THE HIT PROBLEM FOR THE POLYNOMIAL ALGEBRA IN CERTAIN DEGREES

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Abstract. Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$, in k variables $x_1, x_2, \ldots, x_k$, each of degree 1.

We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra. In this paper, we extend our results in [10] on the hit problem in degree $(k-1)(2^d-1)$ with $k \geqslant 6$.

1. Introduction

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$, with the degree of each x_i being 1. This algebra arises as the cohomology with coefficients in $\mathbb{F}_2$ of an elementary abelian 2-group of rank k. Then, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod operations Sq^i and subject to the Cartan formula (see Steenrod and Epstein [14]).

The *Peterson hit problem* in algebraic topology asks for a minimal generating set for the polynomial algebra P_k as a module over the Steenrod algebra. Equivalently, we want to find a vector space basis for $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in each degree, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$.

The vector space QP_k was explicitly calculated by Peterson [9] for $k = 1, 2$, by Kameko [4] for $k = 3$, and by us [15] for $k = 4$. Recently, the hit problem and it's applications to representations of general linear groups have been presented in the books of Walker and Wood [18, 19].

From the results of Wood [20] and Kameko [4], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \tag{1.1}$$

where s, d, m are certain non-negative integers, $1 \leqslant s < k$ and $\mu(m) < s$. Here, by $\mu(m)$ one means the smallest number r for which it is possible to write $m = \sum_{1 \leqslant i \leqslant r}(2^{u_i} - 1)$ with $u_i > 0$. For $s = k-1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [8], Repka and Selick [12], Walker and Wood [17] and the present author [15]. For $s = k - 1$ and $m = 0$, it is partially studied by Mothebe [5, 6] and by Phúc and Sum [10, 11]. In this case, the problem was explicitly calculated for $k \leqslant 5$.

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In this paper, we extend our results in [10] on the hit problem in degree n of the form (1.1) with $s = k-1$, $m = 0$, $k \geqslant 6$ and $d \geqslant 2$.

Denote by $(QP_k)_n$ the subspace of QP_k consisting of the classes represented by the homogeneous polynomials of degree n in P_k. Carlisle and Wood showed in [1] that the dimension of the vector space $(QP_k)_n$ is uniformly bounded by a number depended only on k. Moreover, base on our results in [15], we can show that for d big enough, this dimension does not depend on d.

For a positive integer a, denote by $\alpha(a)$ the number of ones in dyadic expansion of a and by $\zeta(a)$ the greatest integer u such that a is divisible by 2^u. That means $a = 2^{\zeta(a)}b$ with b an odd integer.

Theorem 1.1. *Let $n = (k-1)(2^d-1)$ with d a positive integer and let $d(k) = k-1-\alpha(k-1)-\zeta(k-1)$. If $d \geqslant d(k)+k-1$ and $k \geqslant 4$, then*

$$\dim(QP_k)_n = (2^k-1)\dim(QP_{k-1})_{(k-1)(2^{d(k)}-1)}.$$

For $k = 4$, we have $d(4) = 1$, $\dim(QP_3)_3 = 7$. Hence, by Theorrem 1.1,

$$\dim(QP_4)_{3(2^d-1)} = (2^4-1)\times 7 = 105, \quad \text{for all } d \geqslant 4, \text{ (see Sum [15])}.$$

For $k = 5$, we have $d(5) = 1$, $\dim(QP_4)_4 = 21$. Hence, $\dim(QP_5)_{4(2^d-1)} = (2^5-1)\times 21 = 651$ for all $d \geqslant 5$, (see Phúc and Sum [11]). For $k = 6$, we have $d(6) = 3$, and $5(2^{d(6)}-1) = 35$.

Proposition 1.2 (Hưng [3])**.** *We have* $\dim(QP_5)_{35} = 1117$.

Hưng proved this result in [3] by using a computer computation. However, the detailed proof were unpublished at the time of the writing. We have also proved this proposition by using Kameko's method in [4]. However, the proof is a hard work. It will be published in detail elsewhere.

Combining Theorem 1.1 and Proposition 1.2, we obtain the following.

Corollary 1.3. *Let $n = 5(2^d-1)$ with d a positive integer. If $d \geqslant 8$, then*

$$\dim(QP_6)_n = (2^6-1)\times 1117 = 70371.$$

For any $k \geqslant 7$ and $d \geqslant 2$, we extend our result in [10] on a lower bound for $\dim(QP_k)_n$.

Let ω be a weight vector of degree $\deg\omega = m$ and $QP_k(\omega)$ be the quotient of $(QP_k)_m$ associated with ω (see Section 2.) We prove the following.

Theorem 1.4. *Let $n = (k-1)(2^d-1)$ with d a positive integer. If $d \geqslant 2$, then*

$$\dim(QP_k)_n > \Big(\sum_{\deg\omega = k-1}\dim QP_{k-1}(\omega)\Big)\sum_{u=1}^{\min\{k,d-1\}}\binom{k}{u} + \binom{k}{\min\{k,d\}}.$$

By explicitly computing the space $QP_{k-1}(\omega)$ for some ω we see that this result implies our result in [10], hence it is also implies Mothebe's result in [5, 6].

In Section 2, we recall some needed information on admissible monomials in P_k and Singer's criterion on hit monomials. The proofs of the main results will be presented in Section 3. At the end of Section 3, we show that if $d \geqslant d(k)+k-1$, then Theorem 1.1 implies Theorem 4.13.

2. Preliminaries

In this section, we recall some results on the admissible monomials and the hit monomials from Kameko [4], Mothebe and Uys [7] and Singer [13], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1,j_2,\ldots,j_s\}} = \prod_{j\in\mathbb{N}_k\setminus\mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

In particular, $X_{\mathbb{N}_k} = 1$, $X_\emptyset = x_1x_2\ldots x_k$, $X_j = x_1\ldots\hat{x}_j\ldots x_k$, $1 \leqslant j \leqslant k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$.

For a monomial $x \in P_k$, we write $x = x_1^{\nu_1(x)}x_2^{\nu_2(x)}\ldots x_k^{\nu_k(x)}$. Set $\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\}$, for $t \geqslant 0$. Then, we have $x = \prod_{t\geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. A weight vector ω is a sequence of non-negative integers $(\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ such that $\omega_i = 0$ for $i \gg 0$. For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x.

The sets of the weight vectors and the exponent vectors are given the left lexicographical order. For weight vectors $\omega = (\omega_1, \omega_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$, we define $\deg\omega = \sum_{i>0} 2^{i-1}\omega_i$, the length $\ell(\omega) = \max\{i : \omega_i > 0\}$, the concatenation $\omega|\eta = (\omega_1, \ldots, \omega_r, \eta_1, \eta_2, \ldots)$ if $\ell(\omega) = r$ and $(a)|^b = (a)|(a)|\ldots|(a)$, ($b$ times of (a)'s), where a, b are positive integers. Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y = \deg\omega$ and $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of $P_k(\omega)$ spanned by monomials y such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$, then f is said to be hit.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have $(QP_k)_n \cong \bigoplus_{\deg\omega=n} QP_k(\omega)$ (see Walker and Wood [18]).

Let GL_n be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_n by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_n upon P_n commute with each other, there is an inherited action of GL_n on QP_n.

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ is an Σ_k-module, where $\Sigma_k \subset GL_k$ is the symmetric group. Furthermore, we have the following.

Proposition 2.4 (See Sum [16]). *For any weight vector ω, the space $QP_k(\omega)$ is an GL_k-module.*

For a polynomial $f \in P_k(\omega)$, we denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

Definition 2.5. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:
i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.6. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_m$ such that $y_t < x$ for $t = 1, 2, \ldots, m$ and $x - \sum_{t=1}^{m} y_t \in \mathcal{A}^+P_k$.
A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

For $1 \leqslant i \leqslant k$, define a homomorphism $f_i : P_{k-1} \to P_k$ of $\mathcal{A}$-algebras by substituting $f_i(x_j) = x_j$ for $1 \leqslant j < i$ and $f_i(x_j) = x_{j+1}$ for $i \leqslant j < k$.

