OPTIMAL CONTROL OF A THREE-DIMENSIONAL MAGNETOHYDRODYNAMIC-α MODEL

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Abstract. In this paper we study an optimal control problem for the three-dimensional magnetohydrodynamic-α model (MHD-α) in bounded domains with distributed controls. We first prove the existence of optimal solutions, and then we establish the first-order necessary as well as second-order sufficient optimality conditions.

1. Introduction

Magnetohydrodynamics (MHD) is the branch of continuum mechanics that studies the macroscopic interaction of electrically conducting fluids and electromagnetic fields. The subject is of great interest for its numerous practical applications which includes motion of liquid metals, fusion technology, design of novel submarine propulsion devices and plasma physics. The motion of Newtonian fluids is governed by the Navier-Stokes equations and electromagnetic effects are governed by Maxwell's equations. Under a number of physical assumptions valid for the problems of interest, these two general systems can be reduced to the MHD system, see e.g. [9, 19, 23].

Because of the success of Navier-Stokes-α model in producing solutions in excellent agreement with empirical data for a wide range of large Reynolds numbers and flow in infinite channels or pipes, it is natural to consider such a kind of regularization for magnetohydrodynamic models as well. In [20], Linshiz and Titi have suggested several MHD-α models. For instance, filtering the velocity field but not the magnetic field, we get the following MHD-α model

$$\begin{cases}\partial_t v-\nu\Delta v+\nabla\left(p+\dfrac{|B|^2}{2}\right)=u\times(\nabla\times v)+(B\cdot\nabla)B+h_1, & \text{in } \Omega\times(0,T),\\ \partial_t B-\eta\Delta B+(u\cdot\nabla)B-(B\cdot\nabla)u=h_2, & \text{in } \Omega\times(0,T),\\ v=u-\alpha^2\Delta u, & \text{in } \Omega\times(0,T),\\ \nabla\cdot u=\nabla\cdot v=\nabla\cdot B=0, & \text{in } \Omega\times(0,T),\\ u=\Delta u=0,\quad B\cdot n=0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad B(x,0)=B_0(x), & \text{in } \Omega.\end{cases} \tag{1.1}$$

Here $u=u(x,t)$ is the velocity of the particle of fluid which is at point x at time t, $B=B(x,t)$ is the magnetic field at point x at time t, $p=p(x,t)$ stands for the pressure of the fluid, $\nu>0$ is the kinematic viscosity coefficient, $\eta>0$ is

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the constant magnetic diffusivity, n is the outer normal to $\partial\Omega$ and α is a length scale parameter. When $\alpha = 0$ we formally recover the 3D classical MHD equations in [23].

In recent years, the existence and long-time behavior of solutions to this MHD-α model have attracted the attention of many mathematicians. In [20], Linshiz and Titi have shown a global existence result in a three-dimensional periodic box when $\nu > 0$ and $\eta > 0$, while Fan and Ozawa [10] and Liu [21] have achieved the same result in the whole space $\mathbb{R}^2$ for both cases $(\nu = 1, \eta = 0)$ and $(\nu = 0, \eta = 1)$. More recently, in [32], Zhou and Fan also established the regularity criteria to guarantee the existence of smooth solutions for higher dimensional case. For the long-time behavior of solutions, the existence and regularity of a finite-dimensional global attractor were proved by Catania in [6] and Anh et. al. in [3] in the case of three-dimensional periodic box, and the time decay rate in $L^2(\mathbb{R}^3)$ of solutions was proved by Jiang and Fan in [17]. When $B = 0$, the above MHD-α model reduces to the well-known Navier-Stokes-α equations, for which many results on the existence of solutions and global attractor were achieved, see e.g. [8, 11, 16, 22] and references therein, and decay rates of solutions on the whole space were investigated in [5]. We also refer the interested reader to [7, 18, 31] for results related to other MHD-α models.

The main goal of this paper is to prove the existence of optimal solutions and establish the first-order necessary as well as the second-order sufficient optimality conditions for an optimal control problem for the 3D MHD-α model in bounded domains with Dirichlet boundary conditions, the situation has a more physical meaning than the case of periodic boundary conditions. Optimal control of fluids to alter flows to achieve a desired effect remains an active research area due to its importance for the design and performance of fluid dynamical systems. The past decade has seen significant developments in theoretical and computational analysis in this area, see e.g. [1, 12, 25]. Especially, optimal control problems for the Navier-Stokes equations and 2D MHD equations have been studied extensively during the past years, see e.g. [13, 14, 15, 28, 29, 30] and references therein. However, to the best of our knowledge, optimal control of 3D MHD-α models has not been studied before. This is the main motivation of the present paper.

The mathematical description of the optimal control problem we study is as follows. Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^2, and we denote the space-time cylinder by $Q = \Omega \times (0, T)$. Let $h = (h_1, h_2)$ denote the control belongs to an admissible set U_{ad}, which is an arbitrary non-empty closed convex subset in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$. For given $T > 0$, the cost functional is defined by

$$
\begin{aligned}
J(z,h) = \frac{\alpha_T}{2}\int_\Omega |z(x,T) - z_T(x)|^2 dx + \frac{\alpha_Q}{2}\iint_Q |z(x,t) - z_Q(x,t)|^2 dxdt \\
+ \frac{\gamma}{2}\iint_Q |h(x,t)|^2 dxdt,
\end{aligned}
\tag{1.2}
$$

where $z = (u, B)$, z_T and z_Q denote some desired statements of pair of velocity and magnetic fields, the coefficients α_T, α_Q are non-negative real numbers, where at least one is positive to get a non-trivial objective functional, and the regularization parameter γ measuring the cost of the control is a positive number. We wish to minimize the functional (1.2) subject to the 3D MHD-α model (1.1).

Our first goal in this paper is to show the existence of optimal solutions of problem (1.1) and establish the first-order necessary optimality conditions. Our second goal is to derive the second-order sufficient optimality conditions. It is worthy noticing that our approach allows that the set of admissible controls is an arbitrary non-empty closed convex subset in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$, not necessary a box constraint as in [28, 29], and it seems to be more natural and does not require the use of Lagrange functional. Such an approach has also been used recently in [2] for an optimal control problem with control constraints of the 3D Navier-Stokes-Voigt equations. Although the approach and techniques we use in the present paper are similar to those in [2] (and in fact they are standard techniques in optimal control of PDEs), due to the complexity of nonlinear terms in the MHD-α model, our arguments here are more involved.

The rest of the paper is structured as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the MHD-α model and the existence and uniqueness of solutions to problem (1.1). Section 3 proves the existence of optimal solutions. The first-order necessary optimality condition is given in Section 4. In the last section we derive a second-order sufficient optimality condition.

2. Preliminaries

2.1. **Function spaces and inequalities for the nonlinear terms.** We denote

$$\mathbb{L}^2(\Omega) = \left(L^2(\Omega)\right)^3, \quad \mathbb{H}^m(\Omega) = \left(H^m(\Omega)\right)^3, \quad \mathbb{H}_0^m(\Omega) = \left(H_0^m(\Omega)\right)^3.$$

The spaces used in the theory of the MHD-α model are a combination of spaces used for the Navier-Stokes equations and spaces used in the theory of Maxwell equations. They are

$$\begin{aligned}
\mathcal{V}_1 &= \{v \in \left(C_0^\infty(\Omega)\right)^3 : \nabla\cdot v = 0\},\\
V_1 &= \text{ closure of } \mathcal{V}_1 \text{ in the } \mathbb{H}_0^1(\Omega) \text{ norm},\\
H_1 &= \text{ closure of } \mathcal{V}_1 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm},\\
\mathcal{V}_2 &= \{\theta \in \left(C^\infty(\bar\Omega)\right)^3 : \nabla\cdot\theta = 0;\ \theta\cdot n|_{\partial\Omega} = 0\},\\
V_2 &= \text{ closure of } \mathcal{V}_2 \text{ in the } \mathbb{H}^1(\Omega) \text{ norm},\\
H_2 &= \text{ closure of } \mathcal{V}_2 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm} = H_1.
\end{aligned}$$

So that $V_k' := \mathbb{H}^{-1}(\Omega)$ is the topological dual of $V_k\,(k = 1, 2)$. The spaces $H_k\,(k = 1, 2)$ are endowed with the inner product and the norm of $\mathbb{L}^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively.

