PRODUCTS OF VOLTERRA TYPE OPERATORS AND COMPOSITION OPERATORS BETWEEN FOCK SPACES

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ABSTRACT. We show that entire functions φ, which induce bounded products of Volterra integral operators V_g (Volterra companion operators J_g) and composition operators C_φ acting between different Fock spaces, must be affine functions, i.e. $\varphi(z) = az + b$. Then, using this special form of φ, we characterize boundedness and compactness of these products in term of new quantities, which are much simpler than the Berezin type integral transforms in the previous papers.

1. Introduction

For an entire function g, the Volterra integral operator V_g and its companion operator J_g are defined as follows

$$V_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \text{ and } J_g f(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta.$$

These operators V_g and J_g have been intensively investigated on various function spaces after the works of Pommerenke [17], Aleman and Siskakis on Hardy and Bergman spaces [2, 3].

Given an entire function φ, the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$. Such operators C_φ have become an attractive subject for many researchers during the past few decades (see [5] and [18] for an overview).

In this paper we are interested in the products of Volterra integral operators V_g (Volterra companion operators J_g) and composition operators C_φ. In details, we will investigate boundedness and compactness of the following operators:

$$V_g^\varphi f(z) = V_g \circ C_\varphi f(z) = \int_0^z f(\varphi(\zeta))g'(\zeta)d\zeta,$$

$$C_\varphi^g f(z) = C_\varphi \circ V_g f(z) = \int_0^{\varphi(z)} f(\zeta)g'(\zeta)d\zeta,$$

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$$J_{g,\varphi}f(z) = J_g \circ C_\varphi f(z) = \int_0^z f'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)d\zeta,$$
$$C_{\varphi,g}f(z) = C_\varphi \circ J_g f(z) = \int_0^{\varphi(z)} f'(\zeta)g(\zeta)d\zeta.$$

The study of these operators naturally comes from the isometry of some function spaces (see, for instance, [6] and [7]) and has been carried out on different spaces of holomorphic functions on the unit disc such as Bergman spaces, Bloch spaces, Zygmund spaces, spaces H^∞ [11, 12, 13, 20]. Recently, much progress was made in the study of these products acting between Fock spaces $\mathcal{F}^p(\mathbb{C})$ and $\mathcal{F}^q(\mathbb{C})$ with $0 < p, q \leq \infty$ by Mengestie [14, 15, 16], and Abakumov and Doubtsov [1]. In details, in [14, 15, 16] the author characterized boundedness and compactness of the products V_g^φ, C_φ^g, $C_{\varphi,g}$, and the generalized Volterra companion operator $T_{g,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ with $p, q \in (0, \infty)$, where

$$T_{g,\varphi}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta,$$

in term of the following Berezin type integral transforms, respectively:

$$B_{V_g^\varphi}(w) = \int_{\mathbb{C}} e^{\frac{q}{2}(2\mathrm{Re}(\overline{w}\varphi(z)) - |z|^2 - |w|^2)} \frac{|g'(z)|^q}{(1+|z|)^q} dA(z),$$
$$B_{C_\varphi^g}(w) = \int_{\mathbb{C}} e^{\frac{q}{2}(2\mathrm{Re}(\overline{w}\varphi(z)) - |z|^2 - |w|^2)} \frac{|g'(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} dA(z),$$
$$B_{C_{\varphi,g}}(w) = (1+|w|)^q \int_{\mathbb{C}} e^{\frac{q}{2}(2\mathrm{Re}(\overline{w}\varphi(z)) - |z|^2 - |w|^2)} \frac{|g(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} dA(z),$$
$$B_{T_{g,\varphi}}(w) = (1+|w|)^q \int_{\mathbb{C}} e^{\frac{q}{2}(2\mathrm{Re}(\overline{w}\varphi(z)) - |z|^2 - |w|^2)} \frac{|g(z)|^q}{(1+|z|)^q} dA(z).$$

However, these results are quite difficult to use, even for the special operators

$$T_{\varphi',\varphi}f(z) = C_\varphi f(z) - f(\varphi(0)) \quad \text{and} \quad C_{\varphi,\mathbf{1}}f(z) = C_\varphi f(z) - f(0),$$

for which boundedness and compactness can be easily reduced from the ones of composition operators C_φ established by Tien and Khoi in [19, Corollaries 3.5 and 3.6]. Note that the product $J_{g,\varphi}$ and the operator $T_{g,\varphi}$ are slightly differently defined and we also consider the operator $T_{g,\varphi}$ at the end of this paper.

On the other hand, we recall that in [19] it was shown that entire functions φ, which induce bounded composition operators $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ and weighted composition operators $\psi C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$, must be affine functions, i.e. $\varphi(z) = az + b$. Using this special form of functions φ, the authors completely solved several important questions for both operators C_φ and ψC_φ. Based on this idea, in this paper we also prove that this statement is valid for the products V_g^φ, C_φ^g, $J_{g,\varphi}$, and $C_{\varphi,g}$. That is, these products are bounded from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$ only when $\varphi(z) = az + b$. This allows us to characterize boundedness and compactness of the products V_g^φ, C_g^φ, $J_{g,\varphi}$ and $C_{\varphi,g}$ in terms of the following simpler quantities, respectively:

$$M_{V_g^\varphi}(z) = \frac{|g'(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}},$$

$$M_{C_\varphi^g}(z) = \frac{|g'(\varphi(z))\varphi'(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}},$$

$$M_{J_{g,\varphi}}(z) = \frac{(1+|\varphi(z)|)|g(z)\varphi'(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}},$$

$$M_{C_{\varphi,g}}(z) = \frac{(1+|\varphi(z)|)|g(\varphi(z))\varphi'(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}}.$$

The paper is organized as follows. After the Introduction, in Section 2 we give some preliminary results about Fock spaces $\mathcal{F}^p(\mathbb{C})$, operators defined on them, Fock Carleson measure, and an extension of [10, Proposition 2.1], which plays a crucial role in this work. Section 3 is devoted to the products V_g^φ and C_φ^g, while the products $J_{g,\varphi}$ and $C_{\varphi,g}$ are studied in Section 4. In these sections, firstly we prove that the considered products, denoted by T, are bounded from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$ only if $\varphi(z) = az + b$ with $|a| \le 1$ (Propositions 3.2 and 4.2). Then, in the case $0 < p \le q < \infty$, we show that the product $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $M_T(z) \in L^\infty(\mathbb{C}, dA)$ (or, $M_T(z) \to 0$ as $|z| \to \infty$, respectively) (Theorems 3.4 and 4.4). When $0 < q < p < \infty$, boundedness and compactness of the product $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ are equivalent, and by using Fock Carleson measure we prove that these properties are equivalent to that $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ (Theorems 3.5 and 4.5).

Based on the simpler criteria established in Sections 3 and 4, we can give a clear overall picture on the interplay between two entire functions g and φ in inducing bounded and compact products V_g^φ, C_φ^g, $J_{g,\varphi}$, and $C_{\varphi,g}$ (Remarks 3.6 and 4.7).

Notations: Throughout this paper, we use the notation $A \lesssim B$ (and $A \gtrsim B$) for nonnegative quantities A and B to mean that there is a constant $C > 0$ dependent only on p, q such that $A \le CB$ (and $A \ge CB$, respectively); similarly the notation $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

2.1. **Fock spaces.** For a number $p \in (0, \infty)$, the Fock space $\mathcal{F}^p(\mathbb{C})$ is defined as follows

$$\mathcal{F}^p(\mathbb{C}) = \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\|_p = \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} \, dA(z) \right)^{\frac{1}{p}} < \infty \right\},$$

where $\mathcal{O}(\mathbb{C})$ is the space of entire functions on $\mathbb{C}$ with the usual compact open topology and dA is the Lebesgue measure on $\mathbb{C}$. Furthermore, the space $\mathcal{F}^\infty(\mathbb{C})$ consists of all entire functions $f \in \mathcal{O}(\mathbb{C})$ for which

$$\|f\|_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{|z|^2}{2}} < \infty.$$

It is well known that $\mathcal{F}^p(\mathbb{C})$ with $p \geq 1$ and $\mathcal{F}^\infty(\mathbb{C})$ are Banach spaces. When $0 < p < 1$, $\mathcal{F}^p(\mathbb{C})$ is a complete metric space with the distance $d(f,g) = \|f-g\|_p^p$.

For each $w \in \mathbb{C}$, we define the function

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}, \ z \in \mathbb{C}.$$

These functions play an important role in the study of Fock spaces and operators in them. Obviously, $\|k_w\|_p = 1$ for every $w \in \mathbb{C}$ and $p \in (0,\infty)$; and k_w converges to 0 in $\mathcal{O}(\mathbb{C})$ as $|w| \to \infty$.

Lemma 2.1. *Let $p \in (0,\infty)$ be given. For every $f \in \mathcal{O}(\mathbb{C})$ and $z \in \mathbb{C}$,*

$$|f(z)|e^{-\frac{|z|^2}{2}} \leq \|f\|_p.$$

Lemma 2.2. *For $0 < p < q < \infty$, $\mathcal{F}^p(\mathbb{C}) \subset \mathcal{F}^q(\mathbb{C})$, and the inclusion is proper and continuous. Moreover,*

$$\|f\|_q \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_p \ \textit{ for all } f \in \mathcal{F}^p(\mathbb{C}).$$

The next lemma proved in [19, Lemma 2.3] is a key ingredient in the study of compactness of linear operators on Fock spaces.