Proposition 2.7 (See Mothebe and Uys [7]). *Let i, d be positive integers such that $1 \leqslant i \leqslant k$. If x is an admissible monomial in P_{k-1} then $x_i^{2^d-1} f_i(x)$ is also an admissible monomial in P_k.*

Now, we recall Singer's criterion on the hit monomials in P_k.

Definition 2.8. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $d_1 > d_2 > \ldots > d_{r-1} \geqslant d_r > 0$ and $d_j = 0$ for $j > r$, then it is called a minimal spike.

In [13], Singer showed that if $\mu(n) \leqslant k$, then there exists a unique minimal spike of degree n in P_k.

Theorem 2.9 (See Singer [13]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

This result implies the one of Wood [20].

Theorem 2.10 (See Wood [20]). *Let n be a positive integer. If $\mu(n) > k$, then $(QP_k)_n = 0$.*

For $1 \leqslant r \leqslant k$, set $P_r^+ = \langle\{x = x_1^{\nu_1(x)} x_2^{\nu_2(x)} \ldots x_r^{\nu_r(x)} : \nu_i(x) > 0, 1 \leqslant i \leqslant r\}\rangle$. Then, P_r^+ is an $\mathcal{A}$-submodule of P_k. For $J = (j_1, j_2, \ldots, j_r) : 1 \leqslant j_1 < \ldots < j_r \leqslant k$, we define a monomorphism $\theta_J : P_r \to P_k$ of $\mathcal{A}$-algebras by substituting $\theta_J(x_t) = x_{j_t}$ for $1 \leqslant t \leqslant r$. It is easy to see that, for any weight vector ω, $Q\theta_J(P_r^+)(\omega) \cong QP_r^+(\omega)$. So, by a simple computation using Theorem 2.10, we get the following.

Proposition 2.11 (See Walker and Wood [18]). *For a weight vector ω of degree n, we have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$QP_k(\omega) = \bigoplus_{\mu(n) \leqslant r \leqslant k} \bigoplus_{\ell(J)=r} Q\theta_J(P_r^+)(\omega),$$

where $\ell(J)$ is the length of J. Consequently

$$\dim QP_k(\omega) = \sum_{\mu(n) \leqslant r \leqslant k} \binom{k}{r} \dim QP_r^+(\omega).$$

3. Proofs of main results

First of all, we recall a construction for $\mathcal{A}$-generators of P_k. Denote

$$\mathcal{N}_k = \big\{(i;I); I=(i_1,i_2,\ldots,i_r), 1\leqslant i<i_1<\ldots<i_r\leqslant k,\ 0\leqslant r<k\big\}.$$

Definition 3.1 (See Sum [15])**.** Let $(i;I)\in\mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}}\prod_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r=\ell(I)>0$, For any monomial x in P_{k-1}, we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} f_i(x), & \text{if } r=\ell(I)=0,\\ (x_i^{2^r-1}f_i(x))/x_{(I,u)}, & \text{if there exists } 1\leqslant u\leqslant r \text{ such that}\\ & \quad \nu_{i_1-1}(x)=\ldots=\nu_{i_{(u-1)}-1}(x)=2^r-1,\\ & \quad \nu_{i_u-1}(x)>2^r-1,\\ & \quad \alpha_{r-t}(\nu_{i_u-1}(x))=1,\ \forall t,\ 1\leqslant t\leqslant u,\\ & \quad \alpha_{r-t}(\nu_{i_t-1}(x))=1,\ \forall t,\ u<t\leqslant r,\\ 0, & \text{otherwise.}\end{cases}$$

The following is needed for the proof of Theorem 1.1.

Theorem 3.2 (See Sum [15, Proposition 3.3])**.** *Let $n=\sum_{i=1}^{k-1}(2^{d_i}-1)$ with d_i positive integers such that $d_1>d_2>\ldots>d_{k-2}\geqslant d_{k-1}:=d\geqslant k-1\geqslant 3$, and let $m=\sum_{i=1}^{k-2}(2^{d_i-d_{k-1}}-1)$. If $B_{k-1}(m)$ is a minimal set of generators for $\mathcal{A}$-module P_{k-1} in degree m, then*

$$B_k(n) = \bigcup_{(i;I)\in\mathcal{N}_k}\big\{\phi_{(i;I)}(X_k^{2^d-1}z^{2^d}):\ z\in B_{k-1}(m)\big\}.$$

is also a minimal set of generators for $\mathcal{A}$-module P_k in degree n. Consequently $\dim(QP_k)_n = (2^k-1)\dim(QP_{k-1})_m$.

Let n,m be as is Theorem 3.2. Walker and Wood [19] defined a duplication map $\delta:(QP_k)_n\to(QP_k)_{2n+k-1}$. It is induced by a linear map $\bar\delta:(P_k)_n\to(P_k)_{2n+k-1}$ determined on monomials by $\bar\delta(x)=X_{\mathbb{J}_0(x)}x^2$ if $\omega_1(x)=k-1$ and $\bar\delta(x)=0$ if $\omega_1(x)<k-1$. They have proved in [19, Theorem 1.3] that if $d_{k-1}\geqslant 2$, then δ is an epimorphism.

According to Theorem 3.2, if $d_{k-1}\geqslant k-1\geqslant 3$, then

$$\dim(QP_k)_n = \dim(QP_k)_{2n+k-1} = (2^k-1)\dim(QP_{k-1})_m.$$

Hence, one gets the following.

Corollary 3.3. *Let $k\geqslant 4$ and n be as is Theorem 3.2. If $d_{k-1}\geqslant k-1$, then the duplication map $\delta:(QP_k)_n\to(QP_k)_{2n+k-1}$ is an isomorphism.*

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Set $s=\alpha(k-1)$. Then

$$k-1 = 2^{c_1}+2^{c_2}+\ldots+2^{c_{s-1}}+2^{c_s},$$

where $c_1>c_2>\ldots>c_{s-1}>c_s=\zeta(k-1)\geqslant 0$. Then, we have

$$\begin{aligned} n = (k-1)(2^d-1) &= 2^{d+c_1}+2^{d+c_2}+\ldots+2^{d+c_{s-1}}+2^{d+c_s}-k+1\\ &= \sum_{1\leqslant i\leqslant k-1}(2^{d_i}-1),\end{aligned}$$

where

$$d_i = \begin{cases} d + c_i, & 1 \leqslant i < s,\\ d + c_s - i + s - 1, & s \leqslant i \leqslant k-2,\\ d_{k-2} = d + c_s - k + s + 1 = d - d(k), & i = k-1. \end{cases}$$

It is easy to see that $d_1 > d_2 > \ldots > d_{k-2} = d_{k-1} = d - d(k)$. If $d \geqslant d(k) + k - 1$ and $k \geqslant 4$, then $d_{k-1} = d - d(k) \geqslant k - 1 \geqslant 3$. According to Theorem 3.2, we have

$$\dim(QP_k)_n = (2^k - 1)\dim(QP_{k-1})_m,$$

where

$$\begin{aligned} m &= \sum_{1\leqslant i\leqslant k-2} (2^{d_i - d_{k-1}} - 1)\\ &= 2^{c_1+d(k)} + 2^{c_2+d(k)} + \ldots + 2^{c_s+d(k)} - k + 1\\ &= (k-1)(2^{d(k)} - 1). \end{aligned}$$

The theorem is proved. $\square$

For $1 \leqslant q \leqslant k$, we set $\mathcal{N}_{k,q} = \{(i;I) \in \mathcal{N}_k : \ell(I) < q\}$, then $|\mathcal{N}_{k,q}| = \sum_{u=1}^{q}\binom{k}{u}$.

Proposition 3.4. *Let b be a positive integer. If ω is a weight vector of degree m with $\mu(m) \leqslant k-1$, then the set*

$$\bigcup_{(i;I)\in\mathcal{N}_{k,q}} \big\{\big[\phi_{(i;I)}(X^{2^b-1}z^{2^b})\big]_{(k-1)|^b|\omega} :\ z \in B_{k-1}(\omega)\big\}$$

is linearly independent in $QP_k((k-1)|^b|\omega)$, where $B_{k-1}(\omega)$ is the set of all the admissible monomials of weight vector ω in P_{k-1} and $q = \min\{k, b\}$. Consequently

$$\dim QP_k((k-1)|^b|\omega) \geqslant \dim(QP_{k-1}(\omega))\sum_{u=1}^{q}\binom{k}{u}.$$

We recall a result in our work [10] which is used for the proof of the proposition.

