Threshold of a stochastic SIQS epidemic model with isolation

V. H. Sam [*] N.H. Du,[†] N. T. Dieu[‡]

December 18, 2019

Abstract

The aim of this paper is to give sufficient conditions, very close to the necessary one, to classify the strongly stochastic permanence of SIQS epidemic model with isolation via a threshold value $\widehat{R}$. Precisely, we show that if $\widehat{R} < 1$ then the stochastic SIQS system goes to the disease free case in sense $I_z(t), Q_z(t)$ converges to 0 at the exponential rate and the density of susceptible class $S_z(t)$ converges to the solution of boundary equation almost surely at the exponential rate. In the case $\widehat{R} > 1$, the model is strongly stochastically permanent. We also show the existence of a unique invariant probability measure and prove the convergence in total variation norm of transition probability to the invariant measures. Some numerical examples are also provided to illustrate our findings.

Keywords. SIQS model; Extinction; Permanence; Stationary Distribution; Ergodicity.

Subject Classification. 34C12, 60H10, 92D25.

Running Title. Classification in a Stochastic SIQS Model

[*]HUS Hight School for Gifted Student, VNU University of Science-VNU 182 Luong The Vinh, Thanh Xuan, Hanoi, Vietnam, vuhaisam@yahoo.com

[†]Department of Mathematics, Mechanics and Informatics, Hanoi National University, 334 Nguyen Trai, Thanh Xuan, Hanoi Vietnam, dunh@vnu.edu.vn. This research was supported in part by Vietnam National Foundation for Science and Technology Development (NAFOSTED) n_0 01.03-2017.308.

[‡]School of Natural Sciences Education, Vinh University, 182 Le Duan, Vinh city, Nghe An province, Vietnam, dieunguyen2008@gmail.com. The author would also like to thank Vietnam Institute for Advance Study in Mathematics (VIASM) for supporting and providing a fruitful research environment and hospitality.

1 Introduction

An important issue in mathematical models of disease dynamics is to find out under which conditions the disease can be endemic (permanent over long periods of time) or go extinct. The mathematical study of these types of questions commenced with the early work of Kermack and McKendrick in [11, 12]. Their work is in fact the root of two classical epidemic models, namely, SIR and SIRS models and their variants. Then much attention has been drawn to analyzing, predicting the spread, and designing controls of infectious diseases in host populations (see, for example, [1, 2, 8, 9, 21] for the deterministic models, and [5, 4, 6, 10] for the stochastic cases). As is well-known, the quarantine/isolation is an important strategy for the control and elimination of infectious diseases. Such as, in order to control SARS, two-strain avian influenza, childhood diseases, the Middle East respiratory syndrome, Ebola epidemics, Dengue epidemic, H1N1 flu epidemic, Hepatitis B and C, Tuberculosis, the government of nations are the first to use isolation. Therefore, there are various types of classical epidemic models with quarantine/isolation are introduced and investigated (see [3, 17, 22, 23] and the references therein). X. Zhang et al. [23] considered the stochastic SIQS epidemic model with isolation

$$\begin{cases} dS(t) = \big(\alpha - \beta S(t)I(t) - \mu S(t) + \gamma_1 I(t) + \gamma_2 Q(t)\big)dt + \sigma_1 S(t)dB_1(t) \\ dI(t) = \big(\beta S(t)I(t) - (\mu + \rho_1 + \gamma_1 + \gamma_3)I(t)\big)dt + \sigma_2 I(t)dB_2(t) \\ dQ(t) = (\gamma_3 I(t) - (\mu + \rho_2 + \gamma_2)Q(t))dt + \sigma_3 Q(t)dB_3(t), \end{cases} \tag{1.1}$$

where α the per capita birth and immigration rate of the susceptibles; μ is the per capita disease-free death rate; ρ_1 and ρ_2 are the excess per capita death rate of infective and quarantine class respectively; γ_1 and γ_2 are the rates at which individuals recover and return to susceptible from infective and quarantine class respectively; γ_3 is the rate for individuals leaving the infective class for the quarantine class; $B_1(t)$, $B_2(t)$ and $B_3(t)$ are mutually independent Brownian motions and $\sigma_1, \sigma_2, \sigma_3$ are the intensities of the white noises. In that paper, authors provided the classification for the extinction and persistence of the disease based on the reproduction number $\widehat{R} = \frac{\alpha\beta}{\mu(\mu+\rho_1+\gamma_1+\gamma_3)} - \frac{\sigma_2^2}{2(\mu+\rho_1+\gamma_1+\gamma_3)}$. They showed that if $(S(t), I(t), Q(t))$ is any solution of (1.1) with initial value $(u, v, w) \in \mathbb{R}^3_+$, then

- when $\mu > \frac{\sigma^2}{2}$ and $\widehat{R} < 1$,

$$\lim \frac{\ln I(t)}{t} \leq (\mu + \rho_1 + \gamma_1 + \gamma_3)(\widehat{R} - 1), \lim_{t\to\infty} \langle S(t)\rangle = \frac{\alpha}{\mu}; \lim_{t\to\infty} \langle Q(t)\rangle = 0;$$

- in case $\mu > \frac{\sigma^2}{2}$ and $\widehat{R} > 1$,

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{\alpha}{\mu} - \frac{(\mu+\rho_1+\gamma_1+\gamma_3)(\widehat{R}-1)}{\beta}; \ \lim_{t\to\infty}\langle I(t)\rangle = \frac{(\mu+\rho_1+\gamma_1+\gamma_3)(\widehat{R}-1)}{\beta(\frac{(\mu+\rho_2)\gamma_3}{\mu(\mu+\rho_2+\gamma_2)} + \frac{\mu+\rho_1}{\mu})};$$

$$\lim_{t\to\infty}\langle Q(t)\rangle = \frac{\gamma_3}{\mu+\rho_2+\gamma_2}\frac{(\mu+\rho_1+\gamma_1+\gamma_3)(\widehat{R}-1)}{\beta(\frac{(\mu+\rho_2)\gamma_3}{\mu(\mu+\rho_2+\gamma_2)} + \frac{\mu+\rho_1}{\mu})},$$

where $< f(t) >= \frac{1}{t}\int_0^t f(\tau)d\tau$ and $\sigma^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$.

Condition $\mu > \frac{\sigma^2}{2}$ tells us that the intensities of noises must be small enough. In our opinion, this assumption is rather restrictive and it is easy to give examples where it can not be satisfied.

The aim of this paper is to remove this condition and improve obtained results. We classify the model by using the same reproduction number $\widehat{R}$ but without the condition $\mu > \frac{\sigma^2}{2}$. Moreover, in case $\widehat{R} < 1$, we consider not only the average permanence of susceptible individuals and the average extinction of quarantine individuals as in [23] but also study almost sure convergence to 0 at the exponential rate. In the case of permanence of the disease $\widehat{R} > 1$, the strongly stochastic permanence and existence of ergodic stationary probability measure are proved. Only the critical case $\widehat{R} = 0$ is not studied in this paper. Our findings are considered as a sufficient conditions and almost surely necessary conditions for permanence and extinction of diseases. Similar results hold true for some stochastic SIR models can be seen in [4, 5] and for stochastic SIRS model in [16, 20].

Noting that, in almost existent literature on this topic they often derived the sufficient condition for the permanence and extinction of diseases, there is a gap between the necessary and sufficient conditions for the permanence of the system.

The most difficulty we have to face to obtain these results is unable to use the stochastic comparison theorem to dominate the solutions of (1.1) with the solution of the boundary equation. Therefore, the technique used in [4, 5] is no longer valid and it need being improved.

The rest of the paper is arranged as follows. Section 2 derives a threshold that is used to classify the extinction and permanence of the disease. To establish the desired results, via the dynamics on the boundary we obtain a threshold $\widehat{R}$ that enables us to determine the long term behavior of the solution. Precisely, it is shown that if $\widehat{R} < 1$, the disease will decay in an exponential rate. In case $\widehat{R} > 1$, the solution converges to a stationary distribution in total variation, that means the disease is permanent. The ergodicity of the solution process is also proved. Finally, Section 3 is devoted to some discussion and comparison to existing results in the literature. Some numerical examples are provided to illustrate our results.

