipartite component G_1. Let (A, B) be the bipartition of G_1. Construct the vectors $\boldsymbol{\alpha}', \boldsymbol{\alpha}''$ as follows: $\boldsymbol{\alpha}'_i = \begin{cases} \boldsymbol{\alpha}_i & \text{if } i \notin A \cup B, \\ 0, & \text{if } i \in A, \\ 1, & \text{if } i \in B \end{cases}$ and

$$\boldsymbol{\alpha}''_i = \begin{cases} \boldsymbol{\alpha}_i & \text{if } i \notin A \cup B, \\ 1, & \text{if } i \in A, \\ 0, & \text{if } i \in B. \end{cases}$$

We show that $\boldsymbol{\alpha}', \boldsymbol{\alpha}'' \in \mathcal{SP}(J(G))$. Indeed, take an edge $\{i, j\} \in E(G)$. If neither i nor j belong to $A \cup B$, then $\boldsymbol{\alpha}'_i + \boldsymbol{\alpha}'_j = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j \geq 1$. If exactly one of i and j belongs to $A \cup B$, we can assume that i does. Then $j \in S_1$, since by (ii), $V(G_1) \subseteq S_{1/2} \subseteq V(G) \setminus N(S_0)$. In this case $\boldsymbol{\alpha}'_i + \boldsymbol{\alpha}'_j = \boldsymbol{\alpha}'_i + \boldsymbol{\alpha}_j = 1 + \boldsymbol{\alpha}'_i \geq 1$. If both i and j belong to $A \cup B$, then we can assume that $i \in A, j \in B$, so $\boldsymbol{\alpha}'_i + \boldsymbol{\alpha}'_j = 1$. Hence in any case $\boldsymbol{\alpha}' \in \mathcal{SP}(J(G))$, and the same argument works for $\boldsymbol{\alpha}''$.

But then the convex decomposition $\boldsymbol{\alpha} = (\boldsymbol{\alpha}' + \boldsymbol{\alpha}'')/2$ shows that $\boldsymbol{\alpha}$ is not a vertex of $\mathcal{SP}(J(G))$, a contradiction. Thus (iii) is true.

Step 5: Assume that $u \in S_1$. Since $v \notin S_0$, either $v \in S_1$ or $v \in S_{1/2}$. If $v \in S_1$ then by (ii), $v \in N(S_0)$, a contradiction with v is a leaf and its unique neighbor is $u \in S_1$. Hence $v \in S_{1/2}$. Define the vectors $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$ as follows:

$$\boldsymbol{\alpha}^1_i = \begin{cases} \boldsymbol{\alpha}_i & \text{if } i \neq v, \\ 0, & \text{if } i = v, \end{cases}$$

and

$$\boldsymbol{\alpha}^2_i = \begin{cases} \boldsymbol{\alpha}_i & \text{if } i \neq v, \\ 1, & \text{if } i = v. \end{cases}$$

Since $\boldsymbol{\alpha}^2 \geq \boldsymbol{\alpha}$ componentwise, $\boldsymbol{\alpha}^2 \in \mathcal{SP}(J(G))$. We show that $\boldsymbol{\alpha}^1 \in \mathcal{SP}(J(G))$. Take any edge $\{i, j\} \in E(G)$. If $i \neq v$ and $j \neq v$, then $\boldsymbol{\alpha}^1_i + \boldsymbol{\alpha}^1_j = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j \geq 1$. If say $i = v$, then necessarily $j = u$, and

$$\boldsymbol{\alpha}^1_i + \boldsymbol{\alpha}^1_j = \boldsymbol{\alpha}^1_v + \boldsymbol{\alpha}^1_u = 0 + \boldsymbol{\alpha}_u = 1,$$

noting that $u \in S_1$. Hence $\boldsymbol{\alpha}^1 \in \mathcal{SP}(J(G))$. But then the convex decomposition $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1 + \boldsymbol{\alpha}^2)/2$ shows that $\boldsymbol{\alpha}$ is not a vertex of $\mathcal{SP}(J(G))$, a contradiction. Thus (iv) is true and the proof is concluded. □

The first main result of this section is

Theorem 4.6. *Let G be a graph with on $[r]$ with no isolated vertex, and $J = J(G)$. Then*

(4.5)

$$\delta(J) = \frac{r}{2} + \frac{1}{2} \max\{|N(S)| - |S| \mid S \in \Delta(G) \text{ and } G \setminus N[S] \text{ has no bipartite component}\}.$$

Proof. Let d be the expression in the last line of (4.5).

Step 1: We show that $d \leq \delta(J)$.

Let S be an independent set of G such that $d = r/2 + (|N(S)| - |S|)/2$ and $G \setminus N[S]$ has no bipartite component.

For $i = 1, \dots, r$, define γ_i as follows

$$\gamma_i = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \in N(S), \\ \frac{1}{2} & \text{if } i \in V(G) \setminus N[S]. \end{cases}$$

Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$. Then $\boldsymbol{\gamma}$ is a point of $\mathcal{SP}(J)$. Since $2\boldsymbol{\gamma} \in \mathbb{N}^r$, $x^{2\boldsymbol{\gamma}} \in J(G)^{(2)}$. Observe that $x^{2\boldsymbol{\gamma}}$ is a minimal generator of $J(G)^{(2)}$, since G has no isolated vertex. Hence $|2\boldsymbol{\gamma}| \leqslant 2\delta(J)$ by Lemma 4.3, namely $\delta(J) \geqslant |\boldsymbol{\gamma}| = d$.

Step 2: To prove the reverse inequality, let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ be any vertex of $\mathcal{SP}(J)$. By Lemma 4.5, $\alpha_i \in \{0, 1/2, 1\}$ for every i. Let $S = S_0 = \{i \mid \alpha_i = 0\}$, $S_1 = \{i \mid \alpha_i = 1\}$ and $S_{1/2} = \{i \mid \alpha_i = 1/2\}$. By the same lemma, $S \in \Delta(G)$ and $G \setminus N[S]$ has no bipartite component.

Thus

$$|\boldsymbol{\alpha}| = |S_1| + \frac{|S_{1/2}|}{2} = \frac{|S| + |S_1| + |S_{1/2}|}{2} + \frac{|S_1| - |S|}{2} = \frac{r}{2} + \frac{|N(S)| - |S|}{2} \leqslant d.$$

Choosing the vertex $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = \delta(J)$, we deduce $\delta(J) \leqslant d$, as required. □

For cover ideals, the symbolic Rees algebra is generated in degree at most 2.