Definition 3.5. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i,\\ \sum_{s\in I} x_{s-1}, & \text{if } j = i,\\ x_{j-1}, & \text{if } i < j \leqslant k. \end{cases}$$

Then, $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = \emptyset$, $p_{(i;\emptyset)}(x_i) = 0$ and $p_{(i;I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 3.6 (See Phúc and Sum [10])**.** *If x is a monomial in P_k, then $p_{(i;I)}(x) \in P_{k-1}(\omega(x))$. So, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$ for any weight vector ω.*

Proof of Proposition 3.4. Suppose there is a linear relation

$$S := \sum_{((i;I),z)\in\mathcal{N}_{k,q}\times B_k(\omega)} \gamma_{(i;I),z}\phi_{(i;I)}(X^{2^{d-1}-1}z^{2^{d-1}}) \equiv_{(k-1)|^b|\omega} 0,$$

where $\gamma_{(i;I),z} \in \mathbb{F}_2$. We prove $\gamma_{(j;J),z} = 0$ for all $(j;J) \in \mathcal{N}_{k,q}$ and $z \in B_k(\omega)$. We prove this by induction on $m = \ell(J)$. Let $(i;I) \in \mathcal{N}_{k,q}$. Since $r = \ell(I) <$

$q = \min\{k, b\}$ and $x_i^{2^r-1} f_i(X^{2^{d-1}-1})$ is divisible by $x_{(I,1)}$, using Definition 3.1, we easily obtain

$$\phi_{(i;I)}(X^{2^b-1}z^{2^b}) = \phi_{(i;I)}(X^{2^{d-1}-1})f_i(z^{2^{d-1}}).$$

It is easy to see that if $g \in P_{k-1}^-((k-1)|^b)$, then $gz^{2^b} \in P_{k-1}^-((k-1)|^b|\omega)$; if $(i;I) \subset (j;\emptyset)$, then $(i;I) = (j;\emptyset)$; by Lemma 3.6, $p_{(j;\emptyset)}(\mathcal{S}) \equiv_{(k-1)|^b|\omega} 0$. Hence, using Lemma 3.7 in [15], we obtain

$$p_{(j,\emptyset)}(\mathcal{S}) \equiv_{(k-1)|^b|\omega} \sum_{z \in C_k} \gamma_{(j;\emptyset),z} X^{2^{d-1}-1} z^{2^{d-1}} \equiv_{(k-1)|^b|\omega} 0.$$

Since z is admissible in P_{k-1}, $X^{2^{d-1}-1}z^{2^{d-1}}$ is also admissible in P_{k-1}. Hence, the last relation implies $\gamma_{(j;\emptyset),z} = 0$ for all $z \in B_k(\omega)$.

Suppose $0 < m < q$ and $\gamma_{(i;I),z} = 0$ for all $z \in B_k(\omega)$ and $(i;I) \in \mathcal{N}_{k,q}$ with $\ell(I) < m$. Let $(j;J) \in \mathcal{N}_{k,q}$ with $\ell(J) = m$. According to Lemma 3.6, $p_{(j;J)}(\mathcal{S}) \equiv_{(k-1)|^b|\omega} 0$; if $(i;I) \in \mathcal{N}_{k,q}$, $\ell(I) \geqslant m$ and $(i;I) \subset (j;J)$, then $(i;I) = (j;J)$. Hence, using Lemma 3.7 in [15] and the inductive hypothesis, we obtain

$$p_{(j,J)}(\mathcal{S}) \equiv_{(k-1)|^b|\omega} \sum_{z \in B_k(\omega)} \gamma_{(j;J),z} X^{2^{d-1}-1} z^{2^{d-1}} \equiv_{(k-1)|^b|\omega} 0.$$

From this equality, one gets $\gamma_{(j;J),z} = 0$ for all $z \in B_k(\omega)$. The proposition is proved. $\square$

Proof of Theorem 1.4. Set $\omega(d) = (k-1)|^{d-2}|(k-3, k-4, 2)$, we have $\deg(\omega(d)) = (k-1)(2^d-1)$. Observe that for any $k \geqslant 7$, the monomials

$$z = x_1^{2^{d+1}-1} x_2^{2^{d+1}-1} x_3^{2^d-1} \ldots x_{k-4}^{2^d-1} x_{k-3}^{2^{d-1}-1} x_{k-2}^{2^{d-2}-1} x_{k-1}^{2^{d-2}-1} \in P_{k-1} \subset P_k$$

and $f_1(z) \in P_k$ are the spikes of the same weight vector $\omega(d)$, hence we get $\dim QP_k(\omega(d)) \geqslant 2$. If ω is a weight vector of degree $k-1$, then $\deg((k-1)|^{d-1}|\omega) = (k-1)(2^d-1)$. If $d > k$, then $\min\{k, d-1\} = \min\{k, d\} = k$ and $\binom{k}{\min\{k,d\}} = 1 < \dim QP_k(\omega(d))$. Hence, from the above equalities and Proposition 3.4, we get

$$\begin{aligned}
\dim(QP_k)_n &= \sum_{\deg \eta = n} \dim QP_k(\eta) \\
&\geqslant \sum_{\deg \omega = k-1} \dim QP_k((k-1)|^{d-1}|\omega) + \dim QP_k(\omega(d)) \\
&> \Bigg(\sum_{\deg \omega = k-1} \dim QP_{k-1}(\omega)\Bigg) \sum_{u=1}^{k} \binom{k}{u} + 1 \\
&= \Bigg(\sum_{\deg \omega = k-1} \dim QP_{k-1}(\omega)\Bigg) \sum_{u=1}^{\min\{k,d-1\}} \binom{k}{u} + \binom{k}{\min\{k,d\}}.
\end{aligned}$$

Suppose $d \leqslant k$, then $\min\{k, d-1\} = d-1$, $\min\{k, d\} = d$ and $(k-1)|^{d-1}|(k-1) = (k-1)|^d$. According to Phúc and Sum [10, Proposition 3.7], we have

$$\dim QP_k((k-1)|^d) = \sum_{t=1}^{d} \binom{k}{t} = \sum_{t=1}^{d-1} \binom{k}{t} + \binom{k}{d}.$$

Since $\dim QP_k(\omega(d)) > 0$ and $\dim QP_{k-1}((k-1)) = 1$, combining the above equalities and Proposition 3.4 gives

$$\begin{aligned}\dim(QP_k)_n &\geqslant \sum_{\deg\omega=k-1}\dim QP_k((k-1)|^{d-1}|\omega) + \dim QP_k(\omega(d))\\ &> \Big(\sum_{\deg\omega=k-1}\dim QP_{k-1}(\omega)\Big)\sum_{u=1}^{d-1}\binom{k}{u} + \binom{k}{d}.\end{aligned}$$

The theorem is proved. □

4. Some applications

Base on Theorem 1.4, we can extend our results in [10] by explicitly computing the spaces $QP_{k-1}(\omega)$ with some weight vectors ω of degree $k-1$.

Consider the weight vectors $(k-1-2t-4\varepsilon, t, \varepsilon)$ with $\epsilon = 0, 1$ and $k-1-2t-4\varepsilon \geqslant t$.

We recall the following result in our work [10] for the case $t = 1$, $\varepsilon = 0$.

Proposition 4.1 (Phúc and Sum [10])**.** *For any $k \geqslant 4$,*

$$\dim QP_{k-1}(k-3, 1) = (k-3)\binom{k}{2}.$$

Now we compute $QP_{k-1}(k-5, 2)$ for the case $t = 2$, $\varepsilon = 0$.