2 Threshold between extinction and permanence

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let B_1, B_2, B_3 be independent $\mathcal{F}_t$-adapted, Brownian motions, $\mathbb{R}^{3,\circ}_+ := \{(x, y, z) : x > 0, y > 0, > 0\}$. We consider the equation (1.1). The existence of uniquely a global solution $(S_{u,v,w}(t), I_{u,v,w}(t), Q_{u,v,w}(t)), t \geq 0$ to (1.1) has been provided in [23]. Moreover, the domain $\mathbb{R}^{3,\circ}$ is invariant in the sense that if $(S_{u,v,w}(t), I_{u,v,w}(t), Q_{u,v,w}(t)) \in \mathbb{R}^{3,\circ}, \ \forall\, t \geq 0$ provided $(u, v, w) \in \mathbb{R}^{3,\circ}$.

In the following, we denote $z = (u, v, w)$ and write $(S_z(t), I_z(t), Q_z(t))$ or $(S(t), I(t), Q(t))$ for $(S_{u,v,w}(t), I_{u,v,w}(t), Q_{u,v,w}(t))$ if there is no confusion.

It is easy to see that if $I(0) = 0$ then $I(t) = 0$ for all $t > 0$ and the first equation of (1.1) becomes the equation on the boundary,

$$d\widetilde{S}^0(t) = (\alpha - \mu\widetilde{S}^0(t))dt + \sigma_1\widetilde{S}^0(t)dB_1(t). \tag{2.1}$$

Let $\widetilde{S}^0_u(t)$ be the solution to (2.1) with initial value $\widetilde{S}^0_u(0) = u > 0$. It is noted that we can not use the comparison theorem to get $S_z(t) \leq \widetilde{S}^0_u(t)\ \forall t \geq 0$ as in [5]. By solving the Fokker-Planck equation, it is easy to see that (2.1) has a unique stationary distribution, say μ^0, with density

$$f^*(x) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{\frac{-b}{x}}, x > 0 \tag{2.2}$$

where $a = \frac{2\mu+\sigma_1^2}{\sigma_1^2}, b = \frac{2\alpha}{\sigma_1^2}$ and $\Gamma(\cdot)$ is the Gamma function. The strong law of large numbers [18, Theorem 3.16, p. 46] says that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{S}^0_u(s)ds = \int_0^\infty xf^*(x)dx = \frac{\alpha}{\mu} \quad \text{a.s.} \tag{2.3}$$

From the second equation of (1.1) we have

$$\frac{\ln I_z(t)}{t} = \frac{\ln I_z(0)}{t} + \frac{\beta}{t}\int_0^t \big(S_z(s)ds - (\mu + \rho_1 + \gamma_1 + \gamma_3) - \frac{\sigma_2^2}{2}\big) + \sigma_2\frac{B_2(t)}{t}.$$

The Lyapunov exponent $\lambda = \limsup_{t\to\infty} \frac{\ln I_z(t)}{t}$ of $I_z(t)$ can be calculated from this equation. Intuitively, if $\lambda < 0$ then $\lim_{t\to\infty} I_z(t) = 0$, which implies that $S_z(t) \approx \widetilde{S}^0_u(t)$. Thus, we expect that

$$\lim_{s\to\infty}\frac{1}{t}\int_0^t S_z(s)ds \approx \lim_{s\to\infty}\frac{1}{t}\int_0^t \widetilde{S}_u^0(s)ds = \frac{\alpha}{\mu}.$$

Therefore, a candidate of λ can be recommended as

$$\lambda := \lim_{t\to\infty}\frac{\beta}{t}\int_0^t \widetilde{S}_u^0(s)ds - \left(\mu+\rho_1+\gamma_1+\gamma_3+\frac{\sigma_2^2}{2}\right) = \frac{\alpha\beta}{\mu} - \left(c_2+\frac{\sigma_2^2}{2}\right), \tag{2.4}$$

where $c_2 = \mu+\rho_1+\gamma_1+\gamma_3$, when it is negative.

In the following we study λ because it plays the same role as $\widehat{R}$ since $\widehat{R} < 1$ or $\widehat{R} > 1$ is equivalent to $\lambda < 0$ or $\lambda > 0$.

2.1 Case 1: $\lambda < 0$ (or $\widehat{R} < 1$)

Theorem 2.1. *If $\lambda < 0$, then for any initial value $z = (u, v, w) \in \mathbb{R}_+^{3,\circ}$ we have*

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{\ln I_z(t)}{t} = \lambda < 0, \lim_{t\to\infty}\frac{\ln Q_z(t)}{t} = \max\left\{-\big(c_3+\frac{\sigma_3^2}{2}\big), \lambda\right\}\right\} = 1, \tag{2.5}$$

and

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{\ln|S_z(t)-\widetilde{S}_u^0(t)|}{t} \le \max\left\{-\left(\mu+\frac{\sigma_1^2}{2}\right); -\left(\mu+\frac{\sigma_3^2}{2}\right); \lambda\right\}\right\} = 1, \tag{2.6}$$

where $c_3 = \mu+\rho_2+\gamma_2$.

To proof this theorem, firstly, we present the following lemmas.

Lemma 2.2. *For $0 < p < \dfrac{2\mu}{\sigma^2}$, there exists $M_p > 0$ such that*

$$\limsup_{t\to\infty}\mathbb{E}(S_z(t)+I_z(t)+Q_z(t))^{1+p} \le M_p. \tag{2.7}$$

Further, for any $H, \varepsilon, T > 0$, there exists $M_{H,\varepsilon,T} > 0$ such that

$$\mathbb{P}\{S_z(t)+I_z(t)+Q_z(t) \le M_{H,\varepsilon,T}, t\in[0,T]\} \ge 1-\varepsilon, z = (u,v,w)\in[0,H]^3. \tag{2.8}$$

Proof. Consider the Lyapunov function $V(u,v,w) = (u+v+w)^{1+p}$. The differential operator $LV(u,v,w)$ associated to the equation (1.1) is given by

$$\begin{aligned}LV(u,v,w) &= (1+p)(u+v+w)^p(\alpha-\mu(u+v+w)-\rho_1 v-\rho_2 w)\\ &\quad+\frac{(1+p)p}{2}(u+v+w)^{p-1}(\sigma_1^2x^2+\sigma_2^2y^2+\sigma_3^3z^2)\\ &\le (1+p)(u+v+w)^p(\alpha-\mu(u+v+w))+\frac{(1+p)p\sigma^2}{2}(u+v+w)^{1+p}.\end{aligned}$$

Therefore,

$$LV(u,v,w) \le K_1 - K_2 V(u,v,w), \tag{2.9}$$

where $0 < p < \dfrac{2\mu}{\sigma^2}$ and $0 < K_2 < (1+p)\left(\mu - \frac{p\sigma^2}{2}\right)$ and

$$K_1 = \sup_{(u,v,w)\in\mathbb{R}^3_+} \{LV(u,v,w) + K_2 V(u,v,w)\} < \infty.$$

By using standard arguments as proof of [5, Lemma 2.3], we obtain

$$\limsup_{t\to\infty} \mathbb{E}(V(S_z(t), I_z(t), Q_z(t))) \le \frac{K_1}{K_2} := M_p.$$

Thus we get (2.7).

