FORMAL LOCAL HOMOLOGY

TRAN TUAN NAM[1,4], DO NGOC YEN[2,3], AND NGUYEN MINH TRI[4,5]

Abstract. We introduce a concept of formal local homology modules which is in some sense dual to P. Schenzel's concept of formal local cohomology modules. The dual theorem and the non-vanishing theorem of formal local homology modules will be shown. We also give some conditions for formal local homology modules being finitely generated or artinian.

2000 Mathematics subject classification: Local cohomology 13D45, Topological rings and modules 13J99.
Key words: Formal local cohomology, formal local homology.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ will be a local noetherian (commutative) ring with the $\mathfrak{m}-$adic topology. Let I be an ideal of $(R, \mathfrak{m})$ and M an $R-$module. In [15], P. Schenzel introduced the concept of *formal cohomology* and the $i-th$ $I-$formal cohomology module of M with respect to $\mathfrak{m}$ can be defined by

$$\mathfrak{F}^i_I(M) = \varprojlim_t H^i_{\mathfrak{m}}(M/I^tM).$$

In the paper, we introduce the concept of *formal local homology* which is in some sense dual to P. Schenzel's concept of formal local cohomology. The $i-$th $I-$*formal local homology module* $\mathfrak{F}^I_{i,J}(M)$ of an $R-$module M with respect to J is defined by

$$\mathfrak{F}^I_{i,J}(M) = \varinjlim_t H^J_i(0 :_M I^t).$$

In the case of $J = \mathfrak{m}$ we set $\mathfrak{F}^I_{i,\mathfrak{m}}(M) = \mathfrak{F}^I_i(M)$ and speak simply about the $i-$th $I-$formal local homology module.

We also study some basic properties of formal local homology modules $\mathfrak{F}^I_i(M)$ when M is a linearly compact $R-$module, in particular when M is an artinian $R-$module. The organization of the paper is as follows.

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 101.04-2018.304.

In Section 2, we give the definition of the formal local homology modules $\mathfrak{F}^I_{i,J}(M)$ of an $R-$module M with respect to J by the formula:

$$\mathfrak{F}^I_{i,J}(M) = \varprojlim_t H^J_i(0 :_M I^t).$$

It is shown that $H^0_I(\mathfrak{F}^I_{j,J}(M)) \cong \mathfrak{F}^I_{j,J}(M)$ and $H^i_I(\mathfrak{F}^I_{j,J}(M)) = 0$ for all $i \neq 0$ (Theorem 2.3). The dual theorem (Theorem 2.15) establishes the isomorphisms

$$\mathfrak{F}^I_i(M^*) \cong \mathfrak{F}^i_I(M)^*, \ \mathfrak{F}^i_I(M^*) \cong \mathfrak{F}^I_i(M)^*$$

provided M is a linearly compact module over the complete ring $(R, \mathfrak{m})$. In Theorem 2.18 the short exact sequence of artinian modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

gives rise to a long exact sequence of $I-$formal local homology modules

$$... \longrightarrow \mathfrak{F}^I_i(M') \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \mathfrak{F}^I_i(M'') \longrightarrow \mathfrak{F}^I_{i-1}(M') \longrightarrow ....$$

This section is closed by the non-vanishing theorem (Theorem 2.20) which says that if M is a non zero semi-discrete linearly compact $R-$module such that $0 \leq \mathrm{Ndim}(0 :_M I) \neq 1$, then

$$\mathrm{Ndim}(0 :_M I) = \max\{i \mid \mathfrak{F}^I_i(M) \neq 0\}.$$

On the other hand if M is a semi-discrete linearly compact $R-$module such that $\mathrm{Ndim}(0 :_{\Gamma_\mathfrak{m}(M)} I) \neq 0$, then

$$\mathrm{Ndim}(0 :_{\Gamma_\mathfrak{m}(M)} I) = \max\big\{i \mid \mathfrak{F}^I_i(M) \neq 0\big\}.$$

The last section is devoted to study the finiteness of formal local homology modules. Theorems 3.2 and 3.3 give us the equivalent conditions for the formal local homology modules $\mathfrak{F}^I_i(M)$ being finitely generated. In Theorem 3.4, if I is a principal ideal of $(R, \mathfrak{m})$ and M is an artinian $R-$module, then $\mathfrak{F}^I_i(M)/I\mathfrak{F}^I_i(M)$ is a noetherian $\widehat{R}-$module for all i. Theorem 3.5 shows that if M is an artinian $R-$module and s a non-negative integer such that $\mathfrak{F}^I_i(M)$ is a noetherian $\widehat{R}-$module for all $i < s$, then $\mathfrak{F}^I_s(M)/I\mathfrak{F}^I_s(M)$ is also a noetherian $\widehat{R}-$module. There is a question: When are the formal local homology modules $\mathfrak{F}^I_i(M)$ artinian? Theorem 3.6 answers that if M is an artinian $R-$module with $\mathrm{Ndim}\, M = d$, then $\mathfrak{F}^I_{d-1}(M)$ is an artinian $R-$module. Finally, Theorem 3.7 provides that if M is an artinian $R-$module and s a non-negative integer, then the following statements are equivalent: (i) $\mathfrak{F}^I_i(M)$ is artinian for all $i > s$, (ii) $\mathfrak{F}^I_i(M) = 0$ for all $i > s$ and $\mathrm{Ass}(\mathfrak{F}^I_i(M)) \subseteq \{\mathfrak{m}\}$ for all $i > s$.

2. Formal local homology modules

We first recall the concept of *linearly compact modules* defined by I. G. Macdonald [6]. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection. A Hausdorff linearly topologized $R-$module M is called *semi-discrete* if every submodule of M is closed. Thus a discrete $R-$module is semi-discrete. It is clear that artinian $R-$modules are linearly compact with the discrete topology. So the class of semi-discrete linearly compact modules contains all artinian modules. Moreover, if $(R, \mathfrak{m})$ is a complete ring, then the finitely generated $R-$modules are also linearly compact and semi-discrete.

Let I be an ideal of $(R, \mathfrak{m})$ and M an $R-$module. It is well-known that the $i-th$ local cohomology module $H^i_I(M)$ of M with respect to I can be defined by

$$H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t; M).$$

When $i = 0$, $H^0_I(M) \cong \bigcup\limits_{t>0} (0 :_M I^t) = \Gamma_I(M)$.

In [15], P. Schenzel introduced the concept of *formal cohomology* and the $i-th$ $I-$formal cohomology module of M with respect to $\mathfrak{m}$ can be defined by

$$\mathfrak{F}^i_I(M) = \varinjlim_t H^i_{\mathfrak{m}}(M/I^tM).$$

Note that the $i-th$ local homology module $H^I_i(M)$ of an $R-$module M with respect to I can be is defined by

$$H^I_i(M) = \varprojlim_t \mathrm{Tor}^R_i(R/I^t, M) \ ([3]).$$

When $i = 0$, $H^I_0(M) \cong \varprojlim\limits_t M//I^tM = \Lambda_I(M)$ the $I-$adic completion of M. This suggests the following definition.

Definition 2.1. Let I, J be ideals of R. The $i-$th $I-$*formal local homology module* $\mathfrak{F}^I_{i,J}(M)$ of an $R-$module M with respect to J is defined by

$$\mathfrak{F}^I_{i,J}(M) = \varinjlim_t H^J_i(0 :_M I^t).$$

In the case of $J = \mathfrak{m}$ we set $\mathfrak{F}^I_{i,\mathfrak{m}}(M) = \mathfrak{F}^I_i(M)$ and speak simply about the $i-$th $I-$formal local homology module.

Remark 2.2. (i). It should be mentioned from [4, 3.1 (i)] that $H_i^J(0:_M I^t)$ has a natural structure as a module over the ring $\Lambda_J(R)$, then $\mathfrak{F}_{i,J}^I(M)$ also has a natural structure as a module over the ring $\Lambda_J(R)$. In particular, $\mathfrak{F}_i^I(M)$ has a natural structure as a module over the ring $\widehat{R}$.