Proposition 4.2. *For $k \geqslant 7$, $\dim QP_{k-1}(k-5, 2) = \dfrac{(k-1)(k-6)}{2}\dbinom{k}{4}$.*

Proof. Observe that $P_r^+(k-5, 2) = 0$ for either $r < k-5$ or $r > k-3$. We denote

$$\tilde{B}^+_{(k-5,2)} = \{x_1x_2\ldots x_{k-5}x_i^2x_j^2 : 1 \leqslant i < j \leqslant k-5\},$$
$$\tilde{B}^+_{(k-4,2)} = \{x_1\ldots x_i^2\ldots x_{k-4}x_j^2 : 1 \leqslant i, j \leqslant k-4, 2 \leqslant i \neq j\} \setminus \{x_1^3x_2^2x_3\ldots x_{k-4}\},$$
$$\tilde{B}^+_{(k-3,2)} = \{x_1\ldots x_i^2\ldots x_j^2\ldots x_{k-3} : 2 \leqslant i < j \leqslant k-3\} \setminus \{x_1x_2^2x_3^2x_4\ldots x_{k-3}\}.$$

It is easy to see that $\tilde{B}^+_{(r,2)} \subset P_r^+(k-5, 2)$ for $k-5 \leqslant r \leqslant k-3$.

If $x \in \tilde{B}^+_{(k-5,2)}$, then x is a spike. According to Phúc and Sum [10, Lemma 2.7], x is admissible. Obviously, if x is a monomial in P^+_{k-5}, then $x \in \tilde{B}^+_{(k-5,2)}$. Hence, $\tilde{B}^+_{(k-5,2)}$ is the set of all the admissible monomials in $P^+_{k-5}(k-5, 2)$. If x is a monomial in $P^+_{k-4}(k-5, 2)$, then $x = x_1\ldots x_i^2\ldots x_{k-4}x_j^2$ with $1 \leqslant i, j \leqslant k-4, i \neq j$. If $i = 1$ then

$$x = \sum_{2\leqslant t\leqslant k-4} x_1\ldots x_t^2\ldots x_{k-4}x_j^2 + Sq^1(x_1\ldots x_{k-4}x_j^2).$$

Hence, x is inadmissible. If $j = 1, i = 2$, then

$$x = \sum_{3\leqslant t\leqslant k-4} x_1^3x_2\ldots x_t^2\ldots x_{k-4} + x_1^4x_2\ldots x_{k-4} + Sq^1(x_1^3x_2\ldots x_{k-4}).$$

This equality shows that x is inadmissible. If $i > 1$ and $x \neq x_1^3x_2^2x_3\ldots x_{k-4}$, then x is of the form $x = x_tx_i^2(f_tf_{i-1})(z)$ with $1 \leqslant t < i \leqslant k-4$ and z a spike in P_{k-6}. According to Peterson [9], $x_tx_i^2$ is admissible. So, by Proposition 2.7, x is also admissible. Hence, $\tilde{B}^+_{(k-4,2)}$ is the set of all the admissible monomials in $P^+_{k-4}(k-5, 2)$.

If x is a monomial in $P_{k-3}^+(k-5,2)$, then $x = x_1 \ldots x_i^2 \ldots x_j^2 \ldots x_{k-3}$ with $1 \leqslant i < j \leqslant k-3$. If $i = 1$, then

$$x = \sum_{2\leqslant t\leqslant k-3,\ t\neq j} x_1 \ldots x_t^2 \ldots x_j^2 \ldots x_{k-3} + Sq^1(x_1 \ldots x_j^2 \ldots x_{k-3}).$$

Hence, x is inadmissible. If $x = x_1x_2^2x_3^2x_4 \ldots x_{k-3}$, then

$$x = \sum_{2\leqslant s<t\leqslant k-3,\,(s,t)\neq(2,3)} x_1 \ldots x_s^2 \ldots x_t^2 \ldots x_{k-3} \\ + Sq^1(x_1^2x_2 \ldots x_{k-3}) + Sq^2(x_1x_2 \ldots x_{k-3}).$$

So, x is inadmissible. If $i > 1$ and $x \neq x_1x_2^2x_3^2x_4 \ldots x_{k-3}$, then the monomial x is of the form $x = y(f_1f_{s-1}f_{t-2}f_{u-3})(z)$ with $z = x_1 \ldots x_{k-7} \in P_{k-7}$, $1 < s < t < u \leqslant k-3$ and either $y = x_1x_s^2x_tx_u^2$ or $y = x_1x_sx_t^2x_u^2$. We have proved in [15] that y is admissible. Hence, using Proposition 2.7, x is also admissible.

Thus, we have proved that $\tilde{B}^+_{(r,2)}$ is the set of all the admissible monomials in $P_r^+(k-5,2)$, hence $\dim QP_r^+(k-5,2) = |\tilde{B}^+_{(r,2)}|$ for $k-5 \leqslant r \leqslant k-3$. By a direct computation, we obtain $|\tilde{B}^+_{(k-5,2)}| = \binom{k-5}{2}$, $|\tilde{B}^+_{(k-4,2)}| = (k-5)^2 - 1$ and $|\tilde{B}^+_{(k-3,2)}| = \binom{k-4}{2} - 1$. Hence, using Proposition 2.11, we get

$$\dim QP_{k-1}(k-5,2) = \sum_{k-5\leqslant r\leqslant k-3} \binom{k-1}{r} \dim QP_r^+(k-5,2) \\ = \frac{(k-1)(k-6)}{2}\binom{k}{4}.$$

The proposition is proved. □

By combining Theorem 1.4, Propositions 4.1, 4.2 we obtain a lower bound for $\dim(QP_k)_n$ which extends the one in [10].

Theorem 4.3. *Let $n = (k-1)(2^d-1)$ with d a positive integer. If $k \geqslant 7$ and $d \geqslant 2$, then*

$$\dim(QP_k)_n > \sum_{u=1}^{p}\binom{k}{u} + \left((k-3)\binom{k}{2} + \frac{(k-1)(k-6)}{2}\binom{k}{4}\right)\sum_{v=1}^{q}\binom{k}{v},$$

where $p = \min\{k,d\}$ and $q = \min\{k, d-1\}$.

This result implies the one in our work [10] for $k \geqslant 7$.

Proposition 4.4. *If $k \geqslant 9$, then* $\dim QP_{k-1}(k-7,1,1) = \binom{k-6}{2}\binom{k+1}{6}$.

Proof. We observe that $P_r^+(k-7,1,1) = 0$ for either $r < k-7$ or $r > k-5$. Hence, using Proposition 2.11 we have

$$\dim QP_{k-1}(k-7,1,1) = \sum_{k-7\leqslant r\leqslant k-5}\binom{k-1}{r}\dim QP_r^+(k-7,1,1).$$

Suppose that $k \geqslant 9$. Then we set

$$\begin{aligned}
\bar{B}^+_{(k-7,1)} &= \{x_1x_2\ldots x_{k-7}x_{i_1}^2x_{i_2}^4 : 1 \leqslant i_1 \leqslant i_2 \leqslant k-7\} \subset P^+_{k-7}(k-7,1,1),\\
\bar{B}^+_{(k-6,1)} &= \{x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^4 : 2 \leqslant i_1 \leqslant i_2 \leqslant k-6\}\\
&\qquad \cup \{x_1\ldots x_{i_2}^4\ldots x_{k-6}x_{i_1}^2 : 1 \leqslant i_1 < i_2 \leqslant k-6\} \subset P^+_{k-6}(k-7,1,1),\\
\bar{B}^+_{(k-5,1)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^4\ldots x_{k-5} : 2 \leqslant i_1 < i_2 \leqslant k-5\} \subset P^+_{k-5}(k-7,1,1).
\end{aligned}$$

Let x be a monomial in $P^+_{k-7}(k-7,1,1)$, then $x = x_1x_2\ldots x_{k-7}x_{i_1}^2x_{i_2}^4$ with $1 \leqslant i_1, i_2 \leqslant k-7$. If $i_1 > i_2$, then $x = Sq^2(x_1x_2\ldots x_{k-7}x_{i_1}^2x_{i_2}^2)+$ smaller monomials. Hence, x is inadmissible. If $i_1 = i_2$ then x is a spike, hence x is admissible. If $i_1 < i_2$, then $x = x_{i_1}^3x_{i_2}^5(f_{i_1}f_{i_2-1})(z)$ with $z = x_1\ldots x_{k-9} \in P_{k-9}$. According to Peterson [9], $x_{i_1}^3x_{i_2}^5$ is admissible, so using Proposition 2.7, x is also admissible. This means that $\bar{B}^+_{(k-7,1)}$ is the set of all admissible monomials in $P^+_{k-7}(k-7,1,1)$.