To prove (2.8) we consider a sequence of stopping times

$$\eta_k = \inf\{t \ge 0 : S_z(t) + I_z(t) + Q_z(t) \ge k\}; k \in \mathbb{N}.$$

From (2.9) we have

$$\begin{aligned}
k^{1+p}\mathbb{P}\{\eta_k \le T\} &= \mathbb{E}\left[V(S_z(\eta_k \wedge T), I_z(\eta_k \wedge T), Q_z(\eta_k \wedge T))\right] \\
&\le V(u,v,w) + \mathbb{E}\int_0^T LV(S_z(\eta_k \wedge t), I_z(\eta_k \wedge t), Q_z(\eta_k \wedge t))dt \\
&\le V(u,v,w) + K_1\mathbb{E}[\eta_k \wedge T] \le V(u,v,w) + K_1 T.
\end{aligned}$$

It implies that there exists a k large enough such that $\mathbb{P}\{\eta_k \ge T\} \ge 1-\varepsilon$. Hence,

$$\mathbb{P}\left\{S_z(t) + I_z(t) + Q_z(t) \le k,\ \forall\, t \in [0,T]\right\} = \mathbb{P}\{\eta_k \ge T\} \ge 1-\varepsilon.$$

It means that (2.8) holds. Lemma is proved. □

Lemma 2.3. *Let* $\theta = \dfrac{\widetilde{\lambda}\mu}{10\beta(\gamma_1+\gamma_2)}$ *with* $\widetilde{\lambda} = \min\{-\lambda, c_3 + \frac{\sigma_3^2}{2}\}$ *and assume that* $\lambda < 0$. *Then, for any* $\varepsilon > 0$ *and* $H > 0$, *there exists an* $0 < h \le \frac{\theta}{2}$ *such that for all* $z = (u,v,w) \in [0,H]\times[0,h]\times[0,h]$, *we have*

$$\mathbb{P}\left\{\lim_{t\to\infty} I_z(t) = \lim_{t\to\infty} Q_z(t) = 0\right\} \ge 1-\varepsilon, \tag{2.10}$$

$$\mathbb{P}\left\{\limsup_{t\to\infty} \frac{1}{t}\int_0^t S_z(\tau)d\tau \le \frac{\alpha + (\gamma_1+\gamma_2)\theta}{\mu}\right\} \ge 1-\varepsilon. \tag{2.11}$$

Proof. Consider a perturbed equation of (2.1)

$$d\widetilde{S}^\theta(t) = \Big(\alpha + (\gamma_1+\gamma_2)\theta - \mu\widetilde{S}^\theta(t)\Big)\,dt + \sigma_1\widetilde{S}^\theta(t)dB_1(t). \tag{2.12}$$

By a similar argument, the equation (2.12) has an invariant probability measure, say μ^θ, with density

$$f^*_\theta(x) = \frac{b_1^{a_1}}{\Gamma(a_1)}x^{-(a_1+1)}e^{\frac{-b_1}{x}}, x > 0,$$

where $a_1 = \frac{2\mu+\sigma_1^2}{\sigma_1^2}, b_1 = \frac{2(\alpha+(\gamma_1+\gamma_2)\theta)}{\sigma_1^2}$. Therefore, the solution $\widetilde{S}^\theta_u$ with the initial condition $\widetilde{S}^\theta(0) = u > 0$ satisfies the estimates

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \widetilde{S}^\theta_u(s)ds = \frac{\alpha+(\gamma_1+\gamma_2)\theta}{\mu} \quad \text{a.s.} \tag{2.13}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{1}{\widetilde{S}^\theta_u(\tau)}d\tau = \frac{2\mu+\sigma_1^2}{2(\alpha+(\gamma_1+\gamma_2)\theta)}, \quad \text{a.s.} \tag{2.14}$$

From (2.13), for any $\varepsilon > 0$, there exists a $T_1 > 0$ such that $\mathbb{P}(\Omega_1^H) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_1^u = \left\{\omega : \frac{1}{t}\int_0^t \widetilde{S}^\theta_u(\tau)d\tau \leq \frac{\alpha+(\gamma_1+\gamma_2)\theta}{\mu} + \frac{\widetilde{\lambda}}{10} \text{ for all } t \geq T_1\right\}.$$

Since $\widetilde{S}^\theta_u(s) \leq \widetilde{S}^\theta_H(s)$, $s \geq 0$ almost surely for $u \leq H$, $\mathbb{P}(\Omega_1^u) \geq 1 - \frac{\varepsilon}{4}$ for all $u \in [0, H]$. The strong law of large numbers for Brownian motion

$$\lim_{t\to\infty}\frac{B_k(t)}{t} = 0 \text{ a.s. for } k = 1,2,3, \tag{2.15}$$

implies that $\mathbb{P}(\Omega_2) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_2 = \left\{\omega : \frac{|\sigma_k B_k|}{t} \leq \frac{\widetilde{\lambda}}{10}, \text{ for all } t \geq T_2, k = 1,2,3\right\} \text{ for some } T_2 > 0.$$

We can choose $T \geq \max\{T_1, T_2\}$ such that

$$\frac{\gamma_3\exp\{-\frac{1}{2}\widetilde{\lambda}T\}}{c_3 + \frac{1}{2}\sigma_3^2 - \frac{3}{5}\widetilde{\lambda}} \leq 1. \tag{2.16}$$

By virtue of Lemma 2.2, there exists an $M = M(\varepsilon, T, H) > 0$ such that

$$\mathbb{P}(\Omega_3) \geq 1 - \frac{\varepsilon}{4}, \text{ for all, } (u,v,w) \in \mathbb{R}^3_+ \text{ with } u+v+w \leq H,$$

where

$$\Omega_3 = \left\{\omega : \int_0^T \beta S_z(\tau)d\tau \leq M\right\}. \tag{2.17}$$

Moreover, we can choose M sufficiently large such that

$$\mathbb{P}(\Omega_4) \geq 1 - \frac{\varepsilon}{4} \text{ where } \Omega_4 = \{\omega : \ |\sigma_k B_k(t)| \leq M, \text{ for all } t \in [0,T] \text{ and } k = 1,2,3\}. \quad (2.18)$$

Let $h > 0$ be sufficiently small such that

$$he^M \left(1 + \gamma_3 \frac{e^{(c_3 + \frac{\sigma_3^2}{2})T + 3M}}{c_3 + \frac{\sigma_3^2}{2}}\right) \leq \frac{\theta}{2}. \quad (2.19)$$

Define a stopping time

$$\widetilde{\tau} = \inf\{t \geq 0 : Q_z(t) \geq \theta\}. \quad (2.20)$$

It yields from the second equation of (1.1) that

$$I_z(t) = I_z(0) \exp\left\{\int_0^t \beta S_z(\tau)d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right\}. \quad (2.21)$$

Therefore, on $\Omega_3 \cap \Omega_4$, we have

$$I_z(0)e^{-M-\left(c_2+\frac{\sigma_2^2}{2}\right)T} \leq I_z(t) \leq I_z(0)e^{2M} \text{ for all } t \in [0,T]. \quad (2.22)$$

Furthermore,

$$Q_z(t) = \Theta(t)\left[Q_z(0) + \int_0^t \gamma_3 I_z(\tau)\Theta^{-1}(\tau)d\tau\right] \quad (2.23)$$

with

$$\Theta(t) = e^{-\left(c_3+\frac{\sigma_3^2}{2}\right)t+\sigma_3 B_3(t)}. \quad (2.24)$$

It is seen that on Ω_4 there holds

$$\Theta(t) \leq e^{\sigma_3 B_3(t)} \leq e^M, \text{ for all } t \in [0,T].$$

Hence, by (2.22) it yields that

$$\int_0^t \gamma_3 I_z(\tau)\Theta^{-1}(\tau)d\tau \leq I_z(0)\gamma_3 \int_0^t e^{(c_3+\frac{\sigma_3^2}{2})\tau+3M}d\tau \leq I_z(0)\gamma_3 \frac{e^{(c_3+\frac{\sigma_3^2}{2})T+3M}}{c_3+\frac{\sigma_3^2}{2}} \quad (2.25)$$

for all $t \in [0,T]$ and $\omega \in \Omega_3 \cap \Omega_4$. As a result, if $I_z(0), Q_z(0) \in [0,h)$ then

$$Q_z(t) \leq e^M \left(Q_z(0) + I_z(0)\gamma_3 \frac{e^{(c_3+\frac{\sigma_3^2}{2})T+3M}}{c_3+\frac{\sigma_3^2}{2}}\right) \leq e^M \left(h + h\gamma_3 \frac{e^{(c_3+\frac{\sigma_3^2}{2})T+3M}}{c_3+\frac{\sigma_3^2}{2}}\right) \leq \frac{\theta}{2} \quad (2.26)$$

for all $t \in [0,T]$, $\omega \in \Omega_3 \cap \Omega_4$. Thus, if $z \in [0,H] \times [0,h) \times [0,h)$ then

$$\widetilde{\tau} > T, \text{ for all } \omega \in \Omega_3 \cap \Omega_4. \quad (2.27)$$