Theorem 4.7 (Herzog-Hibi-Trung [17, Theorem 5.1])**.** *Let G be a graph and $J = J(G)$. Then for every $s \geqslant 1$,*

1. *$J^{(2s)} = (J^{(2)})^s$.*
2. *$J^{(2s+1)} = J(J^{(2)})^s$.*

An immediate corollary is

Corollary 4.8. *Let G be a graph. Then $\operatorname{reg} J(G)^{(n)}$ is a quasi-linear function of n of period at most 2 for n large enough.*

Proof. Follows from Theorem 4.7 and [34, Theorem 3.2]. □

The next main result in this section is

Theorem 4.9. *Let G be a graph and $J = J(G)$. Then*

1. *$d(J^{(2s)}) = \delta(J)2s$ for every $s \geqslant 1$.*
2. *There are $m_1 \in \mathcal{G}(J^{(2)})$ and $m_2 \in \mathcal{G}(J)$ such that $m_1 m_2 \in \mathcal{G}(J^{(3)})$. Let e be the maximal degree of such an m_2 (among all of its possible choices). Then*
$$d(J^{(2s+1)}) = \delta(J)2s + e, \quad \textit{for every } s \geqslant d(J) - e.$$
3. *If $\delta(J) = d(J)$ or $\delta(J) = r/2$, then $d(J^{(2s+1)}) = \delta(J)2s + d(J)$ for $s \geqslant 0$.*

Proof. (1) By Lemma 4.3, $d(J^{(2s)}) \leq \delta(J)2s$.

For the reverse inequality, let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ be a vertex of $\mathcal{SP}(J)$ such that $\delta(J) = |\boldsymbol{\alpha}|$. By [33, Formula 23 in Page 104], $\boldsymbol{\alpha}$ is a unique solution of the following system

$$\begin{cases} x_i + x_j = 1, \text{ for } \{i, j\} \in E_1, \\ x_i = 0, \text{ for } i \in V_1, \end{cases}$$

where $E_1 \subseteq E(G)$ and $V_1 \subseteq \{1, \dots, r\}$ with $|E_1| + |V_1| = r$. By Lemma 4.5, $\alpha_i \in \{0, 1/2, 1\}$ for every i.

Since $2s\boldsymbol{\alpha} \in \mathbb{N}^r$, we get $x^{2s\boldsymbol{\alpha}} \in J(G)^{(2s)}$. Note that $2s\boldsymbol{\alpha}$ is a vertex of $\mathcal{SP}_{2s}(J)$, so $x^{2s\boldsymbol{\alpha}}$ is a generator of $J(G)^{(2s)}$. It follows that $d(J^{(2s)}) \geqslant 2s|\boldsymbol{\alpha}| = \delta(J)2s$, as desired.

(2) Let $I = J^{(2)}$. By Theorem 4.7 we have $J^{(2s+1)} = I^s J$. Note that $d(I) = 2\delta(J)$ by part (1) above. Therefore, we can write $I = I_1 + I_2$ where I_2 is generated by elements of $\mathcal{G}(I)$ of degree exactly $2\delta(J)$ and I_1 is generated by the remaining elements.

We first prove following

Claim: $I_2J \nsubseteq I_1J$.

Indeed, if $I_2J \subseteq I_1J$, we will derive a contradiction. Since $IJ = (I_1 + I_2)J = I_1J + I_2J = I_1J$, for every $n \geqslant 1$, from this equality and Theorem 4.7 we get $J^{(2n+1)} = I^nJ = I_1^nJ$. In particular, $d(J^{(2n+1)}) \leqslant d(I_1)n + d(J)$, so $d(I_1) \geqslant 2\delta(J)$ by Theorem 3.3. On the other hand, $d(I_1) < 2\delta(J)$ by the definition of I_1, a contradiction.

We now return to proving part (2). Recall that by Theorem 4.7, $J^{(3)} = J^{(2)}J = IJ$ and $J^{(2s+1)} = I^sJ$. Since $J^{(2s+1)} = I^sJ$, there exist $g \in \mathcal{G}(I^s)$ and $f \in \mathcal{G}(J)$ such that $gf \in \mathcal{G}(J^{(2s+1)})$ and $d(J^{(2s+1)}) = \deg(gf)$.

Represent g as $g = g_1 \cdots g_s$, where each g_i is in I. Since $gf \in \mathcal{G}(J^{(2s+1)})$, we deduce from Lemma 4.2 that $g_i \in \mathcal{G}(J^{(2)}) = \mathcal{G}(I)$ and $g_if \in \mathcal{G}(J^{(3)}) = \mathcal{G}(IJ)$ for any i. By the same lemma, $g_i^sf \in \mathcal{G}(J^{(2s+1)})$ for all i. Assume that $\deg g_1 \leqslant \cdots \leqslant \deg g_s$. Then

$$\begin{aligned} d(J^{(2s+1)}) &= \deg(gf) = \deg g_1 + \cdots + \deg g_s + \deg f \\ &\leqslant s \deg g_s + \deg f \leqslant d(J^{(2s+1)}), \end{aligned}$$

so that $\deg g_1 = \cdots = \deg g_s$ and

$$\deg(gf) = s\deg(g_i) + \deg(f) \quad \text{for } i = 1, \ldots, s.$$

By the claim, there exist $m_1 \in \mathcal{G}(I)$ with $\deg(m_1) = 2\delta(J)$ and $m_2 \in \mathcal{G}(J)$ such that $m_1m_2 \in \mathcal{G}(IJ)$. Let (m_1, m_2) be a such couple such that $e = \deg(m_2)$ is maximal. By Lemma 4.2, $m_1^sm_2 \in \mathcal{G}(J^{(2s+1)})$. In particular,

$$d(J^{(2s+1)}) \geqslant \deg(m_1)s + \deg(m_2) = \delta(J)2s + e. \tag{4.6}$$

It remains to show that the equality occurs whenever $s \geqslant d(J) - e$. Indeed, if $\deg(g_1) = 2\delta(J)$, then we must have $e = \deg(f)$ by the definition of e. In this case, $d(J^{(2s+1)}) = \delta(J)2s + e$.

