ON THE COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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ABSTRACT. In this paper, we show that the i-th generalized local cohomology module with respect to a pair of ideals$H^i_{I,J}(M,N)$ is (I,J)-cofinite if one of the followings holds

(i) I is a principal;
(ii) $H^i_{I,J}(M,N)$ is minimax;
(iii) $(R,\mathfrak{m})$ is a local ring and $\dim H^i_{\mathfrak{a}}(M,N) \leq 1$ for all $\mathfrak{a} \in \tilde{W}(I,J)$;
(iv) $(R,\mathfrak{m})$ is a local ring and $\dim(M \otimes_R N) \leq 2$.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity and I, J are ideals of R. Let M be an R-module, the i-th local cohomology module of M with respect to I is denoted by $H^i_I(M)$. In 1969, Grothendieck conjectured that the R-module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated for all i. One year later, Hartshorne [11] showed a counter-example which shows that this conjecture is false even when R is regular. Moreover, he also defined an R-module N to be I-cofinite if $\operatorname{Supp}_R N \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all i and he asked when $H^i_I(M)$ are I-cofinite for all i.

Delfino and Marley [6], Yoshida [23], Kawasaki [13] and Melkersson [15, 16] have studied and showed many results on the cofiniteness of the local cohomology modules.

A generalization of local cohomology functors has been given by J. Herzog in [12]. Let i be a nonnegative integer and M a finitely generated R-module. Then the i-th generalized local cohomology module of

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M and N with respect to I is defined by

$$H_I^i(M,N) \cong \varinjlim_n \operatorname{Ext}_R^i(M/I^nM, N).$$

It is clear that $H_I^i(R,N)$ is just the ordinary local cohomology module $H_I^i(N)$. In [8], the authors showed that $H_I^i(M,N)$ is I-cofinite for all i when $\dim R/I \leq 1$. In [9] or [5], if I is a non-zero principal ideal of R, then the R-module $H_I^i(M,N)$ is I-cofinite for all i. Moreover, if $\dim(M) \leq 2$ or $\dim(N) \leq 2$, then $H_I^i(M,N)$ is I-cofinite for all i ([5]).

Another extension of local cohomology modules which is the local cohomology modules with respect to a pair ideals (I,J) was introduced by R. Takahashi, Y. Yoshino and T. Yoshizawa in [21]. Let I, J be two ideals of R, the functor $\Gamma_{I,J}$ from the category of R-modules to itseft is defined by

$$\Gamma_{I,J}(M) = \{m \in M \mid I^n m \subseteq Jm \text{ for some integer } n\},$$

where M is an R-module. The functor $\Gamma_{I,J}$ is R-linear and left exact. For an integer i, the i-th local cohomology functor $H_{I,J}^i$ is the i-th right derived functor of $\Gamma_{I,J}$. It is clear that if $J=0$, then the functor $H_{I,J}^i$ coincides with the ordinary local cohomology functor H_I^i of Grothendieck.

A natural generalization of local cohomology modules with respect to (I,J) was introduced in [17] as follows: Let M, N be two R-modules, the module $\Gamma_{I,J}(M,N) = \Gamma_{I,J}(\operatorname{Hom}_R(M,N))$. For each finitely generated R-module M, the i-th generalized local cohomology functor $H_{I,J}^i(M,-)$ is the i-th right derived functor of the functor $\Gamma_{I,J}(M,-)$. Clearly, whenever $J=0$, the functor $H_{I,J}^i(M,-)$ is the generalized local cohomology functor $H_I^i(M,-)$ of J. Herzog [12]. On the other hand, when $M=R$, the generalized local cohomology module $H_{I,J}^i(R,N)$ is the local cohomology module $H_{I,J}^i(N)$ in [21].

In [22], the authors defined the module (I,J)-cofinite which is an extension of I-cofinite modules. An R-module M is (I,J)-cofinite if $\operatorname{Supp}_R(M) \subseteq W(I,J)$ and $\operatorname{Ext}_R^i(R/I,M)$ is finitely generated for all $i \geq 0$. In [1], if M is a finitely generated R-module, then $H_{I,J}^i(M)$ are (I,J)-cofinite for all i in three cases

(i) $\dim R/\mathfrak{a} \leq 1$ for all $\mathfrak{a} \in \tilde{W}(I,J)$, where

$$\tilde{W}(I,J) = \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ and } I^n \subseteq \mathfrak{a} + J \text{ for some integer } n\};$$

(ii) $\sup\{i \in \mathbb{N}_0 \mid H_{I,J}^i(R) \neq 0\} = 1$;

(iii) $(R,\mathfrak{m})$ is a local ring with $\dim R \leq 2$.