Lemma 2.3. *Let $p, q \in (0,\infty)$ and T be a linear continuous operator from $\mathcal{O}(\mathbb{C})$ into itself and $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ be well-defined. Then $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if for every bounded sequence $(f_n)_n$ in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$, the sequence $(Tf_n)_n$ also converges to 0 in $\mathcal{F}^q(\mathbb{C})$.*

To investigate the operators $J_{g,\varphi}$ and $C_{\varphi,g}$, we need the following weighted Fock space. Given a number $p \in (0,\infty)$ and a weight $W(z) = \frac{1}{2}|z|^2 + \log(1+|z|)$, we define the weighted Fock space $\mathcal{F}_W^p(\mathbb{C})$ as follows

$$\mathcal{F}_W^p(\mathbb{C}) = \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\|_{W,p} = \left(\int_{\mathbb{C}} |f(z)|^p e^{-pW(z)}\, dA(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

By modifying $W(r)$ on some finite interval $[0,R]$, we can suppose that $dd^cW(z) \simeq dd^c|z|^2$. Then $\mathcal{F}_W^p(\mathbb{C})$ is a particular case of the space $\mathcal{F}^p(\varphi)$ in [9]. For each $z \in \mathbb{C}$, let $K_{W,z}(\cdot)$ be the Bergman kernel and $k_{W,z}(\zeta) = \frac{K_{W,z}(\zeta)}{\sqrt{K_{W,z}(z)}}$ the normalized Bergman kernel of Hilbert space $F_W^2(\mathbb{C})$. Then, according to [9, Section 2], we get

$$K_{W,z}(z) \simeq e^{2W(z)} \text{ and } \|k_{W,z}\|_{W,p} \simeq 1, \ z \in \mathbb{C}. \tag{2.1}$$

Similar to Lemma 2.1, we have the following auxiliary result.

Lemma 2.4. *Let $p \in (0,\infty)$ be given. For every $f \in \mathcal{O}(\mathbb{C})$ and every $z \in \mathbb{C}$,*

$$\frac{|f(z)|}{1+|z|}e^{-\frac{|z|^2}{2}} \lesssim \left(\int_{\mathbb{C}} \frac{|f(\zeta)|^p}{(1+|\zeta|)^p} e^{-\frac{p|\zeta|^2}{2}} dA(\zeta)\right)^{\frac{1}{p}}.$$

Proof. For each $z \in \mathbb{C}$ and $f \in \mathcal{O}(\mathbb{C})$ fixed, the inequality $2(1+|z|) \geq 1+|z+\zeta|$ for all $|\zeta| \leq 1$ and the subharmonicity of $|f(z+\zeta)e^{-\overline{z}\zeta}|^p$ imply that

$$\begin{aligned}
\frac{|f(z)|^p}{(1+|z|)^p} &\lesssim \int_{|\zeta|\leq 1} \frac{|f(z+\zeta)e^{-\overline{z}\zeta}|^p}{(1+|z+\zeta|)^p} e^{-\frac{p|\zeta|^2}{2}} dA(\zeta) \\
&= e^{\frac{p|z|^2}{2}} \int_{|\zeta|\leq 1} \frac{|f(z+\zeta)|^p}{(1+|z+\zeta|)^p} e^{-\frac{p|z+\zeta|^2}{2}} dA(\zeta) \\
&\leq e^{\frac{p|z|^2}{2}} \int_{\mathbb{C}} \frac{|f(\zeta)|^p}{(1+|\zeta|)^p} e^{-\frac{p|\zeta|^2}{2}} dA(\zeta).
\end{aligned}$$

From this the desired inequality follows. □

2.2. **Fock Carleson measure.** In the case $0 < q < p < \infty$, we will use Fock Carleson measure for both spaces $\mathcal{F}^p(\mathbb{C})$ and $\mathcal{F}^p_W(\mathbb{C})$ to prove that boundedness and compactness of the considered products are equivalent.

We recall that a positive Borel measure μ on $\mathbb{C}$ is called a (p,q)-*Fock Carleson measure*, if the operator $i : \mathcal{F}^p(\mathbb{C}) \to L^q(\mathbb{C}, e^{-\frac{q|z|^2}{2}} d\mu)$ is bounded, i.e. there exists a constant $C > 0$ such that

$$\left(\int\limits_{\mathbb{C}} |f(z)|^q e^{-\frac{q|z|^2}{2}} d\mu(z)\right)^{\frac{1}{q}} \leq C\|f\|_p \text{ for every } f \in \mathcal{F}^p(\mathbb{C}).$$

We write $\|\mu\|$ for the norm of i from $\mathcal{F}^p(\mathbb{C})$ to $L^q(\mathbb{C}, e^{-\frac{q|z|^2}{2}} d\mu)$. The following characterization can be reduced from [8, Theorem 3.3].

Lemma 2.5. *For $0 < q < p < \infty$, a positive Borel measure μ on $\mathbb{C}$ is a (p,q)-Fock Carleson measure if and only if $\widetilde{\mu} \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$, where*

$$\widetilde{\mu}(z) = \int_{\mathbb{C}} |k_z(\zeta)|^q e^{-\frac{q|\zeta|^2}{2}} d\mu(\zeta),\ z \in \mathbb{C}.$$

Furthermore, $\|\mu\| \simeq \|\widetilde{\mu}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}}$.

Next, we say that a positive Borel measure μ on $\mathbb{C}$ is a (p,q,W)-*Fock Carleson measure*, if the operator $i : \mathcal{F}^p_W(\mathbb{C}) \to L^q\left(\mathbb{C}, e^{-qW(z)} d\mu\right)$

is bounded, i.e. there exists a constant $C > 0$ such that

$$\left(\int\limits_{\mathbb{C}} |f(z)|^q e^{-qW(z)} d\mu(z)\right)^{\frac{1}{q}} \leq C\|f\|_{W,p} \text{ for every } f \in \mathcal{F}^p_W(\mathbb{C}).$$

We denote by $\|\mu\|_W$ the norm of i from $\mathcal{F}^p_W(\mathbb{C})$ to $L^q\left(\mathbb{C}, e^{-qW(z)}d\mu\right)$. Applying [9, Theorem 2.8] to the weight $W(z) = \frac{1}{2}|z|^2 + \log(1+|z|)$, we get the following characterization.

Lemma 2.6. *For $0 < q < p < \infty$, a positive Borel measure μ on $\mathbb{C}$ is a (p,q,W)-Fock Carleson measure if and only if $\widetilde{\mu}_W \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$, where*

$$\widetilde{\mu}_W(z) = \int_{\mathbb{C}} |k_{W,z}(\zeta)|^q e^{-qW(\zeta)} d\mu(\zeta), \ z \in \mathbb{C}.$$

Furthermore, $\|\mu\|_W \simeq \|\widetilde{\mu}_W\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}}$.

We end this section with two lemmas, which play an essential role in this paper. The first lemma was proved in [4, Proposition 1] and the second one can be obtained by slightly modifying [10, Proposition 2.1].

Lemma 2.7. *Let $0 < p < \infty$. The following inequality holds*

$$\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z) \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{p|z|^2}{2}} dA(z),$$

for every function f in $\mathcal{O}(\mathbb{C})$.

Lemma 2.8. *Let ψ and φ be two entire functions such that ψ is not identically zero. If*

$$\sup_{z\in\mathbb{C}} \frac{|\psi(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2 - |z|^2}{2}} < \infty,$$

then $\varphi(z) = az + b$ with $|a| \leq 1$.

3. The products of Volterra integral operators and composition operators

In this section we give characterization for boundedness and compactness of the products V^{φ}_g and C^g_{φ} in terms of $M_{V^{\varphi}_g}(z)$ and $M_{C^g_{\varphi}}(z)$, respectively. First we prove the following auxiliary lemma, which allows us to study both operators V^{φ}_g and C^g_{φ} simultaneously. We put

$$m_{V^{\varphi}_g}(z) = \frac{|g'(z)|}{1+|z|} e^{-\frac{|z|^2}{2}} \text{ and } m_{C^g_{\varphi}}(z) = \frac{|g'(\varphi(z))\varphi'(z)|}{1+|z|} e^{-\frac{|z|^2}{2}}, z \in \mathbb{C}.$$

Lemma 3.1. *Let $q \in (0,\infty)$ and T be either V_g^φ or C_φ^g induced by entire functions g and φ, where g is non-constant. For each $f \in \mathcal{O}(\mathbb{C})$ and $z \in \mathbb{C}$,*

$$\begin{aligned}\|Tf\|_q^q &\simeq |(Tf)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q e^{-\frac{q|\varphi(\zeta)|^2}{2}} M_T(\zeta)^q dA(\zeta) \\ &= |(Tf)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta) \\ &\gtrsim |(Tf)(0)|^q + |f(\varphi(z))|^q m_T(z)^q.\end{aligned} \tag{3.1}$$

In particular,

$$\int_{\mathbb{C}} m_T(\zeta)^q dA(\zeta) \lesssim \|T\mathbf{1}\|_q^q. \tag{3.2}$$

Proof. For each $f \in \mathcal{O}(\mathbb{C})$, by Lemma 2.7, we get

$$\begin{aligned}\|V_g^\varphi f\|_q^q &\simeq |(V_g^\varphi f)(0)|^q + \int_{\mathbb{C}} \frac{|f(\varphi(\zeta))g'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\ &= |(V_g^\varphi f)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q e^{-\frac{q|\varphi(\zeta)|^2}{2}} M_{V_g^\varphi}(\zeta)^q dA(\zeta) \\ &= |(V_g^\varphi f)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q m_{V_g^\varphi}(\zeta)^q dA(\zeta)\end{aligned}$$

and

$$\begin{aligned}\|C_\varphi^g f\|_q^q &\simeq |(C_\varphi^g f)(0)|^q + \int_{\mathbb{C}} \frac{|f(\varphi(\zeta))g'(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\ &= |(C_\varphi^g f)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q e^{-\frac{q|\varphi(\zeta)|^2}{2}} M_{C_\varphi^g}(\zeta)^q dA(\zeta) \\ &= |(C_\varphi^g f)(0)|^q + \int_{\mathbb{C}} |f(\varphi(\zeta))|^q m_{C_\varphi^g}(\zeta)^q dA(\zeta).\end{aligned}$$

