Some properties of h-extendible domains in $\mathbb{C}^{n+1}$

NINH VAN THU[1,2] AND NGUYEN QUANG DIEU[3,2]

ABSTRACT. The purpose of this article is twofold. The first aim is to characterize h-extendibility of smoothly bounded pseudoconvex domains in $\mathbb{C}^{n+1}$ by their noncompact automorphism groups. Our second goal is to show that if the squeezing function tends to 1 or the Fridman invariant tends to 0 at an h-extendible boundary point of a smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, then this point must be strongly pseudoconvex.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and let us denote by $\mathrm{Aut}(\Omega)$ the group of biholomorphic self-maps of Ω with the compact-open topology. It is proved by H. Cartan (see [Nar71]) that if Ω is a bounded domain in $\mathbb{C}^n$ and the $\mathrm{Aut}(\Omega)$ is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(x) \to p$. In this circumstance, we call p a *boundary orbit accumulation point*. Moreover, if $\partial\Omega$ enjoys some sort of convexity at p then φ_j converges uniformly on compact sets of Ω to p.

It is known that the local geometry of the so-called "boundary orbit accumulation point" p in turn gives global information about the characterization of model of the domain. We refer the reader to the recent survey [IK99] and the references therein for the development in related subjects. For instance, B. Wong and J. P. Rosay (see [Won77], [Ros79]) proved the following remarkable theorem.

Theorem (Wong-Rosay). *Any bounded domain $\Omega \Subset \mathbb{C}^n$ with a C^2 strongly pseudoconvex boundary orbit accumulation point is biholomorphic to the unit ball in $\mathbb{C}^n$.*

After that, by using the scaling technique, introduced by S. Pinchuk [Pin91], E. Bedford and S. Pinchuk [BP91], F. Berteloot [Ber94] proved several results about the characterization of the complex ellipsoids and models. In [DN09], Do Duc Thai and the first author showed that if Ω is pseudoconvex finite type and smooth of class $\mathcal{C}^\infty$ in some neighborhood of a boundary orbit accumulation point, $\xi_0 \in \partial\Omega$, and the Levi form has corank at most one at ξ_0, then Ω is biholomorphically equivalent to a model

$$M_H = \{(z_1, \cdots, z_n, w) \in \mathbb{C}^n \times \mathbb{C} : \mathrm{Re}(w) + H(z_1, \bar{z}_1) + \sum_{k=1}^{n} |z_k|^2 < 0\},$$

where H is a homogeneous subharmonic polynomial with $\Delta H \not\equiv 0$.

To give a statement of our result, we recall that a smooth pseudoconvex boundary point $p \in \partial\Omega$ is called h-extendible [Yu94, Yu95] (or semiregular [DH94]) if Catlin's

2010 *Mathematics Subject Classification.* Primary 32M05; Secondary 32H02, 32H50, 32T25.

Key words and phrases. Automorphism groups, finite type, h-extendible models, squeezing function, Fridman invariant.

multitype and D'Angelo multitype at p coincide. It is well-known that the class of h-extendible points includes pseudoconvex finite points in $\mathbb{C}^2$, strongly pseudoconvex points in $\mathbb{C}^n$, and convex finite type points $\mathbb{C}^n$. In particular, any pseudoconvex finite type boundary point in $\mathbb{C}^n$ with corank of the Levi form at most one is h-extendible.

The first aim in this paper is to prove the following theorem, which gives a characterization of h-extendible domains with noncompact automorphism groups.

Theorem 1.1. *Assume that Ω is a pseudoconvex domain in $\mathbb{C}^{n+1}$ with $\mathcal{C}^\infty$-smooth boundary $\partial\Omega$. Let $\xi_0 \in \partial\Omega$ be h-extendible with Catlin's finite multitype $(1, m_1, \ldots, m_n)$ and let $\Lambda = (1/m_1, \ldots, 1/m_n)$. Suppose that there exists a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\eta_j := \varphi_j(a)$ converges Λ-nontangentially to ξ_0 for some $a \in \Omega$ (cf. Definition 3.4). Then there exists a biholomorphic mapping $\sigma : \Omega \to M_P$. Here M_P is a domain of the form*

$$M_P := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \colon \mathrm{Re}(w) + P(z) < 0\},$$

where P is a Λ-homogeneous plurisubharmonic real-valued polynomial which contains no pluriharmonic monomials (cf. Definition 3.2). Moreover, the map σ satisfies the following properties:
(a) $\sigma(a) = (0', -1)$.
(b) There exist sequences $\{\xi_j\} \subset \partial\Omega$ and $\{\tilde{\xi}_j\} \subset \partial M_P$ such that $\xi_j \to \xi_0$ as $j \to \infty$ and that σ extends continuously to a homeomorphism near ξ_j and $\tilde{\xi}_j$.

Remark 1.1. Recently, F. Rong and B. Zhang [RZ16] gave a characterization of h-extendible model in which the sequence $\{\eta_j\} \subset \Omega$ converges nontangentially to an h-extendible boundary point $\xi_0 \in \partial\Omega$. Their proof is based on the Pinchuk scaling method. However, the equation (3.6) in page 905 of [RZ16], which plays a crucial role to ensure the normality of the scaling sequence, is unclear to us. Fortunately, by using the attraction property of analytic discs based deeply on the existence of a plurisubharmonic peak function at the origin of the above model M_P, the normality of the scaling sequence is eventually verified (see Proposition 4.3), and then the proof of Theorem 1.1 follows. As a consequence, the above-mentioned result of F. Rong and B. Zhang is obtained.
2. Notice that we do not know if the sequence $\{\tilde{\xi}_j\}$ can be chosen to be *bounded* even when $\partial\Omega$ is *algebraic*. If this is the case then by using results in [Ber95] or [CP01] we can prove that σ extends *holomorphically* through ξ_0.

Now we move to the definition of squeezing function of a domain. Let Ω be a domain in $\mathbb{C}^n$ and $p \in \Omega$. For a holomorphic embedding $f \colon \Omega \to \mathbb{B}^n := \mathbb{B}(0; 1)$ with $f(p) = 0$, we set

$$s_{\Omega,f}(p) := \sup\{r > 0 \colon B(0; r) \subset f(\Omega)\},$$

where $\mathbb{B}^n(z; r) \subset \mathbb{C}^n$ denotes the ball of radius r with center at z. Then the *squeezing function* $s_\Omega : \Omega \to \mathbb{R}$ is defined in [DGZ12] as

$$s_\Omega(p) := \sup_f \{s_{\Omega,f}(p)\}.$$

Note that $0 < s_\Omega(z) \le 1$ for any $z \in \Omega$ and the squeezing function is clearly invariant under biholomorphic mappings.

Next, let us recall the Fridman invariant. Let M be a Kobayashi hyperbolic complex manifold of dimension n and let $B_M(p, r)$ be the Kobayashi ball around

p of radius $r > 0$. Let $\mathcal{R}$ be the set of all $r > 0$ such that there is an injective holomorphic map $f\colon \mathbb{B}^n \to M$ with $B_M(p,r) \subset f(\mathbb{B}^n)$. Note that $\mathcal{R}$ is non-empty (cf. [MV19]). Then the Fridman invariant is defined by

$$h_M(p) = \inf_{r\in\mathcal{R}} \frac{1}{r}.$$

In recent works [DGZ16, DFW14, KZ16] the authors proved that if p is a strongly pseudoconvex boundary point, then $\lim\limits_{\Omega\ni z\to p\in\partial\Omega} s_\Omega(z) = 1$. Conversely to this result, J. E. Fornæss and F. E. Wold posed the following problem (see [FW18, Problem 4.1]).