The inner product and norm in V_1 are given by

$$\begin{aligned}
((u,\tilde u))_1 &= \sum_{i=1}^3 \int_\Omega \nabla u_i\cdot\nabla\tilde u_i dx, \quad \forall u,\tilde u \in V_1,\\
\|u\|_1 &= ((u,u))_1^{1/2}, \qquad \forall u \in V_1.
\end{aligned}$$

Due to the Poincaré inequality, this norm is equivalent to the usual one in $\mathbb{H}_0^1(\Omega)$.

The inner product and norm in V_2 are given by

$$\begin{aligned}
((B,\tilde B))_2 &= \sum_{i=1}^3 \int_\Omega \nabla B_i\cdot\nabla\tilde B_i dx, \qquad \forall B,\tilde B \in V_2,\\
\|B\|_2 &= ((B,B))_2^{1/2}, \qquad \forall B \in V_2.
\end{aligned}$$

By using the Poincaré-Wirtinger inequality, the above bilinear form defines a norm which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on V_2.

We denote by A_1 the Stokes operator, with domain $D(A_1) = \mathbb{H}^2(\Omega) \cap V_1$, defined by $A_1 u = -\mathcal{P}(\Delta u), \forall u \in D(A_1)$, where $\mathcal{P}$ is the Leray projection, i.e. the projection operator from $\mathbb{L}^2(\Omega)$ onto H_1. Furthermore, A_1^{-1} is a compact linear operator on H_1 and $|A_1 \cdot|$ is a norm on $D(A_1)$ that is equivalent to $\mathbb{H}^2$-norm.

Then we introduce the linear nonnegative unbounded operator on H_2

$$A_2 B = -\Delta B, \qquad \forall B \in D(A_2) = \mathbb{H}^2(\Omega).$$

Observe that A_2^{-1} is a compact linear operator on H_2 and we endow $D(A_2)$ with the norm $|A_2 \cdot|$ which is equivalent to the $\mathbb{H}^2$-norm.

By the classical spectrum theorem, there exist sequences $\{\lambda_j^{(i)}\}_{j=1}^\infty, i = 1, 2,$

$$0 < \lambda_1^{(i)} \leq \lambda_2^{(i)} \leq \cdots \leq \lambda_j^{(i)} \leq \cdots, \lambda_j^{(i)} \to +\infty, \text{as } j \to \infty,$$

and family of elements $\{e_j^{(1)}\}_{j=1}^\infty$ of V_1 which are orthogonal in H_1 and $\{e_j^{(2)}\}_{j=1}^\infty$ of V_2 which are orthogonal in H_2 such that

$$A_i e_j^{(i)} = \lambda_j^{(i)} e_j^{(i)}, \quad \forall j \in \mathbb{N}, \quad i = 1, 2.$$

Similarly, when $A_1 u = 0$ on $\partial\Omega$, the operator A_1^2 can be defined on $D(A_1)$ with values in $D(A_1)'$, the dual space of the Hilbert space $D(A_1)$, such that

$$\langle A_1^2 u, \widetilde{u} \rangle_{D(A_1)'} = (A_1 u, A_1 \widetilde{u}), \quad \text{for every } u, \widetilde{u} \in D(A_1).$$

We consider the trilinear form b given by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall w \in V_k, k = 1, 2, \tag{2.1}$$

whenever the integrals make sense. Then we define a continuous bilinear operator $\mathcal{B} : V_k \times V_k \to V_k', k = 1, 2,$ with

$$\langle \mathcal{B}(u, v), w \rangle = b(u, v, w), \forall u, v, w \in V_k.$$

It is easy to check that if $u, v, w \in V_k, k = 1, 2,$ then

$$b(u, v, w) = -b(u, w, v). \tag{2.2}$$

Hence

$$b(u, v, v) = 0. \tag{2.3}$$

Using Hölder's inequality, Sobolev's inequalities in $\mathbb{R}^3$,

$$\begin{aligned} &\|u\|_{\mathbb{L}^4(\Omega)} \leq c\|u\|_{\mathbb{L}^2(\Omega)}^{1/4} \|u\|_{\mathbb{H}^1(\Omega)}^{3/4}, \\ &\|u\|_{\mathbb{L}^3(\Omega)} \leq c\|u\|_{\mathbb{L}^2(\Omega)}^{1/2} \|u\|_{\mathbb{H}^1(\Omega)}^{1/2}, \qquad \text{and} \\ &\|u\|_{\mathbb{L}^6(\Omega)} \leq c\|u\|_{\mathbb{H}^1(\Omega)}, \qquad \text{for every } u \in \mathbb{H}^1(\Omega), \end{aligned}$$

as in [26, 27], one can prove the following lemma.

Lemma 2.1. *For $k=1,2$, we have*

$$|b(u,v,w)| \le c \begin{cases} \|u\|_k \|v\|_k \|w\|_k, & \forall u,v,w \in V_k, \\ \|u\|_k |v| |A_k w|, & \forall u \in V_k, v \in H_k, w \in D(A_k), \\ \|u\|_k^{1/2} |A_k u|^{1/2} |v| \|w\|_k, & \forall u \in D(A_k), v \in H_k, w \in V_k, \\ \|u\|_k^{1/2} |A_k u|^{1/2} \|v\|_k |w|, & \forall u \in D(A_k), v \in V_k, w \in H_k, \\ \|u\|_k \|v\|_k^{1/2} |A_k v|^{1/2} |w|, & \forall u \in V_k, v \in D(A_k), w \in H_k, \\ |u| \|v\|_k |A_k w|, & \forall u \in H_k, v \in V_k, w \in D(A_k). \end{cases} \tag{2.4}$$

Now, if $u \in D(A_k)$, then $\nabla u^T \in \big(H^1(\Omega)\big)^{3\times 3} \hookrightarrow \big(L^6(\Omega)\big)^{3\times 3}$, and consequently, for $v \in \mathbb{L}^2(\Omega)$, we have that $v \cdot \nabla u^T = \sum_{j=1}^3 v_j \nabla u_j \in \mathbb{L}^{3/2}(\Omega) \hookrightarrow \mathbb{H}^{-1}(\Omega)$, with

$$\big\langle v \cdot \nabla u^T, w \big\rangle = \sum_{i,j=1}^3 \int_\Omega w_i \frac{\partial u_j}{\partial x_i} v_j, \quad \forall w \in V_k, k=1,2.$$

We now consider the trilinear form defined by

$$\overline{b}(u,v,w) = b(u,v,w) - b(w,v,u), \quad \forall (u,v,w) \in D(A_k) \times H_k \times V_k, k=1,2, \tag{2.5}$$

and we define a continuous bilinear operator $\widetilde{\mathcal{B}}$ from $V_1 \times V_1$ into V_1' with

$$\langle \widetilde{\mathcal{B}}(u,v), w \rangle = \overline{b}(u,v,w).$$

Next, using the identity

$$(u\cdot\nabla)v + \sum_{j=1}^3 v_j \nabla u_j = -u \times (\nabla \times v) + \nabla(u \cdot v)$$

and using that $\nabla \cdot u = 0$, it is immediate to check that

$$\begin{aligned} \big(-u \times (\nabla \times v), w\big) &= \big((u\cdot\nabla)v, w\big) + \big(v \cdot \nabla u^T, w\big) \\ &= b(u,v,w) + b(w,u,v) = \overline{b}(u,v,w). \end{aligned}$$

Next, we have the following result.