Moreover, for every $f \in \mathcal{O}(\mathbb{C})$ and $z \in \mathbb{C}$, applying Lemma 2.4 to entire functions $(f\circ\varphi)g'$ and $(f\circ\varphi)(g'\circ\varphi)\varphi'$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \frac{|f(\varphi(\zeta))g'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) &\gtrsim \frac{|f(\varphi(z))g'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} \\ &= |f(\varphi(z))|^q m_{V_g^\varphi}(z)^q\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{C}} \frac{|f(\varphi(\zeta))g'(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) &\gtrsim \frac{|f(\varphi(z))g'(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} \\ &= |f(\varphi(z))|^q m_{C_\varphi^g}(z)^q.\end{aligned}$$

From these inequalities the assertions follow. $\square$

Proposition 3.2. *Let $p, q \in (0,\infty)$ and T be either V_g^φ or C_φ^g induced by entire functions φ and g, where g is non-constant. If the operator*

$T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded, then $M_T(z) \in L^\infty(\mathbb{C}, dA)$. In this case, $\varphi(z) = az + b$ with $|a| \leq 1$ and

$$M_T(z) \lesssim \|Tk_{\varphi(z)}\|_q \leq \|T\| \quad \textit{for all } z \in \mathbb{C}. \tag{3.3}$$

Proof. For every $w \in \mathbb{C}$, using $\|k_w\|_p = 1$ and (3.1), we get

$$\|T\|^q \geq \|Tk_w\|_q^q \gtrsim \int_{\mathbb{C}} |k_w(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta) \gtrsim |k_w(\varphi(z))|^q m_T(z)^q,$$

for all $z \in \mathbb{C}$. In particular, with $w = \varphi(z)$ the last inequality means that

$$\|T\| \geq \|Tk_{\varphi(z)}\|_q \gtrsim e^{\frac{|\varphi(z)|^2}{2}} m_T(z) = M_T(z) \quad \text{for all } z \in \mathbb{C}.$$

Hence, by Lemma 2.8, $\varphi(z) = az + b$ with $|a| \leq 1$. $\square$

In view of Proposition 3.2, throughout this section we suppose that $\varphi(z) = az + b$ with $|a| \leq 1$. In the case $a = 0$ we get the following simple result.

Corollary 3.3. *Let $p, q \in (0, \infty)$, $\varphi(z) \equiv b$ and g be a non-constant function in $\mathcal{O}(\mathbb{C})$.*

(a) *The operator $V_g^b : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if $g \in \mathcal{F}^q(\mathbb{C})$. Moreover,*

$$\|V_g^b\| = e^{\frac{|b|^2}{2}} \|g - g(0)\|_q.$$

(b) *The operator $C_b^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact and*

$$\|C_b^g\| \leq |b| e^{\frac{|b|^2}{2}} \max_{\zeta \in [0,b]} |g'(\zeta)|.$$

Proof. (a) For each $f \in \mathcal{F}^p(\mathbb{C})$,

$$V_g^b f(z) = f(b)(g(z) - g(0)), z \in \mathbb{C}.$$

Hence, by Lemma 2.1, we obtain

$$\|V_g^b f\|_q = |f(b)| \|g - g(0)\|_q \leq e^{\frac{|b|^2}{2}} \|g - g(0)\|_q \|f\|_p.$$

Moreover, $\|k_b\|_p = 1$ and

$$\|V_g^b k_b\|_q = |k_b(b)| \|g - g(0)\|_q = e^{\frac{|b|^2}{2}} \|g - g(0)\|_q.$$

From these the assertions follow.

(b) For each $f \in \mathcal{F}^p(\mathbb{C})$,

$$C_b^g f(z) = \int_0^b f(\zeta) g'(\zeta) d\zeta, \; z \in \mathbb{C}.$$

Hence, by Lemma 2.1, we get

$$\begin{aligned}\|C_b^g f\|_q &= \left|\int_0^b f(\zeta)g'(\zeta)d\zeta\right| \\ &\le |b| \max_{\zeta\in[0,b]} |f(\zeta)g'(\zeta)| \le |b|e^{\frac{|b|^2}{2}}\|f\|_p \max_{\zeta\in[0,b]} |g'(\zeta)|.\end{aligned}$$

From these the assertions follow. □

The case $0 < |a| \le 1$ is more complicated. We firstly investigate the operators V_g^φ and C_φ^g acting from a smaller Fock space $\mathcal{F}^p(\mathbb{C})$ to a larger one $\mathcal{F}^q(\mathbb{C})$.

Theorem 3.4. *Let $0 < p \le q < \infty$, T be either V_g^φ or C_φ^g induced by a non-constant entire function g and $\varphi(z) = az + b$ with $0 < |a| \le 1$.*

(a) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $M_T(z) \in L^\infty(\mathbb{C}, dA)$. Moreover,*

$$\|M_{V_g^\varphi}(z)\|_{L^\infty} \lesssim \|V_g^\varphi\| \lesssim |a|^{-\frac{2}{q}}\|M_{V_g^\varphi}(z)\|_{L^\infty},$$

and

$$\|M_{C_\varphi^g}(z)\|_{L^\infty} \lesssim \|C_\varphi^g\| \lesssim \|C_b^g\| + |a|^{-\frac{2}{q}}\|M_{C_\varphi^g}(z)\|_{L^\infty}.$$

(b) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if*

$$\lim_{|z|\to\infty} M_T(z) = 0.$$

Proof. (a) The necessity follows from Proposition 3.2. Suppose that $M_T(z) \in L^\infty(\mathbb{C}, dA)$. Then, by (3.1) and Lemma 2.2, for every $f \in \mathcal{F}^p(\mathbb{C})$, we get

$$\begin{aligned}\|Tf\|_q^q &\simeq |(Tf)(0)|^q + \int_{\mathbb{C}} |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\ &\le |(Tf)(0)|^q + \|M_T(z)\|_{L^\infty}^q \int_{\mathbb{C}} |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\ &\le |(Tf)(0)|^q + |a|^{-2}\|M_T(z)\|_{L^\infty}^q \int_{\mathbb{C}} |f(\zeta)|^q e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\ &\simeq |(Tf)(0)|^q + |a|^{-2}\|M_T(z)\|_{L^\infty}^q \|f\|_q^q \\ &\lesssim |(Tf)(0)|^q + |a|^{-2}\|M_T(z)\|_{L^\infty}^q \|f\|_p^q.\end{aligned}$$

From this, (3.3) and the fact that $(V_g^\varphi f)(0) = 0$ and $(C_\varphi^g f)(0) = C_b^g f$, it follows that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded; moreover

$$\|M_{V_g^\varphi}(z)\|_{L^\infty} \lesssim \|V_g^\varphi\| \lesssim |a|^{-\frac{2}{q}}\|M_{V_g^\varphi}(z)\|_{L^\infty},$$

and

$$\|M_{C_\varphi^g}(z)\|_{L^\infty} \lesssim \|C_\varphi^g\| \lesssim \|C_b^g\| + |a|^{-\frac{2}{q}}\|M_{C_\varphi^g}(z)\|_{L^\infty}.$$

(b) **Necessary.** Suppose that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact. Then for every sequence $(z_n)_n$ with $z_n \to \infty$, $\varphi(z_n) \to \infty$, and hence, the sequence $k_{\varphi(z_n)}$ converges to 0 in $\mathcal{O}(\mathbb{C})$ as $n \to \infty$. Therefore, by (3.3) and Lemma 2.3,

$$M_T(z_n) \lesssim \|Tk_{\varphi(z_n)}\|_q \to 0 \text{ as } n \to \infty.$$

Thus, $\lim_{|z|\to\infty} M_T(z) = 0$.

Sufficiency. By part (a), the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Let (f_n) be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. Then, for every $n \in \mathbb{N}$ and $R > 0$, using (3.1), (3.2), and Lemma 2.2, we get

$$\begin{aligned}
\|Tf_n\|_q^q &\simeq |(Tf_n)(0)|^q + \int_{|z|\le R} |f_n(\varphi(z))|^q m_T(z)^q dA(z) \\
&\quad + \int_{|z|>R} |f_n(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \max_{|z|\le R} |f_n(\varphi(z))|^q \int_{|z|\le R} m_T(z)^q dA(z) \\
&\quad + \sup_{|z|>R} M_T(z)^q \int_{|z|>R} |f_n(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{1}\|_q^q \max_{|z|\le R} |f_n(\varphi(z))|^q + |a|^{-2}\|f_n\|_q^q \sup_{|z|>R} M_T(z)^q \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{1}\|_q^q \max_{|z|\le R} |f_n(\varphi(z))|^q + |a|^{-2} \sup_n \|f_n\|_p^q \sup_{|z|>R} M_T(z)^q.
\end{aligned}$$

Since $(V_g^\varphi f_n)(0) = 0$ for all n and $(C_\varphi^g f_n)(0) \to 0$ as $n \to \infty$, letting $n \to \infty$ and then $R \to \infty$ in the last inequality, we observe that $(Tf_n)_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$. From this and Lemma 2.3, the assertion follows. □

To study the case $q < p$, we define the following positive pull-back measures $\mu_{V_g^\varphi,q}$ and $\mu_{C_\varphi^g,q}$ on $\mathbb{C}$:

$$\mu_{V_g^\varphi,q}(B) = \int_{\varphi^{-1}(B)} \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} dA(z) = \int_{\varphi^{-1}(B)} m_{V_g^\varphi}(z)^q dA(z)$$

and

$$\mu_{C_\varphi^g,q}(B) = \int_{\varphi^{-1}(B)} \frac{|g'(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} dA(z) = \int_{\varphi^{-1}(B)} m_{C_\varphi^g}(z)^q dA(z)$$

for every Borel subset B of $\mathbb{C}$. Thus,

$$\mu_{T,q}(B) = \int_{\varphi^{-1}(B)} m_T(z)^q dA(z) \text{ for } T = V_g^\varphi \text{ or } T = C_\varphi^g. \tag{3.4}$$

Theorem 3.5. *Let $0 < q < p < \infty$, T be either V_g^φ or C_φ^g induced by a non-constant entire function g and $\varphi(z) = az + b$ with $0 < |a| \le 1$. The following statements are equivalent:*

(i) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded.*
(ii) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.*

(iii) $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.