Assume that $\deg(g_1) < 2\delta(J)$. Since $s \geqslant d(J) - e \geqslant \deg(f) - \deg(m_2)$, we have

$$\begin{aligned} \delta(J)2s + \deg(m_2) &\geqslant (\deg(g_1) + 1)s + \deg(f) - s \\ &= \deg(g_1)s + \deg(f) = d(J^{(2s+1)}), \end{aligned}$$

so thanks to (4.6), $d(J^{(2s+1)}) = \delta(J)2s + e$, as required.

(3) If $\delta(J) = d(J)$, then there exists $m \in \mathcal{G}(J)$ of degree $d(J)$. By Lemma 4.2, $m^{(2s+1)}$ is a minimal generator of $J^{(2s+1)}$, so $d(J^{(2s+1)}) \geqslant \delta(J)(2s+1)$. The reverse inequality follows from Lemma 4.3.

If $\delta(J) = r/2$, then for $\boldsymbol{\alpha} = (1, \ldots, 1) \in \mathbb{N}^r$, $x^{\boldsymbol{\alpha}} \in \mathcal{G}(I)$ and $|\boldsymbol{\alpha}| = 2\delta(J)$. Let $x^{\boldsymbol{\gamma}} \in \mathcal{G}(J)$ be such that $|\boldsymbol{\gamma}| = d(J)$. Then $x^{s\boldsymbol{\alpha}}x^{\boldsymbol{\gamma}} \in \mathcal{G}(J^{(2s+1)})$ by Lemma 4.2 and Theorem 4.7. Therefore, $d(J^{(2s+1)}) \geqslant \delta(J)2s + d(J)$. The reverse inequality follows from the equality $J^{(2s+1)} = (J^{(2)})^sJ$. □

The following example shows that $d(J(G)^{(2n+1)})$ need not be a linear function in n from $n = 0$.

Example 4.10. Let G be a graph with the vertex set

$$\{x_i, y_i, z_i \mid i = 1, \ldots, 5\} \ \cup\ \{u, v, w\}$$

which is depicted in Figure 2. Using the EdgeIdeals package in Macaulay2 [12], the graph G and its cover ideal are given as follows.

```
R=ZZ/32003[x_1..x_5,y_1..y_5,z_1..z_5,u,v,w];
G=graph(R,{x_1*x_2,x_1*x_3,x_1*x_4,x_1*x_5,x_2*x_3,x_2*x_4,
x_2*x_5,x_3*x_4,x_3*x_5,x_4*x_5,x_1*y_1,x_1*z_1,x_2*y_2,x_2*z_2,
x_3*y_3,x_3*z_3,x_4*y_4,x_4*z_4,x_5*y_5,x_5*z_5,x_3*u,x_4*u,y_5*u,
u*v,u*w,v*w});
J=dual edgeIdeal G
```

In particular, G has 18 vertices and 26 edges.

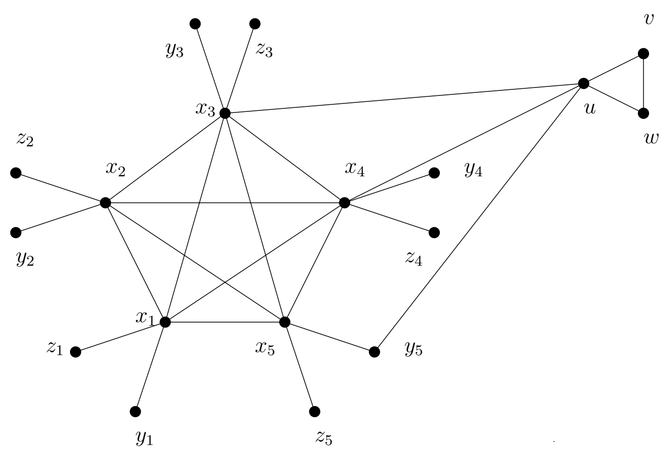

FIGURE 2. The graph G

Let $J = J(G)$. By using Macaulay2 [12] we get

(1) $d(J) = 9$, $d(J^{(2)}) = 19$, and $d(J^{(3)}) = 27$. By Theorem 4.9, $\delta(J) = 19/2$.

(2) The monomials $m_1 = u^2v^2 \prod_{i=1}^5 (x_iy_iz_i) \in \mathcal{G}(J^{(2)})$ and

$$m_2 = x_2x_3x_4x_5y_1z_1uv \in \mathcal{G}(J)$$

satisfy $m_1m_2 \in \mathcal{G}(J^{(3)})$. Note that $\deg(m_1) = 19, \deg(m_2) = 8$.

In the notation of Theorem 4.9, we deduce $8 \le e \le d(J) = 9$. If $e = 9$, then by *ibid.* we have $d(J^{(2n+1)}) = \delta(J)2n + 9 = 19n + 9$ for $n \geqslant d(J) - 9 = 0$. Setting $n = 1$, we get $d(J^{(3)}) = 28$, a contradiction. Hence $e = 8$ and $d(J^{(2n+1)}) = 19n + 8$ if (and only if) $n \geqslant d(J) - 8 = 1$.

5. The Koszul property of symbolic powers of cover ideals

The following result is our main tool in the study of the Koszul property of symbolic powers.

Theorem 5.1. *Let $(R, \mathfrak{m})$ be a standard graded k-algebra. Let x be a non-zero linear form and I', T be non-trivial homogeneous ideals of R such that the following conditions are fulfilled:*

(i) *I' is a Koszul module and x is I'-regular (e.g. x is an R-regular element),*
(ii) *$T \subseteq \mathfrak{m}I'$,*
(iii) *x is a regular element with respect to R/T and $\operatorname{gr}_{\mathfrak{m}} T$.*

Denote $I = xI' + T$. Then the decomposition $I = xI' + T$ is a Betti splitting, and there is a chain

$$\operatorname{ld}_R T \le \operatorname{ld}_R I = \operatorname{ld}_R(T + (x)) = \operatorname{ld}_R(T/xT) \le \operatorname{ld}_R T + 1.$$

Moreover, I is a Koszul module if and only if so is T.

Before proving Theorem 5.1, we recall the following result.

