Convergence, non-negativity and stability of a new tamed Euler-Maruyama scheme for stochastic differential equations with Hölder continuous diffusion coefficient*

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Abstract

We propose and analyze a new tamed Euler-Maruyama approximation scheme for stochastic differential equations with Hölder continuous diffusion. This new scheme preserves the stability and non-negativity of the exact solution.

Keywords Exponential stability Hölder continuous diffusion Non-negativity Stochastic differential equation Tamed Euler-Maruyama approximation

Mathematics Subject Classification 65C30 65L20 60H10

1 Introduction

Stochastic differential equations (SDEs) appear in many applied areas such as mathematical physics, mathematical biology, mathematical finance... In these areas, it is often necessary to compute the expectation of some function of the solution. Since both the explicit form and the probability distribution of X_t are unknown in general, one needs to develop computable discrete approximation schemes that could be used in some kinds of Monte-Carlo simulation.

Convergence and stability of these schemes are well studied for SDEs with globally Lipschitz continuous coefficients (see [10], [14]). During the last few years, there are numerous efforts to construct effective numerical approximations for SDEs with locally Lipschitz continuous coefficients. In particular, Hutzenthaler et. al [6], [7] showed that the explicit EulerMaruyama (EM) scheme fails to converge strongly to the exact solution of some SDEs with non-globally Lipschitz continuous coefficients. Moreover, they introduced a new numerical method called tamed Euler scheme and showed that it converges in L^p-norm with a standard rate of order 1/2 for a class of SDEs with superlinearly growing, one-sided Lipschitz continuous drift and Lipschitz continuous diffusion coefficients. The tamed EM scheme then has been developed by many authors, see [19], [6], [20], [15], [12], for example.

Since SDEs with Hölder continuous diffusion coefficient appears in many models in mathematical finance and mathematical biology, its numerical approximation has been also considered extensively. In [2], Gyöngy and Rásonyi showed that for the SDEs with $\frac{1}{2} + \alpha$-Hölder continuous diffusion coefficient and Lipschitz continuous drift coefficient, the EulerMaruyama scheme converges in L^1-norm at the rate of order α. Their work was later developed in [1], [16], [17], [15].

In many applications, one needs to evaluate the value of a stable process in a long time period even though it may be very small. Therefore, the stability of SDEs was studied extensively by many authors (see [11], [9] for example). Recently, there have been a number of studies focusing on the stability of the approximated solution of SDEs whose exact solution is stable. The first result in this direction was presented in [21], where Saito and

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Mitsui considered SDEs with linear coefficients. Then the results for general SDEs with Lipschitz and locally Lipschitz coefficients were shown in [4], [5], [11], [13]. Since classical approximation schemes (EM or Milstein, for example) do not preserve the stability of the solution, new approximation schemes such as the implicit θ-EM scheme ([3], [4], [13]) the tamed EM scheme ([22], [24]) have been developed.

In this paper, we will construct a new tamed Euler-Maruyama approximation scheme for SDEs whose diffusion coefficient is Hölder continuous. We will show that the new scheme converges in L^1-norm at the same rate as the plain EM scheme given in [2]. Furthermore, the new scheme preserves the exponential stability of the exact solution. In addition, if the exact solution is non-negative, we can easily modify our scheme to get another approximation which is also non-negative. To the best of our knowledge, this is the first stable numerical approximation scheme for SDEs with Hölder continuous diffusion coefficient. The difficulty arising when studying the stability for such SDEs is that near zero, the size of the diffusion coefficient is of order $|x|^{\frac{1}{2}+\alpha}$ which is much greater than $|x|$, the order of the size of Lipschitz continuous diffusion coefficients (see Assumption **A3** bellow).

The rest of this paper is organized as follows. Section 2 presents the new tamed EM scheme together with its strong convergence, exponential stability, and non-negativity. All proofs are deferred to the Section 3. The last section provides some numerical experiments which compare our new scheme with some other well-known ones.

2 Main Results

2.1 Assumptions

Let $(W_t)_{0\leq t\leq T}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual condition. Let b and σ be real valued, $\mathcal{B}(\mathbb{R})$-measurable functions. We consider a stochastic differential equation given by

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad x_0 \in \mathbb{R}, t \in [0, +\infty). \tag{1}$$

We consider the following assumptions on the coefficients b and σ.

A1. There exists a positive constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq -L_1|x-y|^2,$$

for any $x, y \in \mathbb{R}$.

A2. There exists a positive constant L_2 such that

$$|b(x)-b(y)| \leq L_2|x-y|,$$

for any $x, y \in \mathbb{R}$.

A3. There exist positive constants L_3 and $\alpha \in \left[0, \frac{1}{2}\right]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_3|x-y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$.

A4. For each $R > 0$, there exists a positive constant L_R such that

$$|b(x)-b(y)| \leq L_R|x-y|$$

and

$$|\sigma(x)-\sigma(y)| \leq L_R|x-y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$ such that $|x| \leq R$ and $|y| \leq R$.

A5. There exists a positive constant L such that

$$|b(x)|^2 \vee |\sigma(x)|^2 \leq L(1+|x|^2),$$

for any $x \in \mathbb{R}$.

Under conditions **A4** and **A5**, the equation (1) has a unique solution in the strong sense (see [15], Theorem 3.1).

2.2 Tamed Euler-Maruyama scheme

Suppose that assumptions **A1** and **A2** hold. For each $h \in \left(0, \frac{L_1}{L_2^2}\right)$, we denote $\eta_h(t) = kh$ if $t \in [kh, (k+1)h)$ for some $k = 0, 1, \ldots$, and

$$b_h(x) = \frac{b(x)}{1 - L_2^2 L_1^{-1} h}, \text{ and } \sigma_h(t,x) = \frac{\sigma(x)}{1 + h^{1/2} e^{2L_1 t}(|\sigma(x)| + 1)}.$$

A tamed Euler-Maruyama approximation of equation (1) is defined as follows

$$X_t^h = x_0 + \int_0^t b_h(X_{\eta_h(s)}^h) ds + \int_0^t \sigma_h(\eta_h(s), X_{\eta_h(s)}^h) dW_s, \quad t \in [0, +\infty). \tag{2}$$

This implies that for any $t \geq 0$,

$$X_t^h = X_{\eta_h(t)}^h + b_h(X_{\eta_h(t)}^h)\left(t - \eta_h(t)\right) + \sigma_h(\eta_h(t), X_{\eta_h(t)}^h)\left(W_t - W_{\eta_h(t)}\right). \tag{3}$$

In this paper, we are interested in not only the convergence of the approximation scheme but also its stability. Therefore, we adjust coefficients b and σ in both time and space variables whereas the tamed EM schemes presented in [6], [19], [20] adjust the coeficients in only space variable. Note that when $h \to 0$, both terms $\frac{1}{1 - L_2^2 L_1^{-1} h}$ and $\frac{1}{1 + h^{1/2} e^{2L_1 \eta_h(t)}\left(1 + |\sigma(X_{\eta_h(t)}^h)|\right)}$ tend to 1, which ensures the convergence of X_t^h to X_t for each t fixed.