Furthermore,

$$|a|^{\frac{2(p-q)}{pq}} \|M_{V_g^\varphi}(z)\|_{L^{\frac{pq}{p-q}}} \lesssim \|V_g^\varphi\| \lesssim |a|^{-\frac{2}{p}} \|M_{V_g^\varphi}(z)\|_{L^{\frac{pq}{p-q}}}$$

and

$$|a|^{\frac{2(p-q)}{pq}} \|M_{C_\varphi^g}(z)\|_{L^{\frac{pq}{p-q}}} \lesssim \|C_\varphi^g\| \lesssim \|C_b^g\| + |a|^{-\frac{2}{p}} \|M_{C_\varphi^g}(z)\|_{L^{\frac{pq}{p-q}}}.$$

Proof. (ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (iii). Suppose that $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Then for every $f \in \mathcal{F}^p(\mathbb{C})$, by (3.1) and (3.4), we get

$$\begin{aligned}\|T\|\|f\|_p \geq \|Tf\|_q &\gtrsim \left(\int_{\mathbb{C}} |f(\varphi(z))|^q m_T(z)^q dA(z)\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{C}} |f(z)|^q d\mu_{T,q}(z)\right)^{\frac{1}{q}} = \left(\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q|z|^2}{2}} d\lambda_{T,q}(z)\right)^{\frac{1}{q}},\end{aligned}$$

where $d\lambda_{T,q}(z) = e^{\frac{q|z|^2}{2}} d\mu_{T,q}(z)$. From this it follows that $\lambda_{T,q}$ is a (p,q)-Fock Carleson measure. Then, by Lemma 2.5, we obtain

$$\widetilde{\lambda_{T,q}}(w) = \int_{\mathbb{C}} |k_w(\zeta)|^q e^{-\frac{q|\zeta|^2}{2}} d\lambda_{T,q}(\zeta) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

Moreover, by (3.1), for every $w, z \in \mathbb{C}$, we have

$$\begin{aligned}\widetilde{\lambda_{T,q}}(w) &= \int_{\mathbb{C}} |k_w(\zeta)|^q d\mu_{T,q}(\zeta) = \int_{\mathbb{C}} |k_w(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta) \\ &\gtrsim |k_w(\varphi(z))|^q m_T(z)^q.\end{aligned}$$

In particular, with $w = \varphi(z)$ we get that for every $z \in \mathbb{C}$,

$$\widetilde{\lambda_{T,q}}(\varphi(z)) \gtrsim e^{\frac{q|\varphi(z)|^2}{2}} m_T(z)^q = M_T(z)^q.$$

Hence

$$\begin{aligned}\int_{\mathbb{C}} M_T(z)^{\frac{pq}{p-q}} dA(z) &\lesssim \int_{\mathbb{C}} \left(\widetilde{\lambda_{T,q}}(\varphi(z))\right)^{\frac{p}{p-q}} dA(z) \\ &= |a|^{-2} \int_{\mathbb{C}} \left(\widetilde{\lambda_{T,q}}(w)\right)^{\frac{p}{p-q}} dA(w) < \infty.\end{aligned}$$

From this it follows that $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$; moreover, again by Lemma 2.5,

$$\begin{aligned}\|M_T(z)\|_{L^{\frac{pq}{p-q}}} &\lesssim |a|^{-\frac{2(p-q)}{pq}} \left(\|\widetilde{\lambda_{T,q}}\|_{L^{\frac{p}{p-q}}}\right)^{\frac{1}{q}} \\ &\simeq |a|^{-\frac{2(p-q)}{pq}} \|\lambda_{T,q}\| \lesssim |a|^{-\frac{2(p-q)}{pq}} \|T\|. \qquad (3.5)\end{aligned}$$

(iii) $\Longrightarrow$ (ii). For each function $f \in \mathcal{F}^p(\mathbb{C})$, using (3.1) and Hölder's inequality, we get

$$\begin{aligned}
\|Tf\|_q^q &\simeq |(Tf)(0)|^q + \int_{\mathbb{C}} |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\leq |(Tf)(0)|^q + \left(\int_{\mathbb{C}} |f(\varphi(z))|^p e^{-\frac{p|\varphi(z)|^2}{2}} dA(z)\right)^{\frac{q}{p}} \left(\int_{\mathbb{C}} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \\
&\simeq |(Tf)(0)|^q + |a|^{-\frac{2q}{p}} \|f\|_p^q \|M_T(z)\|^q_{L^{\frac{pq}{p-q}}}.
\end{aligned}$$

This and the fact that $(V_g^\varphi f) = 0$ and $(C_\varphi^g f)(0) = C_b^g f$ imply that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded; moreover

$$\|V_g^\varphi\| \lesssim |a|^{-\frac{2}{p}} \|M_{V_g^\varphi}(z)\|_{L^{\frac{pq}{p-q}}} \tag{3.6}$$

and

$$\|C_\varphi^g\| \lesssim \|C_b^g\| + |a|^{-\frac{2}{p}} \|M_{C_\varphi^g}(z)\|_{L^{\frac{pq}{p-q}}}. \tag{3.7}$$

Let $(f_n)_n$ be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. Then for every $R > 0$ and $n \in \mathbb{N}$, using (3.1), (3.2), and again Hölder's inequality, we get

$$\begin{aligned}
\|Tf_n\|_q^q &\simeq |(Tf_n)(0)|^q + \int_{|z|\leq R} |f_n(\varphi(z))|^q m_T(z)^q dA(z) \\
&\quad + \int_{|z|>R} |f_n(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \max_{|z|\leq R} |f_n(\varphi(z))|^q \int_{|z|\leq R} m_T(z)^q dA(z) \\
&\quad + \left(\int_{|z|>R} |f_n(\varphi(z))|^p e^{-\frac{p|\varphi(z)|^2}{2}} dA(z)\right)^{\frac{q}{p}} \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{1}\|_q^q \max_{|z|\leq R} |f_n(\varphi(z))|^q \\
&\quad + |a|^{-\frac{2q}{p}} \|f_n\|_p^q \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{1}\|_q^q \max_{|z|\leq R} |f_n(\varphi(z))|^q \\
&\quad + |a|^{-\frac{2q}{p}} \sup_n \|f_n\|_p^q \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}}.
\end{aligned}$$

Since $(V_g^\varphi f_n)(0) = 0$ for all n and $(C_\varphi^g f_n)(0) \to 0$ as $n \to \infty$, letting $n \to \infty$ and then $R \to \infty$ in the last inequality, and using the fact that $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, we conclude that the sequence Tf_n converges to 0 in $\mathcal{F}^q(\mathbb{C})$.

Consequently, the assertion follows from Lemma 2.3. Moreover, the desired estimates for $\|V_g^\varphi\|$ and $\|C_\varphi^g\|$ can be obtained from (3.5), (3.6) and (3.7). $\square$

We summarize all situations of the function $\varphi(z) = az + b$ with $|a| \le 1$ for both operators V_g^φ and C_φ^g.

Remark 3.6. (1) If $a = 0$, then by Corollary 3.3, for every $0 < p, q < \infty$, $V_g^\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every non-constant function $g \in \mathcal{F}^q(\mathbb{C})$ and $C_\varphi^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every non-constant function $g \in \mathcal{O}(\mathbb{C})$. This means that in this case there exist entire functions g, say $g(z) = e^{\frac{\alpha z^2}{2}}$ with $0 < \alpha < 1$, such that both products $V_g^\varphi = V_g \circ C_\varphi$ and $C_\varphi^g = C_\varphi \circ V_g$ are compact from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$, but the Volterra operator $V_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is unbounded by [4, Theorem 1]. It should be noted that by [19, Corollary 3.2], the composition operator $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact when $a = 0$.

(2) If $0 < |a| < 1$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every $0 < p, q < \infty$. However, in this case there are non-constant entire functions g, which induce an unbounded operator $V_g^\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ or $C_\varphi^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$. Indeed, for $g_1(z) = e^{\frac{\alpha z^2}{2}}$ in $\mathcal{F}^q(\mathbb{C})$ with $1 - |a|^2 < \alpha < 1$, we have

$$\sup_{z\in\mathbb{C}} M_{V_{g_1}^\varphi}(z) = \sup_{z\in\mathbb{C}} \frac{\alpha|z|}{1+|z|} e^{\frac{(|a|^2-1)|z|^2+\alpha \mathrm{Re}(z^2)+2\mathrm{Re}(\bar{b}az)+|b|^2}{2}} = \infty,$$

and for $g_2(z) = e^{\frac{\alpha((z-b)/a)^2}{2}}$ with $1 - |a|^2 < \alpha < 1$, we get

$$C_\varphi^{g_2}(\mathbf{1})(z) = g_2(\varphi(z)) - g_2(0) = e^{\frac{\alpha z^2}{2}} - e^{\frac{\alpha b^2}{2a^2}} \in \mathcal{F}^q(\mathbb{C}),$$

but

$$\sup_{z\in\mathbb{C}} M_{C_\varphi^{g_2}}(z) = \sup_{z\in\mathbb{C}} \frac{\alpha|z|}{1+|z|} e^{\frac{(|a|^2-1)|z|^2+\alpha \mathrm{Re}(z^2)+2\mathrm{Re}(\bar{b}az)+|b|^2}{2}} = \infty.$$

Furthermore, if $b = 0$ and $\alpha = 1 - |a|^2$, then

$$\limsup_{|z|\to\infty} M_{V_{g_1}^\varphi}(z) = \limsup_{|z|\to\infty} M_{C_\varphi^{g_2}}(z) = \limsup_{|z|\to\infty} \frac{\alpha|z|}{1+|z|} e^{(|a|^2-1)(\mathrm{Im} z)^2} = \alpha.$$

In this case by Theorem 3.4, both operators $V_{g_1}^\varphi$ and $C_\varphi^{g_2}$ are bounded, but not compact from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$ with $0 < p \le q < \infty$; moreover, by Theorem 3.5, these operators are unbounded when $0 < q < p < \infty$.

