INSTABILITY OF SOLUTIONS TO KIRCHHOFF TYPE PROBLEMS IN LOW DIMENSION

NHAT VY HUYNH AND PHUONG LE

ABSTRACT. In this paper, we study the Kirchhoff type problem

$$\begin{cases} -m\left(\int_\Omega w_1|\nabla u|^p\,dx\right)\operatorname{div}\left(w_1|\nabla u|^{p-2}\nabla u\right) &= w_2 f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $p \geq 2$, Ω is a C^1 domain of $\mathbb{R}^N$, w_1, w_2 are nonnegative functions, m is a positive function and f is an increasing one. Under some assumptions on Ω, w_1, w_2, m and f, we prove that the problem has no nontrivial stable solution in dimension $N < N^\#$. Moreover, additional assumptions on Ω, m or the boundedness of solutions can boost this critical dimension $N^\#$ to infinity.

1. Introduction and main results

The aim of this paper is to establish the nonexistence of nontrivial stable solutions of the Kirchhoff type problem

$$\begin{cases} -m\left(\|u\|_{w_1}^p\right)\operatorname{div}\left(w_1|\nabla u|^{p-2}\nabla u\right) &= w_2 f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\|u\|_{w_1} = \left(\int_\Omega w_1|\nabla u|^p\,dx\right)^{\frac{1}{p}}$. Here and throughout the paper, we always assume that

(i) $p \geq 2$ and Ω is a (bounded or unbounded) domain of $\mathbb{R}^N$ with C^1 boundary,
(ii) $w_1, w_2 \in L^1_{\text{loc}}(\Omega) \setminus \{0\}$ are nonnegative functions,
(iii) $m : [0, +\infty) \to \mathbb{R}$, $m > 0$ in $(0, +\infty)$ and $m \in C^1((0, +\infty))$,
(iv) $f : (a, b) \to \mathbb{R}$ is an increasing function and $f \in C^1((a, b)) \cap C^2((a, b) \setminus Z_f)$, where $-\infty \leq a < b \leq +\infty$ and Z_f is the set of zeros of f.

Clearly, Z_f has at most one element since f is increasing. We denote by z_f the unique zero of f when $Z_f \neq \emptyset$.

Problem (1.1) has the origin in a physical model introduced by Kirchhoff in 1883. Indeed, Kirchhoff [19] proposed a model for vibration of an elastic

2010 *Mathematics Subject Classification.* 35J92, 35J25, 35B53, 35B35.
Key words and phrases. Kirchhoff problems, stable solutions, nonexistence, Liouville theorems.

Correspondence should be addressed to Phuong Le; lephuong@tdtu.edu.vn.

string given by the equation

$$\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L u_x^2 \, dx \right) u_{xx} = 0. \tag{1.2}$$

Here ρ, P_0, h, E, L are positive constants which have the following physical meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the string length during the vibrations. It is worth mentioning that equation (1.2) received much attention after Lions [28] introduced a functional analysis framework for the problem.

In problem (1.1), w_1 and w_2 are usually regarded as weights, f is the nonlinearity and m is the nonlocal term. Because of the presence of this nonlocal term, the first equation in (1.1) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which make the study of such problem particularly interesting. Moreover, due to the degenerate nature of the weighted p-Laplace operator $\Delta_{p,w_1} u = \operatorname{div}(w_1 |\nabla u|^{p-2} \nabla u)$ when $p > 2$, solutions to this problem must be understood in the weak sense. It was proved in the well known papers [6, 27, 32] that the best regularity of solutions to (1.1) when $p > 2$ is $C^{1,\alpha}(\Omega)$. The existence and multiplicity of solutions to (1.1) and related problems was studied intensively in recent years via the variational method, see, for instance, [17, 18, 24, 25, 29–31] and references therein.

In this paper, we study solutions of (1.1) in the following weak sense.

Definition 1.1. Let $u \in C^1(\overline{\Omega})$ such that u vanishes on $\partial\Omega$. We also assume $\|u\|_{w_1} < \infty$ if m is not constant. We say that

(i) u is a *solution* of (1.1) if

$$m\left(\|u\|_{w_1}^p\right) \int_\Omega w_1 |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_\Omega w_2 f(u) \varphi \, dx \tag{1.3}$$

for all $\varphi \in C_c^1(\overline{\Omega})$ which vanishes on $\partial\Omega$,

(ii) u is a *stable solution* of (1.1) if u is a solution and

$$\begin{aligned} m\left(\|u\|_{w_1}^p\right) \int_\Omega w_1 \left[|\nabla u|^{p-2} |\nabla \varphi|^2 + (p-2) |\nabla u|^{p-4} (\nabla u \nabla \varphi)^2 \right] \\ + p m'\left(\|u\|_{w_1}^p\right) \left(\int_\Omega w_1 |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \right)^2 \geq \int_\Omega w_2 f'(u) \varphi^2 \, dx \end{aligned} \tag{1.4}$$

for all $\varphi \in C_c^1(\overline{\Omega})$ which vanishes on $\partial\Omega$.

Moreover, if $\Omega = \mathbb{R}^N$, then each solution of (1.1) is called an *entire solution*. We also call a solution *nontrivial* if it is not constant in Ω. Since $w_2 \not\equiv 0$ in Ω, it is clear from the definition that (1.1) has no trivial stable solution if f has no zero or m has no derivative at 0. Let us recall that nontrivial solutions of (1.1) may be found by searching for critical points u of the associated energy functional

$$E(u) = \frac{1}{p} M\left(\|u\|_{w_1}^p\right) - \int_\Omega w_2 F(u)\, dx,$$

where $M(t) = \int_0^t m(s)\, ds$ and $F(t) = \int_0^t f(s)\, ds$, i.e., $E'(u)[\varphi] = 0$. One may also observe that (1.4) can be rewritten as $E''(u)[\varphi, \varphi] \geq 0$. Hence, the stability condition translates into the fact that the second variation at u of E is nonnegative. Consequently, all local minima of E are stable solutions of (1.1). For more mathematical background and physical motivation, we refer to the excellent monograph [9] by Dupaigne and references therein.

This paper is concerned with nonexistence results, which are also called Liouville type theorems, for problem (1.1). This type of theorem was first established for bounded analytic functions by Cauchy [3] in 1844. From then on, several Liouville theorems were proved for elliptic equations in bounded and unbounded domains. One of the most well-known Liouville theorems for nonlinear problems is the one in [14]. In this pioneering paper, Gidas and Spruck proved that the Lane-Emden equation $-\Delta u = |u|^{q-1}u$ has no positive entire solution if $1 < q < q_S(N)$, where

$$q_S(N) = \begin{cases} +\infty & \text{if } N = 2, \\ \frac{N+2}{N-2} & \text{if } N > 2 \end{cases}$$

is the Sobolev exponent.