2.3 Strong convergence

The convergence of the tamed EM scheme in L^p-norm and L^p-sup norm are stated in the following theorem.

Theorem 1. *(i) Let assumptions **A4** and **A5** hold. For any $T > 0$,*

$$\lim_{h \to 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^h - X_t|\right] = 0. \tag{4}$$

*(ii) If $0 < h < \frac{L_1}{2L_2^2} \wedge 1$, and assumptions **A2, A3** hold, then there exists a positive constant $C = C(x_0, L_2, L_3, T, \alpha)$ such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t^h - X_t|] \leq \begin{cases} Ch^{\alpha} & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{C}{\log(1/h)} & \text{if } \alpha = 0, \end{cases} \tag{5}$$

and

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^h - X_t|\right] \leq \begin{cases} Ch^{2\alpha^2} & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{C}{\sqrt{\log(1/h)}} & \text{if } \alpha = 0. \end{cases} \tag{6}$$

*(iii) If $0 < h < \frac{L_1}{2L_2^2} \wedge 1$, and assumptions **A2,A3** hold. For any $p \geq 2$ there exists a constant $C = C(x_0, L_2, L_3, T, p, \alpha)$ such that*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t - X_t^h|^p\right] \leq \begin{cases} \frac{C}{\log(1/h)} & \text{if } \alpha = 0, \\ Ch^{p/2} & \text{if } \alpha = \frac{1}{2}, \\ Ch^{\alpha} & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

The tamed EM scheme (2) converges in L^1-norm, L^1-sup norm, and L^p-sup norm at the same rates as the plain EM scheme does when applying for SDEs with Hölder continuous coefficients (see [2]).

2.4 Exponential stability in L^p-norm

In [24], the authors showed the exponential stability of the exact solution X_t and its EM approximation in L^2-norm when the diffusion coefficient σ is locally Lipschitz continuous. Here, we will show the exponential stability of X_t and X_t^h when σ is Hölder continuous.

Let $\mathcal{T}$ denote the set of all finite stopping times. The following theorem states the exponential stability of the exact solution X_t. It seems to be a known result but we could not find it in the literature.

Theorem 2. *Let **A1**–**A3** hold and* $b(0) = \sigma(0) = 0$.

(i) $(X_t)_{t\geq 0}$ *is exponentially stable in* L^1*-norm in the sense that*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[|X_\tau| e^{L_1\tau}\right] \leq |x_0|.$$

Moreover, for any $q \in (0,1)$,

$$\mathbb{E}\left[\sup_{t\geq 0}\left(|X_t|^q e^{L_1 qt}\right)\right] \leq \frac{(2-q)|x_0|^q}{1-q}. \tag{7}$$

(ii) For any $p > 1$, *it holds that*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[|X_\tau|^p e^{\kappa\tau}\right] \leq |x_0|^p + \frac{p(p-1)(1-2\alpha)L_3^2|x_0|^\lambda}{2(p-\lambda)(\lambda L_1-\kappa)}, \tag{8}$$

where $\lambda = (p-1+2\alpha)\wedge 1$ *and* κ *is any positive constant satisfying* $\kappa < \lambda L_1$, *and* $0 < \kappa \leq pL_1 - \dfrac{L_3^2 p(p-1)(p-1+2\alpha-\lambda)}{2(p-\lambda)}$.

The next result states that the tamed EM approximated solution X_t^h is also exponentially stable under the same assumption as in Theorem 2.

Theorem 3. *Let **A1**–**A3** hold,* $b(0) = \sigma(0) = 0$, *and* $0 < h < \frac{L_1}{2L_2^2} \wedge \frac{1}{2L_1}$. *Then there exists a positive constant* $C = C(x_0, L_1, L_2, L_3)$ *such that*

$$\mathbb{E}\left[|X_t^h|^2 e^{2L_1 t}\right] \leq \frac{C}{h}. \tag{9}$$

In particular, for any $\epsilon > 0$, *it holds that*

$$\lim_{t\to+\infty} \mathbb{E}\left[|X_t^h|^2 e^{(2L_1-\epsilon)t}\right] = 0. \tag{10}$$

2.5 Non-negative approximation

In many practical models, the exact solution X_t is almost surely non-negative. For these models, we would like to construct an approximate solution which is non-negative, stable, and converges to the exact solution at the same rate as X_t^h. Indeed, we will show that $\hat{X}_t^h = |X_t^h|$ is such an approximation.

Corollary 4. *Assume that* $X_t \geq 0$ *almost surely for any* $t \geq 0$.

*(i) Let assumptions **A1**–**A3** hold and* $0 < h < \frac{L_1}{L_2^2} \wedge \frac{1}{2L_1}$. *Then for any* $\epsilon > 0$, *it holds that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[|\hat{X}_t^h|^2 e^{(2L_1-\epsilon)t}\right] = 0.$$

*(ii) Let assumptions **A4** and **A5** hold. For any* $T > 0$, *it holds that*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\hat{X}_t^h - X_t|\right] = 0.$$

*(iii) If $0 < h < \frac{L_1}{2L_2^2} \wedge 1$, and assumptions **A2** and **A3** hold, then there exists a positive constant $C = C(x_0, L_2, L_3, T)$ such that*

$$\sup_{0\le t\le T} \mathbb{E}[|\hat{X}_t^h - X_t|] \le \begin{cases} Ch^\alpha & \text{if } 0 < \alpha \le \frac{1}{2}, \\ \dfrac{C}{\log(1/h)} & \text{if } \alpha = 0. \end{cases}$$

and

$$\mathbb{E}\left[\sup_{0\le t\le T} |\hat{X}_t^h - X_t|\right] \le \begin{cases} Ch^{2\alpha^2} & \text{if } 0 < \alpha \le \frac{1}{2}, \\ \dfrac{C}{\sqrt{\log(1/h)}} & \text{if } \alpha = 0. \end{cases}$$

Moreover, for any $p \ge 2$ there exists a constant $C = C(x_0, L_2, L_3, T, p, \alpha)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t - X_t^h|^p\right] \le \begin{cases} \dfrac{C}{\log(1/h)} & \text{if } \alpha = 0, \\ Ch^{p/2} & \text{if } \alpha = \frac{1}{2}, \\ Ch^\alpha & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Proof. Part (i) follows directly from Theorem 2. Part (ii) and (iii) follow from Theorem 1 and a remark that

$$|\hat{X}_t^h - X_t| = \big||X_t^h| - |X_t|\big| \le |X_t^h - X_t|.$$

□

3 Proofs

3.1 Some auxiliary estimates

Lemma 5 ([18])**.** *Let $\xi = (\xi_t)_{t\ge0}$ be a positive, adapted right continuous process, and A be a continuous increasing process such that*

$$\mathbb{E}[\xi_\tau|\mathcal{F}_0] \le \mathbb{E}[A_\tau|\mathcal{F}_0] \quad a.s.,$$

for any bounded stopping time τ. Then for any $\lambda \in (0,1)$,

$$\mathbb{E}\left[\left(\sup_{t\ge0} \xi_t\right)^\lambda\right] \le \left(\frac{2-\lambda}{1-\lambda}\right) \mathbb{E}\left[\left(\sup_{t\ge0} A_t\right)^\lambda\right].$$