The purpose of this paper is to investigate the cofiniteness of the generalized local cohomology modules with respect to a pair of ideals. The module $H^i_{I,J}(M,N)$ is (I,J)-cofinite if one of the followings holds

(i) I is a principal (Theorem 2.4);
(ii) $H^i_{I,J}(M,N)$ is minimax (Theorem 2.5);
(iii) $(R,\mathfrak{m})$ is a local ring and $\dim(H^i_{\mathfrak{a}}(M,N)) \leq 1$ for all $\mathfrak{a} \in \tilde{W}(I,J)$ (Theorem 2.9);
(iv) $(R,\mathfrak{m})$ is a local ring and $\dim(M \otimes_R N) \leq 2$ (Theorem 2.11).

Moreover, there are some properties on the top of generalized local cohomology modules with respect to a pair of ideals. In a local ring, $H^{\operatorname{pd} M+\dim(M\otimes_R N)}_{I,J}(M,N)$ is I-cofinite artinian (Theorem 2.14). In this paper, we assume that M is a finitely generated R-module.

2. Main results

We recall the definition of (I,J)-cofinite modules which was introduced in [22].

Definition 2.1. An R-module K is (I,J)-cofinite if $\operatorname{Supp}_R(K) \subseteq W(I,J)$ and $\operatorname{Ext}^i_R(R/I,K)$ is finitely generated for all $i \geq 0$.

The concept of (I,J)-cofinite modules is a generalization of I-cofinite modules. It is well-known that it I is a principal ideal of R, then $H^i_I(M)$ is I-cofinite for all $i \geq 0$ (see [13, Theorem 1]). The first result is an extension of this property.

Theorem 2.2. *Let M be a finitely generated R-module, I a principal ideal of R. Then $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i \geq 0$.*

Proof. It follows from [21, 2.4] that $H^i_{I,J}(M) = 0$ for all $i > 1$. Since $H^0_{I,J}(M)$ is a submodule of M, it is (I,J)-cofinite.

Let $F = \operatorname{Hom}_R(R/I,-)$ and $G = \Gamma_{I,J}(-)$ be functors from the category of R-modules to itself. By [19, 10.47], we have a Grothendieck spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_R(R/I, H^j_{I,J}(M)) \underset{i}{\Longrightarrow} \operatorname{Ext}^{i+j}_R(R/I,M).$$

For each $i \geq 0$, from the homomorphisms of spectral sequence

$$0 \to E_2^{i,1} \overset{d_2^{i,1}}{\to} E_2^{i+2,0} \to 0 \text{ and } 0 \to E_3^{i,1} \overset{d_3^{i,1}}{\to} E_3^{i+3,-1} = 0$$

we deduce that $\operatorname{Ker} d_2^{i,1} = E_3^{i,1} = \ldots = E_\infty^{i,1}$. There is a filtration Φ of submodules of $H^{i+1} = \operatorname{Ext}_R^{i+1}(R/I, M)$

$$0 = \Phi^{i+2}H^{i+1} \subseteq \ldots \subseteq \Phi^0 H^{i+1} = H^{i+1}$$

such that $E_\infty^{j,i-j+1} \cong \Phi^j H^{i+1}/\Phi^{j+1}H^{i+1}$ for all $0 \le j \le i+1$. Note that H^{i+1} and $E_2^{i+2,0}$ are finitely generated, so are $\operatorname{Ker} d_2^{i,1}$ and $\operatorname{Im} d_2^{i,1}$. This clearly forces that $E_2^{i,1}$ is finitely generated. Consequently, $H^1_{I,J}(M)$ is (I,J)-cofinite and the proof is complete. □

Now, there is a minor result on the cofiniteness of $H^1_{I,J}(M,N)$ which will be used in the next theorem.