Lemma 5.2 (Nguyen [26, Theorem 3.1])**.** *Let $0 \to M' \to P' \to N' \to 0$ be a short exact sequence of non-zero finitely generated R-modules where*

(i) *M' is a Koszul module;*
(ii) *$M' \cap \mathfrak{m}P' = \mathfrak{m}M'$.*

Then there are inequalities $\operatorname{ld}_R P' \le \operatorname{ld}_R N' \le \max\{\operatorname{ld}_R P', 1\}$. In particular, $\operatorname{ld}_R N' = \operatorname{ld}_R P'$ if $\operatorname{ld}_R P' \ge 1$ and $\operatorname{ld}_R N' \le 1$ if $\operatorname{ld}_R P' = 0$.

Moreover, $\operatorname{ld}_R N' = 0$ if and only if $\operatorname{ld}_R P' = 0$ and for all $s \ge 1$, we have $M' \cap \mathfrak{m}^s P' = \mathfrak{m}^s M$.

We also have an easy observation.

Lemma 5.3. *Let $(R, \mathfrak{m})$ be a standard graded k-algebra, and $x \in \mathfrak{m}$ a non-zero linear form. Let T be a homogeneous ideal of R such that x is (R/T)-regular. Then the following are equivalent:*

(1) *x is $\operatorname{gr}_{\mathfrak{m}} T$-regular,*
(2) *$\mathfrak{m}^s T : x = \mathfrak{m}^{s-1} T$ for all $s \ge 1$.*

Proof. Clearly x is $\operatorname{gr}_{\mathfrak{m}} T$-regular if and only if

$$(\mathfrak{m}^{s+2} T : x) \cap \mathfrak{m}^s T = \mathfrak{m}^{s+1} T, \quad \text{for all } s \ge 0. \tag{5.1}$$

Hence (2) $\Longrightarrow$ (1).

Conversely, assume that (1) is true. Since $\mathfrak{m}^{s-1} T \subseteq \mathfrak{m}^s T : x$, it suffices to show for all $s \ge 1$ that $\mathfrak{m}^s T : x \subseteq \mathfrak{m}^{s-1} T$. Induct on $s \ge 1$.

For $s = 1$,

$$\mathfrak{m} T : x \subseteq T : x = T,$$

where the equality follows from the hypothesis x is (R/T)-regular.

Assume that the statement holds true for $s \ge 1$. Using the induction hypothesis, we have

$$\mathfrak{m}^{s+1} T : x \subseteq (\mathfrak{m}^{s+1} T : x) \cap (\mathfrak{m}^s T : x) \subseteq (\mathfrak{m}^{s+1} T : x) \cap \mathfrak{m}^{s-1} T = \mathfrak{m}^s T.$$

The equality in the chain follows from (5.1). The proof is concluded. □

Proof of Theorem 5.1. We proceed through several steps.

Step 1: First we establish the equalities $\operatorname{ld}_R I = \operatorname{ld}_R T/xT = \operatorname{ld}_R(T + (x))$. Consider the short exact sequence

$$0 \longrightarrow xI' \longrightarrow I \longrightarrow \frac{T}{xI' \cap T} = \frac{T}{xT} \longrightarrow 0.$$

The equality holds since $T : x = T \subseteq I'$.

We claim that

$$xI' \cap \mathfrak{m}^s I = \mathfrak{m}^s xI' \quad \text{for all } s \ge 1. \tag{5.2}$$

The inclusion "$\supseteq$" is clear. For the other inclusion, take $a \in xI' \cap \mathfrak{m}^s I = xI' \cap \mathfrak{m}^s(xI'+T)$. Subtracting to an element in $\mathfrak{m}^s xI'$, we may assume that $a \in xI' \cap \mathfrak{m}^s T$. The last module is contained in

$$(x) \cap \mathfrak{m}^s T = x(\mathfrak{m}^s T : x) \subseteq x\mathfrak{m}^{s-1}T \subseteq x\mathfrak{m}^s I'.$$

In the last chain, the first inclusion follows from Lemma 5.3 and the hypothesis x is $\operatorname{gr}_{\mathfrak{m}} T$-regular. The second inclusion follows from the hypothesis $T \subseteq \mathfrak{m}I'$. Thus the claim follows.

Recall that $xI' \cong I'$ is Koszul by the hypothesis. Hence using Lemma 5.2 for the above exact sequence, and Equality (5.2), we get

$$\operatorname{ld}_R I = \operatorname{ld}_R(T/xT).$$

Arguing similarly as above for the ideal $T + (x) = xR + T$, we have $\operatorname{ld}_R(T + (x)) = \operatorname{ld}_R T/xT$. Hence $\operatorname{ld}_R I = \operatorname{ld}_R(T/xT) = \operatorname{ld}_R(T + (x))$, as claimed.

Step 2: Note that x is T-regular, since it is $\operatorname{gr}_{\mathfrak{m}} T$-regular. Let $K(x;T)$ denote the Koszul complex $0 \to T(-1) \xrightarrow{\cdot x} T \to 0$, then $K(x;T)$ is quasi-isomorphic to T/xT. Hence using (the graded analogue of) a result of Iyengar and Römer [22, Remark 2.12], we obtain

$$\operatorname{ld}_R T \le \operatorname{ld}_R K(x;T) = \operatorname{ld}_R(T/xT) \le \operatorname{ld}_R T + 1.$$

Hence we get the desired chain

$$\operatorname{ld}_R T \le \operatorname{ld}_R I = \operatorname{ld}_R(T/xT) \le \operatorname{ld}_R T + 1.$$

Step 3: For the assertion on Betti splitting, note that $xI' \cap T = xT$. Since x is T-regular, then the morphism $\operatorname{Tor}_i^R(k, xT) \longrightarrow \operatorname{Tor}_i^R(k,T)$ is the multiplication by x of $\operatorname{Tor}_i^R(k,T)$, which is trivial. Since $xT \subseteq \mathfrak{m}xI'$ and xI' is Koszul, the map $\operatorname{Tor}_i^R(k, xT) \longrightarrow \operatorname{Tor}_i^R(k, xI')$ is also trivial thanks to [28, Lemma 4.10(b1)]. Hence by Lemma 2.4, the decomposition $I = xI' + T$ is a Betti splitting.

Step 4: As shown above, $\operatorname{ld}_R T \le \operatorname{ld}_R I$, hence if I is Koszul then so is T. Conversely, assume that T is Koszul. Now x is $\operatorname{gr}_{\mathfrak{m}} T$-regular, so by (the graded analogue of) [22, Theorem 2.13(a)], we deduce that T/xT is also Koszul. It remains to use the equality $\operatorname{ld}_R I = \operatorname{ld}_R T/xT$. The proof is concluded. $\square$

Example 5.4. The following example shows that the condition x is $\operatorname{gr}_{\mathfrak{m}} T$-regular in Theorem 5.1 is critical, even when the base ring is regular.