Lemma 6. *Let assumption **A5** hold.*

(i) For any $p > 0$, there exists a positive constant $C_1 = C_1(p, x_0, T, L)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t|^p\right] \le C_1. \tag{11}$$

(ii) If $h < \frac{L_1}{2L_2^2}$, then for any $p \ge 2$, there exist positive constants $C_2 = C_2(p, x_0, T, L)$ and $C_3 = C_3(p, x_0, T, L)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t^h|^p\right] \le C_2, \tag{12}$$

and

$$\sup_{0\le t\le T} \mathbb{E}\left[|X_t^h - X_{\eta_h(t)}^h|^p\right] \le C_3 h^{p/2}. \tag{13}$$

Proof. Since the estimate (11) is well-known, we can omit its proof. The estimate (12) is also followed from classical arguments and the fact that

$$|b_h(x)|^2 \vee |\sigma_h(t,x)|^2 \leq 4L(1+|x|^2).$$

To show (13), we write

$$\begin{aligned}\left|X_t^h - X_{\eta_h(t)}^h\right|^p &\leq 2^{p-1}\left(\left|b_h(X_{\eta_h(t)}^h)h\right|^p + \left|\sigma_h(\eta_h(t), X_{\eta_h(t)}^h)(W_t - W_{\eta_h(t)})\right|^p\right) \\ &\leq 2^{2p-1}L^{p/2}(1+|X_{\eta_h(t)}^h|^2)(h^p + |W_t - W_{\eta_h(t)}|^p).\end{aligned}$$

This fact together with (12) implies the desired result. □

3.2 A modification of the Yamada and Watanabe approximation

In order to show the exponential stability of the exact solution, we propose a modification of the well-known approximation technique of Yamada and Watanabe (see [23], [2]). First, note that for each $p \geq 1, \delta > 1$ and $\varepsilon > 0$ there exist a positive constant $\overline{C}(p,\delta)$ and a continuous function $\psi_{\delta\varepsilon}(p,.) : \mathbb{R} \to \mathbb{R}^+$ such that

(i) $\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta\varepsilon}(p,z)dz = p\varepsilon^{p-1}$,

(ii) $0 \leq \psi_{\delta\varepsilon}(p,z) \leq \overline{C}(p,\delta)z^{p-2}$ for $z \in [\frac{\varepsilon}{\delta}, \epsilon]$; $\psi_{\delta\varepsilon}(p,z) = 0$ for $z \in (0, \frac{\varepsilon}{\delta})$; and $\psi_{\delta\varepsilon}(p,z) = p(p-1)z^{p-2}$ for $z \in (\varepsilon, +\infty)$.

We will approximate the function $x \mapsto |x|^p$ by the function $\phi_{\delta\epsilon}$ defined by

$$\phi_{\delta\varepsilon}(p,x) := \int_0^{|x|}\int_0^y \psi_{\delta\varepsilon}(p,z)dzdy, \quad x \in \mathbb{R}.$$

It is easy to verify that $\phi_{\delta\epsilon}$ has the following properties: for any $x \in \mathbb{R}$

(T1) $\phi'_{\delta\varepsilon}(p,x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(p,|x|)$, where $\phi'_{\delta\varepsilon}(p,x) = \frac{\partial}{\partial x}\phi_{\delta\varepsilon}(p,x)$;

(T2) $p|x|^{p-1}\mathbb{I}_{(\varepsilon;+\infty)}(x) \leq |\phi'_{\delta\varepsilon}(p,x)| \leq p\varepsilon^{p-1}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(x) + p|x|^{p-1}\mathbb{I}_{(\varepsilon;+\infty)}(x)$;

(T3) $\phi_{\delta\varepsilon}(p,x) - p\varepsilon^p \leq |x|^p \leq \varepsilon^p + \phi_{\delta\varepsilon}(p,x)$;

(T4) $\frac{\phi'_{\delta\varepsilon}(p,|x|)}{|x|^p} \leq \frac{p\delta^p}{\varepsilon}$;

(T5) $\phi''_{\delta\varepsilon}(p,|x|) = \psi_{\delta\varepsilon}(p,|x|) \leq \overline{C}(p,\delta)|x|^{p-2}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) + p(p-1)|x|^{p-2}\mathbb{I}_{(\varepsilon;+\infty)}(x)$, where $\phi''_{\delta\varepsilon}(p,x) = \frac{\partial^2}{\partial x^2}\phi_{\delta\varepsilon}(p,x)$.

In the case that $p = 1$, we can choose $\overline{C}(1,\delta) = \frac{2}{\log\delta}$. Moreover, we denote $\phi_{\delta\varepsilon}(x) = \phi_{\delta\varepsilon}(1,x)$ for simplicity.

3.3 Proof of Theorem 1

Note that if b and σ satisfy **A2** and **A3** then they also satisfy **A4** and **A5**. For the moment, we suppose that b and σ satisfy **A4** and **A5**.

In the following, constants are denoted by C which may change from line to line, and which are independent of the time step h, L_R and R, but may depend on L, T, α and x_0.

Consider $h < \frac{L_1}{2L_2^2} \wedge 1$. Set $\tau_R = \inf\{t \geq 0 : |X_t| \geq R\}, \tau_R^h = \inf\{t \geq 0 : |X_t^h| \geq R\}, \tau = \tau_R \wedge \tau_R^h$ and $A = \{\tau_R \leq T\} \cup \{\tau_R^h \leq T\}$. By Lemma 6, we have

$$\mathbb{P}(A) \leq \mathbb{P}(\tau_R \leq T) + \mathbb{P}(\tau_R^h \leq T) \leq \frac{\mathbb{E}(\sup_{0\leq t\leq T}|X_t|^2)}{R^2} + \frac{\mathbb{E}(\sup_{0\leq t\leq T}|X_t^h|^2)}{R^2} \leq \frac{C}{R^2},$$

which, together with Hölder's inequality and Lemma 6, yields

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t - X_t^h|\mathbb{I}_A\right] \le \frac{C}{R}.$$

Set $Y_t^h = X_t - X_t^h$. Since $\tau > T$ on the set $\bar{A} = \Omega \setminus A$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^h|\right] &= \mathbb{E}\left[\sup_{0\le t\le T}|Y_{t\wedge\tau}^h|\mathbb{I}_{\bar{A}}\right] + \mathbb{E}\left[\sup_{0\le t\le T}|Y_t^h|\mathbb{I}_A\right] \\ &\le \mathbb{E}\left[\sup_{0\le t\le T}|Y_{t\wedge\tau}^h|\right] + \frac{C}{R}. \end{aligned} \tag{14}$$

Similarly, we also have

$$\mathbb{E}\left[|Y_t^h|\right] \le \mathbb{E}\left[|Y_{t\wedge\tau}^h|\right] + \frac{C}{R}. \tag{15}$$