Problem. If Ω is a bounded pseudoconvex domain with smooth boundary, and if $\lim\limits_{\Omega\ni z\to p\in\partial\Omega} s_\Omega(z) = 1$, then is the boundary of Ω strongly pseudoconvex at p?

The main results around this problem are due to A. Zimmer [Zim18a, Zim18b], J. E. Fornæss and F. E. Wold [FW18], S. Joo and K.-T. Kim [JK18], P. Mahajan and K. Verma [MV19]. More precisely, in [Zim18a, Zim18b] A. Zimmer proved that the answer is affirmative if the domain is bounded convex with $\mathcal{C}^{2,\alpha}$-smooth boundary. In [FW18], J. E. Fornæss and F. E. Wold constructed a counter-example to this problem, that is, they constructed a bounded convex $\mathcal{C}^2$-smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but

$$\lim_{\Omega\ni z\to\partial\Omega} s_\Omega(z) = 1.$$

Now let us consider a sequence $\{\eta_j\} \subset \Omega$ converging to an h-extendible boundary point $\xi_0 \in \partial\Omega$. Suppose that Ω is pseudoconvex of finite type near ξ_0 and $\lim\limits_{j\to\infty} s_\Omega(\eta_j) = 1$ or $\lim\limits_{j\to\infty} h_\Omega(\eta_j) = 0$. It is known that if the sequence $\{\eta_j\} \subset \Omega$ converges to ξ_0 along the inner normal line to $\partial\Omega$ at ξ_0, then ξ_0 must be strongly pseudoconvex (see [JK18] for $n = 2$ and [MV19] for general case). Moreover, this result was obtained in [Nik18] for the case that $\{\eta_j\} \subset \Omega$ converges nontangentially to ξ_0.

The second aim in this paper is to prove the following theorem.

Theorem 1.2. *Let ξ_0 be an h-extendible boundary point of a $\mathcal{C}^\infty$-smooth, bounded pseudoconvex domain Ω in $\mathbb{C}^{n+1}$. Assume that $\lim\limits_{j\to\infty} s_\Omega(\eta_j) = 1$ or $\lim\limits_{j\to\infty} h_\Omega(\eta_j) = 0$ for some sequence $\{\eta_j\} \subset \Omega$ converging Λ-nontangentially to ξ_0. Then ξ_0 is a strongly pseudoconvex point.*

The organization of this paper is as follows: In Sections 2 and 3, we recall some basic definitions and results needed later. In Section 4, we verify the normality of the scaling sequence and then we give a proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 5.

2. The normality of sequences of biholomorphisms

First of all, we recall the following definition (see [GK87] or [DN09]).

Definition 2.1. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of open sets in a complex manifold M and Ω_0 be an open set of M. The sequence $\{\Omega_i\}_{i=1}^\infty$ is said to converge to Ω_0 (written $\lim \Omega_i = \Omega_0$) if and only if

(i) For any compact set $K \subset \Omega_0$, there is an $i_0 = i_0(K)$ such that $i \geq i_0$ implies that $K \subset \Omega_i$; and

(ii) If K is a compact set which is contained in Ω_i for all sufficiently large i, then $K \subset \Omega_0$.

Next, we need the following proposition, which is a generalization of the theorem of H. Cartan (see [GK87, DT04, DN09]).

Proposition 2.2. *Let $\{A_i\}_{i=1}^\infty$ and $\{\Omega_i\}_{i=1}^\infty$ be sequences of domains in a complex manifold M with $\lim A_i = A_0$ and $\lim \Omega_i = \Omega_0$ for some (uniquely determined) domains A_0, Ω_0 in M. Suppose that $\{f_i : A_i \to \Omega_i\}$ is a sequence of biholomorphic maps. Suppose also that the sequence $\{f_i : A_i \to M\}$ converges uniformly on compact subsets of A_0 to a holomorphic map $F : A_0 \to M$ and the sequence $\{g_i := f_i^{-1} : \Omega_i \to M\}$ converges uniformly on compact subsets of Ω_0 to a holomorphic map $G : \Omega_0 \to M$. Then either of the following assertions holds.*

(i) *The sequence $\{f_i\}$ is compactly divergent, i.e., for each compact set $K \subset \Omega_0$ and each compact set $L \subset \Omega_0$, there exists an integer i_0 such that $f_i(K) \cap L = \emptyset$ for $i \geq i_0$; or*
(ii) *There exists a subsequence $\{f_{i_j}\} \subset \{f_i\}$ such that the sequence $\{f_{i_j}\}$ converges uniformly on compact subsets of A_0 to a biholomorphic map $F : A_0 \to \Omega_0$.*

In addition, we prepare the following proposition (see [Ber94, Proposition 2.1] or [DN09, Proposition 2.2]).

Proposition 2.3. *Let M be a domain in a complex manifold X of dimension n and $\xi_0 \in \partial M$. Assume that ∂M is pseudoconvex and of finite type near ξ_0.*

(a) *Let Ω be a domain in a complex manifold Y of dimension m. Then every sequence $\{\varphi_j\} \subset \mathrm{Hol}(\Omega, M)$ converges unifomly on compact subsets of Ω to ξ_0 if and only if $\lim \varphi_j(a) = \xi_0$ for some $a \in \Omega$.*
(b) *Assume, moreover, that there exists a sequence $\{\varphi_j\} \subset Aut(M)$ such that $\lim \varphi_j(a) = \xi_0$ for some $a \in M$. Then M is taut.*

Remark 2.1. By Proposition 2.3 and by the hypothesis of Theorem 1.1, for each compact subset $K \Subset \Omega$ and each neighborhood U of ξ_0, there exists an integer j_0 such that $\varphi_j(K) \subset \Omega \cap U$ for all $j \geq j_0$. Moreover, Ω is taut.

3. Catlin's multitype and the h-extendibility

3.1. **Catlin's multitype.** For the convenience of the exposition, let us recall *Catlin's multitype* (for more details, we refer to [Cat84, Yu92] and the references therein). Let Ω be a domain in $\mathbb{C}^n$ and ρ be a defining function for Ω near $z_0 \in \partial\Omega$. Let us denote by Γ^n the set of all n-tuples of numbers $\mu = (\mu_1, \dots, \mu_n)$ such that

(i) $1 \leq \mu_1 \leq \cdots \leq \mu_n \leq +\infty$;
(ii) For each j, either $\mu_j = +\infty$ or there is a set of non-negative integers $k_1, \dots, k_j$ with $k_j > 0$ such that

$$\sum_{s=1}^{j} \frac{k_s}{\mu_s} = 1.$$

A weight $\mu \in \Gamma^n$ is called *distinguished* if there exist holomorphic coordinates $(z_1, \dots, z_n)$ about z_0 with z_0 maps to the origin such that

$$D^\alpha \overline{D}^\beta \rho(z_0) = 0 \text{ whenever } \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\mu_i} < 1.$$

Here D^α and $\overline{D}^\beta$ denote the partial differential operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1}\cdots\partial \bar{z}_n^{\beta_n}},$$

respectively.

Definition 3.1. The *multitype* $\mathcal{M}(z_0)$ is defined to be the smallest weight $\mathcal{M} = (m_1, \dots, m_n)$ in Γ^n (smallest in the lexicographic sense) such that $\mathcal{M} \geq \mu$ for every distinguished weight μ.

3.2. **The h-extendibility.** In what follows, we call a multiindex $(\lambda_1, \lambda_2, \dots, \lambda_n)$ a *multiweight* if $1 \geq \lambda_1 \geq \cdots \geq \lambda_n$. Now let us recall the following definitions (cf. [Yu94, Yu95]).