Lemma 2.3. *Let M, N be finitely generated R-modules, I a principal ideal of R. Then $H^1_{I,J}(M,N)$ is (I,J)-cofinite.*

Proof. Let $F = \Gamma_{I,J}(-)$ and $G = \operatorname{Hom}_R(M,-)$ be functors from the category of R-modules to itself. By [19, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{I,J}(\operatorname{Ext}^q_R(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{I,J}(M,N).$$

For each $r \ge 2$, from the homomorphisms of the spectral sequence

$$0 \to E_r^{1,0} \to E_r^{1+r,1-r} = 0$$

we see that $E_2^{1,0} = \ldots = E_\infty^{1,0}$. There is a filtration Φ of submodules of $H^1 = H^1_{I,J}(M,N)$

$$0 = \Phi^2 H^1 \subseteq \Phi^1 H^1 \subseteq \Phi^0 H^1 = H^1$$

such that

$$E_\infty^{1,0} \cong \Phi^1 H^1/\Phi^2 H^1 = \Phi^1 H^1.$$

Note that by Theorem 2.2, $E_2^{1,0} = H^1_{I,J}(\operatorname{Hom}(M,N))$ is (I,J)-cofinite and so is $\Phi^1 H^1$. Since $E_2^{0,1}$ is finitely generated, it follows that $\Phi^0 H^1/\Phi^1 H^1$ is (I,J)-cofinite.
From the short exact sequence

$$0 \to \Phi^1 H^1 \to \Phi^0 H^1 \to \Phi^0 H^1/\Phi^1 H^1 \to 0$$

we see that $H^1_{I,J}(M,N)$ is (I,J)-cofinite. □

We will state and prove the first main result of this paper. The following theorem can be considered as a generalization of Theorem 2.2 or [5, Theorem 1.1] or [9, Theorem 2.8].

Theorem 2.4. *Let M, N be finitely generated R-modules, I a principal ideal of R. Then $H^i_{I,J}(M,N)$ is (I,J)-cofinite for all $i \ge 0$.*

Proof. Note that $\mathrm{Supp}_R(H^i_{I,J}(M,N)) \subseteq W(I,J)$ for all $i \geq 0$. Since $H^0_{I,J}(M,N)$ is finitely generated, it is (I,J)-cofinite. The module $H^1_{I,J}(M,N)$ is (I,J)-cofinite by Lemma 2.3. Now, we prove that $H^i_{I,J}(M,N)$ is (I,J)-cofinite for all $i \geq 2$.
Consider the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{I,J}(\mathrm{Ext}^q_R(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{I,J}(M,N).$$

For each $i \geq 2$, there is a filtration Φ of submodules of $H^i = H^i_{I,J}(M,N)$

$$0 = \Phi^{i+1}H^i \subseteq \ldots \subseteq \Phi^0 H^i = H^i$$

such that

$$E_\infty^{j,i-j} \cong \Phi^j H^i/\Phi^{j+1}H^i \text{ for all } j \leq i.$$

By [21, 2.4], $E_2^{j,i-j} = H^j_{I,J}(\mathrm{Ext}^{i-j}_R(M,N)) = 0$ for all $j \geq 2$. Thus $\Phi^2 H^i = \ldots = \Phi^{i+1}H^i = 0$.
For each $r \geq 2$, from the homomorphisms of spectral sequence

$$0 \to E_r^{1,i-1} \to E_r^{r+1,i-r} = 0$$

we see that

$$E_2^{1,i-1} = \ldots = E_\infty^{1,i-1} \cong \Phi^1 H^i/\Phi^2 H^i = \Phi^1 H^i.$$

By 2.2, $E_2^{1,i-1}$ is (I,J)-cofinite. Since $E_2^{0,i}$ is finitely generated, it follows that $E_\infty^{0,i} \cong \Phi^0 H^i/\Phi^1 H^i$ is finitely generated. Therefore $\Phi^0 H^i = H^i_{I,J}(M,N)$ is (I,J)-cofinite. □

The following result establishes the relation between the minimaxness and the cofiniteness. Recall that an R-module M is called minimax if there is a finite submodule N of M such that M/N is Artinian (see [24]). The class of minimax modules includes all finitely generated and all Artinian modules.

Theorem 2.5. *Let M, N be two finitely generated R-modules and t a non-negative integer. Assume that $H^i_{I,J}(M,N)$ is minimax for all $i < t$. Then $H^i_{I,J}(M,N)$ is (I,J)-cofinite for all $i < t$ and $\mathrm{Hom}_R(R/I, H^t_{I,J}(M,N))$ is finitely generated.*