(3) If $|a| = 1$ and $b \neq 0$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is not bounded for every $0 < p, q < \infty$. In this case, we have

$$M_{V_g^\varphi}(z) = e^{\frac{|b|^2}{2}} \frac{\left|g'(z)e^{\bar{b}az}\right|}{1+|z|} \text{ and } M_{C_\varphi^g}(z) = e^{\frac{|b|^2}{2}} \frac{\left|g'(az+b)e^{\bar{b}az}\right|}{1+|z|}, z \in \mathbb{C}.$$

Using this and Liouville's theorem, we can check that

- $M_{V_g^\varphi}(z) \in L^\infty(\mathbb{C}, dA)$ if and only if $g(z) = (Az+B)e^{-\bar{b}az} + C$; moreover, $M_{V_g^\varphi}(z) \to 0$ as $|z| \to \infty$ precisely when $A = 0$, i.e. $g(z) = Be^{-\bar{b}az} + C$.
- $M_{C_\varphi^g}(z) \in L^\infty(\mathbb{C}, dA)$ if and only if $g(z) = (Az+B)e^{-\bar{b}z} + C$; moreover, $M_{C_\varphi^g}(z) \to 0$ as $|z| \to \infty$ precisely when $A = 0$, i.e. $g(z) = Be^{-\bar{b}z} + C$.

Consequently, for $0 < p \le q < \infty$, by Theorem 3.4,

- the operator $V_g^\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = (Az+B)e^{-\bar{b}az} + C$ (or, $g(z) = Be^{-\bar{b}az} + C$, respectively);
- the operator $C_\varphi^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = (Az+B)e^{-\bar{b}z} + C$ (or, $g(z) = Be^{-\bar{b}z} + C$, respectively).

While for $0 < q < p < \infty$, by Theorem 3.5,

- the operator $V_g^\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = Be^{-\bar{b}az} + C$ and $q > \frac{2p}{p+2}$;
- the operator $C_\varphi^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = Be^{-\bar{b}z} + C$ and $q > \frac{2p}{p+2}$.

(4) If $|a| = 1$ and $b = 0$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded for $0 < p \le q < \infty$ and unbounded for $0 < q < p < \infty$. In this case, we have

$$M_{V_g^\varphi}(z) = \frac{|g'(z)|}{1+|z|} \text{ and } M_{C_\varphi^g}(z) = \frac{|g'(az)|}{1+|z|}, z \in \mathbb{C}.$$

Using this and Liouville's theorem, we can check that for $T = V_g^\varphi$ or $T = C_\varphi^g$, $M_T(z) \in L^\infty(\mathbb{C}, dA)$ if and only if $g(z) = Az^2 + Bz + C$; moreover, $M_T(z) \to 0$ as $|z| \to \infty$ precisely when $A = 0$, i.e. $g(z) = Bz + C$. Therefore,

- if $0 < p \le q < \infty$, then, by Theorem 3.4, the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = Az^2 + Bz + C$ (or, $g(z) = Bz + C$, respectively);
- if $0 < q < p < \infty$, then, by Theorem 3.5, the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $g(z) = Bz + C$ and $q > \frac{2p}{p+2}$.

From this and [4, Theorem 1], in this case V_g^φ and $C_\varphi^g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ are bounded (or, compact) if and only if $V_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact, respectively).

Remark 3.7. From Remark 3.6 we can see that [15, Corollary 1] seems to be incorrect. In fact, the reasoning before [15, Corollary 1] that *the Berezin type integral transform $B_{V_g^\varphi}(w)$ is bounded only when $\varphi(z) =$*

$az+b$ with $|a|\le 1$; moreover, if $|a|=1$, then $b=0$ and compactness is achieved only when $|a|<1$, may be false.

4. The products of Volterra companion operators and composition operators

In this section we establish criteria for boundedness and compactness of the products $J_{g,\varphi}$ and $C_{\varphi,g}$ in terms of $M_{J_{g,\varphi}}(z)$ and $M_{C_{\varphi,g}}(z)$, respectively. Similarly to Section 3, we firstly prove the following auxiliary lemma, which allows us to give the study of both products $J_{g,\varphi}$ and $C_{\varphi,g}$ simultaneously. We put

$$m_{J_{g,\varphi}}(z)=\frac{|g(z)\varphi'(z)|}{1+|z|}e^{-\frac{|z|^2}{2}} \text{ and } m_{C_{\varphi,g}}(z)=\frac{|g(\varphi(z))\varphi'(z)|}{1+|z|}e^{-\frac{|z|^2}{2}}.$$

Lemma 4.1. *Let $q\in(0,\infty)$ and T be either $J_{g,\varphi}$ or $C_{\varphi,g}$ induced by entire functions g and φ, where g is nonzero. For each $f\in\mathcal{O}(\mathbb{C})$ and $z\in\mathbb{C}$,*

$$\begin{aligned}\|Tf\|_q^q &\simeq |(Tf)(0)|^q+\int_{\mathbb{C}}\frac{|f'(\varphi(\zeta))|^q}{(1+|\varphi(\zeta)|)^q}e^{-\frac{q|\varphi(\zeta)|^2}{2}}M_T(\zeta)^q dA(\zeta) \\ &= |(Tf)(0)|^q+\int_{\mathbb{C}}|f'(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta) \\ &\gtrsim |(Tf)(0)|^q+|f'(\varphi(z))|^q m_T(z)^q. \end{aligned}\tag{4.1}$$

In particular,

$$\int_{\mathbb{C}} m_T(\zeta)^q dA(\zeta)\lesssim \|T\boldsymbol{z}\|_q^q.\tag{4.2}$$

Proof. For each $f\in\mathcal{O}(\mathbb{C})$, by Lemma 2.7, we get

$$\begin{aligned}\|J_{g,\varphi}f\|_q^q &\simeq |(J_{g,\varphi}f)(0)|^q+\int_{\mathbb{C}}\frac{|f'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)|^q}{(1+|\zeta|)^q}e^{-\frac{q|\zeta|^2}{2}}dA(\zeta) \\ &= |(J_{g,\varphi}f)(0)|^q+\int_{\mathbb{C}}\frac{|f'(\varphi(\zeta))|^q}{(1+|\varphi(\zeta)|)^q}e^{-\frac{q|\varphi(\zeta)|^2}{2}}M_{J_{g,\varphi}}(\zeta)^q dA(\zeta) \\ &= |(J_{g,\varphi}f)(0)|^q+\int_{\mathbb{C}}|f'(\varphi(\zeta))|^q m_{J_{g,\varphi}}(\zeta)^q dA(\zeta)\end{aligned}$$

and

$$\begin{aligned}\|C_{\varphi,g}f\|_q^q &\simeq |(C_{\varphi,g}f)(0)|^q+\int_{\mathbb{C}}\frac{|f'(\varphi(\zeta))g(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q}e^{-\frac{q|\zeta|^2}{2}}dA(\zeta) \\ &= |(C_{\varphi,g}f)(0)|^q+\int_{\mathbb{C}}\frac{|f'(\varphi(\zeta))|^q}{(1+|\varphi(\zeta)|)^q}e^{-\frac{q|\varphi(\zeta)|^2}{2}}M_{C_{\varphi,g}}(\zeta)^q dA(\zeta) \\ &= |(C_{\varphi,g}f)(0)|^q+\int_{\mathbb{C}}|f'(\varphi(\zeta))|^q m_{C_{\varphi,g}}(\zeta)^q dA(\zeta).\end{aligned}$$

Moreover, for every $f \in \mathcal{O}(\mathbb{C})$ and $z \in \mathbb{C}$, applying Lemma 2.4 to entire functions $(f' \circ \varphi)\varphi' g$ and $(f' \circ \varphi)(g \circ \varphi)\varphi'$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \frac{|f'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) &\gtrsim \frac{|f'(\varphi(z))\varphi'(z)g(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} \\ &= |f'(\varphi(z))|^q m_{J_{g,\varphi}}(z)^q\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{C}} \frac{|f'(\varphi(\zeta))g(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) &\gtrsim \frac{|f'(\varphi(z))g(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} \\ &= |f'(\varphi(z))|^q m_{C_{\varphi,g}}(z)^q.\end{aligned}$$

From these inequalities the assertions follow. □

Proposition 4.2. *Let $p, q \in (0, \infty)$ and T be either $J_{g,\varphi}$ or $C_{\varphi,g}$ induced by entire functions φ and g, where g is nonzero. If the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded, then $M_T(z) \in L^\infty(\mathbb{C}, dA)$. In this case, $\varphi(z) = az + b$ with $|a| \le 1$ and $M_T(z) \lesssim \|T\|$ for all $z \in \mathbb{C}$.*

Proof. For every w, using $\|k_w\|_p = 1$ and (4.1), we get

$$\begin{aligned}\|T\|^q \ge \|Tk_w\|_q^q &\gtrsim \int_{\mathbb{C}} |k_w'(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta) \\ &\gtrsim |k_w'(\varphi(z))|^q m_T(z)^q = |w|^q |k_w(\varphi(z))|^q m_T(z)^q,\end{aligned}$$

for all $z \in \mathbb{C}$. In particular, with $w = \varphi(z)$ the last inequality means that

$$\|T\| \ge \|Tk_{\varphi(z)}\|_q \gtrsim |\varphi(z)| e^{\frac{|\varphi(z)|^2}{2}} m_T(z) \quad \text{for all } z \in \mathbb{C}.$$

From this and Lemma 2.8 it follows that $\varphi(z) = az + b$ with $|a| \le 1$.