Let $R = k[a,b,c,d]$, $T = (a,c^2)(b,d^2) = (ab, ad^2, bc^2, c^2d^2)$. Let $x = a - b$, $I = (x) + T$, $\mathfrak{m} = R_+$. We claim that:

(i) x is (R/T)-regular but not $\operatorname{gr}_{\mathfrak{m}} T$-regular,
(ii) T is Koszul but I is not.

(i): We observe that $T : x = T$ because $T = (a,c^2) \cap (b,d^2)$. We also have

$$c^2d^2x = c^2(ad^2) - d^2(bc^2) \in \mathfrak{m}^2T.$$

Hence $c^2d^2 \in (\mathfrak{m}^2T : x) \setminus (\mathfrak{m}T)$, thus x is not $\operatorname{gr}_{\mathfrak{m}} T$-regular.

(ii): Write $T = aJ + L$ where $J = (b,d^2), L = c^2(b,d^2)$. Then J is Koszul, $L \subseteq \mathfrak{m}J$ and $L \cong J(-2)$ is Koszul. Applying Corollary 5.6, T is also Koszul.

We have

$$I = T + (x) = (x) + (a^2, ac^2, ad^2, c^2d^2).$$

Denote $L = (a^2, ac^2, ad^2, c^2d^2) \subseteq S = k[a,c,d]$. Note that $R = S[x]$, so by Corollary 5.6 and Lemma 2.2, $\operatorname{ld}_R I = \operatorname{ld}_R(LR + (x)) = \operatorname{ld}_S L$.

Assume that I is Koszul, then so is L. Denote by $L_{\le s}$ the ideal generated by homogeneous elements of degree at most s of L. Then by [15, Lemma 8.2.11], we also have $L_{\le 3} = (a^2, ac^2, ad^2) \cong (a, c^2, d^2)(-1)$ is Koszul. In particular, by Lemma 2.1, $\operatorname{reg} L_{\le 3} = 3$.

But then $\operatorname{reg}(a, c^2, d^2) = 2$! This contradiction confirms that I is not Koszul.

Remark 5.5. Example 5.4 also shows that even if T is a Koszul ideal in a polynomial ring R, and x is a regular linear form modulo T, the ideal $T + (x)$ need not be Koszul.

Nevertheless, it is not hard to see that this is true if moreover T has a linear resolution. Indeed, in this case $T \cong \operatorname{gr}_{\mathfrak{m}} T$ as R-modules, so x is $\operatorname{gr}_{\mathfrak{m}} T$-regular. Applying Theorem 5.1, we get $\operatorname{ld}_R(T + (x)) = \operatorname{ld}_R T = 0$.

The next consequence of Theorem 5.1 generalizes [28, Lemma 8.2].

Corollary 5.6. *Let $(R, \mathfrak{m})$ be a polynomial ring over k. Let x be a non-zero linear form, I', T be non-trivial homogeneous ideals of R such that the following conditions are satisfied:*

(i) *I' is Koszul,*
(ii) *$T \subseteq \mathfrak{m} I'$,*
(iii) *there exists a polynomial subring S of R such that $R = S[x]$ and T is generated by elements in S.*

Denote $I = xI' + T$. Then the decomposition $I = xI' + T$ is a Betti splitting and $\operatorname{ld}_R I = \operatorname{ld}_R T$.

Proof. First we verify that x, I', and T satisfy the hypotheses of Theorem 5.1. Note that condition (iii) ensures that x is (R/T)-regular. Hence it remains to check that x is $\operatorname{gr}_{\mathfrak{m}} T$-regular. By the proof of Lemma 5.3, we only need to show that for all $s \ge 1$,

$$\mathfrak{m}^s T : x \subseteq \mathfrak{m}^{s-1} T.$$

Take $a \in \mathfrak{m}^s T : x$.

By change of coordinates, we can assume that x is one of the variables. Let $\mathfrak{n}$ be the graded maximal ideal of S extended to R. Then $\mathfrak{m}^s = ((x) + \mathfrak{n})^s = x\mathfrak{m}^{s-1} + \mathfrak{n}^s$, therefore

$$xa \in \mathfrak{m}^s T = x\mathfrak{m}^{s-1} T + \mathfrak{n}^s T.$$

So for some $b \in \mathfrak{m}^{s-1} T$, $x(a - b) \in \mathfrak{n}^s T$, namely

$$a - b \in \mathfrak{n}^s T : x = \mathfrak{n}^s T.$$

Therefore $a \in \mathfrak{m}^{s-1} T + \mathfrak{n}^s T = \mathfrak{m}^{s-1} T$, as claimed.

That $I = xI' + T$ is a Betti splitting follows from Theorem 5.1.

Regarding T as an ideal of S, by Theorem 5.1, we also have

$$\operatorname{ld}_R I = \operatorname{ld}_R T/xT = \operatorname{ld}_R \left(T \otimes_k \frac{k[x]}{(x)} \right) = \operatorname{ld}_S T,$$

where the last equality holds because of [29, Lemma 2.3]. Hence $\operatorname{ld}_R I = \operatorname{ld}_R T$, as desired. $\square$

The main result of this section is as follows.

Theorem 5.7. *Let G be the graph obtained by adding to each vertex of a graph H at least one pendant. Then all the symbolic powers of the cover ideal $J(G)$ of G are Koszul.*

First, we need an auxiliary lemma. If $m = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ is a monomial of R, its support is defined by $\operatorname{supp}(m) = \{x_i \mid \alpha_i \neq 0\}$. For a set B of monomials in R, set $\operatorname{supp} B = \bigcup_{m \in B} \operatorname{supp}(m)$.

Lemma 5.8. *Let I be a momomial ideal of $S = k[x_1, \dots, x_s]$. Let $1 \le t \le s$ be an integer. Assume that for every monomial $m \in S$ with $\operatorname{supp}(m) \subseteq \{x_1, \dots, x_t\}$, the ideal $(m) \cap I$ is Koszul. Denote $R = S[y, z]$. Let $m_1 = y^\alpha f g$, where $\alpha \ge 1$ is an integer and f, g are monomials of S satisfying the following conditions:*

(i) $\operatorname{supp}(g) \subseteq \{x_1, \dots, x_t\}$,
(ii) $\operatorname{supp} f \cap \big(\operatorname{supp} \mathcal{G}(I) \cup \{x_1, \dots, x_t\}\big) = \emptyset$.