Proof. First, we show that $\mathrm{Hom}_R(R/I, H^t_{I,J}(M,N))$ is finitely generated by induction on t.
Combining the isomorphism

$$\mathrm{Hom}_R(R/I, H^0_{I,J}(M,N)) \cong \mathrm{Hom}_R(R/I, \mathrm{Hom}_R(M, H^0_{I,J}(N)))$$

with the hypothesis on N, the assertion holds in the case $t = 0$.
Assume that the statement is true for all $i < t$. The short exact squence

$$0 \to \Gamma_{I,J}(N) \to N \to N/\Gamma_{I,J}(N) \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(M, \Gamma_{I,J}(N)) \to H^i_{I,J}(M, N) \to H^i_{I,J}(M, N/\Gamma_{I,J}(N)) \to \cdots .$$

It follows from [17, 2.6] that $H^i_{I,J}(M, \Gamma_{I,J}(N)) \cong \mathrm{Ext}^i_R(M, \Gamma_{I,J}(N))$ for all $i \geq 0$. Since $\mathrm{Ext}^i_R(M, \Gamma_{I,J}(N))$ is finitely generated and $H^i_{I,J}(M, N)$ is minimax for all $i < t$, we can conclude that $H^i_{I,J}(M, N/\Gamma_{I,J}(N))$ is minimax for all $i < t$. Let $\overline{N} = N/\Gamma_{I,J}(N)$ and note that $\overline{N}$ is I-torsion-free. There is an element $x \in I$ which is regular on $\overline{N}$. Now the short exact sequence

$$0 \to \overline{N} \xrightarrow{.x} \overline{N} \to \overline{N}/x\overline{N} \to 0$$

gives rise to a long exact sequence

$$\cdots H^{t-1}_{I,J}(M, \overline{N}) \xrightarrow{g} H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}) \xrightarrow{f} H^t_{I,J}(M, \overline{N}) \xrightarrow{.x} H^t_{I,J}(M, \overline{N}) \to \cdots .$$

We see that $\mathrm{Im} g$ and $H^i_{I,J}(M, \overline{N}/x\overline{N})$ are minimax for all $i < t - 1$. By the inductive hypothesis, $\mathrm{Hom}_R(R/I, H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}))$ is finitely generated. Now apply the functor $\mathrm{Hom}_R(R/I, -)$ to the short exact

$$0 \to \mathrm{Ker}\, f \to H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}) \to 0 :_{H^t_{I,J}(M,\overline{N})} x \to 0$$

there is an exact sequence
$... \to \mathrm{Hom}_R(R/I, H^{t-1}_{I,J}(M, \overline{N}/x\overline{N})) \to \mathrm{Hom}_R(R/I, 0 :_{H^t_{I,J}(M,\overline{N})} x)$
$\to \mathrm{Ext}^1_R(R/I, \mathrm{Ker}\, f) \to ....$
Combining [14, 5.3] with [15, 2.1] we see that $\mathrm{Ext}^1_R(R/I, \mathrm{Ker}\, f)$ is finitely generated. Consequently, $\mathrm{Hom}_R(R/I, 0 :_{H^t_{I,J}(M,\overline{N})} x)$ is a finitely generated R-module. Moreover

$$\begin{aligned} \mathrm{Hom}_R(R/I, 0 :_{H^t_{I,J}(M,\overline{N})} x) &\cong \mathrm{Hom}_R(R/I, \mathrm{Hom}_R(R/(x), H^t_{I,J}(M, \overline{N}))) \\ &\cong \mathrm{Hom}_R(R/I, H^t_{I,J}(M, \overline{N})) \end{aligned}$$

since $x \in I$. Therefore $\mathrm{Hom}_R(R/I, H^t_{I,J}(M, \overline{N}))$ is finitely generated and so is $\mathrm{Hom}_R(R/I, H^t_{I,J}(M, N))$. It follows from the hypothesis that $\mathrm{Hom}_R(R/I, H^i_{I,J}(M, N))$ is finitely generated for all $i \leq t$.
We use again [14, 5.3] and [15, 2.1] to see that $\mathrm{Ext}^j_R(R/I, H^i_{I,J}(M, N))$ is finitely generated for all $j \geq 0, i < t$. Thus $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i < t$, which completes the proof. □

Corollary 2.6. *Let M, N be finitely generated R-modules and t a non-negative integer. Assume that $H^i_{I,J}(N)$ is minimax for all $i < t$. Then $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_{I,J}(M, N))$ is finitely generated.*

Proof. It follows from [18, 2.7] and 2.5. □

Corollary 2.7. *Let M, N be finitely generated R-modules and t a non-negative integer. Assume that $H^i_{I,J}(M, N)$ is artinian for all $i < t$. Then $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_{I,J}(M, N))$ is finitely generated.*