Moreover, if $|\varphi(z)| \ge 1$, then

$$\|T\| \ge \|Tk_{\varphi(z)}\|_q \gtrsim (1 + |\varphi(z)|) e^{\frac{|\varphi(z)|^2}{2}} m_T(z) = M_T(z). \tag{4.3}$$

In the case $|\varphi(z)| < 1$, with $w = 2 + \varphi(z)$ the above inequality shows that

$$\begin{aligned}\|T\| \ge \|Tk_{2+\varphi(z)}\|_q &\gtrsim |2 + \varphi(z)||k_{2+\varphi(z)}(\varphi(z))| m_T(z) \\ &\ge e^{-2} e^{\frac{|\varphi(z)|^2}{2}} m_T(z) \gtrsim M_T(z).\end{aligned}$$

Thus, $M_T(z) \lesssim \|T\|$ for all $z \in \mathbb{C}$. □

In view of Proposition 4.2, throughout this section we suppose that $\varphi(z) = az + b$ with $|a| \le 1$. In the case $a = 0$, i.e. $\varphi(z) \equiv b$, it is clear that $J_{g,b}$ is the zero operator and we get the following simple result for $C_{b,g}$.

Corollary 4.3. *Let $p, q \in (0, \infty)$, $\varphi(z) \equiv b$ and g be a nonzero entire function. Then the operator $C_{b,g} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact and*

$$\|C_{b,g}\| \lesssim |b|(1 + |b|) e^{\frac{|b|^2}{2}} \max_{\zeta \in [0,b]} |g(\zeta)|.$$

Proof. For every $f \in \mathcal{F}^p(\mathbb{C})$,

$$C_{b,g}f(z) = \int_0^b f'(\zeta)g(\zeta)d\zeta, \ z \in \mathbb{C}.$$

Hence, by Lemmas 2.4 and 2.7, we obtain

$$\begin{aligned}
\|C_{b,g}f\|_q &= \left| \int_0^b f'(\zeta)g(\zeta)d\zeta \right| \le |b| \max_{\zeta \in [0,b]} |f'(\zeta)g(\zeta)| \\
&\lesssim |b|(1+|b|)e^{\frac{|b|^2}{2}} \left(\int_{\mathbb{C}} \frac{|f'(\zeta)|^p}{(1+|\zeta|)^p} e^{-\frac{p|\zeta|^2}{2}} d(\zeta) \right)^{\frac{1}{p}} \max_{\zeta \in [0,b]} |g(\zeta)| \\
&\le |b|(1+|b|)e^{\frac{|b|^2}{2}} \|f\|_p \max_{\zeta \in [0,b]} |g(\zeta)|.
\end{aligned}$$

From these the assertions follow. □

The case $0 < |a| \le 1$ is more complicated. Now we give necessary and sufficient conditions for boundedness and compactness of $J_{g,\varphi}$ and $C_{\varphi,g}$ in the case $0 < p \le q < \infty$.

Theorem 4.4. *Let $0 < p \le q < \infty$, T be either $J_{g,\varphi}$ or $C_{\varphi,g}$ induced by a nonzero entire function g and $\varphi(z) = az + b$ with $0 < |a| \le 1$.*

(a) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $M_T(z) \in L^\infty(\mathbb{C}, dA)$. Moreover,*

$$\|M_{J_{g,\varphi}}(z)\|_{L^\infty} \lesssim \|J_{g,\varphi}\| \lesssim |a|^{-\frac{2}{q}} \|M_{J_{g,\varphi}}(z)\|_{L^\infty}$$

and

$$\|M_{C_{\varphi,g}}(z)\|_{L^\infty} \lesssim \|C_{\varphi,g}\| \lesssim \|C_{b,g}\| + |a|^{-\frac{2}{q}} \|M_{C_{\varphi,g}}(z)\|_{L^\infty}.$$

(b) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if*

$$\lim_{|z| \to \infty} M_T(z) = 0.$$

Proof. (a) The necessity follows from Proposition 4.2. Suppose that $M_T(z) \in L^\infty(\mathbb{C}, dA)$. Then using (4.1) and Lemmas 2.2 and 2.7, for every $f \in \mathcal{F}^p(\mathbb{C})$, we get

$$\begin{aligned}
\|Tf\|_q^q &\simeq |(Tf)(0)|^q + \int_{\mathbb{C}} \frac{|f'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\le |(Tf)(0)|^q + \|M_T(z)\|_{L^\infty}^q \int_{\mathbb{C}} \frac{|f'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\
&\le |(Tf)(0)|^q + |a|^{-2} \|M_T(z)\|_{L^\infty}^q \int_{\mathbb{C}} \frac{|f'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\
&\lesssim |(Tf)(0)|^q + |a|^{-2} \|M_T(z)\|_{L^\infty}^q \|f\|_q^q \\
&\lesssim |(Tf)(0)|^q + |a|^{-2} \|M_T(z)\|_{L^\infty}^q \|f\|_p^q.
\end{aligned}$$

From this, Proposition 4.2, and the fact that $(J_{g,\varphi}f)(0) = 0$ and $(C_{\varphi,g}f)(0) = C_{b,g}f$, it follows that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded; moreover,

$$\|M_{J_{g,\varphi}}(z)\|_{L^\infty} \lesssim \|J_{g,\varphi}\| \lesssim |a|^{-\frac{2}{q}}\|M_{J_{g,\varphi}}(z)\|_{L^\infty}$$

and

$$\|M_{C_{\varphi,g}}(z)\|_{L^\infty} \lesssim \|C_{\varphi,g}\| \lesssim \|C_{b,g}\| + |a|^{-\frac{2}{q}}\|M_{C_{\varphi,g}}(z)\|_{L^\infty}.$$

(b) **Necessary.** Suppose that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact. Then for every sequence $(z_n)_n$ with $z_n \to \infty$, $\varphi(z_n) \to \infty$, and hence, the sequence $k_{\varphi(z_n)}$ converges to 0 in $\mathcal{O}(\mathbb{C})$ as $n \to \infty$. Therefore, by (4.3) and Lemma 2.3,

$$M_T(z_n) \lesssim \|Tk_{\varphi(z_n)}\| \to 0 \text{ as } n \to \infty.$$

Thus, $\lim_{|z|\to\infty} M_T(z) = 0$.

Sufficiency. By part (a), the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Let (f_n) be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. Then, for every $n \in \mathbb{N}$ and $R > 0$, using (4.1), Lemmas 2.2 and 2.7, we get

$$\begin{aligned}
\|Tf_n\|_q^q &\simeq |(Tf_n)(0)|^q + \int_{|z|\le R} |f_n'(\varphi(z))|^q m_T(z)^q dA(z) \\
&\quad + \int_{|z|>R} \frac{|f_n'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \max_{|z|\le R} |f_n'(\varphi(z))|^q \int_{|z|\le R} m_T(z)^q dA(z) \\
&\quad + \sup_{|z|>R} M_T(z)^q \int_{|z|>R} \frac{|f_n'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{z}\|_q^q \max_{|z|\le R} |f_n'(\varphi(z))|^q \\
&\quad + |a|^{-2} \sup_{|z|>R} M_T(z)^q \int_{\mathbb{C}} \frac{|f_n'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{z}\|_q^q \max_{|z|\le R} |f_n'(\varphi(z))|^q + |a|^{-2}\|f_n\|_q^q \sup_{|z|>R} M_T(z)^q \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{z}\|_q^q \max_{|z|\le R} |f_n'(\varphi(z))|^q + |a|^{-2} \sup_n \|f_n\|_p^q \sup_{|z|>R} M_T(z)^q.
\end{aligned}$$

On the other hand, obviously, the sequence $(f_n')_n$ also converges to 0 in $\mathcal{O}(\mathbb{C})$, $(J_{g,\varphi}f_n)(0) = 0$ for all n, and $(C_{\varphi,g}f_n)(0) \to 0$ as $n \to \infty$. Thus, letting $n \to \infty$ and then $R \to \infty$ in the last inequality, we observe that $(Tf_n)_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$.