There has been a recent surge of interest in Liouville theorems for stable solutions of elliptic problems, mainly after Farina's influential papers [10–12]. Such results are important in studying extremal solutions (see [1] and references therein), as well as in establishing Liouville theorems for a larger class of solutions (see [5, 13] for example). In [10, 11], Farina proved that the only stable entire solution of $-\Delta u = |u|^{q-1}u$ is the trivial one provided that $1 < q < q_c(N)$. Here, $q_c(N)$ is the Joseph-Lundgren exponent (see [16]) defined by

$$q_c(N) = \begin{cases} +\infty & \text{if } N \leq 10, \\ \frac{N^2-8N+4+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11. \end{cases}$$

One may observe that $q_c(N) > q_S(N)$ and the assumption $1 < q < q_c(N)$ is equivalent to $N < \frac{4q+4\sqrt{q(q-1)}}{q-1} + 2$. Later, similar results were obtained for p-Laplace equations with or without weights and with several types of nonlinearities. We refer to [5, 21] for p-Laplace equations with Lane-Emden nonlinearity, to [15, 23] for negative exponent nonlinearity and to [2, 26] for exponential nonlinearity. There are also some works dealing with more general nonlinearities. For instance, we refer to [7, 8] for equation $-\Delta u = f(u)$ and to [2, 20] for equation $-\Delta_p u = f(u)$. One crucial assumption in these papers is that f is nonnegative and convex, besides other assumptions. A typical result in [2] is the nonexistence of nontrivial stable entire solutions to p-Laplace equation $-\Delta_p u = f(u)$ in dimension

$$N < \frac{p}{p-1}\frac{2+(p-1)\Gamma+2\sqrt{(p-1)\gamma-(p-2)}}{(p-1)\Gamma-(p-2)}$$

under the assumption that $f \in C^2(\mathbb{R})$ is a positive, increasing, convex function and $\frac{p-2}{p-1} \le \gamma \le \frac{ff''}{(f')^2} \le \Gamma$. A more general result was established recently in [22].

Liouville theorems for stable solutions of Kirchhoff type problems were also obtained recently by some authors. In [24], the authors proved that the equation $-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right) = |x|^\alpha|u|^{q-1}u$ in $\mathbb{R}^N$ has no nontrivial stable solution in dimension $N < \frac{2(p+2)(q+\sqrt{q(q-3)})}{3(q-1)} + 2$ if $\alpha > -2$ and $q > 3$. Similar results were established in [33] for the equation $-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right) = h(x)f(u)$ in $\mathbb{R}^N$, where $f(u) = e^u$ or $f(u) = |u|^{q-1}u$. However, there is currently no result in literature for Kirchhoff type problems involving weighted p-Laplace operators in general domains or general nonlinearities to the best of our knowledge.

In this paper, we extend previous works by establishing Liouville theorems for nontrivial stable solutions of (1.1) for a very large class of nonlocal terms m and nonlinearities f. We also deal with general C^1 domain $\Omega \subset \mathbb{R}^N$ and nonnegative weights w_1, w_2 in our problems. In order to state our main result, let us denote by B_R the ball centered at the origin of $\mathbb{R}^N$ with radius $R > 0$. For $c \in \mathbb{R}$, we define $\operatorname{sign}(c) = 1$ if $c \ge 0$ and $\operatorname{sign}(c) = -1$ otherwise. We also denote $|U|$ the Lebesgue measure of $U \subset \mathbb{R}^N$ and use the convention $\int_U \frac{1}{w}\,dx = +\infty$ if w is a nonnegative measurable function which equals to zero in a subset of U having positive Lebesgue measure. Our main result is the following Liouville type theorem which can be seen as an extension of some results in [2, 4, 5, 11, 22, 24, 33].

Theorem 1.2. *Assume that there exist* $\tau > \frac{1}{p} - 1$, $\Gamma > \frac{p-2}{p-1}$, $\gamma > \frac{p+p\tau-2}{p+p\tau-1}$ *and* $\alpha \in (\max\{\underline{\alpha}, -1, (p-2)\Gamma - p + 1\}, \overline{\alpha})$, *where*

$$\underline{\alpha} = \frac{2-2\sqrt{(p+p\tau-1)(\gamma-1)+1}}{p+p\tau-1} - 1,$$

$$\overline{\alpha} = \frac{2 + 2\sqrt{(p + p\tau - 1)(\gamma - 1) + 1}}{p + p\tau - 1} - 1,$$

such that the following conditions hold

$$(M) \qquad t \mapsto \frac{m(t)}{t^\tau} \text{ is non-increasing in } (0, \infty),$$

(F_1)
$$C|f(t)|^{2\Gamma} \le \gamma f'(t)^2 \le f(t)f''(t) \text{ for some } C > 0 \text{ and all } t \in (a, b) \setminus Z_f,$$

$$(F_2) \qquad |f|^{\frac{\alpha+1}{2}} \operatorname{sign}(f) \in C^1((a, b)),$$

$$(W) \qquad \lim_{R \to +\infty} R^{\frac{-p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \int_{\Omega \cap B_{2R} \setminus B_R} w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}} \, dx = 0,$$

$$(FW) \qquad Z_f = \emptyset \text{ or } w_2 > 0 \text{ a.e. in } \Omega,$$

$$(FD) \qquad \Omega = \mathbb{R}^N \text{ or } f(0) = 0.$$

Then problem (1.1) *has no nontrivial stable solution.*

Remark 1.3. Since $f \in C^1((a, b))$, we have $|f|^{\frac{\alpha+1}{2}} \operatorname{sign}(f) \in C^1((a, b) \setminus Z_f)$. Therefore, one may observe that the assumption (F_2) is automatically satisfied if $\alpha \ge 1$ or f has no zero. Moreover, assumption (F_1) is satisfied if one of the following two conditions hold

(i) $\Gamma \ge \gamma$ and
$$\gamma f'(t)^2 \le f(t)f''(t) \le \Gamma f'(t)^2 \text{ for all } t \in (a, b) \setminus Z_f,$$
$$\liminf_{t \to a^+} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } f(t_1) \le 0 \text{ for some } t_1 \in (a, b),$$
$$\liminf_{t \to b^-} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } f(t_2) \ge 0 \text{ for some } t_2 \in (a, b),$$