Definition 3.2. Let $f(z)$ be a function on $\mathbb{C}^n$ and let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a multiweight. For any real number $t \geq 0$, set

$$\pi_t(z) = (t^{\lambda_1} z_1, t^{\lambda_2} z_2, \dots, t^{\lambda_n} z_n).$$

We say that f is Λ-*homogeneous with weight* α if $f(\pi_t(z)) = t^\alpha f(z)$ for every $t \geq 0$ and $z \in \mathbb{C}^n$. In case $\alpha = 1$, then f is simply called Λ-*homogeneous*.

For a multiweight Λ, the following function

$$\sigma(z) = \sigma_\Lambda(z) := \sum_{j=1}^n |z_j|^{1/\lambda_j}$$

is Λ-homogeneous. Moreover, for a multiweight Λ and a real-valued Λ-homogeneous function P, we define a homogeneous model $D_{\Lambda,P}$ as follows:

$$D_{\Lambda,P} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}\colon \mathrm{Re}(w) + P(z) < 0\}.$$

Definition 3.3. Let $D_{\Lambda,P}$ be a homogeneous model. Then $D_{\Lambda,P}$ is called h-*extendible* if there exists a Λ-homogeneous $\mathcal{C}^1$ function $a(z)$ on $\mathbb{C}^n \setminus \{0\}$ satisfying the following conditions:

(i) $a(z) > 0$ whenever $z \neq 0$;
(ii) $P(z) - a(z)$ is plurisubharmonic on $\mathbb{C}^n$.

We will call $a(z)$ a *bumping function*.

Remark 3.1. In this paper, our model $D_{\Lambda,P}$ is always assumed to be of finite type. So, by [Yu94, Theorem 2.1] the bumping function $a(z)$ must be $\mathcal{C}^\infty$ on $\mathbb{C}^n \setminus \{0\}$ and $P(z) - a(z)$ is strictly plurisubharmonic on $\mathbb{C}^n \setminus \{0\}$. Moreover, $\Lambda = (1/m_1, \dots, 1/m_n)$, where $(1, m_1, \dots, m_n)$ is the multitype of $D_{\Lambda,P}$ at 0. For several equivalent conditions to the h-extendibility, we refer the reader to [Yu94].

Remark 3.2. Let $a(z)$ be a bumping function. Then there is a constant $C > 0$ such that

$$C\sigma(z) \leq a(z) \leq C^{-1}\sigma(z),\ \forall\ z \in \mathbb{C}^n.$$

By a pointed domain (Ω, p) in $\mathbb{C}^{n+1}$ we mean that Ω is a smooth pseudoconvex domain in $\mathbb{C}^{n+1}$ with $p \in \partial\Omega$. Let ρ be a local defining function for Ω near p and let the multitype $\mathcal{M}(p) = (1, m_1, \dots, m_n)$ be finite. Moreover, since Ω is pseudoconvex, the integers $m_1, \dots, m_n$ are all even.

By the definition of multitype, there are distinguished coordinates $(z, w) = (z_1, \dots, z_n, w)$ such that $p = 0$ and $\rho(z, w)$ can be expanded near 0 as follows:

$$\rho(z, w) = \mathrm{Re}(w) + P(z) + R(z, w),$$

where P is a $(1/m_1, \dots, 1/m_n)$-homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, R is smooth and satisfies

$$|R(z, w)| \leq C \left(|w| + \sum_{j=1}^{n} |z_j|^{m_j} \right)^{\gamma},$$

for some constant $\gamma > 1$ and $C > 0$.

In what follows, the weight of any multiindex $K = (k_1, \dots, k_n) \in \mathbb{N}^n$ with respect to $\Lambda = (1/m_1, \dots, 1/m_n)$ is given by

$$wt(K) = \sum_{j=1}^{n} \frac{k_j}{m_j}.$$

We note that $wt(K + L) = wt(K) + wt(L)$ for any $K, L \in \mathbb{N}^n$. In addition, $\lesssim$ and $\gtrsim$ denote inequality up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

Definition 3.4. We call $M_P = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \colon \mathrm{Re}(w) + P(z) < 0\}$ an *associated model* for (Ω, p). If the pointed domain (Ω, p) has an h-extendible associated model, we say that (Ω, p) is *h-extendible*. In this circumstance, we say that a sequence $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ *converges Λ-nontangentially to p* if $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ and $\sigma(\alpha_j) \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$, where

$$\sigma(z) = \sum_{k=1}^{n} |z_k|^{m_k}.$$

Here and in what follows, $\mathrm{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to $\partial\Omega$.

Remark 3.3. It is well-known that $\{\eta_j\} \subset \Omega$ converges nontangentially to p if $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ and $|\alpha_{jk}| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ for every $1 \leq k \leq n$, where $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn})$. Nevertheless, such sequence converges Λ-nontangentially to p if $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ and $|\alpha_{jk}|^{m_j} \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ for every $1 \leq k \leq n$.

We also need the following definition (cf. [Yu95]).

Definition 3.5. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a fixed n-tuple of positive numbers and $\mu > 0$. We denote by $\mathcal{O}(\mu, \Lambda)$ the set of smooth functions f defined near the origin of $\mathbb{C}^n$ such that

$$D^{\alpha} \overline{D}^{\beta} f(0) = 0 \text{ whenever } \sum_{j=1}^{n} (\alpha_j + \beta_j) \lambda_j \leq \mu.$$

If $n = 1$ and $\Lambda = (1)$ then we use $\mathcal{O}(\mu)$ to denote the functions vanishing to order at least μ at the origin.

Now let us recall the following proposition, whose proof easily follows from the Taylor expansion (see [Yu95, Proposition 4.9]).

Proposition 3.6. (i) *If $f \in \mathcal{O}(\mu, \Lambda)$ then $\frac{\partial f}{\partial z_j}$ and $\frac{\partial f}{\partial \bar{z}_j}$ are in $\mathcal{O}(\mu - \lambda_j, \Lambda)$ for $j = 1, \dots, n$.*

(ii) *Suppose that* $f_i,\ 1 \le i \le N$, *are functions with* $f_i \in \mathcal{O}(\mu_i, \Lambda)$. *Then*

$$\prod_{i=1}^{N} f_i \in \mathcal{O}(\mu, \Lambda), \ \text{where } \mu = \sum_{i=1}^{N} \mu_i.$$

(iii) *If* $f \in \mathcal{O}(\mu, \Lambda)$, *then there are constants* $C, \delta > 0$ *such that* $|f(z)| \le C(\sigma_\Lambda(z))^{\mu+\delta}$ *for all* z *in a small neighborhood of* 0.

By Proposition 3.6, one easily obtains the following corollary.

Corollary 3.7. *If* $f \in \mathcal{O}(\mu, \Lambda)$, *then there are constants* $C, \delta > 0$ *such that* $|D^p \overline{D}^q f(z)| \le C(\sigma_\Lambda(z))^{\mu - wt(p) - wt(q) + \delta}$ *for every multi-indices* $p, q \in \mathbb{N}^n$ *with* $wt(p) + wt(q) < \mu$ *and for all* z *in a small neighborhood of* 0.