Lemma 2.2. [20, Lemma 2.1] *For $k=1,2$, the trilinear form $\overline{b}$ satisfies*

$$\overline{b}(u,v,w) = -\overline{b}(w,v,u), \qquad \forall u,v,w \in V_k,$$

and consequently,

$$\overline{b}(u,v,u) = 0, \quad \text{for all } u,v \in V_k. \tag{2.6}$$

Furthermore, we have

$$|\overline{b}(u,v,w)| \le c \begin{cases} |u| \|v\|_k \|w\|_k^{1/2} |A_k w|^{1/2}, & \forall u \in H_k, v \in V_k, w \in D(A_k), \\ \|u\|_k \|v\|_k |w|^{1/2} \|w\|_k^{1/2}, & \forall u,v,w \in V_k, \\ \big(|u|^{1/2} \|u\|_k^{1/2} |v| |A_k w| + \|u\|_k |v| \|w\|_k^{1/2} |A_k w|^{1/2}\big), & \\ \qquad \forall u \in V_k, v \in H_k, w \in D(A_k). & \end{cases}$$

2.2. **Existence and uniqueness of solutions to the 3D MHD-α model.** We rewrite problem (1.1) as a functional equation

$$\begin{cases} \partial_t v + \nu A_1 v + \widetilde{\mathcal{B}}(u,v) = \mathcal{B}(B,B) + h_1 \text{ in } D(A_1)', \\ v = u + \alpha^2 A_1 u, \\ \partial_t B + \eta A_2 B + \mathcal{B}(u,B) - \mathcal{B}(B,u) = h_2 \text{ in } V_2', \\ u(0) = u_0, \quad B(0) = B_0. \end{cases} \tag{2.7}$$

Definition 2.3. *Let $h \in L^2(0,T;\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$ and given $(u_0,B_0) \in (V_1,H_2)$. For any $T>0$, a weak solution of (2.7) on the interval $[0,T]$, with $(u(0),B(0)) = (u_0,B_0)$, is a pair of functions (u,B) such that*

$$u \in C([0,T];V_1)\cap L^2(0,T;D(A_1)) \quad \text{with} \quad \partial_t u \in L^2(0,T;H_1),$$
$$B \in C([0,T];H_2)\cap L^2(0,T;V_2) \quad \text{with} \quad \partial_t B \in L^2(0,T;V_2')$$

satisfying

$$\begin{aligned} \left\langle \frac{d}{dt}(u+\alpha^2 A_1 u), w \right\rangle_{D(A_1)'} &+ \nu \left\langle u+\alpha^2 A_1 u, A_1 w\right\rangle_{D(A_1)'} \\ &+ \left\langle \widetilde{\mathcal{B}}(u, u+\alpha^2 A_1 u), w\right\rangle_{D(A_1)'} = (\mathcal{B}(B,B),w) + (h_1,w), \\ \left\langle \frac{d}{dt}B,\theta\right\rangle_{V_2'} &+ \eta((B,\theta))_2 + (\mathcal{B}(u,B),\theta) - (\mathcal{B}(B,u),\theta) = (h_2,\theta), \end{aligned} \tag{2.8}$$

for every $w \in D(A_1)$, $\theta \in V_2$ and for almost every $t \in [0,T]$.
Here, the equation (2.8) is understood in the following sense: for almost every $t_0, t \in [0,T]$ and for all $(w,\theta) \in D(A)\times V_2$ we have

$$\begin{aligned} &(u(t)+\alpha^2 A_1 u(t), w) - (u(t_0)+\alpha^2 A_1 u(t_0), w) + \nu \int_{t_0}^t (u(s)+\alpha^2 A_1 u(s), A_1 w)ds \\ &+ \int_{t_0}^t \left\langle \widetilde{\mathcal{B}}(u(s), u(s)+\alpha^2 A_1 u(s)), w\right\rangle_{D(A_1)'} ds \\ = &\int_{t_0}^t (\mathcal{B}(B(s),B(s)),w)ds + \int_{t_0}^t (h_1(s),w)ds, \\ &(B(t),\theta) - (B(t_0),\theta) + \eta\int_{t_0}^t ((B(s),\theta))_2 ds \\ &\qquad + \int_{t_0}^t (\mathcal{B}(u(s),B(s)),\theta)ds = \int_{t_0}^t (\mathcal{B}(B(s),u(s)),\theta)ds + \int_{t_0}^t (h_2(s),\theta)ds. \end{aligned}$$

The following global well-posedness result can be proved similarly to the case of periodic boundary conditions in [20].

Theorem 2.4. *Let $(u_0,B_0) \in V_1\times H_2$ and $h = (h_1,h_2) \in L^2(0,T;\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$. Then there exists a unique weak solution (u,B) of (2.7) on the interval $(0,T)$. Moreover, if $(u_0,B_0) \in D(A_1)\times V_2$ then there exists a unique strong solution (u,B) of (2.7) satisfying*

$$u \in C([0,T];D(A_1))\cap L^2(0,T;D(A_1^{3/2})),$$
$$B \in C([0,T];V_2)\cap L^2(0,T;D(A_2)).$$

3. Existence of optimal solutions

We can now reformulate the given optimal control problem by using the above operators.

Problem P: Find min $J(z,h)$, $z=(u,B)$ subject to the state equations

$$\begin{cases} \partial_t v + \nu A_1 v + \widetilde{\mathcal{B}}(u,v) = \mathcal{B}(B,B) + h_1 \quad \text{in } L^2(0,T;D(A_1)'), \\ \partial_t B + \eta A_2 B + \mathcal{B}(u,B) = \mathcal{B}(B,u) + h_2 \quad \text{in } L^2(0,T;V_2'), \\ v = u + \alpha^2 A_1 u, \\ u(0) = u_0 \quad \text{in } V_1, \quad B(0) = B_0 \quad \text{in } H_2, \end{cases} \tag{3.1}$$

and the control $h=(h_1,h_2)$ belongs to the admissible set U_{ad}, which is a non-empty convex and closed subset in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$.

We call a pair (z,h) of states and control admissible if it satisfies **Problem P**. First, we will prove the existence of an optimal solution.

Theorem 3.1. *There exists an optimal control solution to* **Problem P**.

Proof. The proof is very standard in the theory of optimal control of PDEs, so we only sketch it here.

The set of admissible controls is non-empty and bounded in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$. For every control in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$, by Theorem 2.4, there exists a unique weak solution of the state equation in **Problem P**. Furthermore, the functional J is bounded from below, $J(z,h) \geq 0$ for every admissible (z,h). Hence, there exists the infimum of J over all admissible controls and states

$$0 \leq \overline{J} := \inf_{(z,h) \text{ admissible}} J(z,h) \leq \infty.$$

Moreover, there is a minimizing sequence (z_m, h_m) of admissible pairs such that $J(z_m,h_m) \to \overline{J}$ for $m \to \infty$.

From the convergence we see that the set $\{J(z_m,h_m)\}$ is bounded. This implies that the set $\{h_m\}$ is bounded in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$. Consequently, we can assume that it converges weakly to some $h^* \in \left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$. The set of admissible control is convex and closed in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$, so it is weakly closed, thus the control h^* is admissible, i.e., $h^* \in U_{ad}$.

Moreover, it is standard to check that the sequence $\{z_m\}$ belongs to a bounded set in $L^2(0,T;D(A_1)\times V_2)$ and hence we can assume that z_m converges weakly to some $z^* \in L^2(0,T;D(A_1)\times V_2)$. By using arguments as in Part D of the proof of Theorem 3.1 in [20], one can show that the pair (z^*,h^*) satisfies the state equation (3.1), that is, it is admissible.