(ii). If M is finitely generated, then $H_i^J(0:_M I^t)=0$ for all $i>0$ by [3, 3.2 (ii)], then $\mathfrak{F}_{i,J}^I(M)=0$ for all $i>0$.

In the following theorem we compute the local cohomology modules of an $I-$formal local homology module $\mathfrak{F}_{i,J}^I(M)$.

Theorem 2.3. *Let M be an $R-$module. Then*

$$H_I^i(\mathfrak{F}_{j,J}^I(M)) \cong \begin{cases} 0 & i \neq 0 \\ \mathfrak{F}_{j,J}^I(M) & i = 0 \end{cases}$$

for any integer j.

Proof. We have

$$H_I^i(\mathfrak{F}_{j,J}^I(M)) = H_I^i(\varinjlim_t H_i^J(0:_M I^t)) \cong \varinjlim_t H_I^i(H_i^J(0:_M I^t)).$$

Assume that the ideal I is generated by r element $x_1, x_2, ..., x_r$. Set $\underline{x}(s) = (x_1^s, x_2^s, ..., x_r^s)$ and $H^i(\underline{x}(s), N)$ is the ith Koszul cohomology module of an $R-$module N with respect to $\underline{x}(s)$. we have

$$H_I^i(\mathfrak{F}_{j,J}^I(M)) \cong \varinjlim_t \varinjlim_s H^i(\underline{x}(s), H_j^J(0:_M I^t)).$$

Note that $\underline{x}(s)H_j^J(0:_M I^t)=0$ for all $s \geq t$. Then

$$\varinjlim_s H^i(\underline{x}(s), H_j^J(0:_M I^t)) \cong \begin{cases} 0 & i \neq 0 \\ H_j^J(0:_M I^t) & i = 0. \end{cases}$$

By passing to direct limits $\varinjlim_t$ we have the conclusion as required. □

Corollary 2.4. *Let M be an $R-$module and i an integer such that $0:_{\mathfrak{F}_{i,J}^I(M)} I = 0$. Then $\mathfrak{F}_{i,J}^I(M)=0$.*

Proof. It follows from 2.3 that

$$\mathfrak{F}_{i,J}^I(M) = \Gamma_I(\mathfrak{F}_{i,J}^I(M)) = \bigcup_{t>0}(0:_{\mathfrak{F}_{i,J}^I(M)} I^t).$$

As $0:_{\mathfrak{F}_{i,J}^I(M)} I=0$, we conclude that $\mathfrak{F}_{i,J}^I(M)=0$. □

If M is a linearly compact $R-$module, then M has a natural structure of linearly compact module over $\widehat{R}$ by [4, 7.1]. We have the following lemma.

Lemma 2.5. *Let M be a linearly compact $R-$module. Then*

$$\mathfrak{F}^{I}_{i,J}(M) \cong \mathfrak{F}^{I\widehat{R}}_{i,J\widehat{R}}(M)$$

for all $i \geq 0$.

Proof. The natural homomorphism $R \longrightarrow \widehat{R}$ gives by [3, 3.7] isomorphisms

$$H^{I}_{i}(0 :_M I^t) \cong H^{I\hat{R}}_{i}(0 :_M I^t\hat{R})$$

for all $i \geq 0$. By passing to direct limits, we have the isomorphisms

$$\mathfrak{F}^{I}_{i}(M) \cong \mathfrak{F}^{I\hat{R}}_{i}(M)$$

as required. □

It should be noted that the artinian $R-$modules are linearly compact and discrete. Therefore we have an immediate consequence.

Corollary 2.6. *If M is an artinian $R-$module, then*

$$\mathfrak{F}^{I}_{i}(M) \cong \mathfrak{F}^{I\widehat{R}}_{i}(M)$$

for all $i \geq 0$.

Lemma 2.7. *Let I, J be ideals of R and M a linearly compact $R-$module. If M is $J-$separated (it means that $\bigcap\limits_{t>0} J^tM = 0$), then*

$$\mathfrak{F}^{I}_{i,J}(M) \cong \begin{cases} 0 & i \neq 0 \\ \Gamma_I(M) & i = 0. \end{cases}$$

Proof. As M is $J-$separated, $0 :_M I^t$ is also $J-$separated for all $t > 0$. It follows from [4, 3.8] that

$$H^{J}_{i}(0 :_M I^t) \cong \begin{cases} 0 & i \neq 0 \\ 0 :_M I^t & i = 0. \end{cases}$$

By passing to direct limits we have the conclusion. □

It should be noted by [3, 3.3 (i)] and [4, 3.3] that the local homology modules $H^{J}_{i}(M)$ are linearly compact and $J-$separated for all i. Then we have the immediate consequence.

Corollary 2.8. *Let M be a linearly compact $R-$module. Then*

$$\mathfrak{F}^{I}_{i,J}(H^{J}_{j}(M)) \cong \begin{cases} 0 & i \neq 0 \\ \Gamma_I(H^{J}_{j}(M)) & i = 0 \end{cases}$$

for all j.

In the special case when M is an artinian $R-$module, we have the following consequence.

Corollary 2.9. *Let I, J be ideals of R and M an artinian $R-$module. If M is $J-$separated (it means that $\bigcap_{t>0} J^t M = 0$), then*

$$\mathfrak{F}^I_{i,J}(M) \cong \begin{cases} 0 & i \neq 0 \\ M & i = 0. \end{cases}$$

Proof. Note that artinian modules are linearly compact, then

$$\mathfrak{F}^I_{i,J}(M) \cong \begin{cases} 0 & i \neq 0 \\ \Gamma_I(M) & i = 0 \end{cases}$$

by 2.7. Moreover, as M is an artinian module over the local ring $(R, \mathfrak{m})$, [14, 1.4] provides that $\Gamma_I(M) = M$ and we have the conclusion. □

Lemma 2.10. *Let I, J be ideals of R and M an artinian $R-$module. If M is $I-$separated (it means that $\bigcap_{t>0} I^t M = 0$), then*

$$\mathfrak{F}^I_{i,J}(M) \cong H^J_i(M)$$

for all i.

Proof. As M is an $I-$separated artinian $R-$module, there is a positive integer n such that $I^n M = 0$. Then $0 :_M I^n = M$. Therefore

$$\mathfrak{F}^I_{i,J}(M) = \varinjlim_t H^J_i(0 :_M I^t) \cong H^J_i(M)$$

for all i. □

In the case $J = \mathfrak{m}$, it follows from [3, 4.6] that $H^{\mathfrak{m}}_i(M)$ is a noetherian $\widehat{R}-$module. From 2.10 we have an immediate consequence.

Corollary 2.11. *Let M be an artinian $R-$module. If M is $I-$separated (it means that $\bigcap_{t>0} I^t M = 0$), then*

$$\mathfrak{F}^I_i(M) \cong H^{\mathfrak{m}}_i(M)$$

and then $\mathfrak{F}^I_i(M)$ a noetherian $\widehat{R}-$module for all i.

In order to state the dual theorem we recall the concepts of Matlis dual and Macdonald dual. Let M be an $R-$module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $D(M) = \mathrm{Hom}(M, E(R/\mathfrak{m}))$ is called the Matlis dual of M. If M is a Hausdorff linearly topologized $R-$module, then the *Macdonald dual* of M is defined by $M^* = Hom(M, E(R/\mathfrak{m}))$ the set of continuous homomorphisms of $R-$modules ([6, §9]). The topology on M^* is defined as in [6, 8.1]. Moreover, if M is semi-discrete, then the topology of M^* coincides with that induced on it as a submodule of $E(R/\mathfrak{m})^M$, where $E(R/\mathfrak{m})^M = \prod_{x\in M}(E(R/\mathfrak{m}))^x$, $(E(R/\mathfrak{m}))^x = E(R/\mathfrak{m})$ for all $x \in M$ ([6, 8.6]). A Hausdorff linearly topologized $R-$module M is called *linearly discrete* if every $\mathfrak{m}-$primary quotient of M is discrete. It is clear that $M^* \subseteq D(M)$ and the equality holds if and only if M is semi-discrete in the following lemma.