Let $x \in P^+_{k-6}(k-7,1,1)$, then either $x = x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^4$ or $x = x_1\ldots x_{i_2}^4\ldots x_{k-6}x_{i_1}^2$ with $1 \leqslant i_1, i_2 \leqslant k-6$. If $i_1 > i_2$ and $x = x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^4$, then $x = Sq^2(x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^2)+$ smaller monomials; if $i_1 > i_2$ and $x = x_1\ldots x_{i_2}^4\ldots x_{k-6}x_{i_1}^2$, then $x = Sq^2(x_1\ldots x_{i_2}^2\ldots x_{k-6}x_{i_1}^2)+$ smaller monomials; if $x = x_1^2x_2\ldots x_{k-6}x_{i_2}^4$, then $x = Sq^1(x_1\ldots x_{k-6}x_{i_2}^4)+$ smaller monomials, hence x is inadmissible. If $i_1 = i_2 > 1$, then $x = x_1x_{i_1}^6(f_1f_{i_1-1})(x_1\ldots x_{k-8})$. Since $x_1x_{i_1}^6$ is admissible, by Proposition 2.7, x is admissible. If $x = x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^4$ with $1 < i_1 < i_2$, then

$$x = x_1x_{i_1}^2x_{i_2}^5(f_1f_{i_1-1}f_{i_2-2})(z)$$

with $z = x_1\ldots x_{k-9}$. According to Kameko [4], $x_1x_{i_1}^2x_{i_2}^5$ is admissible, so using Proposition 2.7, x is admissible. Suppose $x = x_1x_2\ldots x_{i_2}^4\ldots x_{k-6}x_{i_1}^2$ with $1 \leqslant i_1 < i_2$. If $i_1 = 1, i_2 = 2$, then $x = x_1^3x_2^4x_3(f_1f_1f_1)(x_1\ldots x_{k-9})$, if $i_1 = 1, i_2 > 2$, then $x = x_1^3x_2x_{i_2}^4(f_1f_1f_{i_2-2})(x_1\ldots x_{k-9})$, if $1 < i_1 < i_2$, then $x = x_1x_{i_1}^3x_{i_2}^4(f_1f_{i_1-1}f_{i_2-2})(x_1\ldots x_{k-9})$. According to Kameko [4], $x_1^3x_2^4x_3$, $x_1^3x_2x_{i_2}^4$, $x_1x_{i_1}^3x_{i_2}^4$ are admissible. By Proposition 2.7, x is admissible. Thus, we have proved that $\bar{B}^+_{(k-6,1)}$ is the set of all admissible monomials in $P^+_{k-6}(k-7,1,1)$.

Let x be a monomial in $P^+_{k-5}(k-7,1,1)$, then $x = x_1\ldots x_{i_1}^2\ldots x_{i_2}^4\ldots x_{k-5}$ with $1 \leqslant i_1 < i_2 \leqslant k-5$. If $i_1 = 1$, then $x = Sq^1(x_1\ldots x_{i_2}^4\ldots x_{k-5})+$ smaller monomials, hence x is inadmissible. If $1 < i_1$ then $x = x_1x_{i_1}^2x_{i_2}^4(f_1f_{i_1-1}f_{i_2-2})(x_1\ldots x_{k-8})$. According to Kameko [4], $x_1x_{i_1}^2x_{i_2}^4$ is admissible. So, by Proposition 2.7, x is admissible.

Thus, we have proved that $\bar{B}^+_{(r,1)}$ is the set of all admissible monomials in $P^+_r(k-7,1,1)$, hence $\dim QP^+_r(k-7,1,1) = |\bar{B}^+_{(r,1)}|$, for $k-7 \leqslant r \leqslant k-5$. A direct computation shows that

$$|\bar{B}^+_{(k-7,1)}| = \binom{k-6}{2},\ |\bar{B}^+_{(k-6,1)}| = 2\binom{k-6}{2},\ |\bar{B}^+_{(k-5,1)}| = \binom{k-6}{2}.$$

Now using Proposition 2.11, we obtain

$$\dim P_{k-1}(k-7,1,1) = \sum_{k-7\leqslant r\leqslant k-5}\binom{k-1}{r}|\bar{B}^+_{(r,1)}| = \binom{k-6}{2}\binom{k+1}{6}.$$

The proposition is proved. □

Remark 4.5. We have $\bar{B}^+_{(1,1)} = \{x_1^7\}$, $\bar{B}^+_{(3,1)} = \{x_1x_2^2x_3^4\}$. Since $x_1^3x_2^4 \equiv x_1x_2^6$, we get $\bar{B}^+_{(2,1)} = \{x_1x_2^6\}$, hence $\dim QP_7(1,1,1) = \binom{7}{1} + \binom{7}{2} + \binom{7}{3} = 63 < 84 = \binom{8-6}{2}\binom{8+1}{6}$. So, Proposition 4.4 is not true for $k = 8$.

Proposition 4.6. *If $k \geqslant 10$, then*

$$\dim QP_{k-1}(k-7,3) = \frac{(k-5)(k-7)(k^3-9k^2+14k-36)}{180}\binom{k}{4}.$$

Proof. Note that $P_r^+(k-7,3) = 0$ for either $r < k-7$ or $r > k-4$. Hence, using Proposition 2.11 we have

$$\dim QP_{k-1}(k-7,3) = \sum_{k-7\leqslant r\leqslant k-4}\binom{k-1}{r}\dim QP_r^+(k-7,3).$$

We set

$$\begin{aligned}
\tilde{B}^+_{(k-7,3)} &= \{x_1x_2\ldots x_{k-7}x_{i_1}^2x_{i_2}^2x_{i_3}^2 : 1 \leqslant i_1 < i_2 < i_3 \leqslant k-7\} \subset P^+_{k-7},\\
\tilde{B}^+_{(k-6,3)} &= \{x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^2x_{i_3}^2 : 2 \leqslant i_1 \leqslant k-6, 1 \leqslant i_2 < i_3 \leqslant k-6, i_2, i_3\\
&\qquad \neq i_1\} \setminus \big(\{x_1^3x_2^2x_3\ldots x_{k-6}x_{i_3}^2 : 3 \leqslant i_3 \leqslant k-6\} \cup \{x_1^3x_2^3x_3^2x_4\ldots x_{k-6}\}\big),\\
\tilde{B}^+_{(k-5,3)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{k-5}x_{i_3}^2 : 2 \leqslant i_1 < i_2 \leqslant k-5, 1 \leqslant i_3 \leqslant k-5, i_3\\
&\qquad \neq i_1, i_2\} \setminus \{x_1^3x_2^2x_3\ldots x_{i_2}^2\ldots x_{k-5} : 3 \leqslant i_2 \leqslant k-5\} \subset P^+_{k-5},\\
\tilde{B}^+_{(k-4,3)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{k-4} : 2 \leqslant i_1 < i_2 < i_3 \leqslant k-4\} \subset P^+_{k-4}.
\end{aligned}$$

We have $\tilde{B}^+_{(r,3)} \subset P_r^+$ for $k-7 \leqslant r \leqslant k-4$.

If $x \in \tilde{B}^+_{(k-7,3)}$, then x is a spike, hence x is admissible. Obviously, if x is a monomial in P^+_{k-7} then $x \in \tilde{B}^+_{(k-7,3)}$. Hence, $\tilde{B}^+_{(k-7,3)}$ is the set of all the admissible monomials in $P^+_{k-7}(k-7,3)$.