Using property **T3** and Itô's formula, we have

$$\begin{aligned}|Y_t^h| &\le \varepsilon + \phi_{\delta\epsilon}(Y_t^h) \\ &\le \varepsilon + \int_0^t \phi'_{\delta\varepsilon}(Y_s^h)\left[\sigma(X_s) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right] dW_s \\ &\quad + \int_0^t \left\{\phi'_{\delta\varepsilon}(Y_s^h)\left[b(X_s) - b_h(X_{\eta_h(s)}^h)\right]\right. \\ &\quad \left. + \frac{\phi''_{\delta\varepsilon}(Y_s^h)}{2}\left[\sigma(X_s) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right]^2\right\} ds,\end{aligned}$$

which implies

$$|Y_{t\wedge\tau}^h| \le \varepsilon + J_1(t) + J_2(t) + J_3(t), \tag{16}$$

where

$$\begin{aligned}J_1(t) &= \int_0^{t\wedge\tau} \phi'_{\delta\varepsilon}(Y_s^h)\left[b(X_s) - b_h(X_{\eta_h(s)}^h)\right] ds, \\ J_2(t) &= \frac{1}{2}\int_0^{t\wedge\tau} \phi''_{\delta\varepsilon}(Y_s^h)\left[\sigma(X_s) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right]^2 ds, \\ J_3(t) &= \int_0^{t\wedge\tau} \phi'_{\delta\varepsilon}(Y_s^h)\left[\sigma(X_s) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right] dW_s.\end{aligned}$$

If follows from **T2** that $|\phi'_{\delta\epsilon}(x)| \le 1$ for all $x \in \mathbb{R}$. Therefore, if $0 < s < \tau \wedge t$, then

$$\begin{aligned}&\left|\phi'_{\delta\varepsilon}(Y_s^h)\left[b(X_s) - b_h(X_{\eta_h(s)}^h)\right]\right| \\ &\le \left|b(X_s) - b(X_s^h)\right| + \left|b(X_s^h) - b(X_{\eta_h(s)}^h)\right| + \left|b(X_{\eta_h(s)}^h) - b_h(X_{\eta_h(s)}^h)\right| \\ &\le L_R|X_s - X_s^h| + L_R|X_s^h - X_{\eta_h(s)}^h| + \frac{2\sqrt{L}L_2^2 h}{L_1}(1 + |X_{\eta_h(s)}^h|),\end{aligned}$$

where we use the estimate that $|b(x) - b_h(x)| \le \frac{2L_2^2}{L_1}\sqrt{L}h(1+|x|)$ for the last term. Therefore, it follows from

Lemma 6 that

$$\mathbb{E}[\sup_{0\le s\le t}|J_1(s)|] \le L_R\mathbb{E}\left[\int_0^{t\wedge\tau}|Y_s^h|ds\right] + L_R\mathbb{E}\left[\int_0^{t\wedge\tau}|X_s^h - X_{\eta_h(s)}^h|ds\right]$$
$$+ Ch\mathbb{E}\left[\int_0^{t\wedge\tau}(1+|X_{\eta_h(s)}^h|)ds\right]$$
$$\le L_R\mathbb{E}\left[\int_0^{t\wedge\tau}|Y_{s\wedge\tau}^h|ds\right] + L_R\mathbb{E}\left[\int_0^{t}|X_s^h - X_{\eta_h(s)}^h|ds\right]$$
$$+ Ch\mathbb{E}\left[\int_0^{t}(1+|X_{\eta_h(s)}^h|)ds\right]$$
$$\le L_R\mathbb{E}\left[\int_0^{t}|Y_{s\wedge\tau}^h|ds\right] + C(L_R+1)\sqrt{h}. \tag{17}$$

With $0 < s < t\wedge\tau$, it follows from **A4** that

$$\left[\sigma(X_s) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right]^2$$
$$\le 3\left[\sigma(X_s) - \sigma(X_s^h)\right]^2 + 3\left[\sigma(X_s^h) - \sigma(X_{\eta_h(s)}^h)\right]^2$$
$$+ 3\left[\sigma(X_{\eta_h(s)}^h) - \sigma_h(\eta_h(s), X_{\eta_h(s)}^h)\right]^2$$
$$\le 3L_R^2|Y_s^h|^{1+2\alpha} + 3L_R^2|X_s^h - X_{\eta_h(s)}^h|^{1+2\alpha} + 3Ch\left[|X_{\eta_h(s)}^h|^4+1\right], \tag{18}$$

where we use the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for the first estimate and the inequality $|\sigma(x)-\sigma_h(t,x)|^2 \le Ch(|x|^4+1)$ if $t\in[0,T]$ for the last one. Following **T5** and Lemma 6, we have

$$\mathbb{E}[\sup_{0\le s\le t}|J_2(s)|] \le \frac{3}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha}T + \frac{L_R^2\delta}{\varepsilon}\mathbb{E}\left[\int_0^{t\wedge\tau}|X_s^h - X_{\eta_h(s)}^h|^{1+2\alpha}ds\right] + \frac{Ch\delta}{\varepsilon}\right\}$$
$$\le \frac{3}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha}T + \frac{L_R^2\delta}{\varepsilon}\mathbb{E}\left[\int_0^{t}|X_s^h - X_{\eta_h(s)}^h|^{1+2\alpha}ds\right] + \frac{Ch\delta}{\varepsilon}\right\}$$
$$\le \frac{3C}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha} + \frac{L_R^2h^{1/2+\alpha}\delta}{\varepsilon} + \frac{h\delta}{\varepsilon}\right\}. \tag{19}$$

Combining this fact and the estimates (16), (17) implies that

$$\mathbb{E}\left[|Y_{t\wedge\tau}^h|\right] \le \varepsilon + L_R\int_0^t\mathbb{E}\left[|Y_{s\wedge\tau}^h|\right]ds + C(L_R+1)\sqrt{h}$$
$$+ \frac{3C}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha} + \frac{L_R^2h^{1/2+\alpha}\delta}{\varepsilon} + \frac{h\delta}{\varepsilon}\right\}.$$

The application of Gronwall's inequality yields

$$\mathbb{E}\left[|Y_{t\wedge\tau}^h|\right] \le \left(\varepsilon + C(L_R+1)\sqrt{h} + \frac{3C}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha} + \frac{L_R^2h^{1/2+\alpha}\delta}{\varepsilon} + \frac{h\delta}{\varepsilon}\right\}\right)e^{L_Rt}. \tag{20}$$