From this and Lemma 2.3, the assertion follows. $\square$

Similarly to the products V_g^φ and C_φ^g, in the case $0 < q < p < \infty$, boundedness and compactness of the operators $J_{g,\varphi}$ and $C_{\varphi,g}$ are equivalent. To show this, we define the following positive pull-back measure $\mu_{J_{g,\varphi},q}$ and $\mu_{C_{\varphi,g},q}$ on $\mathbb{C}$:

$$\mu_{J_{g,\varphi},q}(B) = \int_{\varphi^{-1}(B)} \frac{|g(z)\varphi'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} dA(z) = \int_{\varphi^{-1}(B)} m_{J_{g,\varphi}}(z)^q dA(z)$$

and

$$\mu_{C_{\varphi,g},q}(B) = \int_{\varphi^{-1}(B)} \frac{|g(\varphi(z))\varphi'(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} dA(z) = \int_{\varphi^{-1}(B)} m_{C_{\varphi,g}}(z)^q dA(z)$$

for every Borel subset B of $\mathbb{C}$. Thus,

$$\mu_{T,q}(B) = \int_{\varphi^{-1}(B)} m_T(z)^q dA(z) \text{ for } T = J_{g,\varphi} \text{ or } T = C_{\varphi,g}. \tag{4.4}$$

Theorem 4.5. *Let $0 < q < p < \infty$, T be either $J_{g,\varphi}$ or $C_{\varphi,g}$ induced by a nonzero entire function g and $\varphi(z) = az+b$ with $0 < |a| \le 1$. The following statements are equivalent:*

(i) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded.*
(ii) *The operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.*
(iii) *$M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.*

In this case

$$|a|^{\frac{2(p-q)}{pq}} \|M_{J_{g,\varphi}}(z)\|_{L^{\frac{pq}{p-q}}} \lesssim \|J_{g,\varphi}\| \lesssim |a|^{-\frac{2}{p}} \|M_{J_{g,\varphi}}(z)\|_{L^{\frac{pq}{p-q}}}$$

and

$$|a|^{\frac{2(p-q)}{pq}} \|M_{C_{\varphi,g}}(z)\|_{L^{\frac{pq}{p-q}}} \lesssim \|C_{\varphi,g}\| \lesssim \|C_{b,g}\| + |a|^{-\frac{2}{p}} \|M_{C_{\varphi,g}}(z)\|_{L^{\frac{pq}{p-q}}}.$$

Proof. (ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (iii). Suppose that $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Then for every $f \in \mathcal{F}^p_W(\mathbb{C})$, by Lemma 2.7, $F(z) = \int_0^z f(\zeta)d\zeta$ belongs to $\mathcal{F}^p(\mathbb{C})$, and hence, using (4.1) and (4.4), we have

$$\begin{aligned}
\|T\|\|f\|_{W,p} &\simeq \|T\|\|F\|_p \ge \|TF\|_q \\
&\gtrsim \left(\int_{\mathbb{C}} |f(\varphi(z))|^q m_T(z)^q dA(z)\right)^{\frac{1}{q}} = \left(\int_{\mathbb{C}} |f(z)|^q d\mu_{T,q}(z)\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{C}} \frac{|f(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} d\lambda_{T,q}(z)\right)^{\frac{1}{q}},
\end{aligned}$$

where $d\lambda_{T,q}(z) = (1+|z|)^q e^{\frac{q|z|^2}{2}} d\mu_{T,q}(z)$. From this it follows that $\lambda_{T,q}$ is a (p,q,W)- Fock Carleson measure. Then, by Lemma 2.6, we get

$$(\widetilde{\lambda_{T,q}})_W(w) = \int_{\mathbb{C}} \frac{|k_{W,w}(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} d\lambda_{T,q}(\zeta) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

Moreover, by (4.1), for every $w, z \in \mathbb{C}$,

$$\begin{aligned}(\widetilde{\lambda_{T,q}})_W(w) &= \int_{\mathbb{C}} |k_{W,w}(\zeta)|^q d\mu_{T,q}(\zeta) = \int_{\mathbb{C}} |k_{W,w}(\varphi(\zeta))|^q m_T(\zeta)^q dA(\zeta)\\ &\gtrsim |k_{W,w}(\varphi(z))|^q m_T(z)^q.\end{aligned}$$

In particular, with $w = \varphi(z)$ using (2.1), we obtain

$$\begin{aligned}(\widetilde{\lambda_{T,q}})_W(\varphi(z)) &\gtrsim |k_{W,\varphi(z)}(\varphi(z))|^q m_T(z)^q\\ &\simeq (1+|\varphi(z)|)^q e^{\frac{q|\varphi(z)|^2}{2}} m_T(z)^q = M_T(z)^q,\end{aligned}$$

for every $z \in \mathbb{C}$. Hence

$$\begin{aligned}\int_{\mathbb{C}} M_T(z)^{\frac{pq}{p-q}} dA(z) &\lesssim \int_{\mathbb{C}} \left((\widetilde{\lambda_{T,q}})_W(\varphi(z))\right)^{\frac{p}{p-q}} dA(z)\\ &= |a|^{-2} \int_{\mathbb{C}} \left((\widetilde{\lambda_{T,q}})_W(w)\right)^{\frac{p}{p-q}} dA(w) < \infty.\end{aligned}$$

From this it follows that $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, and by Lemma 2.6 again,

$$\begin{aligned}\|M_T(z)\|_{L^{\frac{pq}{p-q}}} &\lesssim |a|^{-\frac{2(p-q)}{pq}} \left(\|(\widetilde{\lambda_{T,q}})_W\|_{L^{\frac{p}{p-q}}}\right)^{\frac{1}{q}}\\ &\simeq |a|^{-\frac{2(p-q)}{pq}} \|\lambda_{T,q}\|_W \lesssim |a|^{-\frac{2(p-q)}{pq}} \|T\|. \qquad (4.5)\end{aligned}$$

(iii) $\Longrightarrow$ (ii). For each function $f \in \mathcal{F}^p(\mathbb{C})$, using (4.1), Lemma 2.7, and Hölder's inequality, we get

$$\begin{aligned}\|Tf\|_q^q &\simeq |(Tf)(0)|^q + \int_{\mathbb{C}} \frac{|f'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z)\\ &\leq |(Tf)(0)|^q + \left(\int_{\mathbb{C}} \frac{|f'(\varphi(z))|^p}{(1+|\varphi(z)|)^p} e^{-\frac{p|\varphi(z)|^2}{2}} dA(z)\right)^{\frac{q}{p}} \left(\int_{\mathbb{C}} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}}\\ &= |(Tf)(0)|^q + |a|^{-\frac{2q}{p}} \left(\int_{\mathbb{C}} \frac{|f'(\zeta)|^p}{(1+|\zeta|)^p} e^{-\frac{p|\zeta|^2}{2}} dA(\zeta)\right)^{\frac{q}{p}} \left(\int_{\mathbb{C}} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}}\\ &\lesssim |(Tf)(0)|^q + |a|^{-\frac{2q}{p}} \|f\|_p^q \|M_T(z)\|^q_{L^{\frac{pq}{p-q}}}.\end{aligned}$$

This and the fact that $(J_{g,\varphi}f)(0) = 0$ and $(C_{\varphi,g}f)(0) = C_{b,g}f$ imply that the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded; moreover,

$$\|J_{g,\varphi}\| \lesssim |a|^{-\frac{2}{p}} \|M_{J_{g,\varphi}}(z)\|_{L^{\frac{pq}{p-q}}} \qquad (4.6)$$

and

$$\|C_{\varphi,g}\| \lesssim \|C_{b,g}\| + |a|^{-\frac{2}{p}} \|M_{C_{\varphi,g}}(z)\|_{L^{\frac{pq}{p-q}}} \qquad (4.7)$$

Let $(f_n)_n$ be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. Then for every $R > 0$ and $n \in \mathbb{N}$, again using (4.1), (4.2), and Hölder's inequality, we obtain

$$\begin{aligned}
\|Tf_n\|_q^q &\simeq |(Tf_n)(0)|^q + \int_{|z|\le R} |f_n'(\varphi(z))|^q m_T(z)^q dA(z) \\
&\quad + \int_{|z|>R} \frac{|f_n'(\varphi(z))|^q}{(1+|\varphi(z)|)^q} e^{-\frac{q|\varphi(z)|^2}{2}} M_T(z)^q dA(z) \\
&\lesssim |(Tf_n)(0)|^q + \max_{|z|\le R} |f_n'(\varphi(z))|^q \int_{|z|\le R} m_T(z)^q dA(z) \\
&\quad + \left(\int_{|z|>R} \frac{|f_n'(\varphi(z))|^p}{(1+|\varphi(z)|)^p} e^{-\frac{p|\varphi(z)|^2}{2}} dA(z)\right)^{\frac{q}{p}} \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{z}\|_q^q \max_{|z|\le R} |f_n'(\varphi(z))|^q \\
&\quad + |a|^{-\frac{2q}{p}} \|f_n\|_p^q \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}} \\
&\lesssim |(Tf_n)(0)|^q + \|T\mathbf{z}\|_q^q \max_{|z|\le R} |f_n'(\varphi(z))|^q \\
&\quad + |a|^{-\frac{2q}{p}} \sup_n \|f_n\|_p^q \left(\int_{|z|>R} M_T(z)^{\frac{pq}{p-q}} dA(z)\right)^{\frac{p-q}{p}}.
\end{aligned}$$

Similarly to the proof of Theorems 3.5 and 4.4, letting $n \to \infty$ and then $R \to \infty$ in the last inequality, and using the fact that $M_T(z) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, we conclude that the sequence Tf_n converges to 0 in $\mathcal{F}^q(\mathbb{C})$.

Consequently, the assertion follows from Lemma 2.3. Moreover, the desired estimates for $\|J_{g,\varphi}\|$ and $\|C_{\varphi,g}\|$ can be reduced from (4.5), (4.6) and (4.7). $\square$

Putting $\varphi(z) = z$ in Theorems 4.4 and 4.5, we get the following characterization for boundedness and compactness of Volterra companion operator J_g.

Corollary 4.6. *Let g be a nonzero entire function on $\mathbb{C}$.*

(i) *If $0 < p \le q < \infty$, then the operator $J_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if g is a constant; however, this operator J_g is not compact.*
(ii) *If $0 < q < p < \infty$, then every operator $J_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is unbounded.*

We summarize all situations of the function $\varphi(z) = az + b$ with $|a| \le 1$ for both operators $J_{g,\varphi}$ and $C_{\varphi,g}$.

Remark 4.7. (1) If $a = 0$, then $J_{g,\varphi}$ is the zero operator and by Corollary 4.3, for every $0 < p, q < \infty$, $C_{\varphi,g} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for any nonzero entire function g. From this and Corollary 4.6, for every nonconstant entire functions g, the product $C_{\varphi,g} = C_\varphi \circ J_g$ is compact from

$\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$, but the Volterra companion operator $J_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is unbounded.