Lemma 2.12. ([6, 5.8]) *Let M be a Hausdorff linearly topologized $R-$module. Then M is semi-discrete if and only if $M^* = D(M)$.*

Lemma 2.13. ([6, 9.3, 9.12, 9.13]) *Let $(R, \mathfrak{m})$ be a complete local noetherian ring.*

(i) *If M is linearly compact, then M^* is linearly discrete (hence semi-discrete). If M is semi-discrete, then M^* is linearly compact;*
(ii) *If M is linearly compact or linearly discrete, then we have a topological isomorphism $\omega : M \xrightarrow{\simeq} M^{**}$.*

Lemma 2.14. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring.*

(i) *If M is finitely generated, then M^* is artinian;*
(ii) *If M is artinian, then M^* is finitely generated.*

Proof. (i). AS M is a finitely generated module over the complete local noetherian ring $(R, \mathfrak{m})$, It follows from [6, 7.3] that M is linearly compact and semi-discrete. Then $M^* = D(M)$ by 2.12. Now, the conclusion follows from [16, 3.4.11].

(i). It should be noted that an artinian $R-$module is a linearly compact $R-$module with the discrete topology. Then $M^* = D(M)$ by 2.12. Finally, the conclusion follows from [16, 3.4.12]. □

We have the following dual theorem.

Theorem 2.15. *Let $(R, \mathfrak{m})$ be a complete ring and M a linearly compact $R-$module. Then*

$$\mathfrak{F}_i^I(M^*) \cong \mathfrak{F}_I^i(M)^*,$$

$$\mathfrak{F}_I^i(M^*) \cong \mathfrak{F}_i^I(M)^*.$$

for for all i.

Proof. It should be noted by [4, 6.7] that

$$M^*/I^tM^* \cong (0 :_M I^t)^*,$$

$$(M/I^tM)^* \cong 0 :_{M^*} I^t$$

for all $t > 0$. The we have

$$\begin{aligned}\mathfrak{F}_i^I(M^*) &= \varinjlim_t H_i^{\mathfrak{m}}(0:_{M^*} I^t)\\ &\cong \varinjlim_t H_i^{\mathfrak{m}}((M/I^tM)^*)\\ &\cong \varinjlim_t (H^i_{\mathfrak{m}}(M/I^tM))^* \text{ (cf. [4, 6.4(ii)])}\\ &\cong (\varprojlim_t H^i_{\mathfrak{m}}(M/I^tM))^* = \mathfrak{F}^i_I(M)^* \text{ (cf. [6, 9.14]).}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}^i_I(M^*) &= \varprojlim_t H^i_{\mathfrak{m}}(M^*/I^tM^*)\\ &\cong \varprojlim_t H^i_{\mathfrak{m}}((0:_M I^t)^*)\\ &\cong \varprojlim_t (H_i^{\mathfrak{m}}(0:_M I^t))^* \text{ (cf. [4, 6.4(ii)])}\\ &\cong (\varinjlim_t H_i^{\mathfrak{m}}(0:_M I^t))^* = \mathfrak{F}_i^I(M)^* \text{ (cf. [6, 2.6]).}\end{aligned}$$

The proof is complete. □

Corollary 2.16. *Let $(R, \mathfrak{m})$ be a complete ring and M a linearly compact $R-$module. Then*

$$\mathfrak{F}_i^I(M) \cong \mathfrak{F}^i_I(M^*)^*,$$
$$\mathfrak{F}^i_I(M) \cong \mathfrak{F}_i^I(M^*)^*.$$

for for all i.

Proof. Combining 2.13 (ii) with 2.15 yields

$$\mathfrak{F}_i^I(M) \cong \mathfrak{F}_i^I(M^{**}) \cong \mathfrak{F}^i_I(M^*)^*,$$
$$\mathfrak{F}^i_I(M) \cong \mathfrak{F}^i_I(M^{**}) \cong \mathfrak{F}_i^I(M^*)^*$$

as required. □

In order to state Theorem 2.18 about the long exact sequence of formal local homology modules we need the following lemma.

Lemma 2.17. (i) *The continuous homomorphism of linearly topology $R-$modules $f : M \longrightarrow N$ induces continuous homomorphisms of formal local cohomology modules*
$\varphi_i : \mathfrak{F}^i_I(M) \longrightarrow \mathfrak{F}^i_I(N)$ for all i.

(ii) *If M is a semi-discrete linearly compact $R-$module, then the formal local cohomology modules $\mathfrak{F}^i_I(M)$ are linearly compact for all i.*

Proof. (i). The continuous homomorphism $f : M \longrightarrow N$ induces continuous homomorphisms $M/I^tM \longrightarrow N/I^tN$ for all $t > 0$. By an argument analogous to that used for the proof of [4, 2.5] we get continuous homomorphisms $\mathrm{Ext}^i_R(R/\mathfrak{m}^s; M/I^tM) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{m}^s; N/I^tN)$ for all $s,\ t > 0$ and i. By passing into direct limits $\varinjlim_s$ we have continuous homomorphisms of local cohomology modules $H^{\mathfrak{m}}_i(M/I^tM) \longrightarrow H^{\mathfrak{m}}_i(N/I^tN)$. Now, by passing into inverse limits $\varprojlim_t$ we get continuous homomorphisms of formal local cohomology modules $\varphi_i : \mathfrak{F}^i_I(M) \longrightarrow \mathfrak{F}^i_I(N)$ for all i.

(ii). By [4, 7.9] the local cohomology modules $H^i_{\mathfrak{m}}(M/I^tM)$ are artinian for all t and i. Note that $\mathfrak{F}^i_I(M) = \varprojlim_t H^i_{\mathfrak{m}}(M/I^tM)$. Therefore, the conclusion follows from [6, 3.7, 3.10]. □

Theorem 2.18. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of artinian modules. Then there is a long exact sequence of $I-$formal local homology modules*

$$... \longrightarrow \mathfrak{F}^I_i(M') \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \mathfrak{F}^I_i(M'') \longrightarrow \mathfrak{F}^I_{i-1}(M') \longrightarrow ....$$

Proof. It should be noted by [14, 1.11] that an artinian module over a local noetherian ring $(R, \mathfrak{m})$ has a natural structure of artinian module over $\widehat{R}$ and a subset of M is an $R-$submodule if and only if it is an $\widehat{R}-$submodule. Thus, from 2.9 we may assume that $(R, \mathfrak{m})$ is a complete ring. We now consider the short exact sequence of artinian $R-$modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

Note that the artinian $R-$modules are linearly compact and discrete, then the homomorphisms f, g are continuous. Combining [4, 6.5] with 2.14 we have the following short exact sequence of finitely generated $R-$modules

$$0 \longrightarrow M''^* \xrightarrow{f^*} M^* \xrightarrow{g^*} M'^* \longrightarrow 0$$

where the induced homomorphisms $f^*,\ g^*$ are continuous. It gives rise by [15, 3.11] a long exact sequence of $I-$formal local cohomology modules

$$... \longrightarrow \mathfrak{F}^{i-1}_I(M'^*) \longrightarrow \mathfrak{F}^i_I(M''^*) \xrightarrow{g_i} \mathfrak{F}^i_I(M^*) \xrightarrow{f_i} \mathfrak{F}^i_I(M'^*) \longrightarrow ....$$

It should be mentioned from 2.17 (ii) that the $I-$formal local cohomology modules $\mathfrak{F}^i_I(M^*), \mathfrak{F}^i_I(M'^*), \mathfrak{F}^i_I(M''^*)$ are linearly compact and the homomorphisms of the long exact sequence are inverse limits of the homomorphisms of artinian modules. Note that the homomorphisms of artinian modules are continuous, because the topologies on the artinian modules are discrete. Moreover, the inverse limits of the continuous homomorphisms are also continuous. Then the homomorphisms of the long exact sequence are continuous. Therefore the long exact sequence induces an exact sequence by [4, 6.5]

$$...(\mathfrak{F}_I^i(M'^*))^* \longrightarrow (\mathfrak{F}_I^i(M^*))^* \longrightarrow (\mathfrak{F}_I^i(M''^*))^* \longrightarrow (\mathfrak{F}_I^{i-1}(M'^*))^*....$$