(ii) $\gamma \ge \Gamma$ and
$$\gamma f'(t)^2 \le f(t)f''(t) \text{ for all } t \in (a, b) \setminus Z_f,$$
$$\liminf_{t \to a^+} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } f > 0 \text{ in } (a, b),$$
$$\liminf_{t \to b^-} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } f < 0 \text{ in } (a, b),$$
$$\liminf_{t \to z_f} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } Z_f \ne \emptyset.$$

Indeed, we have $(|f(t)|^{-\Gamma}f'(t))' = [f(t)f''(t) - \Gamma f'(t)^2]\,|f(t)|^{-\Gamma-2}f(t)$. Therefore, if f satisfies (i), then $|f|^{-\Gamma}f'$ is non-decreasing in (a,c) and non-increasing in (c,b), where

$$c = \begin{cases} a, & \text{if } f>0 \text{ in } (a,b),\\ b, & \text{if } f<0 \text{ in } (a,b),\\ z_f, & \text{otherwise.} \end{cases}$$

Hence, $f'(t) \ge C|f(t)|^{\Gamma}$ and (F_1) is satisfied. Otherwise, if f satisfies (ii), then $|f|^{-\Gamma}f'$ is non-increasing in (a,c) and non-decreasing in (c,b). Hence, also in this case, $f'(t) \ge C|f(t)|^{\Gamma}$ and (F_1) is satisfied. Let us remark that assumption (i) appeared first in [2]. Some examples of increasing functions f that satisfy (i) and (ii) are

- $f(t) = |t|^{q-1}t$ in $(-\infty,+\infty)$ where $q>1$ and $\Gamma = \gamma = \frac{q-1}{q}$,
- $f(t) = -t^q$ in $(0,+\infty)$ where $q<0$ and $\Gamma = \gamma = \frac{q-1}{q}$,
- $f(t) = e^t$ in $(-\infty,+\infty)$, where $\Gamma = \gamma = 1$.

Now we consider the following condition on w_1 and w_2:

$$(W') \qquad \frac{w_1(x)}{|x|^{q_1}} \le c_1 \text{ and } \frac{w_1(x)}{|x|^{q_1}} \le c_2\frac{w_2(x)}{|x|^{q_2}} \text{ for a.e. } |x|>R_0,$$

where $q_1,q_2 \in \mathbb{R}$ and $c_1,c_2,R_0 > 0$. Clearly, this condition holds when $w_1 \equiv |x|^{q_1}$ and $w_2 \equiv |x|^{q_2}$.

Remark 1.4. Let

$$N_\alpha = \frac{(p-q_1)(\Gamma+\alpha+1) - q_2[(p-2)\Gamma - p - \alpha + 1]}{(p-1)\Gamma - (p-2)}.$$

Then assumption (W) is satisfied if

$$\lim_{R\to+\infty} R^{-N_\alpha}|\Omega \cap B_{2R} \setminus B_R| = 0$$

and (W') holds.

Indeed, since $(p-2)\Gamma - p - \alpha + 1 < 0$ and

$$w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}} = w_1\left(\frac{w_2}{w_1}\right)^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}},$$

we have for $R>R_0$

$$R^{\frac{-p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \int_{\Omega\cap B_{2R}\setminus B_R} w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}}\,dx \le CR^{-N_\alpha}|\Omega\cap B_{2R}\setminus B_R|.$$

In view of (W'), we may get the following consequence from Theorem 1.2.

Proposition 1.5. *Assume (FD), (FW) hold, (M) holds for some $\tau > \frac{1}{p}-1$, (W') holds for some $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$, (F_1) holds for some $\Gamma > \frac{p-2}{p-1}$, $\gamma > \frac{p+p\tau-2}{p+p\tau-1}$ and (F_2) holds for all $\alpha \in (\overline{\alpha}-\varepsilon, \overline{\alpha})$, where $\overline{\alpha}$ is defined as in Theorem 1.2 and $\varepsilon > 0$. Moreover, assume that $(p-2)\Gamma - p - \overline{\alpha} + 1 < 0$ and*

$$\lim_{R\to+\infty} R^{-N^{\#}+\varepsilon}|\Omega \cap B_{2R} \setminus B_R| = 0, \tag{1.5}$$

where

$$N^{\#} = \frac{p+q_2-q_1}{p+p\tau-1}\frac{2+(p+p\tau-1)\Gamma+2\sqrt{(p+p\tau-1)(\gamma-1)+1}}{(p-1)\Gamma-(p-2)} - q_2.$$

Then (1.1) has no nontrivial stable solution.

Remark 1.6. One may choose $\varepsilon < N^{\#} - N$ to see that (1.5) is satisfied if $N < N^{\#}$. If $f(0) = 0$, then (1.5) is satisfied in one of the following cases

- Ω is bounded,
- Ω has finite Lebesgue measure and $N^{\#} > 0$,
- $\Omega \subset \mathbb{R}^K \times \omega$ and $K < \min\{N^{\#}, N\}$, where $\omega \subset \mathbb{R}^{N-K}$ is any domain with finite $(N-K)$-dimensional Lebesgue measure.

Remark 1.7. The case that $\Omega = \mathbb{R}^N$, $m \equiv 1$ and $w_1 \equiv w_2 \equiv 1$ was studied in [2, Theorem 1.3]. However, in [2], f must be positive and convex, b is not allowed to be $+\infty$ and Liouville results there only apply to one-side bounded stable solutions. Therefore, our statement in Proposition 1.5 is new even in this particular case.

Proposition 1.5 claims that problem (1.1) has no nontrivial stable solution in any dimension if Ω is bounded or Ω has finite Lebesgue measure and $N^{\#} > 0$. It is interesting that even in the case $\Omega = \mathbb{R}^N$, we are still able to establish Liouville theorems without any restriction on dimension N if some suitable additional conditions on m or the boundedness of solutions are assumed.

Proposition 1.8. *Assume (FD), (FW) hold, (F_1) holds for some $\Gamma > \frac{p-2}{p-1}$, $\gamma > 0$ and (W') holds for some $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$. Moreover, assume that*

$$t \mapsto t^{1-\frac{1}{p}}m(t) \text{ is non-increasing in } (0,\infty). \tag{1.6}$$

Then (1.1) has no nontrivial stable solution.

Proposition 1.9. *Assume $\Omega = \mathbb{R}^N$, (M) holds for some $\tau > \frac{1}{p}-1$ and (W') holds for some $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$. Moreover, assume that one of the following conditions holds*

(i) $f > 0$ in (a,b) and $a = -\infty$,

(ii) $f < 0$ *in* (a, b) *and* $b = +\infty$

and there exists $\gamma > \frac{p+p\tau-2}{p+p\tau-1}$ *such that*

$$\gamma f'(t)^2 \leq f(t)f''(t) \quad \text{for all } t \in (a, b).$$

Then equation (1.1) *has no bounded below stable solution if (i) holds and no bounded above stable solution if (ii) holds.*

As applications of Proposition 1.5 and Proposition 1.9, we derive the following Liouville theorems for problem (1.1) with Lane-Emden, negative exponent, exponential or singular nonlinearities.