Finally, it remains to show $\overline{J} = J(z^*,h^*)$. The objective functional consists of several norm squares, thus it is weakly lower semicontinuous which implies that

$$J(z^*,h^*) \leq \liminf J(z_m,h_m) = \overline{J}.$$

Since (z^*,h^*) is admissible, and $\overline{J}$ is the infimum over all admissible controls and states, it follows that $\overline{J} = J(z^*,h^*)$. Thus, we have completed the proof. □

4. First-order necessary optimality conditions

First, we recall some definitions from Convex Analysis. Let X be a Hilbert space with the inner product denoted by $(\cdot,\cdot)$ and U be a convex subset of X. Let

$\mathcal{N}_U(h), \mathcal{T}_U(h)$ denote the normal cone and the polar cone of tangents of U at the point $h \in U$ respectively, i.e.,

$$\mathcal{N}_U(h) = \{y \in X : (y, \zeta - h) \le 0, \forall \zeta \in U\},$$
$$\mathcal{T}_U(h) = \{y \in X : (y, \zeta) \le 0, \forall \zeta \in \mathcal{N}_U(h)\}.$$

An element $\omega \in X$ is called a feasible direction at $h \in U$ if there exists $\delta > 0$ such that $h + \varepsilon\omega \in U$ holds for all $\varepsilon \in (0, \delta)$.

The cone of feasible direction at $h \in U$ will be denoted by $\mathcal{F}_U(h)$. Since U is convex, we have that (see [4])

$$\overline{\mathcal{F}_U(h)} = \mathcal{T}_U(h). \tag{4.1}$$

We apply these notations to the case $X = \left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3, U = U_{ad}$ and $h = h^*$.

Definition 4.1. *A control h^* is said to be locally optimal if there exists a constant $\rho > 0$ such that*

$$J(z^*, h^*) \le J(z, h)$$

holds for all $h \in U_{ad}$ with $\|h - h^*\|_{(L^2(Q))^3 \times (L^2(Q))^3} \le \rho$. Here, $z^* = (u^*, B^*)$ and $z = (u, B)$ denote the states associated with h^* and h, respectively.

Next, following the general lines of the approach in [2], we will establish the first-order necessary optimality conditions.

Consider the adjoint equations

$$\begin{cases} -\partial_t(\overline{\lambda} + \alpha^2 A_1\overline{\lambda}) + \nu A_1(\overline{\lambda} + \alpha^2 A_1\overline{\lambda}) - (u^* \cdot \nabla)(\overline{\lambda} + \alpha^2 A_1\overline{\lambda}) \\ \quad +\overline{\lambda} \cdot \nabla(u^* + \alpha^2 A_1 u^*)^T + \alpha^2(\Delta u^* \cdot \nabla)\overline{\lambda} + 2\alpha^2(\widetilde{\nabla} u^* \cdot \nabla)(\widetilde{\nabla}\overline{\lambda}) \\ \quad -\alpha^2(\Delta\overline{\lambda} \cdot \nabla)u^* - 2\alpha^2(\widetilde{\nabla}\overline{\lambda} \cdot \nabla)(\widetilde{\nabla} u^*) \\ \quad +\overline{\varpi} \cdot \nabla(B^*)^T + (B^* \cdot \nabla)\overline{\varpi} = \alpha_Q(u^* - u_Q), & x \in \Omega, t > 0, \\ -\partial_t\overline{\varpi} + \eta A_2\overline{\varpi} - (u^* \cdot \nabla)\overline{\varpi} - \overline{\varpi} \cdot \nabla(u^*)^T \\ \quad +(B^* \cdot \nabla)\overline{\lambda} - \overline{\lambda} \cdot \nabla(B^*)^T = \alpha_Q(B^* - B_Q), & x \in \Omega, t > 0, \\ \nabla \cdot \overline{\lambda} = \nabla \cdot \overline{\varpi} = 0, & x \in \Omega, t > 0, \\ \overline{\lambda} = \Delta\overline{\lambda} = 0, \qquad \overline{\varpi} \cdot n = 0, & x \in \partial\Omega, t > 0, \\ \overline{\lambda}(T) + \alpha^2 A_1\overline{\lambda}(T) = \alpha_T(u^*(T) - u_T), \quad \overline{\varpi}(T) = \alpha_T(B^*(T) - B_T), & x \in \Omega. \end{cases} \tag{4.2}$$

Definition 4.2. *A pair of function $(\overline{\lambda}, \overline{\varpi}) \in L^\infty(0, T; V_1 \times H_2) \cap L^2(0, T; D(A_1) \times V_2)$ with $(\partial_t\overline{\lambda}, \partial_t\overline{\varpi}) \in L^2(0, T; H_1 \times V_2')$ is called a weak solution to the adjoint system (4.2) on the interval $(0, T)$ if it satisfies*

$$\langle \partial_t(\overline{\lambda}(t) + \alpha^2 A_1\overline{\lambda}(t)), w\rangle_{D(A)'} = -\langle G_{u^*}(t), w\rangle_{D(A)'}, \text{ for a.e. } t \in (0, T),$$
$$(\overline{\lambda}(T) + \alpha^2 A_1\overline{\lambda}(T), w) = \alpha_T(u^*(T) - u_T, w),$$
$$\langle \partial_t\overline{\varpi}, \theta\rangle_{V_2'} = -\langle G_{B^*}(t), \theta\rangle_{V_2'}, \text{ for a.e. } t \in (0, T),$$
$$(\overline{\varpi}(T), \theta) = \alpha_T(B^*(T) - B_T, \theta),$$

for all test functions $w \in D(A_1)$ and $\theta \in V_2$. Here

$$\begin{aligned} G_{u^*}(t) := {} & \alpha_Q(u^* - u_Q) - \nu A_1(\overline{\lambda} + \alpha^2 A_1\overline{\lambda}) + (u^* \cdot \nabla)(\overline{\lambda} + \alpha^2 A_1\overline{\lambda}) - \overline{\lambda} \cdot \nabla(u^* + \alpha^2 A_1 u^*)^T \\ & - \alpha^2(\Delta u^* \cdot \nabla)\overline{\lambda} - 2\alpha^2(\widetilde{\nabla} u^* \cdot \nabla)(\widetilde{\nabla}\overline{\lambda}) + \alpha^2(\Delta\overline{\lambda} \cdot \nabla)u^* + 2\alpha^2(\widetilde{\nabla}\overline{\lambda} \cdot \nabla)(\widetilde{\nabla} u^*) \\ & - \overline{\varpi} \cdot \nabla(B^*)^T - (B^* \cdot \nabla)\overline{\varpi}, \end{aligned}$$

$$G_{B^*}(t) := \alpha_Q(B^* - B_Q) - \eta A_2\overline{\omega} + (u^*\cdot\nabla)\overline{\omega} + \overline{\omega}\cdot\nabla(u^*)^T - (B^*\cdot\nabla)\overline{\lambda} + \overline{\lambda}\cdot\nabla(B^*)^T,$$

and we adopt the notation $\widetilde{\nabla} u = (\nabla u_1, \nabla u_2, \nabla u_3)$.

Remark 4.3. Later, we will consider many linearized systems, including systems (4.5), (4.6), (5.2), (5.4) and (5.5) below. The weak solutions of these systems are defined similarly as in Definition 2.3. Using the same arguments as in the proof of Theorem 3.1 in [20], we can prove the existence and uniqueness of a weak solution to these linearized systems and the proof is simpler than that in the nonlinear case due to the linearity of the system, so we will omit the details.

In the proof of the following first-order necessary optimality condition, we particularly show the well-posedness of the adjoint equations (4.2). The proof is based on the operator theory.