(2) If $0 < |a| < 1$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every $0 < p, q < \infty$. However, in this case there are non-constant entire functions g, which induce an unbounded operator $J_{g,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ or $C_{\varphi,g} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$. Indeed, for $g_1(z) = e^{\frac{\alpha z^2}{2}}$ with $1 - |a|^2 < \alpha < 1$, we obtain

$$\sup_{z\in\mathbb{C}} M_{J_{g_1,\varphi}}(z) = \sup_{z\in\mathbb{C}} \frac{|a|(1+|az+b|)}{1+|z|} e^{\frac{(|a|^2-1)|z|^2+\alpha\mathrm{Re}(z^2)+2\mathrm{Re}(\bar{b}az)+|b|^2}{2}} = \infty,$$

and for $g_2(z) = e^{\frac{\alpha((z-b)/a)^2}{2}}$ with $1 - |a|^2 < \alpha < 1$, we get

$$\sup_{z\in\mathbb{C}} M_{C_\varphi,g_2}(z) = \sup_{z\in\mathbb{C}} \frac{|a|(1+|az+b|)}{1+|z|} e^{\frac{(|a|^2-1)|z|^2+\alpha\mathrm{Re}(z^2)+2\mathrm{Re}(\bar{b}az)+|b|^2}{2}} = \infty.$$

Furthermore, if $b = 0$ and $\alpha = 1 - |a|^2$, then

$$\begin{aligned}\limsup_{|z|\to\infty} M_{J_{g_1,\varphi}}(z) &= \limsup_{|z|\to\infty} M_{C_\varphi,g_2}(z)\\ &= \limsup_{|z|\to\infty} \frac{|a|(1+|az|)}{1+|z|} e^{(|a|^2-1)(\mathrm{Im} z)^2} = |a|^2.\end{aligned}$$

In this case by Theorem 4.4, both operators $J_{g_1,\varphi}$ and C_φ, g_2 are bounded, but not compact from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$ with $0 < p \le q < \infty$; moreover, by Theorem 4.5, these operators are unbounded when $0 < q < p < \infty$.

(3) If $|a| = 1$ and $b \ne 0$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is not bounded for every $0 < p, q < \infty$. In this case, we have

$$M_{J_{g,\varphi}}(z) = e^{\frac{|b|^2}{2}} \frac{(1+|az+b|)\left|g(z)e^{\bar{b}az}\right|}{1+|z|}$$

and

$$M_{C_{\varphi,g}}(z) = e^{\frac{|b|^2}{2}} \frac{(1+|az+b|)\left|g(az+b)e^{\bar{b}az}\right|}{1+|z|}, z \in \mathbb{C}.$$

Using this and Liouville's theorem, we can check that

- $M_{J_{g,\varphi}}(z) \in L^\infty(\mathbb{C}, dA)$ if and only if $g(z) = Ae^{-\bar{b}az}$; moreover, $M_{J_{g,\varphi}}(z) \to 0$ as $|z| \to \infty$ precisely when g is the zero function.
- $M_{C_{\varphi,g}}(z) \in L^\infty(\mathbb{C}, dA)$ if and only if $g(z) = Ae^{-\bar{b}z}$; moreover, $M_{C_{\varphi,g}}(z) \to 0$ as $|z| \to \infty$ precisely when g is the zero function.

Consequently, for $0 < p \le q < \infty$, by Theorem 4.4,

- the operator $J_{g,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $g(z) = Ae^{-\bar{b}az}$;
- the operator $C_{\varphi,g} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $g(z) = Ae^{-\bar{b}z}$;

- there does not exist a nonzero entire function g such that $J_{g,\varphi}$ or $C_{\varphi,g} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.

For $0 < q < p < \infty$, by Theorem 4.5, the operators $J_{g,\varphi}$ and $C_{\varphi,g}$: $\mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ are unbounded for any nonzero entire function g.

(4) If $|a| = 1$ and $b = 0$, then by [19, Corollaries 3.5 and 3.6], $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded for $0 < p \leq q < \infty$ and unbounded for $0 < q < p < \infty$. In this case, we have

$$M_{J_{g,\varphi}}(z) = |g(z)| \text{ and } M_{C_{\varphi,g}}(z) = |g(az)|, z \in \mathbb{C}.$$

Using this and Liouville's theorem, we can check that for $T = J_{g,\varphi}$ or $T = C_{\varphi,g}$, $M_T(z) \in L^\infty(\mathbb{C}, dA)$ if and only if g is constant; moreover, $M_T(z) \to 0$ as $|z| \to \infty$ precisely when g is zero. Therefore,

- if $0 < p \leq q < \infty$, then, by Theorem 4.4, the operator T : $\mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if g is constant, and there does not exist a nonzero compact operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$;
- if $0 < q < p < \infty$, then, by Theorem 4.5, the operator T : $\mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is unbounded for any nonzero entire function g.

From this and Corollary 4.6, in this case the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact) if and only if $J_g : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded (or, compact, respectively).

Remark 4.8. In [14, Proposition 2.7] and [16, Theorems 2.1 and 2.2], it was stated that in the case $0 < p \leq q < \infty$,

$$\|C_\varphi^g\| \simeq \|B_{C_\varphi^g}(w)\|_{L^\infty}^{\frac{1}{q}} \text{ and } \|C_{\varphi,g}\| \simeq \|B_{C_{\varphi,g}}(w)\|_{L^\infty}^{\frac{1}{q}}, \tag{4.8}$$

while if $0 < q < p < \infty$, then

$$\|C_\varphi^g\| \simeq \|B_{C_\varphi^g}(w)\|_{L^{\frac{p}{p-q}}} \text{ and } \|C_{\varphi,g}\| \simeq \|B_{C_{\varphi,g}}(w)\|_{L^{\frac{p}{p-q}}}. \tag{4.9}$$

We can see that these estimates are slightly different from our ones obtained in Theorems 3.4, 3.5, 4.4, and 4.5. In fact, to get (4.8) and (4.9) the author used the following inequalities:

$$\|C_\varphi^g f\|_q^q \simeq \int_{\mathbb{C}} \frac{|f(\varphi(\zeta))g'(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta)$$

and

$$\|C_{\varphi,g} f\|_q^q \simeq \int_{\mathbb{C}} \frac{|f'(\varphi(\zeta))g(\varphi(\zeta))\varphi'(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta).$$

From this and Lemma 2.7, it is clear that the terms $|(C_\varphi^g f)(0)|^q$ and $|(C_{\varphi,g} f)(0)|^q$ were omitted in these inequalities. Therefore, the constants C hidden in the estimates (4.8) and (4.9) must depend not only on p and q, but also on functions φ and g.

It should be noted that the constants C hidden in the estimates for $\|C_\varphi^g\|$ and $\|C_{\varphi,g}\|$ in our results are dependent only on p and q.

We end this section with some discussions about the generalized Volterra companion operator $T_{g,\varphi}$ mentioned in the Introduction. Similarly to the operator $J_{g,\varphi}$ and $C_{\varphi,g}$, we put

$$M_{T_{g,\varphi}}(z) = \frac{(1+|\varphi(z)|)|g(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}}, \ z \in \mathbb{C}.$$

Then we can see that Proposition 4.2 also holds for $T_{g,\varphi}$. That is, if the operator $T_{g,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded with $0 < p, q < \infty$, then $M_{T_{g,\varphi}}(z) \in L^\infty(\mathbb{C}, dA)$. In this case, $\varphi(z) = az + b$ with $|a| \leq 1$.

Indeed, for every w, using $\|k_w\|_p = 1$, Lemmas 2.4 and 2.7, we have

$$\begin{aligned}
\|T_{g,\varphi}\|^q \geq \|T_{g,\varphi} k_w\|_q^q &\gtrsim \int_{\mathbb{C}} \frac{|k'_w(\varphi(\zeta))g(\zeta)|^q}{(1+|\zeta|)^q} e^{-\frac{q|\zeta|^2}{2}} dA(\zeta) \\
&\gtrsim \frac{|k'_w(\varphi(z))g(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}} \\
&= |w|^q \frac{|k_w(\varphi(z))g(z)|^q}{(1+|z|)^q} e^{-\frac{q|z|^2}{2}},
\end{aligned}$$

for all $z \in \mathbb{C}$. In particular, with $w = \varphi(z)$ the last inequality means that

$$\|T_{g,\varphi}\| \geq \|T_{g,\varphi} k_{\varphi(z)}\|_q \gtrsim \frac{|\varphi(z)g(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}} \quad \text{for all } z \in \mathbb{C}.$$

From this and Lemma 2.8, it follows that $\varphi(z) = az + b$ with $|a| \leq 1$.

Obviously, if $a = 0$, i.e. $\varphi(z) \equiv b$, then the operator $T_{g,b} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every $0 < p, q < \infty$, otherwise if $\varphi(z) = az + b$ with $0 < |a| \leq 1$, then

$$\begin{aligned}
T_{g,\varphi} f(z) &= \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta \\
&= a^{-1} \int_0^z f'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)d\zeta = a^{-1} J_{g,\varphi} f(z)
\end{aligned}$$

and

$$\begin{aligned}
M_{T_{g,\varphi}}(z) &= \frac{(1+|\varphi(z)|)|g(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}} \\
&= |a|^{-1} \frac{(1+|\varphi(z)|)|g(z)\varphi'(z)|}{1+|z|} e^{\frac{|\varphi(z)|^2-|z|^2}{2}} = |a|^{-1} M_{J_{g,\varphi}}(z),
\end{aligned}$$

for every $f \in \mathcal{O}(\mathbb{C})$ and $z \in \mathbb{C}$.

Consequently, Theorems 4.4 and 4.5 also hold for the operator $T_{g,\varphi}$.

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