Lemma 2.8. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules. Assume that t is a non-negative integer and M has finite projective dimension. If $\dim(H^i_I(M, N)) \leq 1$ for all $i \leq t$, then $H^i_I(M, N)$ is a minimax R-module for all $i \leq t$.*

Proof. By [20, 2.7 and 2.8], we can assume that $\dim R/I \leq 1$ and Bass numbers of $H^i_I(M, N)$ are finite for all $i \leq t$. Let $i \leq t$ and E be an injective hull of $H^i_I(M, N)$. Note that $\dim E \leq 1$ and

$$E = \bigoplus_{\mathfrak{p} \in V(I)} E_R(R/\mathfrak{p})^{\mu^0(\mathfrak{p}, H^i_I(M,N))}.$$

Let $\mathfrak{p} \in \operatorname{Ass}_R(E)$, then $\dim R/\mathfrak{p} \leq 1$. Moreover, $V(I)$ is a finite set. It follows from the proof of (3) $\Rightarrow$ (1) of [23, 3.5] that the injective hull $E(R/\mathfrak{p})$ of $R/\mathfrak{p}$ is minimax. Therefore E is a minimax R-module and so is $H^i_I(M, N)$. □

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $\operatorname{pd}(M) < \infty$. Assume that t is a non-negative integer such that $\dim H^i_{\mathfrak{a}}(M, N) \leq 1$ for all $\mathfrak{a} \in \tilde{W}(I, J)$ and for all $i < t$. Then*

(i) *$\Gamma_I(H^q_{I,J}(M, N))$ and $H^1_I(H^{q-1}_{I,J}(M, N))$ is I-cofinite minimax for all $q < t$;*
(ii) *$\operatorname{Hom}(R/I, H^1_I(H^{t-1}_{I,J}(M, N)))$ is finitely generated;*
(iii) *$H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i < t$.*

Proof. Let $F = \Gamma_I(-)$ and $G = \Gamma_{I,J}(M, -)$ be two functors from the category of R-modules to itself. For an R-module N,

$$FG(N) = \Gamma_I(\Gamma_{I,J}(M, N)) = \Gamma_I(M, N).$$

Let E be an injective R-module, we have

$$\begin{aligned} R^iF(G(E)) &= H^i_I(\Gamma_{I,J}(M,E)) \\ &\cong H^i_I(\mathrm{Hom}_R(M,\Gamma_{I,J}(E))) \end{aligned}$$

Let $\mathbf{F}_\bullet$ be a free resolution of M

$$\mathbf{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Since $\Gamma_I(E)$ is an injective R-module, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M,\Gamma_I(E)) \to \mathrm{Hom}_R(\mathbf{F}_\bullet,\Gamma_I(E))$$

and this is an injective resolution of $\mathrm{Hom}_R(M,\Gamma_I(E))$. On the other hand $\Gamma_I(E) = \Gamma_I(\Gamma_{I,J}(E))$, therefore there is an exact sequence

$$0 \to \mathrm{Hom}_R(M,\Gamma_I(\Gamma_{I,J}(E))) \to \mathrm{Hom}_R(\mathbf{F}_\bullet,\Gamma_I(\Gamma_{I,J}(E))).$$

Since M is finitely generated, the exact sequence can be rewritten

$$0 \to \Gamma_I(\mathrm{Hom}_R(M,\Gamma_{I,J}(E))) \to \Gamma_I(\mathrm{Hom}_R(\mathbf{F}_\bullet,\Gamma_{I,J}(E))).$$

Hence $R^iF(G(E)) \cong H^i_I(\mathrm{Hom}_R(M,\Gamma_{I,J}(E))) = 0$ for all $i > 0$. By [19, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_I(H^q_{I,J}(M,N)) \underset{p}{\Rightarrow} H^{p+q}_I(M,N).$$

(i) Since $\dim H^i_{\mathfrak{a}}(M,N) \le 1$ for all $\mathfrak{a} \in \tilde{W}(I,J)$ and for all $i < t$, we have $\dim H^i_{I,J}(M,N) \le 1$ for all $i < t$. Therefore $E_2^{p,q} = 0$ for all $p > 1$ and for all $q < t$.

Let $q < t$ and $r \ge 2$, the homomorphisms of the spectral sequence

$$0 \to E_r^{0,q} \to E_r^{r,1-r+q} = 0$$

induces that $E_2^{0,q} = E_\infty^{0,q} \cong \Phi^0H^q/\Phi^1H^q$, where

$$0 = \Phi^{q+1}H^q \subseteq \ldots \subseteq \Phi^0H^q = H^q_I(M,N)$$

is a filtration of submodules of $H^q = H^q_I(M,N)$.