Theorem 4.4. *Let* (z^*, h^*) *be an optimal solution to* **Problem P**. *Then there exists* $\overline{y} = (\overline{\lambda}, \overline{\omega})$, *which is the weak solution of the adjoint equations* (4.2). *Moreover, we have*

$$\iint\limits_Q (\overline{y} + \gamma h^*)\cdot \overline{h} dxdt \geq 0, \quad \forall \overline{h} \in \mathcal{T}_{U_{ad}}(h^*). \tag{4.3}$$

As a special case, the variational inequality

$$\iint\limits_Q (\overline{y} + \gamma h^*)\cdot(\zeta - h^*) \geq 0, \quad \forall \zeta \in U_{ad} \tag{4.4}$$

is satisfied.

Proof. Let $\overline{h}$ be a feasible direction at h^*. Taking $h = h^* + \beta\overline{h}$, we have $h \in U_{ad}$ with a small enough $\beta \in \mathbb{R}^+$. Let z be the state associated with h. We can then write (where $s = (w, \theta)$ and $s_\beta = (w_\beta, \theta_\beta)$)

$$z = z^* + \beta s + \beta s_\beta$$

with s is a weak solution of the equations

$$\begin{cases} \partial_t(w + \alpha^2 A_1 w) + \nu A_1(w + \alpha^2 A_1 w) + \widetilde{\mathcal{B}}(u^*, w + \alpha^2 A_1 w) \\ \qquad + \widetilde{\mathcal{B}}(w, u^* + \alpha^2 A_1 u^*) = \mathcal{B}(B^*, \theta) + \mathcal{B}(\theta, B^*) + \overline{h}_1, & x \in \Omega, t > 0, \\ \partial_t\theta + \eta A_2\theta + \mathcal{B}(u^*, \theta) - \mathcal{B}(\theta, u^*) + \mathcal{B}(w, B^*) - \mathcal{B}(B^*, w) = \overline{h}_2, & x \in \Omega, t > 0, \\ \nabla\cdot w = \nabla\cdot\theta = 0, & x \in \Omega, t > 0, \\ w = \Delta w = 0, \quad \theta\cdot n = 0, & x \in \partial\Omega, t > 0, \\ w(x,0) = 0, \quad \theta(x,0) = 0, & x \in \Omega, \end{cases} \tag{4.5}$$

and s_β is a weak solution of the equations

$$\begin{cases} \partial_t(w_\beta + \alpha^2 A_1 w_\beta) + \nu A_1(w_\beta + \alpha^2 A_1 w_\beta) + \widetilde{\mathcal{B}}(u^*, w_\beta + \alpha^2 A_1 w_\beta) \\ \qquad + \widetilde{\mathcal{B}}(w_\beta, u^* + \alpha^2 A_1 u^*) + \beta W_\beta = \mathcal{B}(B^*, \theta_\beta) + \mathcal{B}(\theta_\beta, B^*), & x \in \Omega, t > 0, \\ \partial_t\theta_\beta + \eta A_2\theta_\beta + \mathcal{B}(u^*, \theta_\beta) - \mathcal{B}(\theta_\beta, u^*) + \mathcal{B}(w_\beta, B^*) - \mathcal{B}(B^*, w_\beta) = \beta\Psi_\beta, & x \in \Omega, t > 0, \\ \nabla\cdot w_\beta = \nabla\cdot\theta_\beta = 0, & x \in \Omega, t > 0, \\ w_\beta = \Delta w_\beta = 0, \quad \theta_\beta\cdot n = 0, & x \in \partial\Omega, t > 0, \\ w_\beta(x,0) = 0, \quad \theta_\beta(x,0) = 0, & x \in \Omega. \end{cases} \tag{4.6}$$

Here, we have used the following notations:

$$\begin{aligned} W_\beta :&= \widetilde{\mathcal{B}}(w, w+\alpha^2 A_1 w) + \widetilde{\mathcal{B}}(w_\beta, w_\beta+\alpha^2 A_1 w_\beta) + \widetilde{\mathcal{B}}(w, w_\beta+\alpha^2 A_1 w_\beta) \\ &+ \widetilde{\mathcal{B}}(w_\beta, w+\alpha^2 A_1 w) - \Big[\mathcal{B}(\theta,\theta_\beta) + \mathcal{B}(\theta_\beta,\theta) + \mathcal{B}(\theta,\theta) + \mathcal{B}(\theta_\beta,\theta_\beta)\Big], \\ \Psi_\beta :&= \mathcal{B}(\theta, w) - \mathcal{B}(w,\theta) + \mathcal{B}(\theta_\beta, w) - \mathcal{B}(w,\theta_\beta) \\ &\qquad + \mathcal{B}(\theta, w_\beta) - \mathcal{B}(w_\beta,\theta) + \mathcal{B}(\theta_\beta, w_\beta) - \mathcal{B}(w_\beta,\theta_\beta). \end{aligned}$$

Indeed, by using similar arguments as in Theorem 3.1 in [20], we can prove that (4.5) has a unique weak solution $(w,\theta) \in C([0,T]; V_1 \times H_2) \cap L^2(0,T; D(A_1)\times V_2)$ with $(\partial_t w, \partial_t \theta) \in L^2(0,T; H_1 \times V_2')$, and that for any $\beta > 0$, (4.6) also possesses exactly one weak solution $(w_\beta,\theta_\beta) \in C([0,T]; V_1 \times H_2) \cap L^2(0,T; D(A_1)\times V_2)$ and $(\partial_t w_\beta, \partial_t \theta_\beta) \in L^2(0,T; H_1 \times V_2')$.

Next, we will show $s_\beta \to (0,0)$ in $L^2(0,T; V_1 \times H_2)$ as $\beta \to 0^+$. We begin by taking the inner product of the both first and second equations of (4.6) by w_β and θ_β, respectively, then adding the resulting equations and using (2.2), (2.3), (2.6), we deduce the identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|w_\beta|^2 + \alpha^2\|w_\beta\|_1^2 + |\theta_\beta|^2) + \nu(\|w_\beta\|_1^2 + \alpha^2|A_1 w_\beta|^2) + \eta\|\theta_\beta\|_2^2 \\ &= -\bar{b}(u^*, w_\beta+\alpha^2 A_1 w_\beta, w_\beta) + \bar{b}(\theta_\beta, B^*, w_\beta) + b(\theta_\beta, u^*, w_\beta) + b(\theta,\theta,w_\beta) + b(\theta_\beta, w, \theta_\beta) \\ &- \beta\Big[\bar{b}(w, w+\alpha^2 A_1 w, w_\beta) + \bar{b}(w, w_\beta+\alpha^2 A_1 w_\beta, w_\beta) - \bar{b}(\theta_\beta,\theta,w_\beta) - \bar{b}(w,\theta_\beta,\theta)\Big]. \end{aligned} \tag{4.7}$$

Before we proceed with estimating all the terms on the right-hand side of (4.7). From now on, throughout the paper, c will denote a generic positive constant (depending only on $\nu, \eta, \lambda_1^{(1)}, \alpha$) which can take different values, sometimes even within the same line. Since $u^*, w \in C([0,T]; V_1)$, $B^*, \theta \in C([0,T]; H_2)$, using Lemma 2.2, Poincaré and Cauchy inequalities, we estimate the first, second, sixth, seventh, eighth and nineth terms on the right-hand side of (4.7), as follows:

$$\begin{aligned} |\bar{b}(u^*, w_\beta+\alpha^2 A_1 w_\beta, w_\beta)| &\le c\|u^*\|_1\|w_\beta\|_1^{1/2}|A_1 w_\beta|^{1/2}|w_\beta+\alpha^2 A_1 w_\beta| \\ &\le c\|w_\beta\|_1^2 + \frac{\nu\alpha^2}{16}|A_1 w_\beta|^2, \end{aligned}$$

$$|\bar{b}(\theta_\beta, B^*, w_\beta)| \le c|B^*|\|\theta_\beta\|_2\|w_\beta\|_1^{1/2}|A_1 w_\beta|^{1/2} \le c\|w_\beta\|_1^2 + \frac{\nu\alpha^2}{16}|A_1 w_\beta|^2 + \frac{\eta}{16}\|\theta_\beta\|_2^2.$$