Corollary 1.10. *(Lane-Emden nonlinearity) Let* u *be a stable solution of the problem*

$$\begin{cases} -m\left(\int_\Omega w_1|\nabla u|^p\,dx\right)\operatorname{div}(w_1|\nabla u|^{p-2}\nabla u) &= w_2|u|^{q-1}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Assume that (M) *holds for some* $\tau > \frac{1}{p} - 1$, (W') *holds for some* $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$ *and* $w_2 > 0$ *a.e. in* Ω. *Moreover, assume* $q > p + p\tau - 1$ *and*

$$\lim_{R\to+\infty} R^{-N^\# + \varepsilon}|\Omega \cap B_{2R} \setminus B_R| = 0$$

for some $\varepsilon > 0$, *where*

$$N^\# = \frac{p + q_2 - q_1}{p + p\tau - 1}\frac{2q + (q-1)(p+p\tau-1) + 2\sqrt{q(2q-1)(p+p\tau-1) + q^2}}{q - p + 1} - q_2.$$

Then $u \equiv 0$.

Corollary 1.11. *(Negative exponent nonlinearity) Assume that* (M) *holds for some* $\tau > \frac{1}{p} - 1$ *and* (W') *holds for some* $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$. *Moreover, assume* $q < 0$. *Then equation*

$$m\left(\int_\Omega w_1|\nabla u|^p\,dx\right)\operatorname{div}\left(w_1|\nabla u|^{p-2}\nabla u\right) = w_2 u^q \quad \text{in } \mathbb{R}^N$$

has no positive stable solution in dimension $N < N^\#$ *and has no bounded above positive stable solution in any dimension, where* $N^\#$ *is defined as in Corollary 1.10.*

Corollary 1.12. *(Exponential nonlinearity) Assume that* (M) *holds for some* $\tau > \frac{1}{p} - 1$ *and* (W') *holds for some* $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$. *Then equation*

$$-m\left(\int_\Omega w_1|\nabla u|^p\,dx\right)\operatorname{div}\left(w_1|\nabla u|^{p-2}\nabla u\right) = w_2 e^u \quad \text{in } \mathbb{R}^N$$

has no stable solution in dimension

$$N < N^{\#} := \frac{(p-q_1)(p+p\tau+3)+4q_2}{p+p\tau-1}$$

and has no bounded below stable solution in any dimension.

Corollary 1.13. *(Singular nonlinearities) Assume that (M) holds for some $\tau > \frac{1}{p} - 1$ and (W') holds for some $q_1 \in \mathbb{R}$, $q_2 > q_1 - p$. Moreover, assume that $f : (0, +\infty) \to \mathbb{R}$ has one of the following forms*

(i) $f(t) = -e^{rt^{-s}}$,
(ii) $f(t) = -t^{-r} - t^{-s}$,
(iii) $f(t) = -e^{-rt} - t^{-s}$,

where $r, s > 0$. Then equation

$$-m\left(\int_\Omega w_1|\nabla u|^p\,dx\right)\operatorname{div}\left(w_1|\nabla u|^{p-2}\nabla u\right) = w_2 f(u) \quad \text{in } \mathbb{R}^N$$

has no bounded above positive stable solution.

Remark 1.14. To prove Corollary 1.13, it is sufficient to check that

$$\frac{f(t)f''(t)}{f'(t)^2} > 1$$

for all $t \in (0, \infty)$ and then to apply Proposition 1.9. For instance, if $f(t) = -e^{-rt} - t^{-s}$, we have

$$\begin{aligned}\frac{f(t)f''(t)}{f'(t)^2} &= \frac{\left(r^2e^{-rt} + s(s+1)t^{-s-2}\right)\left(e^{-rt} + t^{-s}\right)}{\left(re^{-rt} + st^{-s-1}\right)^2} \\ &= 1 + \frac{st^{-s-2}\left(e^{-rt} + t^{-s}\right) + \left(r - st^{-1}\right)^2 t^{-s}e^{-rt}}{\left(re^{-rt} + st^{-s-1}\right)^2} > 1.\end{aligned}$$

The rest of the paper is devoted to the proofs of our main results, namely, Theorem 1.2, Proposition 1.5, 1.8 and 1.9. We always denote by C a generic positive constant whose values may vary depending on the situation. If this constant depends on an arbitrary small positive number ε, then we will denote it by C_ε. We also use Young inequality in the form $ab \le \varepsilon a^q + C_\varepsilon b^{q'}$ for $a, b > 0$ and $q, q' > 1$ satisfying $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Proofs of main results

In this section, we define

$$w(t) = (p-1)m(t) + ptm'(t)$$

for $t > 0$.

The following lemma can be easily proved by using Hölder's inequality.

Lemma 2.1. *If u is a nontrivial stable solution of* (1.1)*, then for all $\varphi \in C_c^1(\overline{\Omega})$ which vanishes on $\partial\Omega$, we have*

$$w\left(\|u\|_{w_1}^p\right)\int_\Omega w_1|\nabla u|^{p-2}|\nabla\varphi|^2\,dx \geq \int_\Omega w_2 f'(u)\varphi^2\,dx. \tag{2.1}$$

The main ingredient in the proof of Theorem 1.2 is the following lemma.

Lemma 2.2. *Assume that u is a nontrivial stable solution of* (1.1) *and $\phi,\psi \in C^1(\mathbb{R})$ such that $\psi'(t) \geq \phi'(t)^2$ and $\phi'(t) > 0$ for a.e. $t \in \mathbb{R}$. Then for all nonnegative functions $\eta \in C_c^1(\Omega)$ and $\varepsilon \in (0,1)$ we have*

$$\int_\Omega w_2 H_\varepsilon(u)\eta^p\,dx \leq C_\varepsilon \int_\Omega w_1 G(u)|\nabla\eta|^p\,dx,$$

where

$$\begin{aligned} H_\varepsilon(u) &= \frac{1}{w\left(\|u\|_{w_1}^p\right)} f'(u)\phi(u)^2 - \frac{1+\varepsilon}{m\left(\|u\|_{w_1}^p\right)} f(u)\psi(u),\\ G(u) &= |\phi(u)|^p\phi'(u)^{2-p} + |\psi(u)|^p\psi'(u)^{1-p}. \end{aligned}$$

Moreover, the same conclusion holds for all nonnegative functions $\eta \in C_c^1(\mathbb{R}^N)$ if $\phi(0) = \psi(0) = 0$.