We are going to provide that

$$\widetilde{\tau} = \infty, \quad \text{for almost every } \ \omega \in \cap_{i=1}^{4} \Omega_i. \tag{2.28}$$

or any initial value $z \in [0, H] \times [0, h) \times [0, h)$. Indeed, for $t \in [T, \widetilde{\tau})$ we have $S_z(\tau) \le \widetilde{S}_u^{\theta}(\tau)$ for all $\tau \in [0, t]$, which implies that for almost every $\omega \in \cap_{k=0}^{4} \Omega_k$,

$$\begin{aligned} I_z(t) &= I_z(0) \exp\left\{ \int_0^t \beta S_z(\tau) d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right) t + \sigma_2 B(t) \right\} \\ &\le I_z(0) \exp\left\{ \int_0^t \beta \widetilde{S}_u^{\theta}(\tau) d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right) t + \sigma_2 B(t) \right\} \\ &\le I_z(0) \exp\left\{ \beta \left(\frac{\alpha + (\gamma_1 + \gamma_2)\theta}{\mu} + \frac{\widetilde{\lambda}}{10} \right) t - \left(c_2 + \frac{\sigma_2^2}{2}\right) t + \frac{\widetilde{\lambda}}{10} t \right\} \\ &= I_z(0) \exp\left\{ \lambda t + \frac{3\widetilde{\lambda} t}{10} \right\} \le h \exp\left\{ -\frac{7\widetilde{\lambda} t}{10} \right\}. \end{aligned} \tag{2.29}$$

With $t \ge T$ we can rewrite (2.23) as

$$Q_z(t) = \Theta(t) \left[Q_z(0) + \int_0^T \gamma_3 I_z(\tau) \Theta^{-1}(\tau) d\tau \right] + \Theta(t) \int_T^t \gamma_3 I_z(\tau) \Theta^{-1}(\tau) d\tau. \tag{2.30}$$

On the set $\omega \in \Omega_2$, it is seen

$$\exp\left\{ -\left(c_3 + \frac{\sigma_3^2}{2} + \frac{\widetilde{\lambda}}{10} \right) t \right\} \le \Theta(t) \le \exp\left\{ -\left(c_3 + \frac{\sigma_3^2}{2} - \frac{\widetilde{\lambda}}{10} \right) t \right\}, \quad \forall t \ge T. \tag{2.31}$$

Combining (2.29) and (2.31) obtain that for $z \in [0, H] \times [0, h) \times [0, h)$

$$\begin{aligned} \Theta(t) \int_T^t \gamma_3 I_z(\tau) \Theta^{-1}(\tau) d\tau &\le \gamma_3 I_z(0) \exp\left\{ -\left(c_3 + \frac{\sigma_3^2}{2} - \frac{\widetilde{\lambda}}{10} \right) t \right\} \int_T^t \exp\left\{ \left(c_3 + \frac{\sigma_3^2}{2} - \frac{3\widetilde{\lambda}}{5} \right) \tau \right\} d\tau \\ &\le \gamma_3 I_z(0) \exp\left\{ -\left(c_3 + \frac{\sigma_3^2}{2} - \frac{\widetilde{\lambda}}{10} \right) t \right\} \frac{\exp\left\{ \left(c_3 + \frac{\sigma_3^2}{2} - \frac{3\widetilde{\lambda}}{5} \right) t \right\}}{c_3 + \frac{\sigma_3^2}{2} - \frac{3\widetilde{\lambda}}{5}} \\ &\le \gamma_3 h \frac{\exp\{ -\frac{\widetilde{\lambda} t}{2} \}}{c_3 + \frac{\sigma_3^2}{2} - \frac{3\widetilde{\lambda}}{5}} \le h, \end{aligned} \tag{2.32}$$

where the last inequality follows from (2.16). Let n be any integer greater than T. From (2.26), (2.30) and (2.32), it follows that for any $(u, v, w) \in [0, H] \times [0, h) \times [0, h)$ and for almost every $\omega \in \cap_{k=1}^{4} \Omega_k$ and $t \in [0, n \wedge \widetilde{\tau})$

$$Q_z(t) \le \frac{\theta}{2} + h < \theta.$$

This implies that $\widetilde{\tau} \geq n$ for almost every $\omega \in \cap_{k=1}^4 \Omega_k$. Since n is arbitrary, we get (2.28). Further, it follows from (2.30) (2.31) and (2.32) that

$$Q_z(t) \leq e^{-(c_3+\frac{\sigma_3^2}{2}-\frac{\widetilde{\lambda}}{10})t}\left[h+\gamma_3 h\frac{\exp\left\{(c_3+\frac{\sigma_3^2}{2}-\frac{3\widetilde{\lambda}}{5})T\right\}}{c_3+\frac{\sigma_3^2}{2}-\frac{3\widetilde{\lambda}}{5}}\right]+\gamma_3 h\frac{\exp\{-\frac{1}{2}\widetilde{\lambda}t\}}{c_3+\frac{1}{2}\sigma_3^2-\frac{3}{5}\widetilde{\lambda}} \tag{2.33}$$

on $\cap_{k=1}^4 \Omega_k$. Using (2.28) and letting $t \to \infty$ in (2.29) and (2.33) obtain

$$\lim_{t\to\infty} I_z(t) = \lim_{t\to\infty} Q_z(t) = 0 \tag{2.34}$$

for any initial value $z \in [0, H] \times [0, h) \times [0, h)$ and for almost every $\omega \in \cap_{k=1}^4 \Omega_k$. It is clear that $\mathbb{P}(\cap_{k=1}^4 \Omega_k) \geq 1 - \varepsilon$. Thus, we obtain (2.10). Moreover, from (2.28) and comparison theorem we can see that $S_z(s) \leq \widetilde{S}^\theta(s)$ for all $s \in [0, \infty)$ and $\omega \in \cap_{k=1}^4 \Omega_k$. Therefore,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t S_z(\tau)d\tau \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{S}^\theta(\tau)d\tau = \frac{\alpha + (\gamma_1+\gamma_2)\theta}{\mu} \text{ on } \cap_{k=1}^4 \Omega_k. \tag{2.35}$$

This gets (2.11) and hence the proof of lemma is completed. □

Proof of Theorem 2.1. Since the system (1.1) is non degenerate, it is either recurrent or transient with a probability 1 (see Kliemann [14, Proposition 3.1]). Therefore, (2.34) takes place with a probability 1. Consider random accupation measure

$$\widetilde{\Pi}^t(A) := \frac{1}{t}\int_0^t \mathbf{1}_{\{(S_z(\tau), I_z(\tau), Q_z(\tau)\in A\}} d\tau, \quad t > 0, \quad A \in \mathcal{B}(\mathbb{R}_+^3).$$

By using [7, Lemma 5.7], we can show that with probability 1, any weak limit of $\widetilde{\Pi}^t$ as $t \to \infty$ is an invariant probability measure of process $(S_z(t), I_z(t), Q_z(t)$ in $\mathbb{R}_+^3$. From (2.7) and (2.34), it is seen that the family of measures $\{\widetilde{\Pi}^t(\cdot,\omega), t \geq 0\}$ is tight in $\mathbb{R}_+^3$ for almost sure $\omega \in \Omega$ and any weak limit of $\widetilde{\Pi}^t(\cdot)$ as $t \to \infty$ must have support on $\mathbb{R}_+ \times \{0\} \times \{0\}$. Clearly, on $\mathbb{R}_+ \times \{0\} \times \{0\}$, the process $(S_z(t), I_z(t), Q_z(t)$ has an unique invariant probability measure $\mu^0 \times \delta_{\{0\}} \times \delta_{\{0\}}$, where $\delta_{\{0\}}$ is the Dirac measure with mass at 0. As a result,

$$\lim_{t\to\infty} \widetilde{\Pi}^t(\cdot) = \mu^0 \times \delta_{\{0\}} \times \delta_{\{0\}},\ a.s.$$

On the other hand, the function $x \mapsto x^{1+p}$ is μ^0-integrable whenever $0 < p < \dfrac{2\mu}{\sigma^2}$ and $\left\{\frac{1}{t}\int_0^t S_z(\tau)d\tau : t > 0\right\}$ is uniformly integrable for almost sure $\omega \in \Omega$. Thus, there exists the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S_z(\tau)d\tau = \int_{\mathbb{R}_+} x\mu^0(dx) = \frac{\alpha}{\mu}\ a.s. \tag{2.36}$$