Then for all monomials $m \in R$ with $\operatorname{supp}(m) \subseteq \{x_1, \dots, x_t\}$ and all $p, q \geqslant 0$, the ideal $(z, m_1)^p \cap (mz^q) \cap I$ is Koszul.

Proof. We prove by induction on $p + q$. If $p + q = 0$, then $p = q = 0$. In this case, the conclusion holds true by the assumption.

Assume that $p + q \geqslant 1$. If $p \leqslant q$, then we have

$$(z, m_1)^p \cap (mz^q) \cap I = (mz^q) \cap I.$$

Since $(mz^q) \cap I = z^q((m) \cap I)$, we have $(mz^q) \cap I$ is Koszul.

Assume that $p > q$. Consider two cases.

Case 1: $q = 0$. We have

$$\begin{aligned}(z, m_1)^p \cap (m) \cap I &= (z, m_1)^p \cap ((m) \cap I)\\ &= (z(z, m_1)^{p-1} + (m_1^p)) \cap ((m) \cap I)\\ &= zJ + L\end{aligned}$$

where $J = (z, m_1)^{p-1} \cap (m) \cap I$ and $L = (m_1^p) \cap (m) \cap I$.

Observe that J is Koszul by the induction hypothesis. From the assumptions, $\operatorname{supp}(y^\alpha f) \cap (\operatorname{supp}(m) \cup \operatorname{supp} \mathcal{G}(I)) = \emptyset$, so

$$\begin{aligned}L &= (y^{p\alpha} f^p g^p) \cap (m) \cap I = y^{p\alpha} f^p((g^p) \cap (m) \cap I)\\ &= y^{p\alpha} f^p(\operatorname{lcm}(g^p, m) \cap I).\end{aligned}$$

Since $\operatorname{supp} \operatorname{lcm}(g^p, m) \subseteq \{x_1, \dots, x_t\}$, the assumptions yields that $\operatorname{lcm}(g^p, m) \cap I$ is Koszul. Therefore L is Koszul.

The above arguments also give

$$L \subseteq y^\alpha \left(y^{(p-1)\alpha} f^p g^p \cap (m) \cap I\right) \subseteq y \left((m_1^{p-1}) \cap (m) \cap I\right) \subseteq yJ.$$

Thus $(z, m_1)^p \cap (m) \cap I$ is Koszul by Corollary 5.6.

Case 2: $q \geqslant 1$. Then

$$(z, m_1)^p \cap (mz^q) \cap I = z^q((z, m_1)^{p-q} \cap (m) \cap I),$$

which is Koszul by the induction hypothesis. The proof is complete. $\square$

Now we present the

Proof of Theorem 5.7. Assume that $V(H) = \{x_1, \dots, x_d\}$. Let $R = k[x : x \in V(G)]$. In order to prove the theorem we prove the stronger statement that $(m) \cap J(G)^{(n)}$ is Koszul for every monomial $m \in R$ with $\operatorname{supp}(m) \subseteq \{x_1, \dots, x_d\}$. Choosing $m = 1$, we get the desired conclusion.

Induct on d.

Step 1: If $d=1$, then G is a star with the edge set $E(G)=\{x_1y_1,\dots,x_1y_e\}$, where $e\ge 1$. In this case we have $J(G)=(x_1,y_1\cdots y_e)$ and $J(G)^{(n)}=(x_1,y_1\cdots y_e)^n$. Assume that $m=x_1^p$. Since

$$(m)\cap J(G)^{(n)}=(x_1^p)\cap(x_1,y_1\cdots y_e)^n=x_1^p(x_1,y_1\cdots y_e)^{\max\{0,n-p\}},$$

it suffices to prove that $(x_1,y_1\cdots y_e)^n$ is Koszul for all $n\ge 0$.

If $n=0$, this is clear. Assume that $n\ge 1$ and the statement holds for $n-1$. We write $x=x_1, h=y_1\cdots y_e$. Then

$$(x_1,y_1\cdots y_e)^n=(x,h)^n=x(x,h)^{n-1}+(h^n).$$

By the induction hypothesis, $(x,h)^{n-1}$ is Koszul. Applying Corollary 5.6,

$$\mathrm{ld}_R(x,h)^n=\mathrm{ld}_R(h^n)=0.$$

Step 2: Assume that $d\geqslant 2$. Let $y_1,\dots,y_e$ be the vertices of the pendants of G which are adjacent to x_d, where $e\ge 1$. Let $H'=H\setminus\{x_d\}$ and $G'=G\setminus\{x_d,y_1,\dots,y_e\}$. Then $V(H')=\{x_1,\dots,x_{d-1}\}$ and G' is obtained by adding to each vertex of H' at least one pendant.

Let S be the polynomial ring with variables being the vertices of $G\setminus\{x_d,y_1\}$. Denote $I=J(G')^{(n)}$. By the induction hypothesis and Lemma 2.2, $(m')\cap I$ is Koszul for every monomial $m'\in S$ with $\mathrm{supp}(m')\subseteq\{x_1,\dots,x_{d-1}\}$.

Denote $y=y_1, z=x_d$, then $R=S[y,z]$.

Let $m_1=\prod_{x\in N_G(z)}x$, $f=y_2\cdots y_e$, $g=\prod_{x\in N_H(z)}x$. Then

(i) $m_1=yfg$,
(ii) $\mathrm{supp}(g)\subseteq\{x_1,\dots,x_{d-1}\}$,
(iii) $\mathrm{supp}(f)\cap\big(\mathrm{supp}\,\mathcal{G}(I)\cup\{x_1,\dots,x_{d-1}\}\big)=\emptyset$.

Moreover by Lemma 2.5,

$$\begin{aligned}J(G)^{(n)}&=(z,y_1)^n\cap\cdots\cap(z,y_e)^n\cap\bigcap_{x\in N_H(z)}(z,x)^n\cap J(G')^{(n)}\\&=(z,y_1\cdots y_eg)^n\cap I=(z,m_1)^n\cap I.\end{aligned}$$

The second equality holds by observing that $(z,y_1\cdots y_eg)$ is a complete intersection, or by direct inspection.