The conclusion now follows from 2.16. □

We now recall the concept of *Noetherian dimension* of an $R-$module M denoted by $\operatorname{Ndim} M$. Note that the notion of Noetherian dimension was introduced first by R. N. Roberts [13] by the name Krull dimension. Later, D. Kirby [5] changed this terminology of Roberts and refereed to *Noetherian dimension* to avoid confusion with well-know Krull dimension of finitely generated modules. Let M be an $R-$module. When $M = 0$ we put $\operatorname{Ndim} M = -1$. Then by induction, for any ordinal α, we put $\operatorname{Ndim} M = \alpha$ when (i) $\operatorname{Ndim} M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M, there exists a positive integer m_0 such that $\operatorname{Ndim}(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$. Thus M is non-zero and finitely generated if and only if $\operatorname{Ndim} M = 0$. If $0 \longrightarrow M" \longrightarrow M \longrightarrow M' \longrightarrow 0$ is a short exact sequence of $R-$modules, then $\operatorname{Ndim} M = \max\{\operatorname{Ndim} M", \operatorname{Ndim} M'\}$.

Proposition 2.19. *Let I, J be ideals of R and M an artinian $R-$module. If* $\operatorname{Ndim}(0 :_M I) = 0$, *then*

$$\mathfrak{F}_{i,J}^I(M) \cong \begin{cases} 0 & i \neq 0 \\ M & i = 0. \end{cases}$$

Proof. Since $\operatorname{Ndim}(0 :_M I) = 0$, $0 :_M I$ has finite length by [13, p. 269]. and then $0 :_M I$ is $J-$separated. It follows from [4, 3.8] that

$$H_i^J(0 :_M I^t) \cong \begin{cases} 0 & i \neq 0 \\ 0 :_M I^t & i = 0 \end{cases}$$

for all $t > 0$. By passing to direct limits we have

$$\mathfrak{F}_{i,J}^I(M) \cong \begin{cases} 0 & i \neq 0 \\ \Gamma_I(M) & i = 0. \end{cases}$$

Now the conclusion follows from [14, 1.4], as M is an artinian module over the local ring $(R, \mathfrak{m})$, □

Remember that the *(Krull) dimension* $\dim_R M$ of a non-zero $R-$module M is the supremum of lengths of chains of primes in the support of M if this supremum exists, and ∞ otherwise. For convenience, we set $\dim M = -1$ if $M = 0$. Note that if M is non-zero and artinian, then

$\dim M = 0$. If M is finitely generated, then $\dim M = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M\}$.

In [4, 4.10], if M is a non zero semi-discrete linearly compact $R-$module, then

$$\operatorname{Ndim} \Gamma_{\mathfrak{m}}(M) = \max\left\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\right\} \text{ if } \Gamma_{\mathfrak{m}}(M) \neq 0;$$
$$\operatorname{Ndim} M = \max\left\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\right\} \text{ if } \operatorname{Ndim} M \neq 1.$$

We have the non-vanishing theorem of formal local homology modules.

Theorem 2.20. *Let M be a non zero semi-discrete linearly compact $R-$module. Then*

(i) $\operatorname{Ndim}(0 :_M I) = \max\{i \mid \mathfrak{F}_i^I(M) \neq 0\}$ *if* $0 \leq \operatorname{Ndim}(0 :_M I) \neq 1$;
(ii) $\operatorname{Ndim}(0 :_{\Gamma_{\mathfrak{m}}(M)} I) = \max\left\{i \mid \mathfrak{F}_i^I(M) \neq 0\right\}$
if $\operatorname{Ndim}(0 :_{\Gamma_{\mathfrak{m}}(M)} I) \neq 0$.

Proof. (i). We begin by proving that

$$\operatorname{Ndim}(0 :_M I^t) = \operatorname{Ndim}(0 :_M I)$$

for all $t > 0$. As M is a semi-discrete linearly compact $R-$module, it should be noted by [4, 7.1,7.2] that M has a natural structure of semi-discrete linearly compact module over the ring $\widehat{R}$ and $\operatorname{Ndim}_R M = \operatorname{Ndim}_{\hat{R}} M$. Thus, we may assume that $(R, \mathfrak{m})$ is a complete ring. At first, we prove in the special case when M is artinian. Then $D(M)$ is a finitely generated $R-$module by Matlis dual. We have

$$\dim D(M)/I^t D(M) = \dim D(M)/ID(M).$$

Combining 2.12 with [4, 7.4] yields

$$\begin{aligned}
\operatorname{Ndim}(0 :_M I^t) &= \dim D(0 :_M I^t) \\
&= \dim D(M)/I^t D(M) \\
&= \dim D(M)/ID(M) \\
&= \dim D(0 :_M I) = \operatorname{Ndim}(0 :_M I).
\end{aligned}$$

We now assume that M is a semi-discrete linearly compact $R-$module. By [19, Theorem] there is a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow A \longrightarrow 0,$$

where N is finitely generated and A is artinian. It induces an exact sequence

$$0 \longrightarrow 0 :_N I^t \xrightarrow{f} 0 :_M I^t \xrightarrow{g} 0 :_A I^t \xrightarrow{\delta} \operatorname{Ext}_R^1(R/I^t; N).$$

Then we have two short exact sequence

$$0 \longrightarrow 0 :_N I^t \xrightarrow{f} 0 :_M I^t \longrightarrow \operatorname{Im} g \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} g \longrightarrow 0 :_A I^t \longrightarrow \operatorname{Im} \delta \longrightarrow 0.$$

Since $0 :_N I^t$ and $\operatorname{Im} \delta$ are finitely generated $R-$modules, we get $\operatorname{Ndim}(0 :_N I^t) = \operatorname{Ndim} \operatorname{Im} \delta = 0$. It follows that

$$\begin{aligned} \operatorname{Ndim}(0 :_M I^t) &= \operatorname{Ndim} \operatorname{Im} g \\ &= \operatorname{Ndim}(0 :_A I^t) \\ &= \operatorname{Ndim}(0 :_A I) = \operatorname{Ndim}(0 :_M I). \end{aligned}$$

Now, it follows from [4, 4.10 (ii)] that

$$d = \operatorname{Ndim}(0 :_M I) = \operatorname{Ndim}(0 :_M I^t) = \max \left\{ i \mid H_i^{\mathfrak{m}}(0 :_M I^t) \neq 0 \right\}$$

for all $t > 0$. Then $H_d^{\mathfrak{m}}(0 :_M I^t) \neq 0$ and $H_i^{\mathfrak{m}}(0 :_M I^t) = 0$ for all $i > d$ and $t > 0$. The short exact sequences for all $t > 0$

$$0 \longrightarrow 0 :_M I^t \longrightarrow 0 :_M I^{t+1} \longrightarrow 0 :_M I^{t+1}/0 :_M I^t \longrightarrow 0$$

induce exact sequences

$$...H_{d+1}^{\mathfrak{m}}(0 :_M I^{t+1}/0 :_M I^t) \longrightarrow H_d^{\mathfrak{m}}(0 :_M I^t) \longrightarrow H_d^{\mathfrak{m}}(0 :_M I^{t+1})...$$

by [4, 3.7]. As $\operatorname{Ndim}(0 :_M I^{t+1}/0 :_M I^t) \leq d$, [4, 4.8] shows that $H_{d+1}^{\mathfrak{m}}(0 :_M I^{t+1}/0 :_M I^t) = 0$. Then the homomorphisms

$$H_d^{\mathfrak{m}}(0 :_M I^t) \longrightarrow H_d^{\mathfrak{m}}(0 :_M I^{t+1})$$

are injective for all $t > 0$. Therefore $\varinjlim_t H_d^{\mathfrak{m}}(0 :_M I^t) \neq 0$ and $\varinjlim_t H_i^{\mathfrak{m}}(0 :_M I^t) = 0$ for all $i > d$. Hence (i) is proved.