From the second equation of (1.1) and Itô formula we obtain

$$\frac{\ln I_z(t)}{t} = \frac{\ln I_z(0)}{t} + \beta\frac{1}{t}\int_0^t S_z(\tau)d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right) + \sigma_2\frac{B(t)}{t}.$$

Therefore, it follows from (2.36) that

$$\lim_{t\to\infty} \frac{\ln I_z(t)}{t} = \lambda \ \text{ for almost sure } \omega \in \Omega. \tag{2.37}$$

Thus, for any $\varepsilon > 0$ sufficiently small, there are two random variables η_1, η_2 such that

$$\eta_1 e^{(\lambda-\varepsilon)t} \le I_z(t) \le \eta_2 e^{(\lambda+\varepsilon)t} \text{ for all } t > 0.$$

Further, there exists the limit

$$\lim_{t\to\infty} \frac{\ln \Theta(t)}{t} = -\left(c_3 + \frac{\sigma_3^2}{2}\right) \ a.s,$$

which implies there are two positive random variables ξ_1, ξ_2 satisfying

$$\xi_1 2 e^{-\left(c_3+\frac{\sigma_3^2}{2}+\varepsilon\right)t} \le \Theta(t) \le \xi_2 e^{-\left(c_3+\frac{\sigma_3^2}{2}-\varepsilon\right)t} a.s \text{ for all } t > 0.$$

Hence, from (2.23) we get

$$\begin{aligned} &\xi_1 e^{-\left(c_3+\frac{\sigma_3^2}{2}+\varepsilon\right)t}\left[Q_z(0) + \frac{\eta_1}{\xi_2}\int_0^t e^{\left(\lambda+c_3+\frac{\sigma_3^2}{2}-2\varepsilon\right)\tau} d\tau\right] \\ &\le Q_z(t) \le \xi_2 e^{-\left(c_3+\frac{\sigma_3^2}{2}-\varepsilon\right)t}\left[Q_z(0) + \frac{\eta_2}{\xi_1}\int_0^t e^{\left(\lambda+c_3+\frac{\sigma_3^2}{2}+2\varepsilon\right)\tau} d\tau\right], \end{aligned}$$

which implies that

$$\begin{aligned} &\max\left\{\lambda, -\left(c_3 + \frac{\sigma_3^2}{2}\right)\right\} - 2\varepsilon \le \liminf_{t\to\infty} \frac{\ln Q_z(t)}{t} \\ &\le \limsup_{t\to\infty} \frac{\ln Q_z(t)}{t} \le \max\left\{\lambda, -\left(c_3 + \frac{\sigma_3^2}{2}\right)\right\} + 2\varepsilon \text{ a.s.} \end{aligned}$$

Thus,

$$\lim_{t\to\infty} \frac{\ln Q_z(t)}{t} = \max\left\{\lambda, -\left(c_3 + \frac{\sigma_3^2}{2}\right)\right\} \ \text{ a.s.} \tag{2.38}$$

On the other hand, consider the stopping time

$$\zeta_n = \inf\left\{t \ge n : S_z(t) + I_z(t) + Q_z(t) = \max_{n\le\tau\le n+1}[S_z(\tau) + I_z(\tau) + Q_z(\tau)]\right\}.$$

From (2.9) we have

$$\begin{aligned}&\mathbb{E}\left[\max_{n\le\tau\le n+1}\left((S_z(\tau)+I_z(\tau)+Q_z(\tau)\right)^{1+p}\right]=\mathbb{E}\left[V(S_z(\zeta_n),I_z(\zeta_n),Q_z(\zeta_n)\right]\\&\le\mathbb{E}V(S_z(n),I_z(n),Q_z(n))+\mathbb{E}\int_n^{\zeta_n}LV(S_z(t),I_z(t,Q_z(t))\,dt\\&\le\mathbb{E}V(S_z(n),I_z(n),Q_z(n))+K_1\mathbb{E}[\zeta_n-n]\le\mathbb{E}V(S_z(n),I_z(n),Q_z(n))+K_1.\end{aligned}$$

Thus,

$$\mathbb{E}\sup_{n\le t\le n+1}V(S_z(t),I_z(t),Q_z(t))\le\mathbb{E}V_z(S(n),I_z(n),Q_z(n))+K_1\le K+K_1,\tag{2.39}$$

where $K=\sup_{t\ge0}\mathbb{E}V(S_z(t),I_z(t),Q_z(t))<\infty$ by Lemma 2.2. For any $\varepsilon>0$ and $n\in\mathbb{N}$ put

$$A_n=\left\{\omega:\frac{\ln(\sup_{n\le t\le n+1}S_z(t))}{n}\ge\varepsilon\right\}.$$

By virtue of (2.39)

$$\sum_{n+1}^{\infty}\mathbb{P}(A_n)\le(K+K_1)\sum_{n+1}^{\infty}e^{-\varepsilon n}<\infty.$$

Therefore, Borell-Canteli Lemma says that

$$\limsup_{n\to\infty}\frac{\ln\left[\sup_{n\le t\le n+1}S_z(t)\right]}{n}\le0\text{ a.s.}$$

In particular,

$$\limsup_{t\to\infty}\frac{\ln S_z(t)}{t}\le0\text{ a.s..}\tag{2.40}$$

Hence,

$$\limsup_{t\to\infty}\frac{\ln S_z(t)+\ln I_z(t)}{t}\le\limsup_{t\to\infty}\frac{\ln I_z(t)}{t}\le\lambda<0.$$

As a result, for any $\varepsilon_1>0$, there exists a positive random variable ξ such that

$$S_z(t)I_z(t)\le\xi e^{(\lambda+\frac{\varepsilon_1}{4})t},\forall\,t\ge0.\tag{2.41}$$

In order to show that $S_z(t)$ converges to $\widetilde{S}_u^0(t)$, we consider

$$d(\widetilde{S}_u^0(t)-S_z(t))=[-\mu(\widetilde{S}_u^0(t)-S_z(t))+\beta S_z(t)I_z(t)-\gamma_1I_z(t)-\gamma_2Q_z(t)]dt+\sigma_1(\widetilde{S}_u^0(t)-S_z(t))dB_1(t).\tag{2.42}$$

Denote

$$\vartheta(t)=\exp\left\{\left(\mu+\frac{\sigma_1^2}{2}\right)t-\sigma_1B_1(t)\right\}.$$

Using constant-variation formula, it implies from (2.50) that

$$\widetilde{S}_u^0(t) - S_z(t) = \beta\vartheta^{-1}(t)\int_0^t \vartheta(s)[S_z(\tau)I_z(\tau) - \gamma_1 I_z(\tau) - \gamma_2 Q_z(\tau)]d\tau.$$

It implies that

$$-\beta\vartheta^{-1}(t)\int_0^t \vartheta(s)[\gamma_1 I_z(\tau) + \gamma_2 Q_z(\tau)]d\tau \le \widetilde{S}_u^0(t) - S_z(t) \le \beta\vartheta^{-1}(t)\int_0^t \vartheta(s)S_z(\tau)I_z(\tau)]d\tau. \tag{2.43}$$

Let $\overline{\lambda} = \max\{-(c_3 + \frac{\sigma_3^2}{2}); \lambda\}$ and let $\widehat{\lambda} > \max\{-(\mu + \frac{\sigma_1^2}{2}); \overline{\lambda}\}$ be arbitrary. We choose $\varepsilon_1 > 0$ such that $\widehat{\lambda} - \varepsilon_1 > \max\{-(\mu + \frac{\sigma_1^2}{2}); \overline{\lambda}\}$. Since $\lim_{t\to\infty} \frac{\ln\vartheta(t)}{t} = \mu + \frac{\sigma_1^2}{2}$, there are two positive random variables ξ_1, ξ_2 satisfying

$$\xi_1 e^{(\mu + \frac{\sigma_1^2}{2} - \frac{\varepsilon_1}{4})t} \le \vartheta(t) \le \xi_2 e^{\left(\mu + \frac{\sigma_1^2}{2} + \frac{\varepsilon_1}{4}\right)t}. \tag{2.44}$$