Similarly, the homomorphisms of the spectral sequence

$$0 \to E_r^{1,q-1} \to E_r^{1+r,q-r} = 0$$

induce that

$$E_2^{1,q-1} = E_\infty^{1,q-1} \cong \Phi^1H^q/\Phi^2H^q.$$

Moreover, note that $E_2^{r,q-r} = 0$ for all $r \ge 2$, then

$$\Phi^2H^q = \Phi^3H^q = \ldots = \Phi^{q+1}H^q = 0.$$

Combining 2.8 with [10, 3.2], we see that $H^q_I(M,N)$ is I-cofinite minimax for all $q < t$. By [15, 4.4], $E_2^{0,q}$ and $E_2^{1,q-1}$ are I-cofinite minimax for all $q < t$.

(ii) There is a filtration of submodules of $H^t = H^t_I(M,N)$

$$0 = \Phi^{t+1}H^t \subseteq \ldots \subseteq \Phi^0 H^t = H^t$$

such that

$$E_\infty^{i,t-i} \cong \Phi^i H^t / \Phi^{i+1} H^t$$

for all $i \leq t$. Note that

$$\Phi^2 H^t = \ldots = \Phi^{t+1} H^t = 0$$

because $E_\infty^{i,t-i} = 0$ for all $i > 1$. From the homomorphisms of spectral sequence

$$0 \to E_2^{1,t-1} \to E_2^{3,t-2} = 0$$

we see that

$$E_2^{1,t-1} = E_\infty^{1,t-1} \cong \Phi^1 H^t / \Phi^2 H^t = \Phi^1 H^t.$$

The homomorphisms of spectral sequence

$$0 \to E_2^{0,t} \to E_2^{2,t-1} = 0$$

deduce that

$$E_2^{0,t} = E_\infty^{0,t} \cong \Phi^0 H^t / \Phi^1 H^t = H^t_I(M,N)/E_2^{1,t-1}.$$

Now the short exact sequence

$$0 \to E_2^{1,t-1} \to H^t_I(M,N) \to E_2^{0,t} \to 0$$

gives rise to a long exact sequence

$$0 \to \mathrm{Hom}_R(R/I, E_2^{1,t-1}) \to \mathrm{Hom}_R(R/I, H^t_I(M,N)) \to \ldots .$$

Since $H^i_I(M,N)$ is minimax for all $i < t$, it follows from [2, 3.6] that $\mathrm{Hom}_R(R/I, H^t_I(M,N))$ is finitely generated. We can conclude that $\mathrm{Hom}_R(R/I, H^1_I(H^{t-1}_{I,J}(M,N)))$ is finitely generated.

(iii) It follows from [1, 2.5] that the proof is complete by showing that $\mathrm{Hom}_R(R/I, H^q_{I,J}(M,N))$ and $\mathrm{Ext}^1_R(R/I, H^q_{I,J}(M,N))$ are finitely generated for all $q < t$. By (i), $\mathrm{Hom}_R(R/I, \Gamma_I(H^q_{I,J}(M,N)))$ is finitely generated for all $q < t$. On the other hand

$$\mathrm{Hom}_R(R/I, \Gamma_I(H^q_{I,J}(M,N))) = \mathrm{Hom}_R(R/I, H^q_{I,J}(M,N))$$

hence $\mathrm{Hom}_R(R/I, H^q_{I,J}(M,N))$ is finitely generated for all $q < t$.

We have a Grothendieck spectral sequence

$$D_2^{i,j} = \mathrm{Ext}^i_R(R/I, H^j_I(K)) \underset{i}{\Rightarrow} \mathrm{Ext}^{i+j}_R(R/I, K).$$

Let $q < t, K = H^q_{I,J}(M,N)$ and $T^1 = \mathrm{Ext}^1_R(R/I, K)$. There is a filtration of submodules of T^1

$$0 = \theta^2 T^1 \subseteq \theta^1 T^1 \subseteq \theta^0 T^1 = T^1$$

such that

$$D_\infty^{0,1} \cong \theta^0 T^1/\theta^1 T^1 \text{ and } D_\infty^{1,0} \cong \theta^1 T^1/\theta^2 T^1 = \theta^1 T^1.$$

The homomorphisms of spectral sequence

$$0 \to D_2^{1,0} \to D_2^{3,-1} = 0$$

deduce that

$$D_\infty^{1,0} = D_2^{1,0} = \mathrm{Ext}_R^1(R/I, \Gamma_I(K)) = \mathrm{Ext}_R^1(R/I, \Gamma_I(H_{I,J}^q(M,N))).$$

It follows from (i) that $D_\infty^{1,0}$ is finitely generated.