(ii). It is clear that $0 :_{\Gamma_{\mathfrak{m}}(M)} I = \Gamma_{\mathfrak{m}}(0 :_M I)$. Set $d = \operatorname{Ndim}(0 :_{\Gamma_{\mathfrak{m}}(M)} I) = \operatorname{Ndim}(0 :_{\Gamma_{\mathfrak{m}}(M)} I^t)$. From [4, 4.10(i)] we have

$$d = \operatorname{Ndim}(0 :_{\Gamma_{\mathfrak{m}}(M)} I^t) = \operatorname{Ndim} \Gamma_{\mathfrak{m}}(0 :_M I^t) = \max \left\{ i \mid H_i^{\mathfrak{m}}(0 :_M I^t) \neq 0 \right\}.$$

The rest of the proof is analogous to that in the proof of (i). □

Corollary 2.21. *Let M be an artinian $R-$module such that $0 :_M I \neq 0$. Then*

$$\operatorname{Ndim}(0 :_M I) = \max\{i \mid \mathfrak{F}_i^I(M) \neq 0\}.$$

Proof. It should be noted that an artinian $R-$module is semi-discrete linearly compact. Moreover, $\Gamma_{\mathfrak{m}}(M) = M$. Therefore the conclusion follows from 2.20 (ii). □

3. The finiteness of formal local homology modules

We begin by recalling the concept of *co-support* of an module. The co-support $\mathrm{Cosupp}_R(M)$ of an $R-$module M is the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}(L) \subseteq \mathfrak{p}$ ([18, 2.1]). Note that a module is cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \subset R$.
If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of $R-$modules, then $\mathrm{Cosupp}_R(M) = \mathrm{Cosupp}_R(N) \cup \mathrm{Cosupp}_R(K)$ ([18, 2.7]).

Lemma 3.1. *Let M be an artinian $R-$module. Then*
$$\mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)) \bigcap V(I\hat{R}) \subseteq V(\mathfrak{m}\hat{R}).$$

Proof. By 2.5 we may assume that $(R, \mathfrak{m})$ is a complete ring. It should be noted that M^* is a finitely generated $R-$module. Combining 2.16 with [18, 2.9] yields
$$\begin{aligned}\mathrm{Cosupp}_R(\mathfrak{F}_0^I(M)) &= \mathrm{Cosupp}_R(\mathfrak{F}_I^0(M^*)^*)\\ &\subseteq \mathrm{Cosupp}_R D((\mathfrak{F}_I^0(M^*))\\ &= \mathrm{Supp}_R(\mathfrak{F}_I^0(M^*)).\end{aligned}$$
By the proof of [15, 4.3] we have
$$\begin{aligned}\mathrm{Cosupp}_R(\mathfrak{F}_0^I(M)) \bigcap V(I) &\subseteq \mathrm{Supp}_R(\mathfrak{F}_I^0(M^*)) \bigcap V(I)\\ &\subseteq V(\mathfrak{m}).\end{aligned}$$
The proof is complete. □

We have the following equivalent properties for formal local homology modules $\mathfrak{F}_i^I(M)$ being finitely generated for all $i < s$.

Theorem 3.2. *Let M be an $R-$artinian module and s a positive integer. Then the following statements are equivalent:*

(i) *$\mathfrak{F}_i^I(M)$ is a finitely generated $\widehat{R}-$module for all $i < s$;*
(ii) *$I \subseteq \sqrt{0 :_R \mathfrak{F}_i^I(M)}$ for all $i < s$.*

Proof. $(i) \Rightarrow (ii)$. For $i < s$, as $\mathfrak{F}_i^I(M)$ is a finitely generated $\widehat{R}-$module, the increasing chain of submodules of $\mathfrak{F}_i^I(M)$
$$0 :_{\mathfrak{F}_i^I(M)} I \subseteq 0 :_{\mathfrak{F}_i^I(M)} I^2 \subseteq ... \subseteq 0 :_{\mathfrak{F}_i^I(M)} I^t \subseteq ...$$
is stationary. Thus, there is a positive integer r such that $0 :_{\mathfrak{F}_i^I(M)} I^t = 0 :_{\mathfrak{F}_i^I(M)} I^r$ for all $t \geq r$. It follows from 2.3 that
$$\mathfrak{F}_i^I(M) = \bigcup_{t>0}(0 :_{\mathfrak{F}_i^I(M)} I^t) = 0 :_{\mathfrak{F}_i^I(M)} I^r.$$

Therefore $I^r\mathfrak{F}_i^I(M) = 0$ and then $I \subseteq \sqrt{0 : \mathfrak{F}_i^I(M)}$.

$(ii) \Rightarrow (i)$. We use induction on s. When $s = 1$, we have

$$\mathfrak{F}_0^I(M) = \varinjlim_t \Lambda_{\mathfrak{m}}(0 :_M I^t) = \varinjlim_t \varprojlim_k (0 :_M I^t)/\mathfrak{m}^k(0 :_M I^t).$$

As $0 :_M I^t$ is artinian, there is k_t such that $\mathfrak{m}^k(0 :_M I^t) = \mathfrak{m}^{k_t}(0 :_M I^t)$ for all $k \geq k_t$. Then

$$\varprojlim_k (0 :_M I^t)/\mathfrak{m}^k(0 :_M I^t) = (0 :_M I^t)/\mathfrak{m}^{k_t}(0 :_M I^t) \text{ and}$$

$$\begin{aligned}\mathfrak{F}_0^I(M) &= \varinjlim_t (0 :_M I^t)/\mathfrak{m}^{k_t}(0 :_M I^t) \\ &\cong \varinjlim_t (0 :_M I^t)/\varinjlim_t \mathfrak{m}^{k_t}(0 :_M I^t) = M/\varinjlim_t \mathfrak{m}^{k_t}(0 :_M I^t).\end{aligned}$$

Thus, $\mathfrak{F}_0^I(M)$ is artinian and then $\mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)) = V(0 :_{\hat{R}} \mathfrak{F}_0^I(M))$ by [18, 2.3]. Moreover, from the hypothesis we have $V(0 :_R \mathfrak{F}_0^I(M)) \subseteq V(I)$, so $V(0 :_{\hat{R}} \mathfrak{F}_0^I(M)) \subseteq V(I\hat{R})$. It follows from 3.1 that

$$\mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)) = V(0 :_{\hat{R}} \mathfrak{F}_0^I(M)) \subseteq V(\mathfrak{m}\,\hat{R})$$

and then the $\hat{R}-$module $\mathfrak{F}_0^I(M)$ has finite length.