4. Proof of Theorem 1.1

This section is devoted to a proof of Theorem 1.1. Throughout this section, the domain Ω and the boundary point $\xi_0 \in \partial\Omega$ are assumed satisfy the hypothesis of Theorem 1.1. Let ρ be a local defining function for Ω near ξ and let the multitype $\mathcal{M}(p) = (1, m_1, \ldots, m_n)$ be finite. Especially, because of the pseudoconvexity of Ω, the integers $m_1, \ldots, m_n$ are all even. Let us denote by $\Lambda = (1/m_1, \ldots, 1/m_n)$. By the definition of multitype, there are distinguished coordinates $(\tilde{z}, \tilde{w}) = (\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{w})$ such that $\xi_0 = 0$ and $\rho(\tilde{z}, \tilde{w})$ can be expanded near 0 as follows:

$$\rho(\tilde{z}, \tilde{w}) = \mathrm{Re}(\tilde{w}) + P(\tilde{z}) + Q(\tilde{z}, \tilde{w}),$$

where P is a Λ-homogeneous plurisubharmonic polynomial that contains no pluriharmonic monomials, Q is smooth and satisfies

$$|Q(\tilde{z}, \tilde{w})| \le C \left(|\tilde{w}| + \sum_{j=1}^{n} |\tilde{z}_j|^{m_j} \right)^{\gamma},$$

for some constant $\gamma > 1$ and $C > 0$.

By hypothesis of Theorem 1.1, there exist a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\eta_j := \varphi_j(a)$ converges Λ-nontangentially to ξ_0. Let us write $\eta_j = (\alpha_j, \beta_j) = (\alpha_{j1}, \ldots, \alpha_{jn}, \beta_j)$. Then one has

(a) $|\mathrm{Im}(\beta_j)| \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$;
(b) $|\alpha_{jk}|^{m_k} \lesssim |\mathrm{dist}(\eta_j, \partial\Omega)|$ for $1 \le k \le n$.

By following the proofs of Lemmas 4.10, 4.11 in [Yu95], after a change of variables

$$\begin{cases} z = \tilde{z}; \\ w = \tilde{w} + b_1(\tilde{z})\tilde{w} + b_2(\tilde{z})\tilde{w}^2 + b_3(\tilde{z}), \end{cases}$$

where b_1, b_2, b_3 are smooth functions of $\tilde{z}$ satisfying $b_j = O(|\tilde{z}|^2)$, $j = 1, 2, 3$, there are local holomorphic coordinates (z, w) in which $\xi_0 = 0$ and Ω can be described near 0 as follows:

$$\Omega = \{\rho(z, w) = \mathrm{Re}(w) + P(z) + R_1(z) + R_2(\mathrm{Im} w) + (\mathrm{Im} w) R(z) < 0\}.$$

Here P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic terms, $R_1 \in \mathcal{O}(1, \Lambda)$, $R \in \mathcal{O}(1/2, \Lambda)$, and $R_2 \in \mathcal{O}(2)$. We would like to emphasize that in the new coordinates the sequence $\{\eta_j\}$ still has the properties (a) and (b).

For any sequence $\{\eta_j = (\alpha_j, \beta_j)\}$ of points converging Λ-nontangentially to the origin in $U_0 \cap \{\rho < 0\} =: U_0^-$, we associate with a sequence of points $\eta'_j = (\alpha_{1j}, \cdots, \alpha_{nj}, a_j + \epsilon_j + ib_j)$, where $\epsilon_j > 0$ and $\beta_j = a_j + ib_j$, such that $\eta'_j = (\alpha'_j, \beta'_j)$ is in the hypersurface $\{\rho = 0\}$ for every $j \in \mathbb{N}^*$. We note that $\epsilon_j \approx \mathrm{dist}(\eta_j, \partial\Omega)$. Now let us consider the sequences of dilations Δ^{ϵ_j} and translations $L_{\eta'_j}$, defined respectively by

$$\Delta^{\epsilon_j}(z_1, \dots, z_n, w) = \left(\frac{z_1}{\epsilon_j^{1/m_1}}, \dots, \frac{z_n}{\epsilon_j^{1/m_n}}, \frac{w}{\epsilon_j}\right)$$

and

$$L_{\eta_j}(z, w) = (z, w) - \eta_j = (z - \alpha_j, w - \beta_j).$$

Under the change of variables $(\tilde{z}, \tilde{w}) := \Delta^{\epsilon_j} \circ L_{\eta_j}(z, w)$, i.e.,

$$\begin{cases} w - \beta_j = \epsilon_j \tilde{w} \\ z_k - \alpha_{jk} = \epsilon_j^{1/m_k} \tilde{z}_k,\ k = 1, \dots, n, \end{cases}$$

one sees that $\Delta^{\epsilon_j} \circ L_{\eta'_j}(\alpha_j, \beta_j) = (0, \cdots, 0, -1)$ for every $j \in \mathbb{N}^*$. Moreover, by using Taylor's theorem, the hypersurface $\Delta^{\epsilon_j} \circ L_{\eta'_j}(\{\rho = 0\})$ is defined by an equation of the form

$$\begin{aligned}
0 &= \epsilon_j^{-1} \rho\left(L_{\eta_j}^{-1} \circ (\Delta^{\epsilon_j})^{-1}(\tilde{z}, \tilde{w})\right) \\
&= \mathrm{Re}(\tilde{w}) + R'_2(b_j)\mathrm{Im}(\tilde{w}) + \mathrm{Im}(\tilde{w})R(\alpha_j) + \epsilon_j^{-1} o(\epsilon_j) + P(\tilde{z}) \\
&+ 2\mathrm{Re} \sum_{\substack{|p|>0 \\ wt(p)\le 1}} \frac{D^p P(\alpha_j)}{p!} \epsilon_j^{wt(p)-1} (\tilde{z})^p + \sum_{\substack{|p|,|q|>0 \\ wt(p+q)<1}} \frac{D^p \overline{D}^q P(\alpha_j)}{p!q!} \epsilon_j^{wt(p+q)-1} (\tilde{z})^p (\bar{\tilde{z}})^q \\
&+ 2\mathrm{Re} \sum_{\substack{|p|>0 \\ wt(p)\le 1}} \frac{D^p R_1(\alpha_j)}{p!} \epsilon_j^{wt(p)-1} (\tilde{z})^p + \sum_{\substack{|p|,|q|>0 \\ wt(p+q)\le 1}} \frac{D^p \overline{D}^q R_1(\alpha)}{p!q!} \epsilon_j^{wt(p+q)-1} (\tilde{z})^p (\bar{\tilde{z}})^q \\
&+ \epsilon_j^{-1} b_j \Big(2\mathrm{Re} \sum_{\substack{|p|>0 \\ wt(p)\le 1}} \frac{D^p R(\alpha_j)}{p!} \epsilon_j^{wt(p)} (\tilde{z})^p + \sum_{\substack{|p|,|q|>0 \\ wt(p+q)\le 1}} \frac{D^p \overline{D}^q R(\alpha_j)}{p!q!} \epsilon_j^{wt(p+q)} (\tilde{z})^p (\bar{\tilde{z}})^q\Big).
\end{aligned}$$

Since $\{(\alpha_j, \beta_j)\}_j$ is a sequence of points converging Λ-nontangentially to the origin in U_0^-, without loss of generality, we may assume that

$$\lim_{j\to\infty} \pi_{1/\epsilon_j}(\alpha_j) = \alpha \in \mathbb{C}^n,$$

where $\pi_t(z) = (t^{1/m_1} z_1, \dots, t^{1/m_n} z_n)$ for $t \ge 0$. Hence, by Proposition 3.6 and Corollary 3.7 one has

(i) $\lim_{j\to\infty} \frac{D^p P(\alpha_j)}{p!} \epsilon_j^{wt(p)-1} = \lim_{j\to\infty} \frac{D^p P(\pi_{1/\epsilon_j}(\alpha_j))}{p!} = \frac{D^p P(\alpha)}{p!}$;