Similarly, we have

$$\begin{aligned} |\beta\bar{b}(w, w+\alpha^2 A_1 w, w_\beta)| &\le c\beta\|w\|_1\|w_\beta\|_1^{1/2}|A_1 w_\beta|^{1/2}|w+\alpha^2 A_1 w| \\ &\le \beta\|w_\beta\|_1|A_1 w_\beta| + c\beta|A_1 w|^2 \\ &\le c\|w_\beta\|_1^2 + c\beta^2|A_1 w|^2 + \frac{\nu\alpha^2}{16}|A_1 w_\beta|^2, \end{aligned}$$

$$\begin{aligned} |\beta\bar{b}(w, w_\beta+\alpha^2 A_1 w_\beta, w_\beta)| &\le c\beta\|w\|_1\|w_\beta\|_1^{1/2}|A_1 w_\beta|^{1/2}|w_\beta+\alpha^2 A_1 w_\beta| \\ &\le c\beta^4\|w_\beta\|_1^2 + \frac{\nu\alpha^2}{16}|A_1 w_\beta|^2, \end{aligned}$$

and

$$|\beta\bar{b}(\theta_\beta,\theta,w_\beta)| \le c|\theta|\|\theta_\beta\|_2\|w_\beta\|_1^{1/2}|A_1 w_\beta|^{1/2} \le c\beta^4\|w_\beta\|_1^2 + \frac{\nu\alpha^2}{8}|A_1 w_\beta|^2 + \frac{\eta}{16}\|\theta_\beta\|_2^2,$$

$$|\beta\overline{b}(w,\theta_\beta,\theta)| \le c\beta|\theta|\|\theta_\beta\|_2|A_1w| \le c\beta^2|A_1w|^2 + \frac{\eta}{16}\|\theta_\beta\|_2^2.$$

Beside, by using Lemma 2.1 and the Cauchy inequality, we have

$$|b(\theta_\beta,u^*,w_\beta)| \le c|u^*|\|\theta_\beta\|_2\|w_\beta\|_1^{1/2}|A_1w_\beta|^{1/2} \le c\|w_\beta\|_1^2 + \frac{\nu\alpha^2}{8}|A_1w_\beta|^2 + \frac{\eta}{16}\|\theta_\beta\|_2^2,$$

$$|b(\theta,\theta,w_\beta)| \le c|\theta|\|\theta\|_2\|w_\beta\|_1 \le c\|\theta\|_2^2 + \frac{\nu}{2}\|w_\beta\|_1^2,$$

and

$$|b(\theta_\beta,w,\theta_\beta)| \le c|\theta_\beta|\|\theta_\beta\|_2\|w\|_1 \le c|\theta_\beta|^2 + \frac{\eta}{4}\|\theta_\beta\|_2^2.$$

From all estimates above, inserting all of them on the right-hand side of (4.7), we obtain after straightforward transformations that

$$\begin{aligned}\frac{d}{dt}(|w_\beta|^2 + \alpha^2\|w_\beta\|_1^2 + |\theta_\beta|^2) &\le c_1\beta^2(|A_1w|^2 + \|\theta\|_2^2) + c_2(1+\beta^4)(\|w_\beta\|^2 + |\theta_\beta|^2)\\ &\le c_1(|A_1w|^2 + \|\theta\|_2^2) + c_2(1+\beta^4)(|w_\beta|^2 + \alpha^2\|w_\beta\|_1^2 + |\theta_\beta|^2).\end{aligned}$$

Notice that, with the initial $w_0 = 0 \in D(A_1)$ and $\theta_0 = 0 \in V_2$, from Theorem 2.4, we have that $(w,\theta) \in C([0,T];D(A_1)\times V_2)$. Then, applying Gronwall's inequality, we deduce

$$|w_\beta(t)|^2 + \alpha^2\|w_\beta(t)\|_1^2 + |\theta_\beta|^2 \le \frac{c_1\beta^2}{c_2(1+\beta^4)}\exp(c_2(1+\beta^4)t) - \frac{c_1\beta^2}{c_2(1+\beta^4)}.$$

Hence

$$\|w_\beta\|^2_{C([0,T];H_1)} + \alpha^2\|w_\beta\|^2_{C([0,T];V_1)} + \|\theta_\beta\|^2_{C([0,T];H_2)} \le \frac{c_1\beta^2}{c_2(1+\beta^4)^2}\exp(c(1+\beta^4)T),$$

and so $\|w_\beta\|^2_{C([0,T];V_1)} \to 0$, $\|\theta_\beta\|^2_{C([0,T];H_2)} \to 0$ as $\beta \to 0^+$.

Second, we will show that the linear adjoint equation (4.2) possesses a weak solution $\overline{y} = (\overline{\lambda},\overline{\varpi})$ that belongs to $L^\infty(0,T;V_1\times H_2)\cap L^2(0,T;D(A_1)\times V_2)$ with $\partial_t\overline{y} = (\partial_t\overline{\lambda},\partial_t\overline{\varpi}) \in L^2(0,T;H_1\times V_2')$. We define W_0 as a closed linear space of $W^{1,2}(0,T;H_1\times V_2')\cap L^2(0,T;D(A_1)\times V_2)$ by

$$\begin{aligned}W_0 = \big\{(u,B) \in &L^2(0,T;D(A_1)\times V_2),\\ &(\partial_t u,\partial_t B) \in L^2(0,T;H_1\times V_2') : (u(0),B(0)) = (0,0)\big\}.\end{aligned}$$

Defining an operator $\mathcal{S} : W_0 \to L^2(0,T;D(A_1)'\times V_2')$ by

$$\mathcal{S}\chi := \overline{\chi}, \quad \chi = (w,\theta) \text{ and } \overline{\chi} = (\overline{w},\overline{\theta}),$$

where

$$\begin{aligned}\overline{w} &:= \partial_t(w+\alpha^2A_1w) + \nu A_1(w+\alpha^2A_1w) + \widetilde{\mathcal{B}}(u^*,w+\alpha^2A_1w)\\ &\qquad + \widetilde{\mathcal{B}}(w,u^*+\alpha^2A_1u^*) + \mathcal{B}(w,B^*) - \mathcal{B}(B^*,w),\\ \overline{\theta} &:= \partial_t\theta + \eta A_2\theta + \mathcal{B}(u^*,\theta) - \mathcal{B}(\theta,u^*) - \mathcal{B}(B^*,\theta) - \mathcal{B}(\theta,B^*),\end{aligned}$$

can be consider as elements of $L^2(0,T;D(A_1)')$ and $L^2(0,T;V_2')$, respectively, by

$$\begin{aligned}&\big\langle\partial_t(w+\alpha^2A_1w) + \nu A_1w,\widetilde{w}\big\rangle_{L^2(0,T;D(A_1)')}\\ &:= \int_0^T(\partial_tw(t),\widetilde{w}(t))dt + \alpha^2\int_0^T((\partial_tw(t),\widetilde{w}(t)))_1dt + \nu\int_0^T((w(t),\widetilde{w}(t)))_1dt,\end{aligned}$$

$$\big\langle\widetilde{\mathcal{B}}(u^*,w+\alpha^2A_1w),\widetilde{w}\big\rangle_{L^2(0,T;D(A_1)')} := \int_0^T\overline{b}\big(u^*(t),w(t)+\alpha^2A_1w(t),\widetilde{w}(t)\big)dt,$$

$$\big\langle\widetilde{\mathcal{B}}(w,u^*+\alpha^2A_1u^*),\widetilde{w}\big\rangle_{L^2(0,T;D(A_1)')}:=\int_0^T\overline{b}\big(w(t),u^*(t)+\alpha^2A_1u^*(t),\widetilde{w}(t)\big)dt,$$