Let $s > 1$. As M is artinian, there is a positive integer m such that $I^tM = I^mM$ for all $t \geq m$. Set $K = I^mM$, then the short exact sequence of artinian $R-$modules

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules by 2.18

$$... \longrightarrow \mathfrak{F}_{i+1}^I(M/K) \longrightarrow \mathfrak{F}_i^I(K) \longrightarrow \mathfrak{F}_i^I(M) \longrightarrow \mathfrak{F}_i^I(M/K) \longrightarrow ....$$

It is clear that M/K is $I-$separated, then $\mathfrak{F}_i^I(M/K)$ is a finitely generated $\widehat{R}-$module for all i by 2.11. Thus, the proof will be complete if we show that $\mathfrak{F}_i^I(K)$ is a finitely generated $\widehat{R}-$module for all $i < s$. We know that $I \subseteq \sqrt{0 : \mathfrak{F}_i^I(M/K)}$, as $\mathfrak{F}_i^I(M/K)$ is a finitely generated $\widehat{R}-$module for all i. By the hypothesis, we have $I \subseteq \sqrt{0 : \mathfrak{F}_i^I(K)}$ for all $i < s$. Since $IK = K$, there is an element $x \in I$ such that $xK = K$ by [7, 2.8]. Then there is a positive integer r such that $x^r\mathfrak{F}_i^I(K) = 0$ for all $i < s$. Now the short exact sequence $0 \longrightarrow 0 :_K x^r \longrightarrow K \xrightarrow{x^r} K \longrightarrow 0$ induces a short exact sequence of formal local homology modules

$$0 \longrightarrow \mathfrak{F}_i^I(K) \longrightarrow \mathfrak{F}_{i-1}^I(0 :_K x^r) \longrightarrow \mathfrak{F}_{i-1}^I(K) \longrightarrow 0$$

for all $i < s$. It follows $I \subseteq \sqrt{0 : \mathfrak{F}^I_{i-1}(0 :_K x^r)}$ for all $i < s$. By the inductive hypothesis $\mathfrak{F}^I_{i-1}(0 :_K x^r)$ is a finitely generated $\widehat{R}-$module for all $i < s$. Therefore $\mathfrak{F}^I_i(K)$ is a finitely generated $\widehat{R}-$module for all $i < s$ and the proof is complete. □

The following theorem gives us the equivalent properties for formal local homology modules $\mathfrak{F}^I_i(M)$ being finitely generated for all $i > s$.

Theorem 3.3. *Let M be an artinian $R-$module and s a non-negative integer. Then the following statements are equivalent:*

(i) *$\mathfrak{F}^I_i(M)$ is a finitely generated $\widehat{R}-$module for all $i > s$;*
(ii) *$I \subseteq \sqrt{0 : \mathfrak{F}^I_i(M)}$ for all $i > s$.*

Proof. $(i) \Rightarrow (ii)$. The argument is similar to that used in the proof of 3.2.

$(ii) \Rightarrow (i)$. We now proceed by induction on $d = \operatorname{Ndim} M$. When $d = 0$, we have $\operatorname{Ndim}(0 :_M I) = 0$ and then $\mathfrak{F}^I_i(M) = 0$ for all $i > 0$.

Let $d > 0$. As M is artinian, there is a positive integer m such that $I^t M = I^m M$ for all $t \geq m$. Set $K = I^m M$, by an argument similar to that in the proof of 3.2 we only need to prove that $\mathfrak{F}^I_i(K)$ is a finitely generated $\widehat{R}-$module for all $i > s$. From the hypothesis, we have $I \subseteq \sqrt{0 : \mathfrak{F}^I_i(K)}$ for all $i > s$. Since $IK = K$, there is an element $x \in I$ such that $xK = K$ by [7, 2.8]. Then there is a positive integer r such that $x^r \mathfrak{F}^I_i(K) = 0$ for all $i > s$. Set $y = x^r$, the short exact sequence

$$0 \longrightarrow 0 :_K y \longrightarrow K \xrightarrow{y} K \longrightarrow 0$$

induces a short exact sequence of formal local homology modules

$$0 \longrightarrow \mathfrak{F}^I_i(K) \longrightarrow \mathfrak{F}^I_{i-1}(0 :_K y) \longrightarrow \mathfrak{F}^I_{i-1}(K) \longrightarrow 0$$

for all $i > s$. It follows $I \subseteq \sqrt{0 : \mathfrak{F}^I_{i-1}(0 :_K y)}$ for all $i > s$. It should be noted by [4, 3.7] that $\operatorname{Ndim}(0 :_K y) \leq d - 1$. From the inductive hypothesis $\mathfrak{F}^I_{i-1}(0 :_K y)$ is a finitely generated $\widehat{R}-$module for all $i > s$. Therefore $\mathfrak{F}^I_i(K)$ is a finitely generated $\widehat{R}-$module for all $i > s$ and the proof is complete. □

There is an question: What is the module $\mathfrak{F}^I_i(M)/I\mathfrak{F}^I_i(M)$? When I is a principal ideal we have an answer in the following theorem.

Theorem 3.4. *Let I be a principal ideal of $(R, \mathfrak{m})$ and M an artinian $R-$module. Then $\mathfrak{F}^I_i(M)/I\mathfrak{F}^I_i(M)$ is a noetherian $\widehat{R}-$module for all i.*

Proof. Assume that I is generated by the element x. As M is artinian, there is a positive integer m such that $x^t M = x^m M$ for all $t \geq m$. Set $K = x^m M$, the short exact sequence of artinian $R-$modules

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} M/K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules

$$... \longrightarrow \mathfrak{F}^I_{i+1}(M/K) \xrightarrow{\delta_{i+1}} \mathfrak{F}^I_i(K) \xrightarrow{f_i} \mathfrak{F}^I_i(M) \xrightarrow{g_i} \mathfrak{F}^I_i(M/K) \longrightarrow ....$$

Then we have short exact sequences

$$0 \longrightarrow \operatorname{Im} f_i \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \operatorname{Im} g_i \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} \delta_{i+1} \longrightarrow \mathfrak{F}^I_i(K) \longrightarrow \operatorname{Im} f_i \longrightarrow 0.$$

These short exact sequences induce the following exact sequences

$$\operatorname{Im} f_i/I \operatorname{Im} f_i \longrightarrow \mathfrak{F}^I_i(M)/I\mathfrak{F}^I_i(M) \longrightarrow \operatorname{Im} g_i/I \operatorname{Im} g_i \longrightarrow 0,$$

$$\operatorname{Im} \delta_{i+1}/I \operatorname{Im} \delta_{i+1} \longrightarrow \mathfrak{F}^I_i(K)/I\mathfrak{F}^I_i(K) \longrightarrow \operatorname{Im} f_i/I \operatorname{Im} f_i \longrightarrow 0.$$

It should be mentioned that M/K is $I-$separated. By 2.11, $\mathfrak{F}^I_i(M/K)$ is a noetherian $\widehat{R}-$module and then $\operatorname{Im} g_i/I \operatorname{Im} g_i$ is a noetherian $\widehat{R}-$module for all i. Thus, the proof is complete by showing that $\mathfrak{F}^I_i(K)/I\mathfrak{F}^I_i(K)$ is a noetherian $\widehat{R}-$module for all i. As $xK = K$, there is a short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{x} K \longrightarrow 0.$$

It gives rise to a long exact sequence

$$... \longrightarrow \mathfrak{F}^I_i(0 :_K x) \longrightarrow \mathfrak{F}^I_i(K) \xrightarrow{x} \mathfrak{F}^I_i(K) \longrightarrow \mathfrak{F}^I_{i-1}(0 :_K x) \longrightarrow ....$$

Note that $0 :_K x$ is $I-$separated, then $\mathfrak{F}^I_i(0 :_K x)$ is a noetherian $\widehat{R}-$module for all i by 2.11. It follows from the long exact sequence that $\mathfrak{F}^I_i(K)/x\mathfrak{F}^I_i(K)$ is a noetherian $\widehat{R}-$module for all i. Therefore $\mathfrak{F}^I_i(K)/I\mathfrak{F}^I_i(K)$ is a noetherian $\widehat{R}-$module for all i and the proof is complete. □

The following theorem shows other conditions for the $\widehat{R}-$module $\mathfrak{F}^I_i(M)/I\mathfrak{F}^I_i(M)$ being noetherian.

Theorem 3.5. *Let M be an artinian $R-$module and s a non-negative integer. If $\mathfrak{F}^I_i(M)$ is a noetherian $\widehat{R}-$module for all $i < s$, then $\mathfrak{F}^I_s(M)/I\mathfrak{F}^I_s(M)$ is a noetherian $\widehat{R}-$module.*

Proof. We use induction on s. When $s=0$, it follows from the proof of 3.2 that $\mathfrak{F}_0^I(M)$ is a quotient module of M, so it is artinian. By [18, 2.3],

$$\begin{aligned}\mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)/I\mathfrak{F}_0^I(M)) &= V(0:_{\hat{R}}\mathfrak{F}_0^I(M)/I\mathfrak{F}_0^I(M))\\ &= V(I\hat{R}+(0:_{\hat{R}}\mathfrak{F}_0^I(M))\\ &= V(I\hat{R})\bigcap V(0:_{\hat{R}}\mathfrak{F}_0^I(M))\\ &= V(I\hat{R})\bigcap \mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)).\end{aligned}$$

It follows from 3.1 that $\mathrm{Cosupp}_{\hat{R}}(\mathfrak{F}_0^I(M)/I\mathfrak{F}_0^I(M))\subseteq V(\mathfrak{m}\hat{R})$ and then the $\hat{R}-$module $\mathfrak{F}_0^I(M)/I\mathfrak{F}_0^I(M)$ has finite length.