(i) $\lim_{j\to\infty} \frac{D^p R_1(\alpha_j)}{p!} \epsilon_j^{wt(p)-1} = \lim_{j\to\infty} \frac{D^p R(\alpha_j)}{p!} \epsilon_j^{wt(p)} = 0$ whenever $wt(p) \le 1$;

(ii) $\lim_{j\to\infty} \frac{D^p \overline{D}^q P(\alpha_j)}{p!q!} \epsilon_j^{wt(p+q)-1} = \lim_{j\to\infty} \frac{D^p \overline{D}^q P(\pi_{1/\epsilon_j}(\alpha_j))}{p!q!} = \lim_{j\to\infty} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}$ whenever $wt(p+q) < 1$;

(iii) $\lim_{j\to\infty} \frac{D^p \overline{D}^q R_1(\alpha_j)}{p!q!} \epsilon_j^{wt(p+q)-1} = \lim_{j\to\infty} \frac{D^p \overline{D}^q R(\alpha_j)}{p!q!} \epsilon_j^{wt(p+q)} = 0$ whenever $wt(p) + wt(q) \le 1$;

(iv) $\lim_{j\to\infty} R_2'(b_j) = \lim_{j\to\infty} R(\alpha_j) = 0.$

Therefore, after taking a subsequence if necessary, we may assume that the sequence of domains $\Omega_j := \Delta^{\epsilon_j} \circ L_{\eta_j'}(U_0^-)$ converges normally to the following model

$$M_{P,\alpha} := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C} \colon \mathrm{Re}(\tilde{w}) + P(\tilde{z} + \alpha) - P(\alpha) < 0\},$$

which is obviously biholomorphically equivalent to the model M_P.

Without loss of generality, in what follows we always assume that $\{\Omega_j\}$ converges to M_P.

Now we need the following lemma which precises [Ber95, Lemme de localisation] (see also [Ga99, Lemma 2.1.1]).

Lemma 4.1 (Localization lemma)**.** *Let D be a domain in $\mathbb{C}^n$ and $\zeta_0 \in \partial D$. Suppose that there exists a function φ which is continuous on $\overline{D} \cap \{|z - \zeta_0| \le R\}$ such that*
(i) φ is plurisubharmonic on $D \cap \{|z - \zeta_0| < R\}$.
(ii) $\varphi > 0$ on $\overline{D} \cap \{|z - \zeta_0| \le r\}$ $(r < R)$.
(iii) $\varphi < 0$ on $\overline{D} \cap \{r' \le |z - \zeta_0| \le R'\}$ $(r < r' < R' < R)$.
Let $U := D \cap \{|z - \zeta_0| < \frac{r}{6}\}, V := D \cap \{|z - \zeta_0| < \frac{r}{5}\}$. Then, there exists a constant $\tau_0 \in (0, 1)$ such that every holomorphic maps $f \colon \mathbb{B}^k \to D$, where $\mathbb{B}^k$ is the unit ball in $\mathbb{C}^k$, satisfies

$$f(0) \in U \ \Rightarrow\ f(\mathbb{B}^k(0, \tau_0)) \subset V,$$

where $\mathbb{B}^k(a, \tau_0) := \{z \in \mathbb{C}^k \colon |z - a| < \tau_0\}$ is the open ball of radius τ_0 with center at a.

Proof. We follow closely the proof of the localization lemma given in [Ber95], which in turns is based on Theorem 3 in [Si81]. Using a patching technique as in [Ber95], we can construct a bounded negative plurisubharmonic function $\tilde{\varphi}$ on D such that $\tilde{\varphi} - |z|^2$ is plurisubharmonic on $D \cap \{|z - \zeta_0| < r\}$. Then, by an ingenious argument using the maximum principle we obtain the following lower bound for the infinitesimal Kobayashi metric

$$F_D(z, v) \ge \sqrt{\frac{2}{r}} e^{\frac{M}{2}\tilde{\varphi}(z)} \|v\|, \forall v \in \mathbb{C}^n, \forall z \in D \cap \{|z - \zeta_0| < \frac{r}{4}\}.$$

Now suppose the lemma is false, then there exists a sequence of holomorphic maps $f_j : \mathbb{B}^k \to D$ and $a_j \to 0, a_j \in \mathbb{B}^k$ with $f_j(0) \in V$ but $f_j(a_j) \notin U$. By the decreasing property of the Kobayashi pseudo-distance we obtain

$$d_D(f_j(0), f_j(a_j)) \le d_{\mathbb{B}^k}(0, a_j) \to 0 \text{ as } j \to \infty.$$

On the other hand, we can find $b_j \in D \cap \{|z - \zeta_0| = \frac{r}{5}\}$ such that

$$d_D(f_j(0), f_j(a_j)) + \frac{1}{j} \ge d_D(f_j(0), b_j).$$

For a real smooth curve $\gamma \subset D$ joining $f_j(0)$ and b_j we have

$$k_D(f_j(0), b_j) \ge \int_0^1 F_D(\gamma(t), \gamma'(t)) \ge \sqrt{\frac{2}{r}} e^{\frac{M}{2} \inf_{z \in D} \tilde{\varphi}(z)} \|f_j(0) - b_j\|.$$

It follows that $\varliminf_{j\to\infty} k_D(f_j(0), b_j) > 0$. Putting all these estimates together we obtain a contradiction. □

We need the following technical lemma which plays a key role in the proof of Theorem 1.1.

Lemma 4.2. *Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^{n+1}$ converging to M_P. Let K be a compact subset of M_P. Then there exists a compact subset L of M_P, an index $j(K) \geq 1$, and $\tau \in (0,1)$ having the following properties: If $g : \mathbb{B}^k \to \Omega_j$ is holomorphic for $j \geq j(K)$ and $g(0) \in K$ then $g(\mathbb{B}^k(0,\tau)) \subset L$.*

Proof. We split the proof into two steps.
Step 1. We show that there exist neighborhoods $\widetilde{U}, \widetilde{U}'$ of the origin and $\tau_0 > 0$ such that: *For j large enough, if $f \colon \mathbb{B}^k \to \Omega_j$ is holomorphic and $f(0) \in \widetilde{U}'$ then $f(\mathbb{B}^k_{\tau_0}) \subset \widetilde{U}$.* For this purpose, we note that there exists a plurisubharmonic peak function for M_P at $(0',0)$ (see [Yu94]). Thus we may find $0 < r < r' < R' < R$, a plurisubharmonic function φ on M_P which is continuous on $\overline{M_P}$ such that $\varphi > 0$ on $M_P \cap \{|z| < r\}$ and $\varphi < 0$ on $M_P \cap \{r' < |z| < R'\}$.