On the other hand, using Burkholder-Davis-Gundy's inequality, we get

$$
\begin{aligned}
\mathbb{E}\left[\left|\sup_{0\le t\le T} J_3(t)\right|\right] \le& 3\mathbb{E}\left[\left\{\int_0^{T\wedge\tau} \left|\sigma(X_s)-\sigma_h(\eta_h(s),X^h_{\eta_h(s)})\right|^2 ds\right\}^{1/2}\right] \\
\le& \sqrt{27}L_R\mathbb{E}\left[\left\{\int_0^T |Y^h_{s\wedge\tau}|^{1+2\alpha} ds\right\}^{1/2}\right] \\
&+ \sqrt{27}L_R\left\{\mathbb{E}\left[\int_0^T \left|X^h_s - X^h_{\eta_h(s)}\right|^{1+2\alpha} ds\right]\right\}^{1/2} \\
&+ C\sqrt{h}\left\{\mathbb{E}\left[\int_0^T \left(|X_{\eta_h(s)}|^4+1\right) ds\right]\right\}^{1/2},
\end{aligned}
$$

where we use (18) for the last estimate. It follows from Lemma 6 that

$$
\mathbb{E}\left[\left|\sup_{0\le t\le T} J_3(t)\right|\right] \le \sqrt{27}L_R\mathbb{E}\left[\left\{\int_0^T |Y^h_{s\wedge\tau}|^{1+2\alpha} ds\right\}^{1/2}\right] + C(1+L_R)h^{(1+2\alpha)/4}. \tag{21}
$$

If $\alpha = 0$, by choosing $\varepsilon = h^{1/4}$ and $\delta = h^{-1/4}$ in (20) we have

$$
\sup_{0\le t\le T} \mathbb{E}[|Y^h_{t\wedge\tau}|] \le C\frac{e^{L_RT}(1+L_R^2)}{\log\frac{1}{h}}. \tag{22}
$$

Combining this fact and (16), (17), (19), and (21) implies

$$
\mathbb{E}\left[\sup_{0\le t\le T} |Y^h_{t\wedge\tau}|\right] \le C\frac{e^{L_RT}(1+L_R^3)}{\sqrt{\log\frac{1}{h}}}.
$$

This fact together with (14) yields,

$$
\mathbb{E}\left[\sup_{0\le t\le T} |Y^h_t|\right] \le C\frac{e^{L_RT}(1+L_R^3)}{\sqrt{\log\frac{1}{h}}} + \frac{C}{R}. \tag{23}
$$

Let $h\downarrow 0$ and then let $R\uparrow\infty$ we obtain (4). Similary, it follows from (22) and (15) that

$$
\sup_{0\le t\le T} \mathbb{E}[|Y^h_t|] \le C\frac{e^{L_RT}(1+L_R^2)}{\log\frac{1}{h}} + \frac{C}{R}. \tag{24}
$$

If $\alpha\in(0,\frac{1}{2}]$, it follows from (21) that

$$
\begin{aligned}
\mathbb{E}\left[\left|\sup_{0\le t\le T} J_3(t)\right|\right] \le& \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le T} |Y^h_{t\wedge\tau}|\right] + \frac{27}{2}L_R^2\int_0^T \left(\mathbb{E}[|Y^h_{s\wedge\tau}|]\right)^{2\alpha} ds \\
&+ C(1+L_R)h^{(1+2\alpha)/4}.
\end{aligned}
$$

Combining this fact and estimates (16), (17), (19) yields

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} |Y^h_{t\wedge\tau}|\right] \le& 2\varepsilon + 2L_R\int_0^T \mathbb{E}|Y^h_{s\wedge\tau}|ds + \frac{6C}{\log\delta}\left\{L_R^2\varepsilon^{2\alpha} + \frac{L_R^2h^{1/2+\alpha}\delta}{\varepsilon} + \frac{h\delta}{\varepsilon}\right\} \\
&+ 27L_R^2\int_0^T \left(\mathbb{E}[|Y^h_{s\wedge\tau}|]\right)^{2\alpha} ds + 2C(1+L_R)h^{(1+2\alpha)/4}.
\end{aligned} \tag{25}
$$

By choosing $\delta = 2, \varepsilon = \sqrt{h}$ in (20), we get

$$\mathbb{E}\left[|Y^h_{t\wedge\tau}|\right] \le C(1+L_R^2)e^{L_R t}h^{\alpha}. \tag{26}$$

This fact together with (25) yields

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y^h_{t\wedge\tau}|\right] \le C(1+L_R^{2+4\alpha})e^{L_R T}h^{2\alpha^2}.$$

Then it follows from (14) that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y^h_t|\right] \le C(1+L_R^{2+4\alpha})e^{L_R T}h^{2\alpha^2} + \frac{C}{R}. \tag{27}$$

Similary, it follows from (26) and (15) that

$$\sup_{0\le t\le T}\mathbb{E}\left[|Y^h_t|\right] \le C(1+L_R^2)e^{L_R t}h^{\alpha} + \frac{C}{R}. \tag{28}$$

Again, first let $h\downarrow 0$ and then let $R\uparrow\infty$ in (27), we obtain (4). The proof of Part (i) is completed.

Next, we prove Part (ii). Suppose that assumptions **A2** and **A3** hold, then L_R does not depend on R. If $\alpha = 0$, let $R\uparrow\infty$ in (24) and (23) we obtain (5) and (6), respectively. If $\alpha\in(0,\frac{1}{2}]$, let $R\uparrow\infty$ in (28) and (27) we also obtain (5) and (6), respectively.

The proof of Part (iii) goes along similar lines as the one for Part (i), so we only sketch it. Suppose b and σ satisfy **A2** and **A3**. We denote by C_p a quantity which is independent of h, but may depend on $L, L_2, L3, T, \alpha, x_0$ and p. The value C_p may also change from line to line. For any $p\ge 2$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Y^h_t|^p\right] \le & C_p\epsilon^p + C_p\int_0^t \mathbb{E}\left[|Y^h_s|^p\right]ds + C_p h^{p/2}\\ &+ \frac{C_p}{(\log\delta)^p}\left\{\varepsilon^{2p\alpha} + \frac{h^{p(1+2\alpha)/2}\delta^p}{\varepsilon^p} + \frac{h^p\delta^p}{\varepsilon^p}\right\}\\ &+ C_p\mathbb{E}\left[\int_0^t |Y^h_s|^{p(1+2\alpha)/2}ds\right] + C_p h^{p(1+2\alpha)/4}.\end{aligned}$$

Using Young's inequality $\frac{p(2\alpha+1)-2}{2(p-1)}x^p + \frac{p}{2(p-1)}(1-2\alpha)x \ge x^{p(2\alpha+1)/2}$, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Y^h_t|^p\right] \le C_p\epsilon^p + C_p\int_0^t \Big(\mathbb{E}\left[|Y^h_s|^p\right] + (1-2\alpha)\mathbb{E}[|Y^h_t|]\Big)ds + C_p h^{p/2}\\ + \frac{C_p}{(\log\delta)^p}\left\{\varepsilon^{2p\alpha} + \frac{h^{p(1+2\alpha)/2}\delta^p}{\varepsilon^p} + \frac{h^p\delta^p}{\varepsilon^p}\right\} + C_p h^{p(1+2\alpha)/4}.\end{aligned} \tag{29}$$

By choosing $\varepsilon = h^{1/4}, \delta = h^{-1/4}$ when $\alpha = 0$; $\varepsilon = h^{1/2}, \delta = 2$ when $\alpha\in(0,\frac{1}{2}]$ in (29), using the esimate (5) and applying Gronwall inequality, we get (6).