Take any monomial $m\in R$ with $\mathrm{supp}(m)\subseteq\{x_1,\dots,x_{d-1},z\}$. We can write $m=m'z^p$ where $\mathrm{supp}(m')\subseteq\{x_1,\dots,x_{d-1}\}$. Hence

$$(m)\cap J(G)^{(n)}=(z,m_1)^n\cap(m)\cap I=(z,m_1)^n\cap(m'z^p)\cap I.$$

By Lemma 5.8, the last ideal is Koszul. This finishes the induction on d and the proof. □

The *corona* $\mathrm{cor}(G)$ of a graph G is the graph obtained from G by adding a pendant at each vertex of G. More generally, the generalized corona $\mathrm{cor}(G,s)$ is the graph obtained from G by adding $s\ge 1$ pendant edges to each vertex of G (see Figure 1).

By Alexander duality [15, Chapter 8], we know that the edge ideal $I(G)$ is Koszul (having a linear resolution) if and only if $J(G)$ is sequentially Cohen-Macaulay (respectively, Cohen-Macaulay). Combining this with work of Villarreal [37, Section 4], Francisco and Hà [8, Corollary 3.6], we know that $J(\mathrm{cor}(G))$ has a linear resolution. We generalize this for all symbolic powers of $J(\mathrm{cor}(G))$ as follows.

Corollary 5.9. *Let G be a simple graph. Then all the symbolic powers of the cover ideal $J(\mathrm{cor}(G))$ have linear resolutions.*

We introduce some more notations. Let $V(G) = \{x_1, \ldots, x_d\}$, where it is harmless to assume that $d \geq 1$. Let $y_1, \ldots, y_d$ be the new vertices in $V(\mathrm{cor}(G))$, where y_i is only adjacent to x_i for all $i = 1, \ldots, d$.

Convention 5.10. We denote the coordinates of the ambient $\mathbb{R}^{2d}$ containing $\mathcal{SP}(J(\mathrm{cor}(G)))$ by $x_1, \ldots, x_d, y_1, \ldots, y_d$ instead of $x_1, \ldots, x_d, x_{d+1}, \ldots, x_{2d}$, thus $y_i = x_{d+i}$ for $i = 1, \ldots, d$.

The proof of Corollary 5.9 depends on the following lemma (where Convention 5.10 is in force).

Lemma 5.11. *Denote $J = J(\mathrm{cor}(G))$. Then for any vertex $\boldsymbol{\alpha} \in \mathbb{R}^{2d}$ of $\mathcal{SP}(J)$, up to a relabelling of the variables, there exist integers $0 \leq p \leq q \leq d$ such that $\boldsymbol{\alpha}$ is a solution of the following system:*

$$\begin{cases} x_1 = \cdots = x_p = y_{p+1} = \cdots = y_q = 0, \\ y_1 = \cdots = y_p = x_{p+1} = \cdots = x_q = 1, \\ x_j = y_j = 1/2, \quad \text{if } q+1 \leq j \leq d. \end{cases}$$

In particular, $|\boldsymbol{\alpha}| = d$.

Proof. For the first assertion, note that by Lemma 4.5, $\boldsymbol{\alpha}_i \in \{0, 1, 1/2\}$ for all $i = 1, \ldots, 2d$. Denote $S_0 = \{x_i : \boldsymbol{\alpha}_i = 0\}$, $S_1 = \{x_i : \boldsymbol{\alpha}_i = 1\}$, $S_{1/2} = \{x_i : \boldsymbol{\alpha}_i = 1/2\}$. By Lemma 4.5, we also have S_0 is an independent set of $\mathrm{cor}(G)$.

Without loss of generality, we can assume that $S_0 = \{x_1, \ldots, x_p, y_{p+1}, \ldots, y_q\}$ for some $0 \leq p \leq q \leq d$ (recall Convention 5.10). We have to show that $S_1 = \{y_1, \ldots, y_q, x_{p+1}, \ldots, x_q\}$.

By Lemma 4.5, $\{y_1, \ldots, y_q, x_{p+1}, \ldots, x_q\} \subseteq N(S_0) = S_1$. Clearly $y_{q+1}, \ldots, y_d \notin S_1$ since they do not belong to $N(S_0)$. Hence it remains to show that $x_i \notin S_1$ for $q+1 \leq i \leq d$.

By the definition of S_0, $y_i \notin S_0$. Now y_i is a leaf of $\mathrm{cor}(G)$ and $N(y_i) = \{x_i\}$, so by Lemma 4.5, $x_i \notin S_1$, as desired.

The second assertion now follows from accounting. The proof is concluded. $\square$

Proof of Corollary 5.9. It is harmless to assume that $d = |V(G)| \geq 1$, as mentioned above. By Theorem 5.7 it suffices to show that $J^{(n)} = J(\mathrm{cor}(G))^{(n)}$ generated by monomials of degree dn.

Step 1: Take any vertex $\mathbf{v} \in \mathbb{R}^{2d}$ of $\mathcal{SP}(J)$. By Lemma 5.11, it follows that $|\mathbf{v}| = d$; in particular $\delta(J) = d$.

Step 2: Let $x^{\boldsymbol{\alpha}}$ be a minimal generator of $J^{(n)}$. Since $\boldsymbol{\alpha} \in \mathcal{SP}_n(J)$, we get $\dfrac{1}{n}\boldsymbol{\alpha} \in \mathcal{SP}(J)$. Together with Step 1, it follows that

$$\frac{1}{n}|\boldsymbol{\alpha}| \geqslant \min\{|\mathbf{v}| \mid \mathbf{v} \text{ is a vertex of } \mathcal{SP}(J)\} = d,$$

namely $|\boldsymbol{\alpha}| \geqslant nd$.

On the other hand, by Lemma 4.3,

$$|\boldsymbol{\alpha}| \leqslant d(J^{(n)}) \leqslant \delta(J)n = dn.$$

Thus $|\boldsymbol{\alpha}| = nd$, as required. $\square$

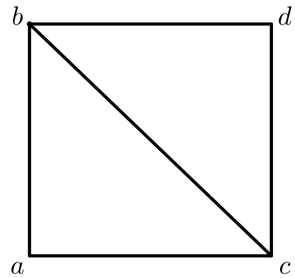

FIGURE 3. The graph G_2

Remark 5.12. A graph G which contains no induced cycle of length at least 4 is called a *chordal* graph. We say that G is a *star graph based on a complete graph* K_m if G is connected and $V(G)=\{1,\dots,m,m+1,\dots,m+g\}$ for some $g\geq 0$ such that:

(1) the complete graph on $\{1,\dots,m\}$ is a subgraph of G, and,
(2) there is no edge in G connecting i and j for all $m+1\leq i<j\leq m+g$.