Let $s>0$. As M is artinian, there is a positive integer m such that $I^tM=I^mM$ for all $t\geq m$. Set $K=I^mM$, the short exact sequence of artinian $R-$modules

$$0\longrightarrow K\stackrel{f}{\longrightarrow}M\stackrel{g}{\longrightarrow}M/K\longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules

$$...\longrightarrow\mathfrak{F}_{i+1}^I(M/K)\stackrel{\delta_{i+1}}{\longrightarrow}\mathfrak{F}_i^I(K)\stackrel{f_i}{\longrightarrow}\mathfrak{F}_i^I(M)\stackrel{g_i}{\longrightarrow}\mathfrak{F}_i^I(M/K)\longrightarrow ....$$

By an argument similar to that in the proof of 3.4 we only need to prove that $\mathfrak{F}_s^I(K)/I\mathfrak{F}_s^I(K)$ is a noetherian $\widehat{R}-$module. Since $IK=K$, there is an element $x\in I$ such that $xK=K$ by [7, 2.8]. Now the short exact sequence

$$0\longrightarrow 0:_K x\longrightarrow K\stackrel{x}{\longrightarrow}K\longrightarrow 0$$

gives rise to a long exact sequence

$$...\mathfrak{F}_i^I(K)\stackrel{x}{\longrightarrow}\mathfrak{F}_i^I(K)\longrightarrow\mathfrak{F}_{i-1}^I(0:_K x)\longrightarrow\mathfrak{F}_{i-1}^I(K)\stackrel{x}{\longrightarrow}\mathfrak{F}_{i-1}^I(K)....$$

It should be noted by 2.11 that $\mathfrak{F}_i^I(M/K)$ is a noetherian $\widehat{R}-$module for all i. From the first long exact sequence, we deduce that $\mathfrak{F}_i^I(K)$ is a noetherian $\widehat{R}-$module for all $i<s$ by the hypothesis. Then $\mathfrak{F}_i^I(0:_K x)$ is a noetherian $\widehat{R}-$module for all $i<s-1$. It follows from the inductive hypothesis that $\mathfrak{F}_{s-1}^I(0:_K x)/I\mathfrak{F}_{s-1}^I(0:_K x)$ is a noetherian $\widehat{R}-$module. From the last exact sequence we have the following short exact sequence

$$0\longrightarrow\mathfrak{F}_s^I(K)/x\mathfrak{F}_s^I(K)\longrightarrow\mathfrak{F}_{s-1}^I(0:_K x)\longrightarrow 0:_{\mathfrak{F}_{s-1}^I(K)}x\longrightarrow 0.$$

It induces an exact sequence

$$\mathrm{Tor}_1^R(R/I;0:_{\mathfrak{F}_{s-1}^I(K)}x)\longrightarrow\mathfrak{F}_s^I(K)/I\mathfrak{F}_s^I(K)\longrightarrow\mathfrak{F}_{s-1}^I(0:_K x)/I\mathfrak{F}_{s-1}^I(0:_K x).$$

Since $0 :_{\mathfrak{F}^I_{s-1}(K)} x$ is a noetherian $\widehat{R}-$module, $\mathrm{Tor}^R_1(R/I; 0 :_{\mathfrak{F}^I_{s-1}(K)} x)$ is also a noetherian $\widehat{R}-$module. It follows that $\mathfrak{F}^I_s(K)/I\mathfrak{F}^I_s(K)$ is a noetherian $\widehat{R}-$module and the proof is complete. □

Let M be an artinian $R-$module. There is an other question: When are the formal local homology modules $\mathfrak{F}^I_i(M)$ noetherian? The following theorem gives us an answer when $i = \operatorname{Ndim} M$.

Theorem 3.6. *Let M be an artinian $R-$module with $\operatorname{Ndim} M = d$. Then $\mathfrak{F}^I_d(M)$ is a noetherian $\widehat{R}-$module.*

Proof. We prove by induction on $d = \operatorname{Ndim} M$. When $d = 0$, M has finite length. It follows from 2.19 that $\mathfrak{F}^I_0(M) \cong M$. Then $\mathfrak{F}^I_0(M)$ is a noetherian $\widehat{R}-$module.

Let $d > 0$. As M is artinian, there is a positive integer m such that $I^t M = I^m M$ for all $t \geq m$. Set $K = I^m M$, the short exact sequence of artinian $R-$modules

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules by 2.18

$$... \longrightarrow \mathfrak{F}^I_{i+1}(M/K) \longrightarrow \mathfrak{F}^I_i(K) \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \mathfrak{F}^I_i(M/K) \longrightarrow ....$$

Since $IK = K$, there is an element $x \in I$ such that $xK = K$ by [7, 2.8]. It should be noted by [4, 3.7] that $\operatorname{Ndim}(0 :_K I) \leq \operatorname{Ndim}(0 :_K x) \leq d - 1$. Hence $\mathfrak{F}^I_d(K) = 0$ by 2.21. Thus, the long exact sequence follows the following exact sequence

$$0 \longrightarrow \mathfrak{F}^I_d(M) \longrightarrow \mathfrak{F}^I_d(M/K).$$

It is clear that M/K is $I-$separated, then $\mathfrak{F}^I_i(M/K)$ is a finitely generated $\widehat{R}-$module for all i by 2.11. Therefore, $\mathfrak{F}^I_d(M)$ is a noetherian $\widehat{R}-$module. The proof is complete. □

The following theorem gives us the equivalent properties for formal local homology modules $\mathfrak{F}^I_i(M)$ being finitely artinian for all $i > s$.

Theorem 3.7. *Let M be an artinian $R-$module and s a non-negative integer. Then the following statements are equivalent:*

(i) *$\mathfrak{F}^I_i(M)$ is artinian for all $i > s$;*
(ii) *$\mathfrak{F}^I_i(M) = 0$ for all $i > s$;*
(iii) *$\operatorname{Ass}(\mathfrak{F}^I_i(M)) \subseteq \{\mathfrak{m}\}$ for all $i > s$.*

Proof. *(i)* ⇒ *(ii)*. We use induction on $d = \operatorname{Ndim} M$. If $d = 0$, then $\operatorname{Ndim}(0 :_M I) = 0$. By 2.21, $\mathfrak{F}^I_i(M) = 0$ for all $i > 0$.