3.4 Proof of Theorem 2

Applying Itô's formula for $e^{\kappa t}\phi_{\delta\varepsilon}(p,x)$ for some $\kappa>0$ and $p\ge 1$, and the property **T3**, we obtain

$$\begin{aligned}|X_t|^p e^{\kappa t} &\le \varepsilon^p e^{\kappa t} + \phi_{\delta\varepsilon}(p,X_t)\,e^{\kappa t}\\ &\le \varepsilon^p e^{\kappa t} + p\varepsilon^p + |x_0|^p + \int_0^t e^{\kappa s}\phi'_{\delta\varepsilon}(p,X_s)\sigma(X_s)dW_s\\ &\quad + \int_0^t e^{\kappa s}\left[\phi'_{\delta\varepsilon}(p,X_s)b(X_s) + \frac{1}{2}\phi''_{\delta\varepsilon}(p,X_s)\sigma^2(X_s) + \kappa|X_s|^p + \kappa p\varepsilon^p\right]ds.\end{aligned} \tag{30}$$

Thanks to **A1, A2, T1, T2**,

$$\begin{aligned}\phi'_{\delta\varepsilon}(p,X_s)b(X_s) &= \phi'_{\delta\varepsilon}(p,X_s)\,b(X_s)\mathbb{I}_{\{|X_s|\le\varepsilon\}} + \frac{\phi'_{\delta\varepsilon}(p,|X_s|)}{|X_s|}X_s b(X_s)\mathbb{I}_{\{|X_s|>\varepsilon\}}\\ &\le p\varepsilon^{p-1}|b(X_s)|\mathbb{I}_{\{|X_s|\le\varepsilon\}} - pL_1|X_s|^p\mathbb{I}_{\{|X_s|>\varepsilon\}}\\ &\le pL_2\varepsilon^p\mathbb{I}_{\{|X_s|\le\varepsilon\}} - pL_1|X_s|^p(1-\mathbb{I}_{\{|X_s|\le\varepsilon\}})\\ &\le p(L_1+L_2)\varepsilon^p - pL_1|X_s|^p. \end{aligned}\tag{31}$$

It follows from condition **A3** and the property **T5** that

$$\begin{aligned}&\phi''_{\delta\varepsilon}(p,X_s)\sigma^2(X_s) = \phi''_{\delta\varepsilon}(p,|X_s|)\sigma^2(X_s)\\ &\le \overline{C}(p,\delta)L_3^2|X_s|^{p-1+2\alpha}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|X_s|) + L_3^2p(p-1)|X_s|^{p-1+2\alpha}\mathbb{I}_{(\varepsilon;+\infty)}(|X_s|)\\ &\le \overline{C}(p,\delta)L_3^2\varepsilon^{p-1+2\alpha} + L_3^2p(p-1)|X_s|^{p-1+2\alpha}. \end{aligned}\tag{32}$$

Combining (30), (31), and (32), we have

$$\begin{aligned}|X_t|^pe^{\kappa t} &\le \varepsilon^pe^{\kappa t} + p\varepsilon^p + |x_0|^p + \int_0^t e^{\kappa s}\phi'_{\delta\varepsilon}(X_s)\sigma(X_s)dW_s\\ &+ \int_0^t e^{\kappa s}\left[p(L_1+L_2)\varepsilon^p - pL_1|X_s|^p + \frac{1}{2}\overline{C}(p,\delta)L_3^2\varepsilon^{p-1+2\alpha}\right]ds\\ &+ \int_0^t e^{\kappa s}\left[\frac{p(p-1)L_3^2}{2}|X_s|^{p-1+2\alpha} + \kappa|X_s|^p + \kappa p\varepsilon^p\right]ds\\ &\le \varepsilon^pe^{\kappa t} + p\varepsilon^p + |x_0|^p + \int_0^t e^{\kappa s}\phi'_{\delta\varepsilon}(X_s)\sigma(X_s)dW_s\\ &+ \left[p(L_1+L_2)\varepsilon^p + \frac{1}{2}\overline{C}(p,\delta)L_3^2\varepsilon^{p-1+2\alpha} + \kappa p\varepsilon^p\right]\frac{e^{\kappa t}-1}{\kappa}\\ &+ \int_0^t e^{\kappa s}\left[(\kappa - pL_1)|X_s|^p + \frac{p(p-1)L_3^2}{2}|X_s|^{p-1+2\alpha}\right]ds. \end{aligned}\tag{33}$$

Part (i): Consider $p=1$. We choose $\overline{C}(1,\delta) = \dfrac{2}{\log\delta}$, and $\kappa = L_1$, then for any $N>0, \varepsilon>0$, and finite stopping time τ, it holds that

$$\begin{aligned}\mathbb{E}[|X_{\tau\wedge N}|e^{L_1(\tau\wedge N)}] &\le \varepsilon\mathbb{E}\left[e^{L_1(\tau\wedge N)}\right] + \varepsilon + |x_0|\\ &\quad + \left[(2L_1+L_2)\varepsilon + \frac{L_3^2\varepsilon^{2\alpha}}{\log\delta}\right]\mathbb{E}\left[\frac{e^{L_1(\tau\wedge N)}-1}{L_1}\right]\\ &\le \varepsilon\left(e^{L_1N}+1\right) + |x_0| + \left[(2L_1+L_2)\varepsilon + \frac{L_3^2\varepsilon^{2\alpha}}{\log\delta}\right]\left(\frac{e^{L_1N}-1}{L_1}\right). \end{aligned}$$

First let $\delta\uparrow\infty$, and then let $\varepsilon\downarrow 0$, we have

$$\mathbb{E}[|X_{\tau\wedge N}|e^{L_1(\tau\wedge N)}] \le |x_0|.$$

Because $|X_{\tau\wedge N}|e^{L_1(\tau\wedge N)} \xrightarrow{a.s} X_\tau e^{L_1\tau}$ as $N\to\infty$, by Fatou's lemma, we obtain

$$\mathbb{E}\left[|X_\tau|e^{L_1\tau}\right] \le |x_0|. \tag{34}$$

This fact together with Lemma 5 implies (7).
Part (ii): Consider $p>1$. Since $0<\lambda<1\wedge(p-1+2\alpha)$, using Young's inequality, we get

$$|X_s|^{p-1+2\alpha} \le \frac{1-2\alpha}{p-\lambda}|X_s|^\lambda + \frac{p-1+2\alpha-\lambda}{p-\lambda}|X_s|^p.$$