If $x \in \tilde{B}^+_{(k-6,3)}$, then $x = x_1x_{i_1}^2f_1(f_{i_1-1}(z))$ with z a spike in P_{k-8}. Since $x_1x_{i_1}^2$ is admissible, by Proposition 2.7, x is also admissible. If x is a monomial in $P^+_{k-6}(k-7,3)$, then $x = x_1\ldots x_{i_1}^2\ldots x_{k-6}x_{i_2}^2x_{i_3}$ with $1 \leqslant i_1, i_2, i_3 \leqslant k-6, i_2, i_3 \neq i_1, i_2 < i_3$. If $i_1 = 1$ then $x = Sq^1(x_1\ldots x_{k-6}x_{i_2}^2x_{i_3}^2) +$ smaller monomials. Hence, x is inadmissible. If $i_2 = 1, i_1 = 2$ then $x = Sq^1(x_1^3x_2\ldots x_{k-6}x_{i_3}^2)+$ smaller monomials. This equality shows that x is inadmissible. If $i_2 = 1, i_3 = 2, i_1 = 3$ then $x = Sq^1(x_1^3x_2^3x_3\ldots x_{k-6}x_{i_3}^2)+$ smaller monomials. So, x is inadmissible. Thus, we have showed that $\tilde{B}^+_{(k-6,3)}$ is the set of all the admissible monomials in $P^+_{k-6}(k-7,3)$.

If $x \in \tilde{B}^+_{(k-5,3)}$, then $x = yf_1(f_{u-1}(f_{v-2}f_{w-3}(z)))$, where $1 < u < v < w$, y is one of the monomials: $x_1^3x_ux_v^2x_w^2$, $x_1x_u^3x_v^2x_w^2$, $x_1x_u^2x_v^3x_w^2$, $x_1x_u^2x_v^2x_w^3$ and $z = x_1\ldots x_{k-9} \in P_{k-9}$. We have proved in [15] that y is admissible. Hence, by Proposition 2.7, x is also admissible. Let x be a monomial in $P^+_{k-5}(k-7,3)$. If $x \notin \tilde{B}^+_{(k-5,3)}$, then either $x = x_1^2x_2\ldots x_{i_2}^2\ldots x_{k-5}x_{i_3}^2$, $i_2, i_3 > 1, i_2 \neq i_3$ or $x = x_1^3x_2^2\ldots x_{i_2}^2\ldots x_{k-5}$, $i_2 > 2$. If $x = x_1^2x_2\ldots x_{i_2}^2\ldots x_{k-5}x_{i_3}^2$, then $x = Sq^1(x_1\ldots x_{i_2}^2\ldots x_{k-5}x_{i_3}^2)+$ smaller monomials. If $x = x_1^3x_2^2\ldots x_{i_2}^2\ldots x_{k-5}$, then $x = Sq^1(x_1^3x_2\ldots x_{i_2}^2\ldots x_{k-5})+$ smaller monomials. Hence, x is inadmissible.

If $x \in \tilde{B}^+_{(k-4,3)}$, then $x = yf_1(f_{u-1}(f_{v-2}f_{w-3}(z)))$, where $1 < u < v < w$, $y = x_1x_u^2x_v^2x_w^2$ and $z = x_1\ldots x_{k-8} \in P_{k-8}$. We have proved in [15] that y is admissible. Hence, by Proposition 2.7, x is also admissible. If $x \in P^+_{k-4}(k-7,3)$

and $x \notin \tilde B^+_{(k-4,3)}$, then $x = x_1^2x_2\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{k-4}$ with $1 < i_1 < i_2 \leqslant k-4$. So, we get $x = Sq^1(x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{k-4})+$ smaller monomials. Hence, x is inadmissible.

We have proved that $\tilde B^+_{(r,3)}$ is the set of all admissible monomials in $P_r^+(k-7,3)$, hence we obtain $\dim QP_r^+(k-7,3) = |\tilde B^+_{(r,3)}|$, for $k-7 \leqslant r \leqslant k-4$. By a direct computation, we get

$$|\tilde B^+_{(k-7,3)}| = \binom{k-7}{3},\ |\tilde B^+_{(k-6,3)}| = (k-9)\binom{k-6}{2} = \frac{(k-6)(k-7)(k-9)}{2},$$
$$|\tilde B^+_{(k-5,3)}| = (k-5)\binom{k-7}{2} = \frac{(k-5)(k-7)(k-8)}{2},\ |\tilde B^+_{(k-4,3)}| = \binom{k-5}{3}.$$

Now, applying Proposition 2.11, we obtain

$$\begin{aligned}\dim QP_{k-1}(k-7,3) &= \sum_{k-7\leqslant r\leqslant k-4}\binom{k-1}{r}|\tilde B^+_{(r,3)}|\\ &= \frac{(k-5)(k-7)(k^3-9k^2+14k-36)}{180}\binom{k}{4} := a(k).\end{aligned}$$

The proof is completed. □

Remark 4.7. Since $\tilde B^+_{(2,3)} = \tilde B^+_{(3,3)} = \emptyset$, Proposition 4.6 holds for $k=9$. We have $\tilde B^+_{(1,3)} = \tilde B^+_{(2,3)} = \tilde B^+_{(3,3)} = \emptyset$ and $|\tilde B^+_{(4,3)}| = 1$, hence $\dim QP_7(1,3) = \binom{7}{4} = 35 > 14 = a(8)$. So, Proposition 4.6 is not true for $k=8$. Since $QP_7(0,3) = 0$, the proposition holds for $k=7$.

Proposition 4.8. *If $k \geqslant 13$, then*

$$\dim QP_{k-1}(k-9,4) = \frac{(k-1)(k-10)(k^4-20k^3+129k^2-354k+840)}{1344}\binom{k}{6}.$$

We need the following for the proof of this proposition.

Lemma 4.9. *The following monomials are admissible in P_6:*

$$a_1 = x_1x_2x_3^2x_4^2x_5^2x_6^2,\ a_2 = x_1x_2^2x_3x_4^2x_5^2x_6^2,$$
$$a_3 = x_1x_2^2x_3^2x_4x_5^2x_6^2,\ a_4 = x_1x_2^2x_3^2x_4^2x_5x_6^2.$$

Proof. We prove the lemma by showing that $\{a_1,a_2,a_3,a_4\}$ is the set of all admissible monomials in $P_6^+(2,4)$. Let x be a monomial in $P_6^+(2,4)$, then

$$x = x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{i_4}^2\ldots x_6,\ 1\leqslant i_1 < i_2 < i_3 < i_4 \leqslant 6.$$

If $i_1 = 1$, then $x = Sq^1(x_1\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{i_4}^2\ldots x_6)+$ smaller monomials. If $i_1 > 1, i_4 < 6$, then

$$\begin{aligned}x = x_1x_2^2x_3^2x_4^2x_5^2x_6 &= Sq^1(x_1^2Sq^2(x_2\ldots x_6))\\ &+ Sq^4(x_1\ldots x_6) + \text{ smaller monomials.}\end{aligned}$$

Hence, x is inadmissible. Thus, we have proved that if x is admissible, then x is one of the monomials a_1,a_2,a_3,a_4. Now we prove the set

$$\{[a_1]_{(2,4)}, [a_2]_{(2,4)}, [a_2]_{(2,4)}, [a_4]_{(2,4)}\}$$

is linearly independent in $QP_6^+(2,4)$. Suppose there is a linear relation

$$S := \gamma_1a_1 + \gamma_2a_2 + \gamma_3a_3 + \gamma_4a_4 \equiv_{(2,4)} 0, \tag{4.1}$$

with $\gamma_u \in \mathbb{F}_2$, $1 \leqslant u \leqslant 4$. By applying the homomorphism $p_{(1,j)} : P_6 \to P_5$ to the relation (4.1) for $1 < j < 6$, we get

$$\begin{aligned}
p_{(1,2)}(S) &\equiv_{(2,4)} (\gamma_2+\gamma_3+\gamma_4)x_1^3x_2x_3^2x_4^2x_5^2 \equiv_{(2,4)} 0,\\
p_{(1,3)}(S) &\equiv_{(2,4)} (\gamma_1+\gamma_3+\gamma_4)x_1x_2^3x_3^2x_4^2x_5^2 \equiv_{(2,4)} 0,\\
p_{(1,4)}(S) &\equiv_{(2,4)} (\gamma_1+\gamma_2+\gamma_4)x_1x_2^2x_3^3x_4^2x_5^2 \equiv_{(2,4)} 0,\\
p_{(1,5)}(S) &\equiv_{(2,4)} (\gamma_1+\gamma_2+\gamma_3)x_1x_2^2x_3^2x_4^3x_5^2 \equiv_{(2,4)} 0.
\end{aligned}$$

We have prove in [15] that the monomial $x_1x_2^2x_3^2x_4^2$ is admissible in P_4. Hence, by Proposition 2.7, the monomials $x_1^3x_2x_3^2x_4^2x_5^2$, $x_1x_2^3x_3^2x_4^2x_5^2$, $x_1x_2^2x_3^3x_4^2x_5^2$, $x_1x_2^2x_3^2x_4^3x_5^2$ are admissible in P_5. So, from the above equalities we get $\gamma_i = 0$ for $1 \leqslant i \leqslant 4$. The lemma is proved. □

We now prove Proposition 4.8.