By setting $\varepsilon_0 := \frac{r}{7}$, since the sequence $\{\Omega_j\}$ converges to M_P as $j \to \infty$, we can find $j_0 \geq 1$ and a large open ball B_r around $\xi_0 := (0, \varepsilon_0)$ such that for $j \geq j_0$ we have

$$\Omega_j \subset \widetilde{\Omega}_r := M_{P,r} \cup (\mathbb{C}^{n+1} \setminus \overline{B_r}),$$

where $M_{P,r} := \{(z,w) : \mathrm{Re}(w) + P(z) < \varepsilon_0\}$. Now consider the following neighborhoods of $(0,0)$

$$\widetilde{U} := \{|z - \xi_0| < \frac{r}{5}\}, \widetilde{U}' := \{|z - \xi_0| < \frac{r}{6}\}.$$

By applying Lemma 1 to $\widetilde{\Omega}_r$, the peaking function $\psi(z,w) := \varphi(z, w - \varepsilon_0)$ and the datum r', r, R', R we obtain $\tau_0 > 0$ satisfying the conclusion of *Step 1.*
Step 2. We argue by contradiction. If the lemma is false then we can find a sequence $\mathbb{B}^k \ni \xi_j \to 0$, holomorphic maps $g_j : \mathbb{B}^k \to \Omega_j$ such that

$$g_j(0) \in K \subset M_P \text{ but } g_j(\xi_j) \to \partial M_P \cup \{\infty\}. \tag{1}$$

The key step in deriving a contradiction is to show that $\{g_j\}$ is locally uniformly near the origin. For this, choose $\lambda_0 > 0$ so big that $\Delta^{\lambda_0}(K) \subset \widetilde{U}'$. Then by *Step 1* we obtain

$$(\Delta^{\lambda_0} \circ g_j)(\mathbb{B}^k_{\tau_0}) \subset \widetilde{U}, \forall j.$$

Hence for every j we have $g_j(\mathbb{B}^k_{\tau_0}) \subset \left(\Delta^{\lambda_0}\right)^{-1}(\widetilde{U})$, a bounded open subset of $\mathbb{C}^{n+1}$. Now, by Montel's theorem, after passing to a subsequence we may assume that g_j converges uniformly on compact sets of $\mathbb{B}^k_{\tau_0}$ to a holomorphic map $g : \mathbb{B}^k_{\tau_0} \to \mathbb{C}^{n+1}$. It follows that

$$\lim_{j\to\infty} g_j(0) = g(0) = \lim_{j\to\infty} g_j(\xi_j).$$

We obtain a contradiction to (1). Hence we get a constant $\tau > 0$ that satisfies both conditions in *Step 1* and *Step 2.* □

The main step in the proof of Theorem 1 is included in the following result. We also use this proposition crucially in the next section.

Proposition 4.3. *Let ω be a domain in $\mathbb{C}^k$, $a \in \omega$ and $\sigma_j : \omega \to \Omega_j$ be a sequence of holomorphic mappings such that $\{\sigma_j(a)\} \Subset M_P$. Then $\{\sigma_j\}$ contains a subsequence that converges locally uniformly to a holomorphic map $\sigma : \omega \to M_P$.*

Proof. Choose $r > 0$ so small such that $\mathbb{B}^k(a, r) \Subset \omega$. Set

$$g_{a,j}(z) := \sigma_j\Big(r(z + \frac{a}{r})\Big)\ j \geq 1.$$

Then $g_{a,j} : \mathbb{B}^k \to \Omega_j$ and satisfies $g_{a,j}(0) = \sigma_j(a)$ is contained in a fixed compact subset K of M_P. It follows, in view of Lemma 3, that $\sigma_j(\mathbb{B}^k(a, \tau r))$ is included in some compact subset L of M_P for j large enough. Now we let ω' be the collection of $x \in \omega$ such that there exists a neighborhood U of x such that $\sigma_j(U)$ is contained in a compact subset of M_P for all j *large enough*. Then ω' is an open subset of ω and $a \in \omega'$. We claim that $\omega' = \omega$. If this is not so, then we can find a point $x_0 \in \omega \cap \partial\omega'$. Choose $x_1 \in \omega'$ closed to x_0 and $r' > 0$ so small that:

$$x_0 \in \mathbb{B}^k(x_1, \tau r') \subset \mathbb{B}^k(x_1, r') \Subset \omega.$$

By considering the new sequence

$$\sigma'_j(z) = \sigma_j\Big(r'(z + \frac{x_1}{r'})\Big),\ z \in \mathbb{B}^k.$$

We may apply Lemma 3 again to infer that $\sigma_j(\mathbb{B}^k(x_1, \tau r'))$ is contained in some compact set of M_P for j large enough. This implies that $x_0 \in \omega'$. We reach a contradiction. Thus $\omega' = \omega$ as claimed.

Finally, in view of Montel's theorem, after passing to a subsequence, we may assume that σ_j uniformly converges on compact sets of ω to a holomorphic map $\sigma : \omega \to \mathbb{C}^n$. By the above reasoning we see that $\sigma(\omega) \subset M_P$. The desired conclusion follows. □

We are now ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that (Ω, ξ_0) is h-extendible. It means that the model M_P is also h-extendible. By the hypothesis, the sequence $\{\eta_j := \varphi_j(a)\}$ converges Λ-nontangentially to $\xi_0 = (0', 0)$. Then one can find a sequence $\{\epsilon_j\} \subset \mathbb{R}^+$ converging to 0^+ such that the sequence of points $\eta'_j = \eta_j + (0', \epsilon_j)$ is in the hypersurface $\{\rho = 0\}$ for every $j \geq 1$. Let us define $T_j := \Delta^{\epsilon_j} \circ L_{\eta'_j}$ and $\sigma_j := T_j \circ \varphi_j \colon \varphi_j^{-1}(U_0^-) \to \Omega_j$. Then one sees that $T_j(\eta_j) = (0', -1)$ and $\{\sigma_j\}$ is a sequence of biholomorphic mappings satisfying

$$\sigma_j(a) = b := (0', -1),\ j \geq 1.$$

Thus, by Proposition 4.3, after passing to a subsequence, we may assume that σ_j converges locally uniformly to a holomorphic map $\sigma : \Omega \to M_P$ which satisfies $\sigma(a) = b$.

On the other hand, since Ω is taut, the sequence $\sigma_j^{-1} \colon \Omega_j \to \varphi_j^{-1}(U_0^-) \subset \Omega$ is also normal. Since $\sigma_j^{-1}(b) = a \in \Omega$, we may also assume, after switching a subsequence that σ_j^{-1} converges locally uniformly to a holomorphic map $\sigma^* : M_P \to \Omega$. It then follows from Proposition 2.2 that σ^* is the inverse of σ and so σ maps Ω biholomorphically onto M_P. It is then obvious that $\sigma(a) = \lim_{j\to\infty} \sigma_j(a) = (0', -1)$. Thus, we have shown the assertion (a).

For (b), we claim that there exists a sequence $\xi_j \to \xi_0$ such that

$$\varliminf_{x \to \xi_j} |\sigma(x)| < \infty\ \forall j.$$

If the claim fails then we may find an open ball B around ξ_0 such that

$$\lim_{x\to\xi} |\sigma(x)| = \infty \ \forall \xi \in B \cap \partial\Omega.$$

Then we choose a *bounded* holomorphic function f on M_P such that $f \not\equiv 0$ and

$$\lim_{|z|\to\infty, z\in M_P} f(z) = 0.$$

Indeed, it suffices to take $N = 1$ in the proof of Theorem 3.4 in [Yu94] to obtain the desired function f. It follows that $\hat{f} := f \circ \sigma$ is bounded holomorphic on Ω and satisfies

$$\lim_{x\to\xi} \hat{f}(x) = 0 \ \forall \xi \in B \cap \partial\Omega.$$

Suppose that $\hat{f} \not\equiv 0$ on Ω. Then $S := \{x \in \Omega : \hat{f}(x) = 0\}$ is a complex hypersurface of Ω. Thus we can find a point $x_0 \in \Omega \setminus S$ that is so close to $\partial\Omega$ such that for some $\xi^0 \in B \cap \partial\Omega$ the open segment connecting ξ^0 and x_0 stays in Ω. Let l be the complex line joining x_0 and ξ^0 and Ω_l be the connected component of $l \cap \Omega$ that contains x_0. Then $\hat{f}|_l$ is a bounded holomorphic function on Ω_l that tends to 0 at an open piece of $\partial\Omega_l$. By applying the two constant theorem to the bounded subharmonic function $\log |\hat{f}|_l|$ we infer that $\log |\hat{f}|_l|$ must be identically $-\infty$ on Ω_l. In particular $\hat{f}(x_0) = 0$, which is absurd. Hence $\hat{f} \equiv 0$ on Ω, which is impossible since σ is biholomorphic. Thus our claim is valid.