Proof of Lemma 2.2. We test (1.3) with $\varphi = \psi(u)\eta^p$ to obtain

$$\begin{aligned} &\int_\Omega w_1\psi'(u)\eta^p|\nabla u|^p\,dx + p\int_\Omega w_1\psi(u)\eta^{p-1}|\nabla u|^{p-2}\nabla u\nabla\eta\,dx\\ &\qquad = \frac{1}{m\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f(u)\psi(u)\eta^p\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_\Omega w_1\psi'(u)\eta^p|\nabla u|^p\,dx - \frac{1}{m\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f(u)\psi(u)\eta^p\,dx\\ &\leq p\int_\Omega w_1|\psi(u)|\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta|\,dx\\ &\leq \int_\Omega \varepsilon\left(w_1^{\frac{p-1}{p}}\psi'(u)^{\frac{p-1}{p}}\eta^{p-1}|\nabla u|^{p-1}\right)^{\frac{p}{p-1}} + C_\varepsilon\left(w_1^{\frac{1}{p}}|\psi(u)|\psi'(u)^{\frac{1-p}{p}}|\nabla\eta|\right)^p\,dx\\ &= \varepsilon\int_\Omega w_1\psi'(u)\eta^p|\nabla u|^p\,dx + C_\varepsilon\int_\Omega w_1|\psi(u)|^p\psi'(u)^{1-p}|\nabla\eta|^p\,dx, \end{aligned}$$

which implies

$$\begin{aligned} (1-\varepsilon)\int_\Omega w_1\psi'(u)\eta^p|\nabla u|^p\,dx &\leq \frac{1}{m\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f(u)\psi(u)\eta^p\,dx\\ &\quad + C_\varepsilon\int_\Omega w_1|\psi(u)|^p\psi'(u)^{1-p}|\nabla\eta|^p\,dx. \end{aligned} \tag{2.2}$$

Now we apply (2.1) with $\varphi = \phi(u)\eta^{\frac{p}{2}}$ to get

$$
\begin{aligned}
&\frac{1}{w\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f'(u)\phi(u)^2\eta^p\,dx \\
&\le \int_\Omega w_1|\nabla u|^{p-2}\left|\phi'(u)\eta^{\frac{p}{2}}\nabla u + \frac{p}{2}\phi(u)\eta^{\frac{p-2}{2}}\nabla\eta\right|^2\,dx \\
&\le \int_\Omega w_1\phi'(u)^2\eta^p|\nabla u|^p\,dx + \frac{p^2}{4}\int_\Omega w_1\phi(u)^2\eta^{p-2}|\nabla u|^{p-2}|\nabla\eta|^2\,dx \\
&\qquad + p\int_\Omega w_1|\phi(u)|\phi'(u)\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta|\,dx.
\end{aligned}
\tag{2.3}
$$

Moreover, we have the estimate

$$
\begin{aligned}
&p\int_\Omega w_1|\phi(u)|\phi'(u)\eta^{p-1}|\nabla u|^{p-1}|\nabla\eta|\,dx \\
&\le \int_\Omega\left\{\frac{\varepsilon}{2}\left(w_1^{\frac{p-1}{p}}\phi'(u)^{\frac{2p-2}{p}}\eta^{p-1}|\nabla u|^{p-1}\right)^{\frac{p}{p-1}}\right. \\
&\qquad\qquad \left. + C_\varepsilon\left(w_1^{\frac{1}{p}}|\phi(u)|\phi'(u)^{\frac{2-p}{p}}|\nabla\eta|\right)^p\right\}dx \\
&= \frac{\varepsilon}{2}\int_\Omega w_1\phi'(u)^2\eta^p|\nabla u|^p\,dx + C_\varepsilon\int_\Omega w_1|\phi(u)|^p\phi'(u)^{2-p}|\nabla\eta|^p\,dx,
\end{aligned}
$$

and if $p>2$ we also have

$$
\begin{aligned}
&\frac{p^2}{4}\int_\Omega w_1\phi(u)^2\eta^{p-2}|\nabla u|^{p-2}|\nabla\eta|^2\,dx \\
&\le \int_\Omega\left\{\frac{\varepsilon}{2}\left(w_1^{\frac{p-2}{p}}\phi'(u)^{\frac{2p-4}{p}}\eta^{p-2}|\nabla u|^{p-2}\right)^{\frac{p}{p-2}}\right. \\
&\qquad\qquad \left. + C_\varepsilon\left(w_1^{\frac{2}{p}}\phi(u)^2\phi'(u)^{\frac{4-2p}{p}}|\nabla\eta|^2\right)^{\frac{p}{2}}\right\}dx \\
&= \frac{\varepsilon}{2}\int_\Omega w_1\phi'(u)^2\eta^p|\nabla u|^p\,dx + C_\varepsilon\int_\Omega w_1|\phi(u)|^p\phi'(u)^{2-p}|\nabla\eta|^p\,dx.
\end{aligned}
$$

Plugging the last two estimates into (2.3) to get

$$
\begin{aligned}
\frac{1}{w\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f'(u)\phi(u)^2\eta^p\,dx &\le (1+\varepsilon)\int_\Omega w_1\phi'(u)^2\eta^p|\nabla u|^p\,dx \\
&\quad + C_\varepsilon\int_\Omega w_1|\phi(u)|^p\phi'(u)^{2-p}|\nabla\eta|^p\,dx.
\end{aligned}
\tag{2.4}
$$

From (2.2), (2.4) and assumption $\psi' \ge \phi'^2$ we obtain

$$
\begin{aligned}
\frac{1}{w\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f'(u)\phi(u)^2\eta^p\,dx &\le \frac{1+\varepsilon}{1-\varepsilon}\frac{1}{m\left(\|u\|_{w_1}^p\right)}\int_\Omega w_2 f(u)\psi(u)\eta^p\,dx \\
&\quad + C_\varepsilon\int_\Omega w_1\left[|\phi(u)|^p\phi'(u)^{2-p} + |\psi(u)|^p\psi'(u)^{1-p}\right]|\nabla\eta|^p\,dx.
\end{aligned}
$$

Finally, we may replace ε with $\frac{\varepsilon}{2+\varepsilon}$ to get the conclusion. ☐