Proof of Proposition 4.8. Observe that $P_r^+(k-9,4) = 0$ for either $r < k-9$ or $r > k-5$. Hence, using Proposition 2.11 we have

$$\dim QP_{k-1}(k-9,4) = \sum_{k-9\leqslant r\leqslant k-5} \binom{k-1}{r} \dim QP_r^+(k-9,4).$$

We set

$$\begin{aligned}
\tilde B^+_{(k-9,4)} &= \{x_1x_2\ldots x_{k-9}x_{i_1}^2x_{i_2}^2x_{i_3}^2x_{i_4}^2 : 1\leqslant i_1<i_2<i_3<i_4\leqslant k-9\},\\
\tilde B^+_{(k-8,4)} &= \{x_1\ldots x_{i_1}^2\ldots x_{k-8}x_{i_2}^2x_{i_3}^2x_{i_4}^2 : 2\leqslant i_1\leqslant k-8, 1\leqslant i_2<i_3<i_4\leqslant k-8,\\
&\qquad i_2,i_3,i_4\neq i_1\}\setminus\big(\{x_1^3x_2^2x_3\ldots x_{k-8}x_{i_3}^2x_{i_4}^2 : 3\leqslant i_3<i_4\leqslant k-8\}\\
&\qquad \cup\{x_1^3x_2^3x_3^2x_4\ldots x_{k-8}x_{i_4}^2 : 4\leqslant i_4\leqslant k-8\}\cup\{x_1^3x_2^3x_3^3x_4^2x_5\ldots x_{k-8}\}\big),\\
\tilde B^+_{(k-7,4)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{k-7}x_{i_3}^2x_{i_4}^2 : 2\leqslant i_1<i_2\leqslant k-7, 1\leqslant i_3<i_4\leqslant\\
&\qquad k-7, i_3,i_4\neq i_1,i_2\}\setminus\big(\{x_1^3x_2^2x_3\ldots x_{i_2}^2\ldots x_{k-7}x_{i_4}^2 : 3\leqslant i_2,i_4\leqslant k-7,\\
&\qquad i_4\neq i_2\}\cup\{x_1^3x_2^3x_3^2x_4\ldots x_{i_4}^2\ldots x_{k-7} : 4\leqslant i_4\leqslant k-7\}\big),\\
\tilde B^+_{(k-6,4)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{k-6}x_{i_4}^2 : 2\leqslant i_1<i_2<i_3\leqslant k-6,\\
&\qquad 1\leqslant i_4\leqslant k-6, i_4\neq i_1,i_2,i_3\}\\
&\qquad\qquad \setminus\{x_1^3x_2^2x_3\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{k-6} : 3\leqslant i_2<i_3\leqslant k-6\},\\
\tilde B^+_{(k-5,4)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{i_4}^2\ldots x_{k-5} : 2\leqslant i_1<i_2<i_3<i_4\leqslant k-5\}\\
&\qquad\qquad \setminus\{x_1x_2^2x_3^2x_4^2x_5^2x_6\ldots x_{k-5}\}.
\end{aligned}$$

By arguments similar to the ones in the proof of Proposition 4.6 we can prove that $\tilde B^+_{(r,4)}$ is the set of all the admissible monomials in $QP_r^+(k-9,4)$ for $k-9\leqslant r\leqslant k-6$.

Let $x \in \tilde B^+_{(k-5,4)}$. Then $x = y(f_1f_{i_1-1}f_{i_2-2}f_{i_3-3}f_{i_4-4}f_{i_5-5})(z)$, where y is one of the monomials:

$$x_1x_{i_1}x_{i_2}^2x_{i_3}^2x_{i_4}^2x_{i_5}^2,\ x_1x_{i_1}^2x_{i_2}x_{i_3}^2x_{i_4}^2x_{i_5}^2,\ x_1x_{i_1}^2x_{i_2}^2x_{i_3}x_{i_4}^2x_{i_5}^2,\ x_1x_{i_1}^2x_{i_2}^2x_{i_3}^2x_{i_4}x_{i_5}^2,$$

with $1 < i_1 < i_2 < i_3 < i_4 < i_5 \leqslant k-5$ and $z = x_1\ldots x_{k-11} \in P_{k-11}$. By Lemma 4.9, y is admissible. So, by Proposition 2.7, x is also admissible.

Now let x be a monomial in $P^+_{k-5}(k-9,4)$, then

$$x = x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^2\ldots x_{i_4}^2\ldots x_{k-5} : 1\leqslant i_1<i_2<i_3<i_4\leqslant k-5$$

If $i_1 = 1$, then $x = Sq^1(x_1 \ldots x_{i_2}^2 \ldots x_{i_3}^2 \ldots x_{i_4}^2 \ldots x_{k-5})+$ smaller monomials. Hence, x is inadmissible. If $x = x_1x_2^2x_3^2x_4^2x_5^2x_6 \ldots x_{k-5}$, then

$$x = Sq^1(x_1^2Sq^2(x_2 \ldots x_{k-5})) + Sq^4(x_1 \ldots x_{k-5}) + \text{smaller monomials.}$$

This equality shows that x is inadmissible.

Thus, we have proved that $\tilde{B}^+_{(r,4)}$ is the set of all the admissible monomials in $QP_r^+(k-9,4)$, so we get $\dim QP_r^+(k-9,4) = |\tilde{B}^+_{(r,4)}|$, for $k-9 \leqslant r \leqslant k-5$. By a direct computation, we obtain

$$\begin{aligned}
&|\tilde{B}^+_{(k-9,4)}| = \binom{k-9}{4},\ |\tilde{B}^+_{(k-8,4)}| = (k-12)\binom{k-8}{3},\\
&|\tilde{B}^+_{(k-7,4)}| = \binom{k-7}{2}\binom{k-10}{2},\ |\tilde{B}^+_{(k-6,4)}| = (k-6)\binom{k-8}{3},\\
&|\tilde{B}^+_{(k-5,4)}| = \binom{k-6}{4} - 1 = \frac{(k-5)(k-10)(k^2-15k+60)}{24}.
\end{aligned}$$

By using Proposition 2.11, we obtain

$$\begin{aligned}
\dim QP_{k-1}(k-9,4) &= \sum_{k-9\leqslant r\leqslant k-5}\binom{k-1}{r}|\tilde{B}^+_{(r,4)}|\\
&= b(k) := \frac{(k-1)(k-10)(k^4-20k^3+129k^2-354k+840)}{1344}\binom{k}{6}.
\end{aligned}$$

The proposition is proved. □

Remark 4.10. We have $\tilde{B}^+_{(3,4)} = \tilde{B}^+_{(4,4)} = \emptyset$, hence Proposition 4.8 holds for $k = 12$. Since $\tilde{B}^+_{(2,4)} = \tilde{B}^+_{(3,4)} = \tilde{B}^+_{(4,4)} = \emptyset$, $|\tilde{B}^+_{(5,4)}| = 5$, $|\tilde{B}^+_{(6,4)}| = 4$, we get $\dim QP_{10}(2,4) = 5\binom{10}{5} + 4\binom{10}{6} = 2100 > 1980 = b(11)$. Hence, Proposition 4.8 is not true for $k = 11$. By a simple computation, we have $QP_9(1,4) = 0$, hence Proposition 4.8 is also true for $k = 10$.