On the other hand, since Ω is of finite type at ξ_0, we may achieve that Ω is of finite type at every point ξ_j. Furthermore, one can also find sequences $\Omega \ni \{x_{k,j}\} \to \xi_j$ such that $\sigma(x_{k,j}) \to \tilde{\xi}_j \in \partial M_P$ as $k \to \infty$. Now we can apply Proposition 3 in [Ber95] to reach the conclusion (b). The proof is thereby complete. □

5. Proof of Theorem 1.2

Throughout this section, let Ω be a domain and $\xi_0 \in \partial\Omega$ be as in the hypothesis of Theorem 1.2. Let ρ be a local smooth defining function for Ω near ξ_0. After a change of coordinates, we can find the coordinate functions $(z_1, \ldots, z_n, w)$ defined on a neighborhood U_0 of ξ_0 such that $\xi_0 = 0$ and Ω can be described locally near 0 as

$$\Omega = \{\rho(z, w) = \mathrm{Re}(w) + P(z) + R_1(z) + R_2(\mathrm{Im} w) + (\mathrm{Im} w)R(z) < 0\}. \qquad (2)$$

Here P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic monomials, $R_1 \in \mathcal{O}(1, \Lambda), R \in \mathcal{O}(1/2, \Lambda)$, and $R_2 \in \mathcal{O}(2)$. Let us fix a small neighborhood U_0 of 0 and consider any point $\eta = (\alpha, \beta) \in U_0$. Now we define an anisotropic dilation Δ^ϵ and a translation L_η, respectively, by

$$\Delta^\epsilon(z_1, \ldots, z_n, w) = \Big(\frac{z_1}{\epsilon^{1/m_1}}, \ldots, \frac{z_n}{\epsilon^{1/m_n}}, \frac{w}{\epsilon}\Big)$$

and

$$L_\eta(z, w) = (z, w) - \eta = (z - \alpha, w - \beta).$$

Let $\{\eta_j\}$ be a sequence in Ω converging Λ-nontangentially to $\xi_0 = 0$. Without loss of generality, we may assume that $\eta_j = (\alpha_j, \beta_j) \in U_0^- := U_0 \cap \{\rho < 0\}$ for all j. For this sequence $\{\eta_j\}$, one associates with a sequence of points $\eta_j' = (\alpha_{1j}, \ldots, \alpha_{n_j}, \beta_j + \epsilon_j)$, $\epsilon_j > 0$, η_j' in the hypersurface $\{\rho = 0\}$. Let us consider the sequences of dilations Δ^{ϵ_j} and translations $L_{\eta_j'}$. Then $\Delta^{\epsilon_j} \circ L_{\eta_j'}(\eta_j) = (0, \ldots, 0, -1)$ and moreover, by

Lemma 1, after taking a subsequence, one can deduce that $\Delta^{\epsilon_j} \circ L_{\eta'_j}(U_0^-)$ converges to the following model

$$M_P := \{\hat{\rho} := \mathrm{Re}(w) + P(z) < 0\},$$

where $P(z)$ is the real Λ-homogeneous polynomial given in (2).

Now we are ready to give a proof of Theorem 1.2. To do this, we split the proof into two cases as follows:

Case 1: $\lim\limits_{j\to\infty} s_\Omega(\eta_j) = 1$.

In this case, let us set $\delta_j = 2(1 - s_\Omega(\eta_j))$ for all j. Then by our assumption, for each j there exists an injective holomorphic map $f_j : \Omega \to \mathbb{B}^{n+1}$ such that $f_j(\eta_j) = (0, \dots, 0)$ and $\mathbb{B}^{n+1}(0; 1 - \delta_j) \subset f_j(\Omega)$. By Proposition 2.3, one sees that $f_j(\Omega \cap U_0)$ converges to $\mathbb{B}^{n+1}$. So, Proposition 4.3 shows that the sequence $T_j \circ f_j^{-1} \colon f_j(\Omega \cap U_0) \to T_j(\Omega \cap U_0)$ is normal and its limits are holomorphic mappings from $\mathbb{B}^{n+1}$ to M_P, where $T_j := \Delta^{\epsilon_j} \circ L_{\eta'_j}$ for every $j \in \mathbb{N}^*$. Moreover, by Montel's theorem the sequence $f_j \circ T_j^{-1} \colon T_j(\Omega \cap U_0) \to f_j(\Omega \cap U_0) \subset \mathbb{B}^{n+1}$ is also normal. We note that since $T_j \circ f_j^{-1}(0) = (0', -1) \in M_P$, it follows that the sequence $T_j \circ f_j^{-1}$ is not compactly divergent. Therefore, by Proposition 2.2, after taking some subsequence we may assume that $T_j \circ f_j^{-1}$ converges uniformly on every compact subset of $\mathbb{B}^{n+1}$ to a biholomorphism from $\mathbb{B}^{n+1}$ onto M_P.

Observe that the unit ball $\mathbb{B}^{n+1}$ is biholomorphic to the Siegel half-space

$$\mathcal{U} := \{(z, w) \in \mathbb{C}^n \colon \mathrm{Re}(w) + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 0\}.$$

Hence, we may assume that there exists a biholomorphism $\psi \colon M_P \to \mathcal{U}$.

As in the end of the proof of Theorem 1, we can find a bounded holomorphic function ϕ on $\mathcal{U}$ which is continuous on $\overline{\mathcal{U}}$, $\phi \not\equiv 0$ and tends to 0 at infinity. (Actually in this concrete situation we may write down explicitly such a function ϕ.) We claim that there exists $t_0 \in \mathbb{R}$ such that $\varliminf\limits_{\substack{x\to 0\\ x<0}} |\psi(0', x + it_0)| < +\infty$. Indeed, if this would not be the case, the function $\phi \circ \psi$ would equal to 0 on the half-plane $\{\mathrm{Re}(w) < 0, z = 0\}$ and this is impossible since $\phi \not\equiv 0$. Therefore, we may assume that there exists a sequence $x_k < 0$ such that $\lim x_k = 0$ and $\lim \psi(0', x_k + it_0) = p_0 \in \partial\mathcal{U}$. Hence, it is proved in [CP01, Theorem 2.1] that under these circumstances ψ extends holomorphically to a neighborhood of $(0', it_0)$. Since the Levi form is preserved under local biholomorphisms around a boundary point, it follows that M_P is strongly pseudoconvex at $(0', it_0) \in \partial M_P$. This yields that $m_1 = \cdots = m_n = 2$ and $P(z) = |z_1|^2 + \cdots + |z_n|^2$, and thus Ω is strongly pseudoconvex at ξ_0, as desired.

Case 2: $\lim\limits_{j\to\infty} h_\Omega(\eta_j) = 0$.

Since the point ξ_0 is a local peak point (cf. [Yu94]), it follows that the Fridman invariant can be localized near ξ_0, that is, $\lim\limits_{j\to\infty} h_{U_0\cap\Omega}(\eta_j) = 0$ (cf. [MV12, Proposition 3.4]). Moreover, by our assumption, there exist a sequence of positive real numbers $R_j \to +\infty$ and a sequence of biholomorphic embeddings $g_j : \mathbb{B}^{n+1} \to U_0 \cap \Omega$ such that $g_j(0) = \eta_j$ and $B_{U_0\cap\Omega}(\eta_j, R_j) \subset g_j(\mathbb{B}^{n+1})$.