Proof of Theorem 1.2. By contradiction, assume that problem (1.1) has a nontrivial stable solution u. In (a,b), we define $\phi(t) = |f(t)|^{\frac{\alpha+1}{2}}\operatorname{sign}(f(t))$ and $\psi(t) = \int_c^t \phi'(s)^2\,ds$, where

$$c = \begin{cases} a, & \text{if } f(t) > 0 \text{ for all } t \in (a,b), \\ b, & \text{if } f(t) < 0 \text{ for all } t \in (a,b), \\ z_f, & \text{otherwise.} \end{cases}$$

We define H_ε and G as in Lemma 2.2. If $t > c$ we have $f(t) > 0$, $\psi(t) > 0$ and for all $d \in (c,t)$,

$$\begin{aligned} &\alpha \int_d^t |f(s)|^{\alpha-1} f'(s)^2\,ds \\ &= |f(t)|^{\alpha-1} f(t) f'(t) - |f(d)|^{\alpha-1} f(d) f'(d) - \int_d^t |f(s)|^{\alpha-1} f(s) f''(s)\,ds \\ &\le |f(t)|^{\alpha-1} f(t) f'(t) - \gamma \int_d^t |f(s)|^{\alpha-1} f'(s)^2\,ds. \end{aligned}$$

This and the fact that

$$\alpha + \gamma > \underline{\alpha} + \gamma = \frac{\left(\sqrt{(p+p\tau-1)(\gamma-1)+1} - 1\right)^2}{p+p\tau-1} \ge 0$$

implies

$$\int_d^t |f(s)|^{\alpha-1} f'(s)^2\,ds \le \frac{1}{\alpha+\gamma} |f(t)|^{\alpha-1} f(t) f'(t).$$

Letting $d \to c^+$ in the above inequality, we deduce that $\psi(t)$ is finite and

$$0 < f(t)\psi(t) \le \frac{(\alpha+1)^2}{4(\alpha+\gamma)} f'(t)\phi(t)^2. \tag{2.5}$$

By similar argument, (2.5) also holds for $t < c$. One the other hand, the assumption (M) implies $\frac{tm'(t)}{m(t)} \le \tau$ for all $t > 0$. From these facts we have

$$\begin{aligned} H_\varepsilon(u) &= \frac{1}{w\left(\|u\|_{w_1}^p\right)} f'(u)\phi(u)^2 - \frac{1+\varepsilon}{m\left(\|u\|_{w_1}^p\right)} f(u)\psi(u) \\ &\ge \left(\frac{1}{w\left(\|u\|_{w_1}^p\right)} - \frac{(\alpha+1)^2}{4(\alpha+\gamma)} \frac{1+\varepsilon}{m\left(\|u\|_{w_1}^p\right)} \right) f'(u)\phi(u)^2 \\ &= \left(\frac{m\left(\|u\|_{w_1}^p\right)}{w\left(\|u\|_{w_1}^p\right)} - \frac{(1+\varepsilon)(\alpha+1)^2}{4(\alpha+\gamma)} \right) \frac{1}{m\left(\|u\|_{w_1}^p\right)} f'(u)\phi(u)^2 \\ &\ge \left(\frac{1}{p+p\tau-1} - \frac{(1+\varepsilon)(\alpha+1)^2}{4(\alpha+\gamma)} \right) \frac{1}{m\left(\|u\|_{w_1}^p\right)} f'(u)\phi(u)^2. \end{aligned}$$

Since $\frac{1}{p+p\tau-1} - \frac{(\alpha+1)^2}{4(\alpha+\gamma)} > 0$ by assumption $\alpha \in (\underline{\alpha}, \overline{\alpha})$, we may fix some $\varepsilon > 0$ such that $\frac{1}{p+p\tau-1} - \frac{(1+\varepsilon)(\alpha+1)^2}{4(\alpha+\gamma)} > 0$. Hence, together with assumption

(F_1), we obtain

$$H_\varepsilon(u) \geq \frac{C}{m\left(\|u\|_{w_1}^p\right)} f'(u)\phi(u)^2 \geq \frac{C}{m\left(\|u\|_{w_1}^p\right)} |f(u)|^{\Gamma+\alpha+1}. \tag{2.6}$$

Now we use (2.5) to estimate $G(u)$ as follow

$$\begin{aligned} G(u) &= |\phi(u)|^p \phi'(u)^{2-p} + |\psi(u)|^p \psi'(u)^{1-p} \\ &\leq |\phi(u)|^p \phi'(u)^{2-p} + C\left(|f(u)|^{-1} f'(u)\phi(u)^2\right)^p \phi'(u)^{2-2p} \\ &= C|f(u)|^{p+\alpha-1} f'(u)^{2-p}. \end{aligned}$$

With the aid of (F_1), we deduce

$$G(u) \leq C|f(u)|^{(2-p)\Gamma+p+\alpha-1}. \tag{2.7}$$

By assumption (FD), either $\Omega = \mathbb{R}^N$ or $f(0) = 0$. The latter case implies $\phi(0) = \psi(0) = 0$. From (2.6), (2.7) and Lemma 2.2, for all nonnegative function $\eta \in C_c^1(\mathbb{R}^N)$, we have

$$\int_\Omega w_2 |f(u)|^{\Gamma+\alpha+1} \eta^p \, dx \leq Cm\left(\|u\|_{w_1}^p\right) \int_\Omega w_1 |f(u)|^{(2-p)\Gamma+p+\alpha-1} |\nabla \eta|^p \, dx.$$

Applying this inequality for $\eta = \xi^m$, where $\xi \in C_c^1(\mathbb{R}^N)$ is nonnegative and $m = \frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}$, to get

$$\begin{aligned} &\int_\Omega w_2 |f(u)|^{\Gamma+\alpha+1} \xi^{pm} \, dx \\ &\qquad \leq Cm\left(\|u\|_{w_1}^p\right) \int_\Omega w_1 |f(u)|^{(2-p)\Gamma+p+\alpha-1} \xi^{p(m-1)} |\nabla \xi|^p \, dx. \end{aligned}$$