Any star graph based on a complete graph is chordal.

Let G be a chordal graph. Francisco and Van Tuyl [10, Proof of Theorem 3.2] showed that for such a G, $J(G)$ is Koszul[1]. In [16], Herzog, Hibi and Ohsugi conjectured that all the powers of $J(G)$ are Koszul. Furthermore, in *ibid.*, Theorem 3.3, they confirmed this in the case G is a star graph based on a complete graph K_m. Hence it is natural to ask: If G is star graph based on a complete graph, is it true that $J(G)^{(n)}$ Koszul for all $n\geqslant 1$?

The answer is "No!" Here is a counterexample. Consider the graph G_2 in Figure 3. It is the complete graph on the vertices $\{a,b,c,d\}$ with one edge removed. The corresponding cover ideal is

$$J=J(G_2)=(bc,abd,acd).$$

Since G_2 is a star graph based on K_2, J and all of its ordinanary powers are Koszul by [16, Theorem 3.3]. But $J^{(n)}$ is not Koszul for all $n\geq 2$ by [4, Page 186].

It is natural to ask

Question 5.13. Classify all star graphs based on a complete graph G such that all the symbolic powers of $J(G)$ are Koszul.

The *cover number* of a graph G is the minimal cardinality of a vertex cover. Observe that a subset $\tau\subseteq V(G)$ is a minimal vertex cover of G if and only if $V(G)\setminus\tau$ is a maximal independent set of G.

Let K_m be a complete graph with m vertices and $G=\operatorname{cor}(K_m,s)$ where $m\geqslant 3$ and $s\geqslant 2$. In the rest of the paper we show that both $d(J(G)^{(n)})$ and $\operatorname{reg}(J(G)^{(n)})$ are not necessarily asymptotic linear functions in n.

Lemma 5.14. *For any $m\geqslant 3$ and $s\geqslant 2$, we have:*

(1) $d(J(\operatorname{cor}(K_m,s)))=m+s-1$.

[1]This result can be proved quicky using Corollary 5.6.

(2) $\delta(J(\operatorname{cor}(K_m, s))) = \frac{1}{2}m(s+1)$.

Proof. Let $G = \operatorname{cor}(K_m, s)$. Then G has $r = m(s+1)$ vertices and ms leaves.

(1) Let S be a maximal independent set of G. Then either S is the set of leaves of G, or S consists of a vertex of K_m and $(m-1)s$ leaves which are incident with the remaining vertices of K_m. Thus, the cover number of G is

$$m(s+1) - (1 + (m-1)s) = m + s - 1,$$

and thus $d(J(G)) = m + s - 1$.

(2) Since $|V(G)| = m(s+1)$, by Theorem 4.6 we only need to prove the following: Let S be an independent set of G such that $G \setminus N[S]$ has no bipartite components. Then $|N(S)| \le |S|$, with equality happens when $S = \emptyset$.

We consider three cases:

Case 1: $S = \emptyset$. Then $N(S) = \emptyset$ and $|N(S)| - |S| = 0$.

Case 2: S contains a vertex of K_m, say v. Then $G \setminus N[S]$ is either empty or totally disconnected, in which case it is bipartite. Since $G \setminus N[S]$ has no bipartite component, the first alternative happens. It follows that S consists of v and all the leaves not adjacent to it. Thus, $|S| = 1 + (m-1)s \ge |N(S)| = s + m - 1$, since

$$1 + (m-1)s - (s + m - 1) = (m-2)(s-1) > 0.$$

Case 3: S contains only leaves of G. Let $v_1, \dots, v_m$ be vertices of K_m. Then $N(S)$ consists only of vertices of K_m, say $v_1, \dots, v_t$ for $1 \le t \le m$. Each v_i requires at least a leaf adjacent to it, so clearly $|S| \ge t = |N(S)|$.

The proof is concluded. $\square$

Finally, we present a family of counterexamples to Question 1.2.

Theorem 5.15. *Let $G = \operatorname{cor}(K_m, s)$ where $m \geqslant 3$ and $s \geqslant 2$. Let $J = J(G)$ be its cover ideal. Then for all $n \geqslant 0$,*

(1) $\operatorname{reg} J^{(2n)} = d(J^{(2n)}) = m(s+1)n$;

(2) $\operatorname{reg}(J^{(2n+1)}) = d(J^{(2n+1)}) = m(s+1)n + m + s - 1$.

In particular, for all n,

$$\operatorname{reg}(J^{(n)}) = d(J^{(n)}) = (m+s-1)n + (m-2)(s-1)\left\lfloor \frac{n}{2} \right\rfloor,$$

which is not an eventually linear function of n.

Proof. By Theorem 5.7, $J^{(n)}$ is Koszul for all $n \ge 1$. Hence by Lemma 2.1, $\operatorname{reg}(J^{(n)}) = d(J^{(n)})$ for all n.

Note that by Lemma 5.14, $\delta(J) = \delta(J(G)) = m(s+1)/2$, namely half the number of vertices of G. Hence by Theorem 4.9, for all $n \geqslant 0$

$$d(J^{(2n)}) = 2n\delta(J),$$
$$d(J^{(2n+1)}) = 2n\delta(J) + d(J).$$

From Proposition 5.14(1), $d(J) = m + s - 1$, so the desired formulas follow. $\square$

Remark 5.16. The smallest counterexample for Question 1.2 that we have lives in embedding dimension 9. If we allows non-squarefree monomial ideals, then there are also counterexamples in embedding dimension 4.

Consider the ideal

$$I = (x, y^2) \cap (x, a) \cap (a^2, ab^2, b^3) \subseteq k[x, y, a, b].$$

Using arguments similar to the proof of Theorem 5.15, we can show that for all $n \geq 1$, $I^{(n)}$ is Koszul, and $\operatorname{reg} I^{(n)} = 4n + \lfloor \frac{n+1}{2} \rfloor$. Hence $\operatorname{reg} I^{(n)}$ is quasi-linear but not eventually linear.

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