$$\big\langle\mathcal{B}(w,B^*)-\mathcal{B}(B^*,w),\widetilde{w}\big\rangle_{L^2(0,T;D(A_1)')}:=\int_0^T\overline{b}\big(B^*(t),\widetilde{w}(t),w(t)\big)dt,$$

for $\widetilde{w}\in L^2(0,T;D(A))$, and

$$\big\langle\partial_t\theta+\eta A_2\theta,\widetilde{\theta}\big\rangle_{L^2(0,T;V_2')}:=\int_0^T(\partial_t\theta(t),\widetilde{\theta}(t))dt+\eta\int_0^T((\theta(t),\widetilde{\theta}(t)))_2dt,$$

$$\big\langle\mathcal{B}(u^*,\theta)-\mathcal{B}(\theta,u^*),\widetilde{\theta}\big\rangle_{L^2(0,T;V_2')}:=\int_0^T\overline{b}(\theta(t),\widetilde{\theta}(t),u^*(t))dt,$$

$$\big\langle\mathcal{B}(B^*,\theta)+\mathcal{B}(\theta,B^*),\widetilde{\theta}\big\rangle_{L^2(0,T;V_2')}:=\int_0^T\big[b(B^*(t),\theta(t),\widetilde{\theta}(t))+b(\theta(t),B^*(t),\widetilde{\theta}(t))\big]dt.$$

Then $\mathcal{S}$ is an isomorphism, so the adjoint operator $\mathcal{S}^*:L^2(0,T;D(A_1)\times V_2)\to W_0^*$ is also an isomorphism. Hence, for any $g\in W_0^*$, there exists a unique $\overline{y}=(\overline{\lambda},\overline{\omega})\in L^2(0,T;D(A_1)\times V_2)$ such that $\mathcal{S}^*\overline{y}=g$ in W_0^*. For any $\phi=(\phi_1,\phi_2)\in W_0$, we have

$$\langle\mathcal{S}^*\overline{y},\phi\rangle_{W_0^*}=\langle g,\phi\rangle_{W_0^*},$$

which implies that

$$\langle\mathcal{S}\phi,\overline{y}\rangle_{L^2(0,T;D(A_1)'\times V_2')}=\langle g,\phi\rangle_{W_0^*}. \tag{4.8}$$

Consider $g_1=(g_{1u^*},g_{1B^*})=\alpha_Q(u^*-u_Q,B^*-B_Q)\in L^2(0,T;\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))\cap W_0^*$ and $g_2=(g_{2u^*},g_{2B^*})=\alpha_T(u^*(T)-u_T,B^*(T)-B_T)\in W_0^*$ defined by

$$\langle g_1,\phi\rangle_{W_0^*}=\alpha_Q\iint_Q\Big[\big(u^*(x,t)-u_Q\big)\phi_1(t)+\big(B^*(x,t)-B_Q\big)\phi_2(t)\Big]dxdt,$$

$$\langle g_2,\phi\rangle_{W_0^*}=\alpha_T\int_\Omega\Big[\big(u^*(T)-u_T\big)\phi_1(T)+\big(B^*(T)-B_T\big)\phi_2(T)\Big]dx,$$

for $\phi\in W_0$. From the equations

$$\begin{aligned}&-\partial_t(\overline{\lambda}+\alpha^2A_1\overline{\lambda})+\nu A_1(\overline{\lambda}+\alpha^2A_1\overline{\lambda})-(u^*\cdot\nabla)(\overline{\lambda}+\alpha^2A_1\overline{\lambda})\\&\quad+\overline{\lambda}\cdot\nabla(u^*+\alpha^2A_1u^*)^T+\alpha^2(\Delta u^*\cdot\nabla)\overline{\lambda}+2\alpha^2(\widetilde{\nabla}u^*\cdot\nabla)(\widetilde{\nabla}\overline{\lambda})\\&\quad-\alpha^2(\Delta\overline{\lambda}\cdot)\nabla u^*-2\alpha^2(\widetilde{\nabla}\overline{\lambda}\cdot\nabla)(\widetilde{\nabla}u^*)+\overline{\omega}\cdot\nabla(B^*)^T+(B^*\cdot\nabla)\overline{\omega}=g_{1u^*}\end{aligned} \tag{4.9}$$

and

$$-\partial_t\overline{\omega}+\eta A_2\overline{\omega}-(u^*\cdot\nabla)\overline{\omega}-\overline{\omega}\cdot\nabla(u^*)^T+(B^*\cdot\nabla)\overline{\lambda}-\overline{\lambda}\cdot\nabla(B^*)^T=g_{1B^*}, \tag{4.10}$$

we will prove that for $g_1\in L^2(0,T;\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$ then $(\partial_t\overline{\lambda},\partial_t\overline{\omega})\in L^2(0,T;H_1\times V_2')$. To this aim we have to estimate some terms. Since $\overline{\lambda}\in L^2(0,T;D(A_1))$, the first term $A_1(\overline{\lambda}+\alpha^2A_1\overline{\lambda})$ is in $L^2(0,T;D(A_1)')$. Next, for all $v\in L^2(0,T;D(A_1))$, by using (2.4), we estimate the rest terms in (4.9) as follows

$$\begin{aligned}&\Big|\langle-(u^*\cdot\nabla)(\overline{\lambda}+\alpha^2A_1\overline{\lambda})+\overline{\lambda}\cdot\nabla(u^*+\alpha^2A_1u^*)^T,v\rangle_{D(A_1)'}\Big|\\&\qquad\le c(\|u^*\|_1|\overline{\lambda}+\alpha^2A_1\overline{\lambda}|+\|\overline{\lambda}\|_1|u^*+\alpha^2A_1u^*)|)\|v\|_{\mathbb{H}^2(\Omega)},\end{aligned}$$

$$\Big|\langle\alpha^2(\Delta u^*\cdot\nabla)\overline{\lambda}+2\alpha^2(\widetilde{\nabla}u^*\cdot\nabla)(\widetilde{\nabla}\overline{\lambda}),v\rangle_{D(A_1)'}\Big|\le c|A_1u^*|\|\overline{\lambda}\|_1\|v\|_{\mathbb{H}^2(\Omega)},$$

$$\Big|\langle-\alpha^2(\Delta\overline{\lambda}\cdot\nabla)u^*-2\alpha^2(\widetilde{\nabla}\overline{\lambda}\cdot\nabla)(\widetilde{\nabla}u^*),v\rangle_{D(A_1)'}\Big|\le c|A_1\overline{\lambda}|\|u^*\|_1\|v\|_{\mathbb{H}^2(\Omega)},$$

and

$$\left|\left(\overline{\omega}\cdot\nabla(B^*)^T+(B^*\cdot\nabla)\overline{\omega},v\right)\right|\le c\|\overline{\omega}\|_2|B^*|\|v\|_{\mathbb{H}^2(\Omega)}.$$

Beside, it is easy to deduce the term $A_2\overline{\omega}$ is in $L^2(0,T;V_2')$ and by using (2.4), for all $\varphi\in L^2(0,T;V_2)$, we also have the estimates of the terms in (4.10)

$$\left|\left(-(u^*\cdot\nabla)\overline{\omega}-\overline{\omega}\cdot\nabla(u^*)^T,\varphi\right)\right|\le\|u^*\|_1\|\overline{\omega}\|_2\|\varphi\|_2,$$

$$\left|\left((B^*\cdot\nabla)\overline{\lambda}-\overline{\lambda}\cdot\nabla(B^*)^T,\varphi\right)\right|\le|B^*||A_1\overline{\lambda}|\|\varphi\|_2.$$