On the other hand, we have

$$D_2^{0,1} = \mathrm{Hom}_R(R/I, H_I^1(K)) = \mathrm{Hom}_R(R/I, H_I^1(H_{I,J}^q(M,N))).$$

It follows from (i) and (ii), $\mathrm{Hom}_R(R/I, H_I^1(H_{I,J}^q(M,N)))$ is finitely generated. The short exact sequence

$$0 \to D_\infty^{1,0} \to T^1 \to D_\infty^{0,1} \to 0$$

shows that $T^1 = \mathrm{Ext}_R^1(R/I, H_{I,J}^q(M,N))$ is finitely generated, which completes the proof. □

Theorem 2.10. *Let $(R,\mathfrak{m})$ be a local ring and N a finitely generated R-module with $\dim(N) \leq 2$. Then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \geq 0$.*

Proof. There is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(R/I, H_{I,J}^q(N)) \underset{p}{\Rightarrow} \mathrm{Ext}_R^{p+q}(R/I, N).$$

It follows from [21, 4.7 (1)] that $H_{I,J}^i(N) = 0$ for all $i > 2$. Combining [3, 2.3], [4, 2.1] with [6, Theorem 3], we see that $H_{I,J}^2(N)$ is I-cofinite. The proof is complete by showing that $H_{I,J}^1(N)$ is (I,J)-cofinite. Let $i \geq 0$, from the homomorphisms of spectral sequence

$$0 \to E_3^{i,1} \to 0$$

we have $E_3^{i,1} = E_\infty^{i,1} \cong \Phi^i H^{i+1}/\Phi^{i+1} H^{i+1}$, where

$$0 = \Phi^{i+2} H^{i+1} \subseteq \ldots \subseteq \Phi^0 H^{i+1} = H^{i+1}$$

is a filtration of submodules of $H^{i+1} = \mathrm{Ext}_R^{i+1}(R/I, N)$.

The module H^{i+1} is finitely generated, so is $E_\infty^{i,1}$. Since $H_{I,J}^2(N)$ is I-cofinite, we have $E_2^{i-2,2}$ is finitely generated. On the other hand, $E_3^{i,1} = \mathrm{Ker}(E_2^{i,1} \to E_2^{i+2,0})/\mathrm{Im}(E_2^{i-2,2} \to E_2^{i,1})$ which gives $E_2^{i,1}$ is finitely generated, and the proof is completed . □

Theorem 2.11. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules. If $\dim(M \otimes_R N) \leq 2$, then $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i \geq 0$.*

Proof. Consider the spectral sequence

$$E_2^{p,q} = H^p_{I,J}(\mathrm{Ext}^q_R(M, N)) \underset{p}{\Rightarrow} H^{p+q}_{I,J}(M, N).$$

Since $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M, N)) \subseteq \mathrm{Supp}_R(M \otimes_R N)$ for all $i \geq 0$, it follows that $\dim(\mathrm{Ext}^i_R(M, N)) \leq 2$ for all $i \geq 0$. Let $n \geq 0$, there is a filtration of submodules of $H^n = H^n_{I,J}(M, N)$

$$0 = \Phi^{n+1}H^n \subseteq \ldots \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$. Since $E_2^{i,n-i} = 0$ for all $i > 2$, we have $\Phi^3 H^n = \ldots = \Phi^{n+1} H^n = 0$. From the homomorphisms of the spectral sequence

$$0 \to E_3^{2,n-2} \to 0, 0 \to E_2^{1,n-1} \to 0 \text{ and } 0 \to E_3^{0,n} \to 0$$

we have $E_3^{2,n-2} = E_\infty^{2,n-2}$, $E_2^{1,n-1} = E_\infty^{1,n-1}$ and $E_3^{0,n} = E_\infty^{0,n}$. That $E_2^{2,n-2}$ is I-cofinite artinian follows from [3, 2.3], [4, 2.1] with [6, Theorem 3]. This implies that $E_3^{2,n-2}$ is I-cofinite artinian. By 2.10, $E_2^{1,n-1}$ is (I, J)-cofinite. Therefore $\Phi^1 H^n$ is (I, J)-cofinite by the short exact sequence

$$0 \to \Phi^2 H^n \to \Phi^1 H^n \to \Phi^1 H^n / \Phi^2 H^n \to 0.$$

Now, since $E_2^{0,n}$ is finitely generated, it follows that $\Phi^0 H^n / \Phi^1 H^n$ is finitely generated. Therefore, we can conclude that $\Phi^0 H^n = H^n_{I,J}(M, N)$ is (I, J)-cofinite, and the proof is completed. □

Next Corollarys immediately follow by 2.11.