Young inequality with $q = \frac{\Gamma+\alpha+1}{(2-p)\Gamma+p+\alpha-1} > 0$, $q' = \frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)} > 0$ and $\varepsilon = \frac{1}{2}$ leads now to

$$\begin{aligned} &\int_\Omega w_2 |f(u)|^{\Gamma+\alpha+1} \xi^{pm} \, dx \\ &\leq \int_\Omega \left\{ \frac{1}{2} \left(w_2^{\frac{1}{q}} |f(u)|^{(2-p)\Gamma+p+\alpha-1} \xi^{p(m-1)} \right)^q \right. \\ &\qquad\qquad \left. + C\left(m\left(\|u\|_{w_1}^p\right) w_1 w_2^{-\frac{1}{q}} |\nabla \xi|^p \right)^{q'} \right\} dx \\ &= \frac{1}{2} \int_\Omega w_2 |f(u)|^{\Gamma+\alpha+1} \xi^{pm} \, dx + Cm\left(\|u\|_{w_1}^p\right)^{q'} \int_\Omega w_1^{q'} w_2^{-\frac{q'}{q}} |\nabla \xi|^{pq'} \, dx, \end{aligned}$$

which implies

$$\begin{aligned} &\int_\Omega w_2 |f(u)|^{\Gamma+\alpha+1} \xi^{\frac{p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \, dx \\ &\qquad \leq Cm\left(\|u\|_{w_1}^p\right)^{q'} \int_\Omega w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}} |\nabla \xi|^{\frac{p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \, dx. \end{aligned}$$

Choosing a test function $\xi \in C_c^1(\mathbb{R}^N)$ such that $0 \le \xi \le 1$ in $\mathbb{R}^N$, $\xi = 1$ in B_R, $\xi = 0$ outside of B_{2R} and $|\nabla \xi| \le CR^{-1}$ in $B_{2R} \setminus B_R$, we obtain

$$\begin{aligned}&\int_{\Omega\cap B_R} w_2|f(u)|^{\Gamma+\alpha+1}\,dx\\ &\qquad \le Cm\left(\|u\|_{w_1}^p\right)^{q'} R^{\frac{-p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \int_{\Omega\cap B_{2R}\setminus B_R} w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}}\,dx.\end{aligned}$$

Letting $R \to +\infty$ and using assumption (W) to get $w_2|f(u)|^{\Gamma+\alpha+1} = 0$ a.e. in Ω. But this contradicts the fact that u is non-constant, $w_2 \not\equiv 0$ and (FW). □

Proof of Proposition 1.5. Since

$$\lim_{\alpha\to\overline{\alpha}^-}((p-2)\Gamma - p - \alpha + 1) = (p-2)\Gamma - p - \overline{\alpha} + 1 < 0,$$

we may choose $\alpha \in (\max\{\underline{\alpha}, -1, \overline{\alpha}-\varepsilon\}, \overline{\alpha})$ sufficiently close to $\overline{\alpha}$ such that $(p-2)\Gamma - p - \alpha + 1 < 0$. Here $\underline{\alpha}$ is defined as in Theorem 1.2. As in Remark 1.4, for $R > R_0$ we have

(2.8)
$$R^{\frac{-p(\Gamma+\alpha+1)}{(p-1)\Gamma-(p-2)}} \int_{\Omega\cap B_{2R}\setminus B_R} w_1^{\frac{\Gamma+\alpha+1}{(p-1)\Gamma-(p-2)}} w_2^{\frac{(p-2)\Gamma-p-\alpha+1}{(p-1)\Gamma-(p-2)}}\,dx \le CR^{-N_\alpha}|\Omega\cap B_{2R}\setminus B_R|,$$

where

$$N_\alpha = \frac{(p-q_1)(\Gamma+\alpha+1) - q_2[(p-2)\Gamma - p - \alpha + 1]}{(p-1)\Gamma-(p-2)}.$$

Since $\lim_{\alpha\to\overline{\alpha}^-} N_\alpha = N^\#$, we may choose α even closer to $\overline{\alpha}$ if necessary such that $N_\alpha > N^\# - \varepsilon$ and then let $R \to +\infty$ in (2.8) to obtain (W). Now the conclusion follows immediately from Theorem 1.2. □

Proof of Proposition 1.8. By contradiction, assume that (1.1) has a nontrivial stable solution. Since

$$\lim_{\tau\to\left(\frac{1}{p}-1\right)^+} \overline{\alpha} = +\infty,$$

where $\overline{\alpha}$ is defined as in Theorem 1.2, we may choose $\tau > \frac{1}{p} - 1$ sufficiently close to $\frac{1}{p}-1$ and choose some $\varepsilon > 0$ sufficiently small such that $(p-2)\Gamma - p - \overline{\alpha} + 1 < 0$ and $\overline{\alpha} - \varepsilon > 1$. Then (M) holds for such τ due to (1.6) while (F_2) holds for all $\alpha \in (\overline{\alpha}-\varepsilon, \overline{\alpha})$ thanks to Remark 1.3.

We may choose τ even closer to $\frac{1}{p}-1$ if necessary such that

$$\begin{aligned}N^\# &:= \frac{p+q_2-q_1}{p+p\tau-1}\,\frac{2+(p+p\tau-1)\Gamma+2\sqrt{(p+p\tau-1)(\gamma-1)+1}}{(p-1)\Gamma-(p-2)} - q_2\\ &> N + \varepsilon.\end{aligned}$$

Then (1.5) is satisfied and we reach a contradiction with Proposition 1.5. □

Proof of Proposition 1.9. We first show that $f' > 0$ in (a,b). Indeed, suppose $f'(t_0) = 0$ for some $t_0 \in (a,b)$. If $f > 0$ in $(-\infty, b)$, then $f'' \geq 0$ in $(-\infty, b)$. Therefore, $f'(t) \leq f'(t_0) = 0$ for all $t \in (-\infty, t_0]$. But this contradicts the fact that f is increasing in $t \in (-\infty, t_0]$. Similar argument can be applied in the case that $f > 0$ in $(a, +\infty)$.

By contradiction, assume that (i) holds and (1.1) has a bounded below stable solution u. Clearly, u is nontrivial since constant functions do not satisfy the equation. Then we may restrict f into the interval $\left(\inf\limits_{(a,b)} u - 1, +\infty\right)$ $\cap (a,b)$ and find out that f satisfies condition (ii) in Remark 1.3 in its new domain for any $\Gamma \in \left(\frac{p-2}{p-1}, \gamma\right]$. Hence, f also satisfies (F_1) in its new domain for any $\Gamma \in \left(\frac{p-2}{p-1}, \gamma\right]$. Moreover, (F_2) holds for any α thanks to the fact that f has no zero and Remark 1.3.