As a result of (2.37) and (2.38), there exists a random variables ξ_3 such that

$$I_z(t) \vee Q_z(t) \le \xi_3 e^{(\overline{\lambda} + \frac{\varepsilon_1}{4})t} \tag{2.45}$$

From (2.41), (2.43), (2.44) and (2.45), we get

$$\begin{cases} e^{-\widehat{\lambda}t}(\widetilde{S}_u^0(t) - S_z(t)) \ge -\Gamma_1 e^{(-\widehat{\lambda} - (\mu + \frac{\sigma_1^2}{2}) + \frac{\varepsilon_1}{4})t}\int_0^t e^{\left(\overline{\lambda} + \mu + \frac{\sigma_1^2}{2} + \frac{\varepsilon_1}{2}\right)s}ds, \\ e^{-\widehat{\lambda}t}(\widetilde{S}_u^0(t) - S_z(t)) \le \Gamma e^{(-\widehat{\lambda} - (\mu + \frac{\sigma_1^2}{2}) + \frac{\varepsilon_1}{4})t}\int_0^t e^{\left(\lambda + \mu + \frac{\sigma_1^2}{2} + \frac{\varepsilon_1}{2}\right)s}ds, \end{cases}$$

where $\Gamma_1 = \frac{\beta(\gamma_1+\gamma_2)\xi_2\xi_3}{\xi_1}$ and $\Gamma_2 = \frac{\beta\xi\xi_2}{\xi_1}$. Using L'Hospital rule obtains

$$\begin{cases} \limsup_{t\to\infty} e^{-\widehat{\lambda}t}(\widetilde{S}_u^0(t) - S_z(t)) \ge -\lim_{t\to\infty} \Gamma_1 e^{(-\widehat{\lambda} - (\mu + \frac{\sigma_1^2}{2}) + \frac{\varepsilon_1}{4})t}\int_0^t e^{\left(\overline{\lambda} + \mu + \frac{\sigma_1^2}{2} + \frac{\varepsilon_1}{2}\right)s}ds = 0 \\ \limsup_{t\to\infty} e^{-\widehat{\lambda}t}(\widetilde{S}_u^0(t) - S_z(t)) \le \lim_{t\to\infty} \Gamma_2 e^{(-\widehat{\lambda} - (\mu + \frac{\sigma_1^2}{2}) + \frac{\varepsilon_1}{4})t}\int_0^t e^{\left(\lambda + \mu + \frac{\sigma_1^2}{2} + \frac{\varepsilon_1}{2}\right)s}ds = 0 \end{cases}$$

Thus,

$$\lim_{t\to\infty} e^{-\widehat{\lambda}t}(\widetilde{S}_u^0(t) - S_z(t)) = 0 \;\; a.s.$$

The proof is complete. □

2.2 Case 2: $\lambda > 0$ (or $\widehat{R} > 1$)

Theorem 2.4. *Let $(S_z(t), I_z(t), Q_z(t))$ be the solution to the equation (1.1) with initial value $z = (u, v, w) \in \mathbb{R}_+^3, v > 0$. If $\lambda > 0$ (or $\widehat{R} > 1$), the model is strongly stochastically permanent in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\liminf_{t\to\infty} \mathbb{P}\{\min\{S_z(t), I_z(t), Q_z(t)\} \ge \delta\} > 1 - \varepsilon, \tag{2.46}$$

To proof Theorem 2.4 we need following lemmas.

Lemma 2.5. *For any* $0 < \hbar < H < \infty, T > 1$ *and* $\varepsilon > 0$, *there exists* $\widehat{\delta} = \widehat{\delta}(H, \hbar, T, \varepsilon)$ *such that*

$$\mathbb{P}\{\min\{S_z(t), I_z(t), Q_z(t)\} \geq \widehat{\delta}, \forall t \in [T, 2T]\} > 1 - \varepsilon \tag{2.47}$$

holds for all initial value $z = (u, v, w) \in [0, H] \times [\hbar, H] \times [0, H]$.

Proof. Recall from the proof of Lemma 2.3 that there exist an $\widetilde{M}$ such that $\mathbb{P}_z(\widetilde{\Omega}) > 1 - \frac{\epsilon}{2}$ where

$$\widetilde{\Omega} = \Big\{\omega : \sup_{t \in [0,2T]} |\sigma_k B_k(t)| \vee \int_0^t S_z(\tau)d\tau < \widetilde{M}\Big\}. \tag{2.48}$$

Similar to (2.22), we have

$$I_z(0)e^{-\widetilde{M}-\left(2c_2+\sigma_2^2\right)T} \leq I_z(t) \leq I_z(0)e^{2\widetilde{M}} \text{ for all } t \in [0, 2T]. \tag{2.49}$$

Therefore, on $\widetilde{\Omega}$ we have

$$I_z(t) \geq I_z(0)e^{-\widetilde{M}-(2c_2+\sigma_2^2)T} := \delta_1 \text{ for all } t \in [0, 2T]. \tag{2.50}$$

It implies from (2.24) and (2.48), that

$$e^{-(2c_3+\sigma_3)T-\widetilde{M}} \leq \Theta(t) \leq e^{\widetilde{M}}, \text{ for almost surely } \omega \in \widetilde{\Omega} \text{ and } t \in [0, 2T]. \tag{2.51}$$

Combining (2.23), (2.51), we obtain

$$\begin{aligned} Q_z(t) &\geq e^{-(2c_3+\sigma_3)T-\widetilde{M}} \left[Q_z(0) + \int_0^t \gamma_3\delta_1 e^{-\widetilde{M}} d\tau\right] \\ &\geq \gamma_3\delta_1 T e^{-(2c_3+\sigma_3)T-2\widetilde{M}} =: \delta_2 \text{ for all } t \in [T, 2T]. \end{aligned} \tag{2.52}$$

On the other hand,

$$S_z(t) = \Psi(t)\left(S_z(0) + \int_0^t \Psi^{-1}(\tau)(\alpha + \gamma_1 I_z(\tau) + \gamma_2 Q_z(\tau))d\tau\right), \tag{2.53}$$

where

$$\Psi(t) = \exp\left\{-\beta\int_0^t I_z(\tau)d\tau - \left(\mu + \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right\}.$$

In view of (2.49), for almost sure $\omega \in \widetilde{\Omega}$, we have

$$\exp\left\{-2\beta H T e^{2\widetilde{M}} - (2\mu + \sigma_1^2)T - \widetilde{M}\right\} \leq \Psi(t) \leq \exp\left\{\widetilde{M}\right\}$$

Therefore,

$$S_z(t) \geq \Psi(t) \int_0^t \alpha \Psi^{-1}(\tau) d\tau \geq T \exp\left\{-2\beta H T e^{2\widetilde{M}} - (2\mu + \sigma_1^2)T - 2\widetilde{M}\right\} := \delta_3, \ \forall t \in [T; 2T].$$

By putting $\widehat{\delta} = \min\{\delta_1, \delta_2, \delta_3\}$, we see that $\min\{S_z(t), I_z(t), Q_z(t)\} > \widehat{\delta}$ for all $t \in [T, 2T]$ on $\widetilde{\Omega}$. The proof is completed. □

Lemma 2.6. *Let $(S_z(t), I_z(t), Q_z(t))$ be the solution to equation (1.1) with initial value $z = (u, v, w) \in \mathbb{R}^3_+$. If $\lambda > 0$ (or $\widehat{R} > 1$), there exist $\theta > 0$ and $H = H(\theta)$ such that*

$$\limsup_{t\to\infty} \mathbb{E} I_z^{-\theta}(t) \leq H \ \text{ for any } (u, v, w) \in \mathbb{R}^3_+, v > 0. \tag{2.54}$$

Proof. Since $I_z(0) = 0$ then $I_z(t) = 0$ and $S_z(t) \equiv \widetilde{S}^0(t))$ for all $t \geq 0$. Therefore from (2.3), (2.4) and Feller properties, there exist $\delta_4 > 0$ such that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \beta \mathbb{E}[S_z(\tau)] d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right) \geq \frac{3\lambda}{4} \text{ uniformly in } z = (u, v, w) \in \mathbb{R}^3_+, v \leq \delta_4.$$