Consider the holomorphic maps

$$G_j := T_j \circ g_j \colon \mathbb{B}^{n+1} \to \Omega_j,$$

where $T_j := \Delta^{\epsilon_j} \circ L_{\eta'_j}$ for every $j \in \mathbb{N}^*$. We note that $G_j(0', 0) = (0', -1)$ for every $j \in \mathbb{N}^*$. Then Proposition 4.3 implies that the sequence $\{G_j\}$ is normal and its limits are holomorphic mappings from $\mathbb{B}^{n+1}$ to M_P. Moreover, by Montel's theorem the sequence $G_j^{-1}\colon \Omega_j \to \mathbb{B}^{n+1}$ is also normal. Therefore, by Proposition 2.2, after taking some subsequence if necessary we may assume that $\{G_j\}$ converges uniformly on every compact subset of $\mathbb{B}^{n+1}$ to a biholomorphism G from $\mathbb{B}^{n+1}$ onto M_P. Using the same argument as in the proof of Theorem 1.2 for the squeezing function, we conclude that Ω is strongly pseudoconvex at ξ_0, as desired.

Altogether, the proof of Theorem 1.2 is finally complete. □

Acknowledgement. Part of this work was done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2019. We would like to thank the VIASM for financial support and hospitality. The first author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.02-2017.311. The second author was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.02-2019.304. It is a pleasure to thank Hyeseon Kim for stimulating discussions.

References

[BP91] Bedford E. and Pinchuk S.: Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group. J. Geom. Anal. **1**, 165–191 (1991).

[Ber95] Berteloot F.: Attraction de disques analytiques et continuité Holdérienne d'applications holomorphes propres. Topics in Compl. Anal., Banach Center Publ., 91–98 (1995).

[Ber94] Berteloot F.: Characterization of models in $\mathbb{C}^2$ by their automorphism groups. Internat. J. Math. **5**, 619–634 (1994).

[Cat84] Catlin D.: Boundary invariants of pseudoconvex domains. Ann. of Math. (2) **120**(3), 529–586 (1984).

[CP01] Coupet B., Pinchuk S.: Holomorphic equivalence problem for weighted homogeneous rigid domains in $\mathbb{C}^{n+1}$. Complex analysis in modern mathematics (Russian), 57–70, FAZIS, Moscow (2001).

[DGZ12] Deng F., Guan Q., Zhang L.: Some properties of squeezing functions on bounded domains. Pacific J. Math. **257**(2), 319–341 (2012).

[DGZ16] Deng F., Guan Q., Zhang L.: Properties of squeezing functions and global transformations of bounded domains. Trans. Amer. Math. Soc. **368**(4), 2679–2696 (2016).

[DFW14] Diederich K., Fornæss J.E., Wold E.F.: Exposing points on the boundary of a strictly pseudoconvex or a locally convexifiable domain of finite 1-type. J. Geom. Anal. **24**, 2124–2134 (2014) .

[DH94] Diederich K., Herbort G.: Pseudoconvex domains of semiregular type, in: Contributions to Complex Analysis and Analytic Geometry, in: Asp. Math., E, vol. 26, Vieweg, Braunschweig, 127–161 (1994).

[DN09] Do D.T., Ninh V.T.: Characterization of domains in $\mathbb{C}^n$ by their noncompact automorphism groups. Nagoya Math. J. **196**, 135–160 (2009).

[DT04] Do D.T., Tran H.M.: Generalizations of the theorems of Cartan and Greene- Krantz to complex manifolds. Illinois J. of Math. **48**, 1367–1384 (2004).

[FW18] Fornæss J.E., Wold E.F.: A non-strictly pseudoconvex domain for which the squeezing function tends to 1 towards the boundary. Pacific J. Math. **297**(1), 79–86 (2018).

[GK87] Greene R.E., Krantz S.G.: Biholomorphic self-maps of domains. Lecture Notes in Math., **1276**, 136–207 (1987).

[Ga99] Gaussier H.: Tautness and complete hyperbolicity of domains in $\mathbb{C}^n$. Proc. Amer. Math. Soc. **127**(1), 105–116 (1999).

[IK99] Isaev A., Krantz S.G.: Domains with non-compact automorphism group: A survey. Adv. Math. **146**, 1–38 (1999).

[JK18] Joo S., Kim K.-T.: On boundary points at which the squeezing function tends to one. J. Geom. Anal. **28**(3), 2456–2465 (2018).

[KZ16] Kim K.-T., Zhang L.: On the uniform squeezing property and the squeezing function. Pac. J. Math. **282**(2), 341–358 (2016).

[MV12] Mahajan P., Verma K.: Some aspects of the Kobayashi and Carathéodory metrics on pseudoconvex domains. J. Geom. Anal. **22**(2), 491–560 (2012).

[MV19] Mahajan P., Verma K.: A comparison of two biholomorphic invariants. Internat. J. Math. **30**(1), 1950012, 16 pp. (2019).

[Nar71] Narasimhan R.: Several Complex Variables. Chicago Lectures in Mathematics, University of Chicago Press (1971).

[Nik18] Nikolov N.: Behavior of the squeezing function near h-extendible boundary points. Proc. Amer. Math. Soc. **146**(8), 3455–3457 (2018).

[Pin91] Pinchuk S.: The scaling method and holomorphic mappings. Proc. Symp. Pure Math. 52, Part 1, Amer. Math. Soc. (1991).

[Si81] Sibony N.: A class of hyperbolic manifolds. Ann. of Math. Stud. 100, 357–372 (1981).

[RZ16] Rong F., Zhang B.: On h-extendible domains and associated models. C. R. Math. Acad. Sci. Paris **354**(9), 901–906 (2016).

[Ros79] Rosay J.P.: Sur une caracterisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes. Ann. Inst. Fourier **29**(4), 91–97 (1979).

[Won77] Wong B.: Characterization of the ball in $\mathbb{C}^n$ by its automorphism group. Invent. Math. **41**, 253–257 (1977).

[Yu92] Yu J.: Multitypes of convex domains. Indiana Univ. Math. J. **41**(3), 837–849 (1992).

[Yu94] Yu J.: Peak functions on weakly pseudoconvex domains. Indiana Univ. Math. J. **43**(4), 1271–1295 (1994).

[Yu95] Yu J.: Weighted boundary limits of the generalized Kobayashi-Royden metrics on weakly pseudoconvex domains. Trans. Amer. Math. Soc. **347**(2), 587–614 (1995).

[Zim18a] Zimmer A.: A gap theorem for the complex geometry of convex domains. Trans. Amer. Math. Soc. **370**(10), 7489–7509 (2018).

[Zim18b] Zimmer A.: Characterizing strong pseudoconvexity, obstructions to biholomorphisms, and Lyapunov exponents. Math. Ann. (2018). https://doi.org/10.1007/s00208-018-1715-7.

Ninh Van Thu

[1] Department of Mathematics, VNU University of Science, Vietnam National University, Hanoi, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

[2] Thang Long Institute of Mathematics and Applied Sciences, Nghiem Xuan Yem, Hoang Mai, HaNoi, Vietnam

Email address: thunv@vnu.edu.vn

Nguyen Quang Dieu

[3] Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

[2] Thang Long Institute of Mathematics and Applied Sciences, Nghiem Xuan Yem, Hoang Mai, HaNoi, Vietnam

Email address: ngquang.dieu@hnue.edu.vn