From (33),

$$
\begin{aligned}
|X_t|^p e^{\kappa t} \le\ & \varepsilon^p e^{\kappa t} + p\varepsilon^p + |x_0|^p + \int_0^t e^{\kappa s} \phi'_{\delta\varepsilon}(X_s)\sigma(X_s) dW_s \\
& + \left[p(L_1 + L_2 + \kappa)\varepsilon^p + \frac{1}{2}\overline{C}(p,\delta) L_3^2 \varepsilon^{p-1+2\alpha} \right] \frac{e^{\kappa t} - 1}{\kappa} \\
& + \int_0^t e^{\kappa s} \left(\kappa - pL_1 + \frac{p(p-1)(p-1+2\alpha-\lambda)L_3^2}{2(p-\lambda)} \right) |X_s|^p ds \\
& + \int_0^t e^{\kappa s} \frac{p(p-1)(1-2\alpha)L_3^2}{2(p-\lambda)} |X_s|^\lambda ds \\
\le\ & \varepsilon^p e^{\kappa t} + p\varepsilon^p + |x_0|^p + \int_0^t e^{\kappa s} \phi'_{\delta\varepsilon}(X_s)\sigma(X_s) dW_s \\
& + \left[p(L_1 + L_2 + \kappa)\varepsilon^p + \frac{1}{2}\overline{C}(p,\delta) L_3^2 \varepsilon^{p-1+2\alpha} \right] \frac{e^{\kappa t} - 1}{\kappa} \\
& + \int_0^t e^{\kappa s} \frac{p(p-1)(1-2\alpha)L_3^2}{2(p-\lambda)} |X_s|^\lambda ds, \qquad (35)
\end{aligned}
$$

where we use the fact that $\kappa \le pL_1 - \dfrac{L_3^2 p(p-1)(p-1+2\alpha-\lambda)}{2(p-\lambda)}$. For any $N > 0$, $\varepsilon > 0$, and finite stopping time τ,

$$
\begin{aligned}
\mathbb{E}\left[|X_{\tau\wedge N}|^p e^{\kappa(\tau\wedge N)} \right] \le\ & \varepsilon^p e^{\kappa N} + p\varepsilon^p + |x_0|^p \\
& + \left[p(L_1 + L_2 + \kappa)\varepsilon^p + \frac{1}{2}\overline{C}(p,\delta) L_3^2 \varepsilon^{p-1+2\alpha} \right] \frac{e^{\kappa N} - 1}{\kappa} \\
& + \int_0^N \frac{p(p-1)(1-2\alpha)L_3^2}{2(p-\lambda)} \mathbb{E}\left[e^{\kappa s} |X_s|^\lambda \right] ds.
\end{aligned}
$$

Let $\varepsilon \downarrow 0$, we have

$$
\mathbb{E}\left[|X_{\tau\wedge N}|^p e^{\kappa(\tau\wedge N)} \right] \le |x_0|^p + \int_0^N \frac{p(p-1)(1-2\alpha)L_3^2}{2(p-\lambda)} \mathbb{E}\left[e^{\kappa s} |X_s|^\lambda \right] ds.
$$

From (34) and Holder's inequality, we get $\mathbb{E}\left[e^{\kappa s} |X_s|^\lambda \right] \le |x_0|^\lambda e^{(\kappa - \lambda L_1)s}$. Since $\kappa < \lambda L_1$, we get

$$
\begin{aligned}
\mathbb{E}\left[|X_{\tau\wedge N}|^p e^{\kappa(\tau\wedge N)} \right] \le |x_0|^p + \int_0^N \frac{p(p-1)(1-2\alpha)L_3^2}{2(p-\lambda)} |x_0|^\lambda e^{(\kappa-\lambda L_1)s} ds \le |x_0|^p \\
+ \frac{p(p-1)(1-2\alpha)L_3^2 |x_0|^\lambda}{2(p-\lambda)(\lambda L_1 - \kappa)}.
\end{aligned}
$$

Let $N \uparrow \infty$ and apply Fatou's lemma, we obtain (8).

3.5 Proof of Theorem 3

It follows from (3) that we can write $\mathbb{E}\left[|X^h_{(k+1)h}|^2 \right]$ as

$$
\mathbb{E}\left[|X^h_{kh}|^2 \right] + 2h\mathbb{E}\left[X^h_{kh} b_h(X^h_{kh}) \right] + h^2 \mathbb{E}\left[|b_h(X^h_{kh})|^2 \right] + h\mathbb{E}\left[|\sigma_h(kh, X^h_{kh})|^2 \right].
$$

Thanks to **A1** and **A2**, and the fact that $|\sigma_h(kh, X^h_{kh})| \le h^{-1/2} e^{-2L_1 kh}$, we get

$$
\mathbb{E}\left[|X^h_{(k+1)h}|^2 \right] \le \left[1 - \frac{2L_1 h}{1 - L_2^2 L_1^{-1} h} + \frac{L_2^2 h^2}{\left(1 - L_2^2 L_1^{-1} h\right)^2} \right] \mathbb{E}\left[|X^h_{kh}|^2 \right] + e^{-4L_1 kh}. \qquad (36)
$$

Because $1-\frac{2L_1h}{1-L_2^2L_1^{-1}h}+\frac{L_2^2h^2}{\left(1-L_2^2L_1^{-1}h\right)^2}\leq 1-2L_1h$ when $h<\frac{L_1}{2L_2^2}\wedge\frac{1}{2L_1}$, it follows from (36) that

$$\mathbb{E}\left[|X_{kh}^h|^2\right]\leq(1-2L_1h)^k|x_0|^2+\sum_{i=0}^{k-1}e^{-4L_1(k-1-i)h}(1-2L_1h)^i.$$

Using the simple estimate $e^x\geq x+1$, we get

$$\mathbb{E}\left[|X_{kh}^h|^2\right]\leq e^{-2L_1kh}|x_0|^2+\sum_{i=0}^{k-1}e^{-4L_1(k-1-i)h-2L_1ih}.$$

After some elementary estimates, we get

$$\mathbb{E}\left[|X_{kh}^h|^2\right]\leq\frac{|x_0|^2+e^2}{2L_1}\frac{e^{-2L_1kh}}{h}. \tag{37}$$

Moreover, it follows from (3) that

$$\mathbb{E}\left[|X_t^h|^2\right]\leq 3\left\{\mathbb{E}\left[|X_{\eta_h(t)}^h|^2\right]+h^2\mathbb{E}\left[|b_h\left(X_{\eta_h(t)}^h\right)|^2\right]+h\mathbb{E}\left[|\sigma_h(kh,X_{kh}^h)|^2\right]\right\}.$$

Using again the estimates $|b_h(x)|\leq 2L_2|x|$ and $|\sigma_h(t,x)|\leq h^{-1/2}e^{-2L_1t}$, we get

$$\mathbb{E}\left[|X_t^h|^2\right]\leq 3(1+4L_2^2h^2)\mathbb{E}\left[|X_{\eta_h(t)}^h|^2\right]+3e^{-4L_1\eta_h(t)}.$$

This fact together with (37) implies (9). The relation (10) is a direct consequence of (9). The proof is complete.