Thus, we can find a constant T sufficient large satisfying

$$\int_0^T \beta \mathbb{E}[S_z(\tau)] d\tau - \left(c_2 + \frac{\sigma_2^2}{2}\right) T \geq \frac{\lambda T}{2} \text{ uniformly in } z = (u, v, w) \in \mathbb{R}^3_+, v \leq \delta_4 \tag{2.55}$$

By using Itô formula and (2.55), for $z = (u, v, w) \in \mathbb{R}^3_+, v \leq \delta_4$, we have

$$\mathbb{E} \ln I_z(T) \geq \ln v + \frac{\lambda T}{2} \tag{2.56}$$

Consider the Lyapunov function $V_\theta(u, v, w) = v^{-\theta}$, where θ is a positive constant. We have

$$\begin{aligned} \mathcal{L}V_\theta(u, v, w) &= -\theta v^{-\theta} \left[\beta u - \left(\mu + \rho_1 + \gamma_1 + \gamma_3 - \frac{\theta + 1}{2} \sigma_2^2\right)\right] \\ &\leq H_\theta V_\theta(u, v, w), \end{aligned}$$

where $H_\theta = \theta\left(\mu + \rho_1 + \gamma_1 + \gamma_3\right)$. Thus, by using Itô's formula and taking expectation both sides, we obtain

$$\mathbb{E} I_z^{-\theta}(t) \leq v^{-\theta} \exp(H_\theta t) \text{ for any } t \geq 0, z = (u, v, w) \in \mathbb{R}^3_+, v > 0. \tag{2.57}$$

From Lemma 2.2 and (2.57), we obtain

$$\mathbb{E}\big(I_z(t) + I_z^{-1}(t)\big) \leq \overline{H} < \infty \quad \text{for all } 0 \leq t \leq T.$$

Applying [7, Lemma 3.5, pp. 1912] yields that the that the log-Laplace transform $\ln \mathbb{E}_{\phi,i} I_z^{-\theta}(T)$ is twice differentiable on $[0, \frac{1}{2})$ and there is a constant H_2 such that

$$\ln \mathbb{E} I_z^{-\theta}(T) \leq -\mathbb{E} \ln I_z(T)\theta + H_2\theta^2. \tag{2.58}$$

By combining (2.56) with (2.58), we arrive at

$$\mathbb{E} I_z^{-\theta}(T) \leq v^{-\theta} \exp(-\frac{\lambda\theta T}{4}) \quad \text{for} \quad (u,v,w) \in \mathbb{R}_+^3, v < \delta_4.$$

with a sufficiently small θ. This inequality and (2.57) imply that

$$\mathbb{E} I_z^{-\theta}(T) \leq qv^{-\theta} + \kappa \quad \text{for all} \quad (u,v,w) \in \mathbb{R}_+^3, v > 0,$$

where $\kappa = \delta_4^{-\theta} \exp\left(\frac{\lambda\theta T}{4}\right)$ and $q = \exp\left(-\frac{\lambda\theta T}{4}\right)$.
By using standard arguments as proof of [20, Theorem 2.2], we obtain (2.54). □

Proof of Theorem 2.4. In view of (2.54), for any $\varepsilon > 0$ there exists $\delta_5 > 0$ and a sufficiently large $k \in \mathbb{N}$ such that

$$\mathbb{P}\{I_z(kT) \geq \delta_5\} \geq 1 - \varepsilon. \tag{2.59}$$

By combining (2.59), Lemma 2.5, and the Markov property of the solution, we have

$$\mathbb{P}\{\min\{S_z(t), I_z(t), Q_z(t)\} \geq \widehat{\delta}, \forall t \in [(k+1)T, (k+2)T]\} > 1-\varepsilon \text{ for any } (u,v,w) \in \mathbb{R}_+^3, v > 0.$$

Letting $k \to \infty$ in this estimate yields (2.54). The proof is completed. □

From (2.46) implies that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\big[\mathbf{1}_{\{S_z(\tau)\geq\delta, I_z(\tau)\geq\delta, Q_z(\tau)\geq\delta\}}\big] d\tau \geq 1 - \frac{\varepsilon}{4} \tag{2.60}$$

Let $H > \frac{4M_p}{\varepsilon}$. We also have from (2.7) that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\mathbf{1}_{\{\max\{S_z(\tau),I_z(\tau),Q_z(\tau)\}\geq H\}} d\tau &\leq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\mathbf{1}_{\{S_z(\tau)+I_z(\tau)+Q_z(\tau)\geq H\}} d\tau \\ &\leq \frac{1}{H^{1+p}} \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}(S_z(\tau) + I_z(\tau) + Q_z(\tau))^{1+p} d\tau \leq \frac{M_p}{H^{1+p}} \leq \frac{\varepsilon}{4}.\end{aligned} \tag{2.61}$$

It follows from (2.60) and (2.61) that for all $\varepsilon \in (0,1)$ we can choose H sufficiently large and δ sufficiently small such that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\mathbf{1}_{\{(S_z(\tau),I_z(\tau),Q_z(t))\in D\}} d\tau \geq 1 - \varepsilon > 0, \tag{2.62}$$

where $D = \{(u,v,w) : \delta \le u,v,w \le H\}$. By virtue of the invariance of $\mathbb{R}^{3,\circ}_+ = \{(x,y,z) : x,y,z > 0\}$ under equation (1.1), we can consider the Markov process $(S_z(t), I_z(t), Q_z(t))$ on the state space $\mathbb{R}^{3,\circ}_+$. It is easy to show that $(S_z(t), I_z(t), Q_z(t))$ has the Feller property. Thus, in view of inequality (2.62) and the compactness of D in $\mathbb{R}^{3,\circ}_+$, we implies that there is an invariant probability measure π^* on $\mathcal{M}$ (see [19] or [15]). By the independence of $B_1(t), B_2(t), B_3(t)$, it implies that $\mathbb{R}^{3,\circ}_+$ is the support of π^*. Hence, the invariant probability is unique and the strong law of large numbers holds; see [13, Theorems 3.1, 3.3]. We have the following result.

Theorem 2.7. *If $\lambda > 0$, soluion $(S_z(t), I_z(t), Q_z(t))$ of the equation (1.1) has a unique invariant probability measure π^* with support $\mathbb{R}^{3,\circ}_+$. Moreover,*

(a) *For any π^*-integrable $f(x,y,z) : \mathbb{R}^{2,\circ}_+ \to \mathbb{R}$, we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(S_z(\tau), I_z(\tau), Q_z(\tau))d\tau = \int f(x,y,z)\pi^*(dx, dy.dz)a.s.$$

(b) *For all initial value $(u,v,w) \in \mathbb{R}^{3,\circ}_+$,*

$$\lim_{t\to\infty} \|P(t,(u,v,w),\cdot) - \pi^*(\cdot)\| = 0,$$

where $P(t,(u,v,w),\cdot)$ is the transition probability of $(S_z(t), I_z(t), Q_z(t))$ and $\|\cdot\|$ is the total variation norm.

3 Discussion and Numerical Examples

We have shown that the extinction and permanence of the disease in a stochastic SIQS model with isolation can be determined by the sign of a threshold value λ. Only the critical case $\lambda = 0$ is not studied in this paper. To illustrate the significance of our results, let us compare our results with those in [23]. Firstly, our results can be proved without the appeal to the condition $\mu > \frac{\sigma^2}{2}$. Moreover, in case $\widehat{R} < 1$, instead of convergence in mean, we prove that the system (1.1) approximates to disease free case and $S_z(t)$ converges almost surely to $\tilde{S}^0(t)$ at an exponential rate. In case $\widehat{R} > 1$ we prove that the disease is permanent and a stationary distribution exists. Of course, this result follows the persistence in mean as in [23, Theorem 4, p. 370]. Let us finish this paper by providing some numerical examples.