Proposition 4.11. *If $k \geqslant 11$, then*

$$\dim QP_{k-1}(k-9,2,1) = \frac{(k-1)(k-8)(k-10)}{3}\binom{k+1}{8}.$$

Proof. Note that $P_r^+(k-9,2,1) = 0$ for either $r < k-9$ or $r > k-6$. Hence, using Proposition 2.11 we have

$$\dim QP_{k-1}(k-9,2,1) = \sum_{k-9\leqslant r\leqslant k-6}\binom{k-1}{r}\dim QP_r^+(k-9,2,1).$$

We set

$$\begin{aligned}
\bar{B}^+_{(k-9,2)} &= \{x_1x_2\ldots x_{k-9}x_{i_1}^2x_{i_2}^2x_{i_3}^4 : 1\leqslant i_1 < i_2 \leqslant k-9, i_1 \leqslant i_3 \leqslant k-9\},\\
\bar{B}^+_{(k-8,2)} &= \big(\{x_1\ldots x_{i_1}^2\ldots x_{k-8}x_{i_2}^2x_{i_3}^4 : 2\leqslant i_1 < i_2 \leqslant k-8, i_1 \leqslant i_3 \leqslant k-8\}\\
&\quad \cup \{x_1\ldots x_{i_2}^2\ldots x_{k-8}x_{i_1}^2x_{i_3}^4 : 1\leqslant i_1 < i_2 \leqslant k-8, i_1 \leqslant i_3 \leqslant k-8\}\\
&\quad \cup \{x_1\ldots x_{i_3}^4\ldots x_{k-8}x_{i_1}^2x_{i_2}^2 : 1\leqslant i_1 < i_2 \leqslant k-8, i_1 < i_3 \leqslant k-8,\\
&\qquad i_3 \neq i_2\}\big) \setminus \{x_1^3x_2^2x_3\ldots x_{k-8}x_i^4 : 1 \leqslant i \leqslant k-8\},
\end{aligned}$$

$$\begin{aligned}\bar{B}^+_{(k-7,2)} &= \big(\{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{k-7}x_{i_3}^4 : 2\leqslant i_1<i_2\leqslant k-7, i_1\leqslant i_3\leqslant k-7\}\\ &\quad\cup\{x_1\ldots x_{i_1}^2\ldots x_{i_3}^4\ldots x_{k-7}x_{i_2}^2 : 1\leqslant i_1<i_2\leqslant k-7, i_1<i_3\leqslant k-7,\\ &\quad i_3\neq i_2\}\cup\{x_1\ldots x_{i_2}^2\ldots x_{i_3}^4\ldots x_{k-7}x_{i_1}^2 : 1\leqslant i_1<i_2\leqslant k-8,\\ &\quad i_1<i_3\leqslant k-7, i_3\neq i_2\}\big)\setminus\big(\{x_1x_2^2x_3^6x_4\ldots x_{k-7}, x_1x_2^6x_3^2x_4\ldots x_{k-7},\\ &\qquad x_1x_2^2x_3^2x_4\ldots x_{k-7}x_i^4 : 4\leqslant i\leqslant k-7\}\\ &\qquad\quad\cup\{x_1^3x_2^4x_3^2x_4\ldots x_{k-7}, x_1^3x_2^2x_3\ldots x_i^4\ldots x_{k-7} : 3\leqslant i\leqslant k-7\}\big),\\ \bar{B}^+_{(k-6,2)} &= \{x_1\ldots x_{i_1}^2\ldots x_{i_2}^2\ldots x_{i_3}^4\ldots x_{k-6} : 2\leqslant i_1<i_2, i_3\leqslant k-6, i_2\neq i_3\}\\ &\qquad\setminus\{x_1x_2^2x_3^4x_4^2x_5\ldots x_{k-6}, x_1x_2^2x_3^2x_4\ldots x_i^4\ldots x_{k-6} : 4\leqslant i\leqslant k-6\}.\end{aligned}$$

By an analogous arguments to the previous ones, we can show that $\bar{B}^+_{(r,2)}$ is the set of all admissible monomials in $P_r^+(k-9,2,1)$ for $k-9\leqslant r\leqslant k-6$. Hence, $\dim QP_r^+(k-9,2,1)=|\bar{B}^+_{(r,2)}|$ for $k-9\leqslant r\leqslant k-6$. By a direct computation, we get

$$\begin{aligned}&|\bar{B}^+_{(k-9,2)}| = 2\binom{k-8}{3},\ |\bar{B}^+_{(k-8,2)}| = (k-8)^2(k-10),\\ &|\bar{B}^+_{(k-7,2)}| = (k-7)(k-8)(k-10),\ |\bar{B}^+_{(k-6,2)}| = \frac{(k-6)(k-8)(k-10)}{3}.\end{aligned}$$

So, we obtain

$$\begin{aligned}\dim QP_{k-1}(k-9,2,1) &= \sum_{k-9\leqslant r\leqslant k-6}\binom{k-1}{r}|\bar{B}^+_{(r,2)}|\\ &= \frac{(k-1)(k-8)(k-10)}{3}\binom{k+1}{8}.\end{aligned}$$

This completes the proof. □

Remark 4.12. For $k=10$, we have proved in [15] that $QP_4(1,2,1)=0$. So, this implies $QP_\ell(1,2,1)=0$, $\ell=1,2,3$. Using Proposition 2.11 one gets $QP_9(1,2,1)=0$. Hence, Proposition 4.11 holds for $k=10$.

By a direct computation using Theorem 1.4, Propositions 4.1, 4.2, 4.4, 4.6, 4.8, 4.11 and the relation $\binom{k+1}{2t}=\binom{k}{2t}+\frac{k-2t+2}{2t-1}\binom{k}{2(t-1)}$ for $t>0$, we easily obtain a new lower bound for $\dim(QP)_n$.

Theorem 4.13. *Let $n=(k-1)(2^d-1)$ with d a positive integer. If $k\geqslant 10$ and $d\geqslant 2$, then*

$$\dim(QP_k)_n > \Big(\sum_{u=0}^{4}C_{k,u}\binom{k}{2u}\Big)\sum_{v=1}^{\min\{k,d-1\}}\binom{k}{v}+\binom{k}{\min\{k,d\}},$$

where

$$C_{k,u}=\begin{cases}1, & u=0,\\ k-3, & u=1,\\ \frac{k^5-21k^4+175k^3-735k^2+1984k-3744}{180}, & u=2,\\ \frac{(k-6)(k-7)}{2}+\frac{(k-1)(k-10)(k^4-20k^3+193k^2-1250k+3912)}{1344}, & u=3,\\ \frac{(k-1)(k-8)(k-10)}{3}, & u=4.\end{cases}$$

Remark 4.14. Let $d(k)$ be as in Theorem 1.1 and let $\omega(d(k))$ be as in the proof of Theorem 1.4. By an elementary computation, we can show that $d(k) \geqslant 3$ for any $k \geqslant 6$. If $d \geqslant d(k)+k-1$, then $d > k$, $\min\{k,d\} = \min\{k, d-1\} = k$ and $\sum_{u=1}^{k}\binom{k}{u} = 2^k - 1$. If ω is a weight vector with $\deg\omega = k-1$, then $\deg((k-1)|^{d(k)-1}|\omega) = (k-1)(2^{d(k)}-1)$, $\dim QP_{k-1}((k-1)|^{d(k)-1}|\omega) = \dim QP_{k-1}(\omega)$, $\dim QP_{k-1}(\omega(d(k))) > 0$ and $\binom{k}{\min\{k,d\}} = 1 < 2^k - 1$. According to Theorem 1.1, we have

$$\begin{aligned}\dim(QP_k)_n &= (2^k-1)\dim(QP_{k-1})_{(k-1)(2^{d(k)}-1)}\\ &\geqslant (2^k-1)\Bigg(\sum_{\deg\omega=k-1}\dim QP_{k-1}((k-1)|^{d(k)-1}|\omega)\\ &\qquad\qquad + \dim QP_{k-1}(\omega(d(k)))\Bigg)\\ &\geqslant (2^k-1)\sum_{\deg\omega=k-1}\dim QP_{k-1}(\omega) + 2^k - 1\\ &> \Bigg(\sum_{\deg\omega=k-1}\dim QP_{k-1}(\omega)\Bigg)\sum_{u=1}^{\min\{k,d-1\}}\binom{k}{u} + \binom{k}{\min\{k,d\}}.\end{aligned}$$

This shows that Theorem 1.1 implies Theorem 1.4, hence it also implies Theorem 4.13.

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