4 Numerical experiments

We consider the following SDE

$$X_t=0.1-\int_0^t X_s ds+\int_0^t |X_s|^{\alpha+\frac{1}{2}}dW_s. \tag{38}$$

It is well-known that such equation has a unique strong non-negative solution (see [8] , [15]). Moreover, it follows from Theorem 2 that $(X_t)_{t\geq 0}$ is exponentially stable in L^1-norm. We consider three numerical schemes for this equation: the plain EM, the backward EM and the non-negative tamed EM $\hat{X}$ introduced in Section 2.5. Recall that the plain EM (PEM) for equation (38) is given by

$$\begin{cases}X_0^{PEM,h}=x_0,\\ X_{(k+1)h}^{PEM,h}=X_{kh}^{PEM,h}-X_{kh}^{PEM,h}h+|X_{kh}^{PEM,h}|^{\alpha+\frac{1}{2}}(W_{(k+1)h}-W_{kh}),\ k\geq 0,\end{cases}$$

and the backward EM (BEM) for it is given by

$$\begin{cases}X_0^{BEM,h}=x_0,\\ X_{(k+1)h}^{BEM,h}=X_{kh}^{BEM,h}-X_{(k+1)h}^{BEM,h}h+|X_{kh}^{BEM,h}|^{\alpha+\frac{1}{2}}(W_{(k+1)h}-W_{kh}),\ k\geq 0.\end{cases}$$

The convergence of the plain EM scheme was considered in [2], where the authors showed that it converges at the same rate as the tamed EM $\hat{X}^h$. The backward EM has been also studied in [3], [14], [13].

In our numerical experiment, we consider the L^1-norm of the approximation error

$$e_{h,T}=\mathbb{E}[|X_t-X_T^h|].$$

In particular, we focus on the errors at two points $T=1$ (short time) and $T=5$ (long time). Since we also concern with the convergence rate with respect to the value of α, we consider $\alpha=0.05$ and $\alpha=0.45$. Although we do not know the explicit form of the solution to (38), Corollary 4 guarantees that the non-negative tamed EM

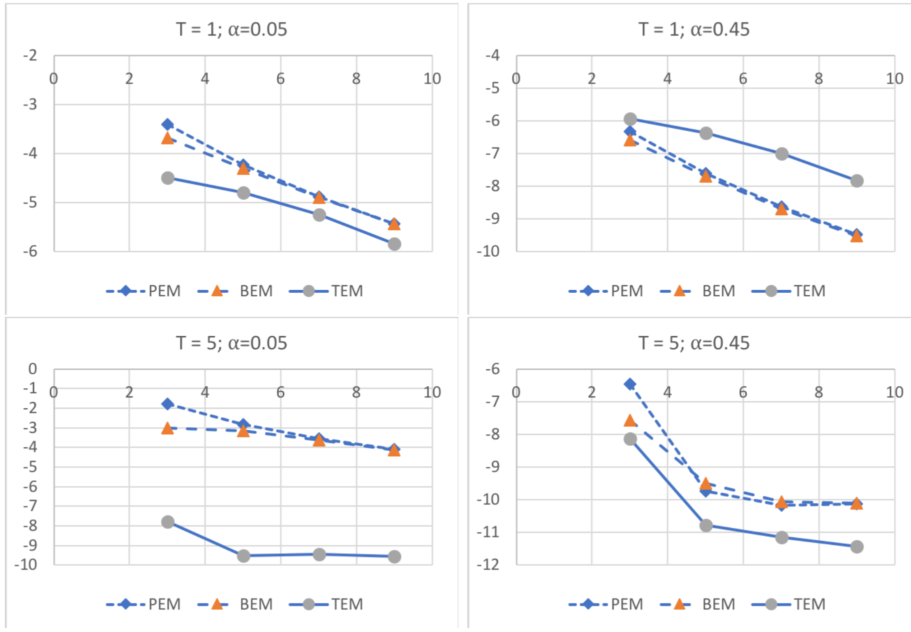

Figure 1: Strong error plots in $\log_2 - \log_2$ scale of plain EM scheme (PEM), backwark EM scheme (BEM) and non-negative tamed EM scheme (TEM)

approximation $\hat{X}_t^h$ strongly converges to the true solution. Therefore, it is reasonable to take the non-negative tamed EM approximation $\hat{X}_t^h$ with a very small time step $h = 2^{-15}$ as a reference solution. We then simulate the sample paths of X_t by using the three numerical approximation schemes with timesteps $2^3h, 2^5h, 2^7h$, and 2^9h. The values of $e_{h,T}$ corresponding to each approximation scheme are computed via the Monte-Carlo method based on $N = 5000$ such sample paths.

In Figure 1, we plot $e_{h,T}$ against h on a $\log_2 - \log_2$ scale. We see that the plain EM scheme and the backward EM scheme converge at almost the same rate. Compare with the non-negative tamed EM scheme, their approximation errors are smaller in the regular case, i.e., when T is small and α is large. However, in irregular case when T is large or α is small, the approximation error of the non-negative tamed EM scheme is much smaller than that of plain and backward EM ones.

Next, we consider the stability of each numerical schemes. In Figure 2, we plot sample paths of $\hat{X}^h, X^{PEM,h}$ and $X^{BEM,h}$ on the interval $[0, 5]$ based on a common sample path of the Brownian motion W. We choose the stepsize $h = 0.01$. We can see that the sample path of the non-negative tamed EM scheme is stable at 0 while the ones of plain and backward EM schemes seem unstable. Moreover, the non-negative tamed EM scheme preserves the non-negativity of the solution while the other schemes do not.

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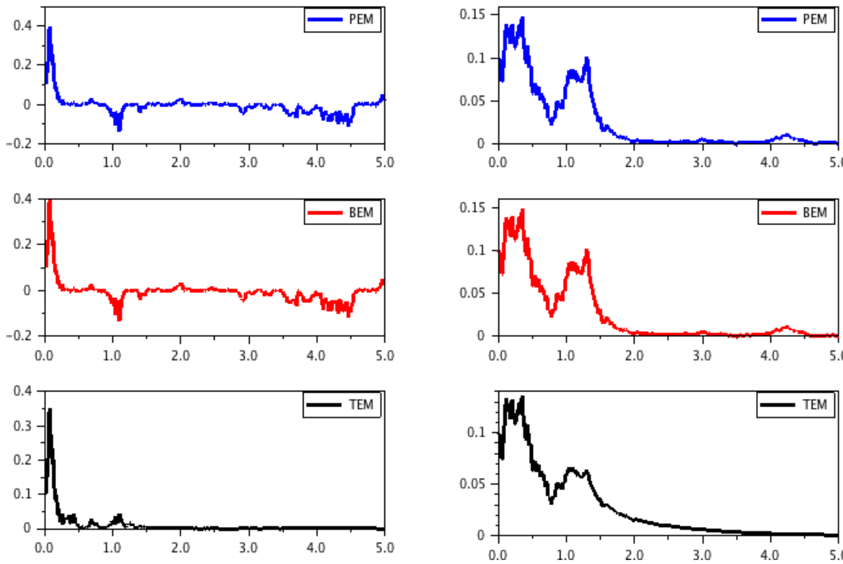

Figure 2: Approximate sample paths of $(X_t)_{0\leq t\leq 5}$, with $\alpha = 0.05$ (left) and $\alpha = 0.45$ (right).

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