Example 3.1. Consider (1.1) with parameters $\alpha = 4$, $\beta = 2$, $\mu = 2$, $\rho_1 = 1, \rho_2 = 0.25$, $\gamma_1 = 0.5, \gamma_2 = 0.25, \gamma_3 = 1$, $\sigma_1 = 1$, and $\sigma_2 = 2.5$ and $\sigma_3 = 2$. With these paramaters condition $\mu > \frac{\max\{\sigma_i^2, i=\overline{1,3}\}}{2}$ is not satisfied. Direct calculation shows that $\lambda = -3.625 < 0$. By virtue of Theorem 2.1, $\lim\limits_{t\to\infty} \dfrac{\ln I_z(t)}{t} = -3.625$, $\lim\limits_{t\to\infty} \dfrac{\ln Q_z(t)}{t} = -3$ and $\lim\limits_{t\to\infty} \dfrac{\ln |S_z(t) - \widetilde{S}^0(t)|}{t} \le -2.5$. This claim is supported by Figures 1. That is, the population will eventually have no disease and $S_z(t)$ convergence to $\widetilde{S}^0(t)$ at exponental rate.

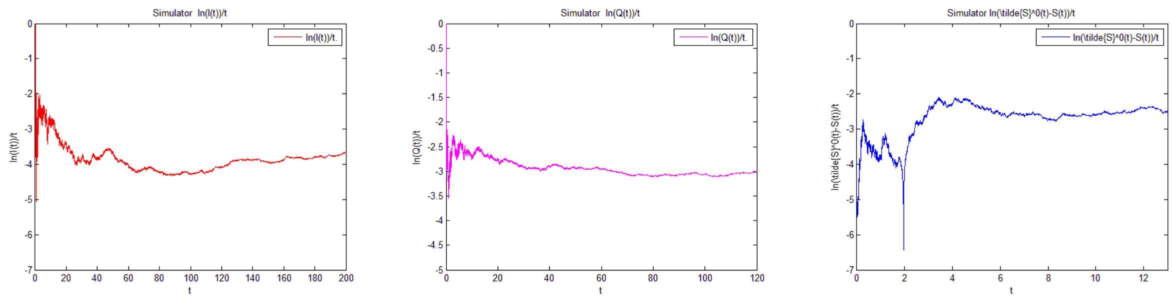

Figure 1: Estimated paths of $\dfrac{\ln I_z(t)}{t}$ (in red line), $\dfrac{\ln Q_z(t)}{t}$ (in ping line) and $\dfrac{\ln |S_z(t) - \widetilde{S}_u^0(t)|}{t}$ (in blue line) in Example 3.1.

Example 3.2. Consider (1.1) with parameters $\alpha = 5$, $\beta = 5$, $\mu = 1$, $\rho_1 = 0.5, \rho_2 = 0.25$, $\gamma_1 = 0.5$, $\gamma_2 = 0.25$, $\gamma_3 = 1.5$, $\sigma_1 = 1$, and $\sigma_2 = 2.5$, $\sigma_3 = 1$, . For these parameters, the conditions $\mu > \frac{\max\{\sigma_i^2, i=\overline{1,3}\}}{2}$ is also not satisfied. We obtain $\lambda = 18.375 > 0$, As a result of Theorem 2.4, model is strongly stochastically permanent. A sample path of $(S_z(t), I_z(t), Q_z(t))$ is described in Figures 2

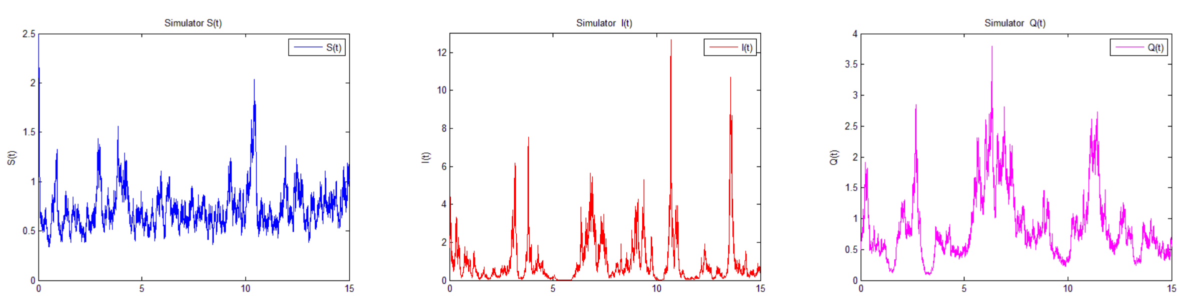

Figure 2: Trajectories of $(S_z(t), I_z(t), Q_z(t))$ in Example 3.2.

References

[1] V. Capasso, *Mathematical Structures of Epidemic Systems*, Springer-Verlag, Berlin, 1993.

[2] V. Capasso and G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model, Math. Biosci., 42 (1978), pp. 41-61.

[3] Y. Chen, ; B. Wen, Z. Teng, The global dynamics for a stochastic SIQS epidemic model with isolation. Phys. A 492 (2018), 1604-1624.

[4] N. H. Du, N. T. Dieu, Long-time behavior of an SIR model with perturbed disease transmission coefficient. *Discrete Contin. Dyn. Syst. Ser. B* **21** (2016), no. 10, 3429-3440.

[5] N. T. Dieu, D. H. Nguyen, N. H. Du, G. Yin, Classification of asymptotic behavior in a stochastic SIR model. SIAM J. Appl. Dyn. Syst. **15** (2016), no. 2, 1062-1084.

[6] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIQS epidemic model, *SIAM J. Appl. Math.,* 71 (2011), no. 3, 876-902.

[7] A. Hening, Nguyen, Dang H. Coexistence and extinction for stochastic Kolmogorov systems. *Ann. Appl. Probab.* 28 (2018), no. 3, 1893-1942.

[8] H. W. Hethcote, D. W. Tudor, Integral equation models for endemic infectious diseases, *J. Math. Biol.,* 9 (1980), no. 1, 37-47.

[9] M. Iannelli, F. A. Milner, A. Pugliese, Analytical and numerical results for the age-structured S-I-S epidemic model with mixed inter-intracohort transmission, *SIAM J. Math. Anal.,* 23 (1992), no. 3, 662688.

[10] D.Q. Jiang, J.J. Yu, C.Y. Ji, N.Z. Shi, Asymptotic behavior of global positive solution to a stochastic SIR model, *Math. Comput. Modell.,* 54(2011), 221-232.

[11] W.O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics, (part I), *Proc. R. Soc. Lond. Ser. A,* 115 (1927), 700-721.

[12] W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics, (part II), *Proc. Roy. Sot. Ser. A*, 138 (1932), 55-83.

[13] R.Z. Khas'minskii, Ergodic properties of recurrent diffusion processes and stabilization of the Cauchy problem for parabolic equations. *Theory Probab. Appl.* **5** (1960), 179-196.

[14] W. Kliemann, Recurrence and invariant measures for degenerate diffusions, *Ann. Probab.,* 15 (1987), no. 2, 690-707.

[15] S. P. Meyn, R. L. Tweedie, *Markov Chains and Stochastic Stability,* Springer, London, 1993.

[16] D.H. Nguyen, G. Yin and C. Zhu, Long-term analysis of a stochastic SIRS model with general incidence, submitted.

[17] M. Nuno, Z. Feng, M. Martcheva, C. Castillo-Chavez, Dynamics of two-strain influenza with isolation and partial cross-immunity, SIAM J. Appl. Math. 65 (2005) 964-982.

[18] A. V. Skorohod, Asymptotic methods in the theory of stochastic differential equations, Providence, RI: Amer. Math. Soc., 1989.

[19] L. Stettner, On the existence and uniqueness of invariant measure for continuous time Markov processes, *LCDS Report* No. 86-16, April 1986, Brown University, Providence.

[20] T. D. Tuong, D.H. Nguyen, N.T. Dieu, T. Ky, Extinction and permanence in a stochastic SIRS model in regime-switching with general incidence rate. *Nonlinear Anal. Hybrid Syst.* 34 (2019), 121-130.

[21] F. J. S. Wang, Asymptotic behavior of some deterministic epidemic models. *SIAM J. Math. Anal.* 9 (1978), 529-534.

[22] X. Zhang, H. Huo, H. Xiang, X. Meng, Dynamics of the deterministic and stochastic SIQS epidemic model with non-linear incidence, Appl. Math. Comput. 243 (2014) 546-558.

[23] X. Zhang, H. Huo, H. Xiang, Q. Shi, D. Li, The threshold of a stochastic SIQS epidemic model. Phys. A 482 (2017), 362-374.