One may observe that

$$\begin{aligned}\lim_{\Gamma \to \left(\frac{p-2}{p-1}\right)^+} [(p-2)\Gamma - p - \overline{\alpha} + 1] &= \frac{2-p}{p-1} - \frac{2 + 2\sqrt{(p+p\tau-1)(\gamma-1)+1}}{p+p\tau-1} \\ &< \frac{2-p}{p-1} \leq 0.\end{aligned}$$

If we choose Γ sufficiently close to $\frac{p-2}{p-1}$ such that $(p-2)\Gamma - p - \overline{\alpha} + 1 < 0$ and

$$\begin{aligned}N^{\#} :=& \frac{p+q_2-q_1}{p+p\tau-1} \frac{2 + (p+p\tau-1)\Gamma + 2\sqrt{(p+p\tau-1)(\gamma-1)+1}}{(p-1)\Gamma - (p-2)} - q_2 \\ &> N,\end{aligned}$$

we reach a contradiction with Proposition 1.5. Similarly, problem (1.1) has no bounded above stable solution if (ii) holds. $\square$

Conclusion

Via integral estimates, we have established Liouville theorems for stable solutions of p-Kirchhoff type problems in bounded or unbounded domains with Dirichlet boundary value condition. The problems in our paper may have very general nonlinearities or nonlocal terms which were not studied in literature before. Our theorems therefore extend and unify previous results in [2, 4, 5, 11, 22, 24, 33]. Moreover, our results also indicate that Liouville type theorem for stable solutions usually hold in low dimensional space.

Acknowledgments

This work was done while the second author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2019. He would like to thank the staff of the institute for their support and hospitality.

References

[1] X. Cabré, *Boundedness of stable solutions to semilinear elliptic equations: A survey*, Adv. Nonlinear Stud. 17 (2017), 355–368.

[2] D. Castorina, P. Esposito, B. Sciunzi, *Low dimensional instability for semilinear and quasilinear problems in* $\mathbb{R}^N$, Commun. Pure Appl. Anal. 8 (2009), 1779–1793.

[3] A. Cauchy, *Mémoire sur les fonctions complémentaires*, C. R. Acad. Sci. Paris 19 (1844), 1377–1384.

[4] C. Cowan, M. Fazly, *On stable entire solutions of semi-linear elliptic equations with weights*, Proc. Amer. Math. Soc. 140 (2012), 2003–2012.

[5] L. Damascelli, A. Farina, B. Sciunzi, E. Valdinoci, *Liouville results for m-Laplace equations of Lane-Emden-Fowler type*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 1099–1119.

[6] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983), 827–850.

[7] L. Dupaigne, A. Farina, *Liouville theorems for stable solutions of semilinear elliptic equations with convex nonlinearities*, Nonlinear Anal. 70 (2009), 2882–2888.

[8] L. Dupaigne, A. Farina, *Stable solutions of $-\Delta u = f(u)$ in* $\mathbb{R}^N$, J. Eur. Math. Soc. 12 (2010), 855–882.

[9] L. Dupaigne, *Stable solutions of elliptic partial differential equations*, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics 143, Boca Raton, FL, 2011.

[10] A. Farina, *Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of* $\mathbb{R}^N$, C. R. Math. Acad. Sci. Paris 341 (2005), 415–418.

[11] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. 87 (2007), 537–561.

[12] A. Farina, *Stable solutions of $\Delta u = e^u$ on* $\mathbb{R}^N$, C. R. Math. Acad. Sci. Paris 345 (2007), 63–66.

[13] J. García-Melián, *Nonexistence of positive solutions for Hénon equation*, arXiv:1703.04353 (2017).

[14] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34 (1981), 525–598.

[15] Z.M. Guo, L.F. Mei, *Liouville type results for a p-Laplace equation with negative exponent*, Acta Math. Sin. (Engl. Ser.) 32 (2016), 1515–1540.

[16] D.D. Joseph, T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Ration. Mech. Anal. 49 (1973), 241–269.

[17] S.M. Khalkhali, A. Razani, *Multiple Solutions for a Quasilinear (p, q)-Elliptic System*, Electron. J. Differential Equations 2013 (2013), No. 144, 1–14.

[18] S.M. Khalkhali, S. Heidarkhani, A. Razani, *Infinitely many solutions for a fourth-order boundary-value problem*, Electron. J. Differential Equations 164 (2012), 1–14.

[19] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[20] P. Le, *Liouville theorems for stable solutions of p-Laplace equations with convex nonlinearities*, J. Math. Anal. Appl. 443 (2016), 431–444.

[21] P. Le, V. Ho, *On stable solutions to weighted quasilinear problems of Lane-Emden type*, Electron. J. Differential Equations 2018 (2018), No. 71, pp. 1–11.

[22] P. Le, V. Ho, *Liouville results for stable solutions of quasilinear equations with weights*, Acta Math. Sci . Ser. B Engl. Ed. 39 (2019), 357–368.

[23] P. Le, H.T. Nguyen, T.Y. Nguyen, *On positive stable solutions to weighted quasilinear problems with negative exponent*, Complex Var. Elliptic Equ. 63, (2018), 1739–1751.

[24] P. Le, N.V. Huynh, V. Ho, *Classification results for Kirchhoff equations in $\mathbb{R}^N$*, Complex Var. Elliptic Equ. 64 (2019), 1146–1157.

[25] P. Le, N.V. Huynh, V. Ho, *Positive solutions of the p-Kirchhoff problem with degenerate and sign-changing nonlocal term*, Z. Angew. Math. Phys. 70 (2019), no. 68, 1–12.

[26] P. Le, D.H. Le, K.A. Le, *On stable solutions to weighted quasilinear problems of Gelfand type*, Mediterr. J. Math. 15 (2018), no. 94, 1–12.

[27] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988), 1203–1219.

[28] J. Lions, *On some questions in boundary value problems of mathematical physics*, North Holl. Math. Stud. 30 (1978), 284–346.

[29] R. Mahdavi Khanghahi, A. Razani, *Solutions for a singular elliptic problem involving the $p(x)$-Laplacian*, Filomat 32 (14) (2018), 4841–4850.

[30] M. Makvand Chaharlang, A. Razani, *Infinitely many solutions for a fourth order singular elliptic problem*, Filomat 32 (14) (2018), 5003–5010.

[31] M. Makvand Chaharlang, A. Razani, *Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition*, Commun. Korean Math. Soc. 34 (2019), 155–167.

[32] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations 51, (1984), 126–150.

[33] Y. Wei, C. Chen, H. Song, H. Yang, *Liouville-type theorems for stable solutions of Kirchhoff equations with exponential and superlinear nonlinearities*, Complex Var. Elliptic Equ. (2018), https://doi.org/10.1080/17476933.2018.1514030.

Nhat Vy Huynh
Faculty of Mathematics & Computer Science, University of Science, Vietnam University of Ho Chi Minh City, Vietnam;
Department of Fundamental Sciences, Ho Chi Minh City University of Transport, Vietnam
Email address: hmnguon@gmail.com

Phuong Le (corresponding author)
Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam;
Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Email address: lephuong@tdtu.edu.vn