Let $d > 0$. As M is artinian, there is a positive integer m such that $\mathfrak{m}^t M = \mathfrak{m}^m M$ for all $t \geq m$. Set $K = \mathfrak{m}^m M$, the short exact sequence of artinian $R-$modules

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules by 2.18

$$... \longrightarrow \mathfrak{F}^I_{i+1}(M/K) \longrightarrow \mathfrak{F}^I_i(K) \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \mathfrak{F}^I_i(M/K) \longrightarrow ....$$

It is clear that M/K is $I-$separated, then $\mathfrak{F}^I_i(M/K) \cong H^{\mathfrak{m}}_i(M/K)$ by 2.11. Moreover, M/K is also $\mathfrak{m}-$separated, so $H^{\mathfrak{m}}_i(M/K) = 0$ for all $i > 0$ by [4, 3.8]. Hence

$$\mathfrak{F}^I_i(K) \cong \mathfrak{F}^I_i(M)$$

for all $i > 0$. Thus, the proof will be complete if we show that $\mathfrak{F}^I_i(K) = 0$ for all $i > s$. By the hypothesis, $\mathfrak{F}^I_i(K)$ is artinian for all $i > s$. As $\mathfrak{m} K = K$, there is an element $x \in \mathfrak{m}$ such that $xK = K$ by [7, 2.8]. Now the short exact sequence $0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{x} K \longrightarrow 0$ gives rise to an exact sequence of formal local homology modules

$$... \longrightarrow \mathfrak{F}^I_{i+1}(K) \longrightarrow \mathfrak{F}^I_i(0 :_K x) \longrightarrow \mathfrak{F}^I_i(K) \xrightarrow{.x} \mathfrak{F}^I_i(K) \longrightarrow ....$$

By [4, 4.7] $\operatorname{Ndim}(0 :_K x) \leq d - 1$. Then the inductive hypothesis gives $\mathfrak{F}^I_i(0 :_K x) = 0$ for all $i > s$ and we have an exact sequence

$$0 \longrightarrow \mathfrak{F}^I_i(K) \xrightarrow{.x} \mathfrak{F}^I_i(K)$$

for all $i > s$. It follows that $0 :_{\mathfrak{F}^I_i(K)} x = 0$ for all $i > s$. Since $\mathfrak{F}^I_i(K)$ is artinian for all $i > s$, we conclude that $\mathfrak{F}^I_i(K) = 0$ for all $i > s$.

$(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (i)$. We use induction on $d = \operatorname{Ndim} M$. If $d = 0$, then $\operatorname{Ndim}(0 :_M I) = 0$. By 2.21, $\mathfrak{F}^I_i(M) = 0$ for all $i > 0$.

Let $d > 0$. As M is artinian, there is a positive integer m such that $\mathfrak{m}^t M = \mathfrak{m}^m M$ for all $t \geq m$. Set $K = \mathfrak{m}^m M$, the short exact sequence of artinian $R-$modules

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules by 2.18

$$... \longrightarrow \mathfrak{F}^I_{i+1}(M/K) \longrightarrow \mathfrak{F}^I_i(K) \longrightarrow \mathfrak{F}^I_i(M) \longrightarrow \mathfrak{F}^I_i(M/K) \longrightarrow ....$$

It is clear that M/K is $I-$separated, then $\mathfrak{F}^I_i(M/K) \cong H^{\mathfrak{m}}_i(M/K)$ by 2.11. Moreover, M/K is also $\mathfrak{m}-$separated, so $H^{\mathfrak{m}}_i(M/K) = 0$ for all $i > 0$ by [4, 3.8]. Hence

$$\mathfrak{F}^I_i(K) \cong \mathfrak{F}^I_i(M)$$

for all $i > 0$. Thus, the proof will be complete if we show that $\mathfrak{F}_i^I(K)$ is artinian for all $i > s$. As $\mathfrak{m}\, K = K$, there is an element $x \in \mathfrak{m}$ such that $xK = K$ by [7, 2.8]. Now, the short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{x} K \longrightarrow 0$$

gives rise to a long exact sequence of formal local homology modules

$$\ldots \longrightarrow \mathfrak{F}_{i+1}^I(K) \longrightarrow \mathfrak{F}_i^I(0 :_K x) \longrightarrow \mathfrak{F}_i^I(K) \xrightarrow{.x} \mathfrak{F}_i^I(K) \longrightarrow \ldots.$$

It should be noted by [4, 4.7] that $\mathrm{Ndim}(0 :_K x) \leq d-1$. On the other hand, $\mathrm{Ass}(\mathfrak{F}_i^I(K)) \subseteq \{\mathfrak{m}\}$ for all $i > s$ by the hypothesis. From the exact sequence we have $\mathrm{Ass}(\mathfrak{F}_i^I(0 :_K x)) \subseteq \{\mathfrak{m}\}$ for all $i > s$. Then $\mathfrak{F}_i^I(0 :_K x)$ is artinian for all $i > s$ by the inductive hypothesis. Therefore $0 :_{\mathfrak{F}_i^I(K)} x$ is artinian for all $i > s$. Moreover, as $\mathrm{Ass}(\mathfrak{F}_i^I(K)) \subseteq \{\mathfrak{m}\}$, we get $\Gamma_m(\mathfrak{F}_i^I(K)) = \mathfrak{F}_i^I(K)$ for all $i > s$. From [10, 1.3] we conclude that $\mathfrak{F}_i^I(K)$ is artinian for all $i > s$ and this finishes the proof. □

Acknowledgments. The final work of this paper was done when the authors visited Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank VIASM for hospitality and financial support.

References

[1] M. Bijan-Zadeh, S. Rezaei, Artinianness and Attached Primes of Formal Local Cohomology Modules," *Algebra Colloquium* 21 : 2 (2014), 307-316.

[2] Brodmann M. P., Sharp R. Y., *Local cohomology: an algebraic introduction with geometric applications,* Cambridge University Press (1998).

[3] N. T. Cuong, T. T. Nam, "The $I-$adic completion and local homology for Artinian modules," *Math. Proc. Camb. Phil. Soc.,* 131 (2001), 61-72.

[4] N. T. Cuong, T. T. Nam, A Local homology theory for linearly compact module, *J. Algebra* 319(2008), 4712-4737.

[5] D. Kirby, *Dimension and length of artinian modules",* Quart, J. Math. Oxford **(2) 41**(1990), 419-429.

[6] I. G. Macdonald, Duality over complete local rings, *Topology* **1**(1962), 213-235.

[7] I. G. Macdonald, Secondary representation of modules over a commutative ring, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 2343. Academic Press, London, 1973.

[8] A. Mafi, "Results of formal local cohomology modules," *Bull. Malays. Math. Sci. Soc.* (2) 36 (2013), no. 1, 173-177.

[9] L. Melkersson, Properties of cofinite modules and applications to local cohomology, Math. Proc. Camb. Phil. Soc.(1999),125–417

[10] L. Melkersson, ”On a symptotic stability for sets of prime ideals connected with the powers of an ideal,” *Math. Proc. Camb. Phil. Soc.* 107(1990), 267-271.

[11] T. T. Nam, “Minimax modules, local homology and local cohomology,” *International Journal of Mathematics* Vol. 26, No. 12 (2015) 1550102 (16 pages).

[12] Rezaei S., “Some results on local homology and local cohomology modules,” *Illinois Journal of Mathematics* Vol. 57 No. 1(2013), 17-23.

[13] R. N. Roberts, *Krull dimension for artinian modules over quasi-local commutative rings,* Quart. J. Math. Oxford (3) **26**(1975), 269-273.

[14] R. Y. Sharp. *A method for the study of artinian modules with an application to asymptotic behavior.* In: Commutative Algebra (Math. Sciences Research Inst. Publ. No. 15, Springer-Verlag, 1989), 443-465.

[15] P. Schenzel, On formal local cohomology and connectedness, *J. Algebra*, 315(2), (2007), 894-923

[16] Strooker J., *Homological questions in local algebra,* Cambridge University Press (1990).

[17] Yan Gu, ”The Artinianness Of Formal Local Cohomology Modules,” *Bull. Malays. Math. Sci. Soc.* (2) 37 (2014), no. 2, 449-456.

[18] S. Yassemi, Coassociated primes, Comm. Algebra 23 (4) (1995) 1473-1498.

[19] H. Zöschinger, Linear-Kompakte Moduln uber noetherschen Ringen, *Arch. Math.* **41**(1983), 121-130.

[1]Department of Mathematics-Informatics, Ho Chi Minh University of Pedagogy, Ho Chi Minh city, Vietnam.

E-mail address: namtuantran@gmail.com, namtt@hcmue.edu.vn

[2]Faculty of Mathematics & Computer Science, University of Science, VNU-HCM, Ho Chi Minh City, Vietnam

[3]Faculty of Fundamental Science 2, Posts and Telecommunications Institute of Technology, Ho Chi Minh City, Vietnam

E-mail address: dongocyen.dhsp@gmail.com

[4]Algebra and Geometry Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

[5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail address: nguyenminhtri@tdtu.edu.vn