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules. If $\dim(R) \leq 2$, then $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i \geq 0$.*

Corollary 2.13. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules. If $\dim(M) \leq 2$ or $\dim(N) \leq 2$, then $H^i_{I,J}(M, N)$ is (I, J)-cofinite for all $i \geq 0$.*

Theorem 2.14. *Let $(R, \mathfrak{m})$ be a local ring and M, N two R-modules with $d = \mathrm{pd}(M) + \dim(M \otimes_R N) < \infty$. Then $H^d_{I,J}(M, N)$ is I-cofinite artinian.*

Proof. Consider the spectral sequence

$$E_2^{p,q} = H^p_{I,J}(\mathrm{Ext}^q_R(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{I,J}(M,N).$$

There is a filtration of submodules of $H^d = H^d_{I,J}(M,N)$

$$0 = \Phi^{d+1}H^d \subseteq \ldots \subseteq \Phi^0 H^d = H^d$$

such that $E_\infty^{i,d-i} \cong \Phi^i H^d/\Phi^{i+1}H^d$ for all $i \leq d$. Since $E_2^{p,q} = 0$ when $q > \mathrm{pd}(M)$ or $p > \dim(M \otimes_R N)$, it follows that $E_\infty^{i,d-i} = 0$ for all $i \neq \dim(M \otimes_R N)$. Hence $\Phi^{\dim(M\otimes_R N)+1}H^d = \ldots = \Phi^{d+1}H^d = 0$ and $\Phi^{\dim(M\otimes_R N)}H^d = \ldots = \Phi^0 H^d = H^d$. Moreover, $E_2^{\dim(M\otimes_R N),\mathrm{pd}(M)} = E_\infty^{\dim(M\otimes_R N),\mathrm{pd}(M)}$. Therefore $H^{\dim(M\otimes_R N)}_{I,J}(\mathrm{Ext}^{\mathrm{pd}(M)}_R(M,N)) \cong H^d_{I,J}(M,N)$. If $\dim(\mathrm{Ext}^{\mathrm{pd}(M)}_R(M,N)) < \dim(M \otimes_R N)$, then $H^d_{I,J}(M,N) = 0$. If $\dim(\mathrm{Ext}^{\mathrm{pd}(M)}_R(M,N)) = \dim(M \otimes_R N)$, then $H^d_{I,J}(M,N)$ is I-cofinite artinian by [3, 2.3], [4, 2.1] and [6, Theorem 3]. □

Corollary 2.15. *Let $(R,\mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $t = \mathrm{pd}(M), c = \dim(M \otimes_R N)$ and $d = t + c < \infty$. Then*

$$\mathrm{Att}_R(H^d_{I,J}(M,N)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{Supp}_R N \cap V(J) \mid \mathrm{cd}(I, R/\mathfrak{p}) = c\}.$$

Proof. By the proof of 2.14, we may assume that $\dim(\mathrm{Ext}^t_R(M,N)) = c$. By [3, 2.1], we have

$$\mathrm{Att}_R(H^c_{I,J}(\mathrm{Ext}^t_R(M,N))) = \{\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Ext}^t_R(M,N)) \cap V(J) \mid \mathrm{cd}(I, R/\mathfrak{p}) = c\}.$$

Therefore

$$\begin{aligned}\mathrm{Att}_R(H^d_{I,J}(M,N)) &= \mathrm{Att}_R(H^c_{I,J}(\mathrm{Ext}^t_R(M,N)))\\ &= \{\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Ext}^t_R(M,N)) \cap V(J) \mid \mathrm{cd}(I, R/\mathfrak{p}) = c\}\\ &\subseteq \{\mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{Supp}_R N \cap V(J) \mid \mathrm{cd}(I, R/\mathfrak{p}) = c\},\end{aligned}$$

as required. □

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