For convenience, we set

$$\begin{aligned}G_{u^*}(t):=&\ g_{1u^*}-\nu A_1(\overline{\lambda}+\alpha^2A_1\overline{\lambda})+(u^*\cdot\nabla)(\overline{\lambda}+\alpha^2A_1\overline{\lambda})-\overline{\lambda}\cdot\nabla(u^*+\alpha^2A_1u^*)^T\\&-\alpha^2(\Delta u^*\cdot\nabla)\overline{\lambda}-2\alpha^2(\widetilde{\nabla}u^*\cdot\nabla)(\widetilde{\nabla}\overline{\lambda})+\alpha^2(\Delta\overline{\lambda}\cdot\nabla)u^*+2\alpha^2(\widetilde{\nabla}\overline{\lambda}\cdot\nabla)(\widetilde{\nabla}u^*)\\&-\overline{\omega}\cdot\nabla(B^*)^T-(B^*\cdot\nabla)\overline{\omega},\\G_{B^*}(t):=&\ g_{1B^*}-\eta A_2\overline{\omega}+(u^*\cdot\nabla)\overline{\omega}+\overline{\omega}\cdot\nabla(u^*)^T-(B^*\cdot\nabla)\overline{\lambda}+\overline{\lambda}\cdot\nabla(B^*)^T.\end{aligned}$$

Then, by using the boundedness of (u^*,B^*) and $(\overline{\lambda},\overline{\omega})$ in $L^\infty(0,T;V_1\times H_2)\cap L^2(0,T;D(A_1)\times V_2)$, we deduce that $(G_{u^*}(t),G_{B^*}(t))\in L^2(0,T;D(A_1)'\times V_2')$. Take $g=g_1+g_2$ in (4.8), we get

$$\begin{aligned}&\langle\partial_t(\phi_1+\alpha^2A_1\phi_1),\overline{\lambda}\rangle_{L^2(0,T;D(A_1)')}\\&=\langle g_{1u^*},\phi_1\rangle_{L^2(0,T;D(A_1)')}+\langle g_{2u^*},\phi_1\rangle_{L^2(0,T;D(A_1)'}-\left\langle\nu A_1(\phi_1+\alpha^2A_1\phi_1),\overline{\lambda}\right\rangle_{L^2(0,T;D(A_1)')}\\&-\left\langle\widetilde{\mathcal{B}}(u^*,\phi_1+\alpha^2A_1\phi_1)+\widetilde{\mathcal{B}}(\phi_1,u^*+\alpha^2A_1u^*)+\mathcal{B}(\phi_1,B^*)-\mathcal{B}(B^*,\phi_1),\overline{\lambda}\right\rangle_{L^2(0,T;D(A_1)')},\\&\langle\partial_t\phi_2,\overline{\omega}\rangle_{L^2(0,T;V_2')}=\langle g_{1B^*},\phi_2\rangle_{L^2(0,T;V_2')}+\langle g_{2B^*},\phi_2\rangle_{L^2(0,T;V_2')}\\&\qquad-\left\langle\eta A_2\phi_2+\mathcal{B}(u^*,\phi_2)-\mathcal{B}(\phi_2,u^*)-\mathcal{B}(B^*,\phi_2)-\mathcal{B}(\phi_2,B^*),\overline{\omega}\right\rangle_{L^2(0,T;V_2')}.\end{aligned}$$

This deduces that

$$\begin{cases}\langle\partial_t(\phi_1+\alpha^2A_1\phi_1),\overline{\lambda}\rangle_{L^2(0,T;D(A_1)')}\\\qquad\qquad=\left\langle G_{u^*}(t),\phi_1\right\rangle_{L^2(0,T;D(A_1)')}+\alpha_T(u^*(T)-u_T,\phi_1(T)),\\\langle\partial_t\phi_2,\overline{\omega}\rangle_{L^2(0,T;V_2')}=\left\langle G_{B^*}(t),\phi_2\right\rangle_{L^2(0,T;V_2')}+\alpha_T(B^*(T)-B_T,\phi_2(T)).\end{cases}\tag{4.11}$$

For $(v,\varphi)\in D(A_1)\times V_2$, set $\phi(t)=(\rho_1(t)v,\rho_2(t)\varphi)$ with $\rho_1(t),\rho_2(t)\in C_0^\infty(0,T)$, we have $\phi\in W_0$. In (4.11), taking $\phi_1(t)=\rho_1(t)v$ and $\phi_2(t)=\rho_2(t)\varphi$, we obtain

$$\begin{aligned}&\int_0^T\rho_1'(t)\langle v+\alpha^2A_1v,\overline{\lambda}\rangle_{D(A_1)'}dt=\int_0^T\rho_1(t)\left\langle G_{u^*}(t),v\right\rangle_{D(A_1)'}dt,\\&\int_0^T\rho_2'(t)\langle\varphi,\overline{\omega}\rangle_{V_2'}dt=\int_0^T\rho_2(t)\left\langle G_{B^*}(t),\varphi\right\rangle_{V_2'}dt.\end{aligned}\tag{4.12}$$

Moreover, we have

$$\begin{aligned}\int_0^T\rho_1'(t)\langle v+\alpha^2A_1v,\overline{\lambda}\rangle_{D(A_1)'}dt&=\int_0^T\rho_1'(t)\langle\overline{\lambda}+\alpha^2A_1\overline{\lambda},v\rangle_{D(A_1)'}dt\\&=-\int_0^T\rho_1(t)\langle\partial_t(\overline{\lambda}(t)+\alpha^2A_1\overline{\lambda}(t)),v\rangle_{D(A_1)'}dt,\end{aligned}$$

$$\int_0^T\rho_2'(t)\langle\varphi,\overline{\omega}\rangle_{V_2'}dt=\int_0^T\rho_2'(t)\langle\overline{\omega},\varphi\rangle_{V_2'}dt=-\int_0^T\rho_2(t)\langle\partial_t\overline{\omega}(t),\varphi\rangle_{V_2'}dt.$$

This and (4.12) give that

$$\begin{aligned}
&\int_0^T \rho_1(t)\langle \partial_t(\overline{\lambda}(t)+\alpha^2 A_1\overline{\lambda}(t)), v\rangle_{D(A_1)'}dt = -\int_0^T \rho_1(t)\big\langle G_{u^*}(t), v\big\rangle_{D(A_1)'}dt,\\
&\int_0^T \rho_2(t)\langle \partial_t\overline{\omega}(t), \varphi\rangle_{V_2'}dt = -\int_0^T \rho_2(t)\big\langle G_{B^*}(t), \varphi\big\rangle_{V_2'}dt
\end{aligned}$$

for all $\rho_1(t), \rho_2(t) \in C_0^\infty(0,T)$. Thus, we obtain

$$\begin{cases}
\langle \partial_t(\overline{\lambda}(t)+\alpha^2 A_1\overline{\lambda}(t)), v\rangle_{D(A_1)'} = -\big\langle G_{u^*}(t), v\big\rangle_{D(A_1)'},\\
\langle \partial_t\overline{\omega}(t), \varphi\rangle_{V_2'} = -\big\langle G_{B^*}(t), \varphi\big\rangle_{V_2'}
\end{cases} \tag{4.13}$$

for all $(v,\varphi) \in D(A_1)\times V_2$ and for a.e. $t\in(0,T)$. In other words, we have the existence and representation of the derivative $\big(\partial_t(\overline{\lambda}(t)+\alpha^2 A_1\overline{\lambda}(t)), \partial_t\overline{\omega}\big) = -(G_{u^*}(t), G_{B^*}(t))$ in the sense of vector-valued distributions. In the previous considerations we found $(G_{u^*}(t), G_{B^*}(t)) \in L^2(0,T;D(A_1)'\times V_2')$, which allows us to conclude that $\partial_t\overline{\lambda} \in L^2(0,T;H_1)$ and $\partial_t\overline{\omega}\in L^2(0,T;V_2')$.

Next, integrating by parts in the left-hand side of the first and second equations in (4.11), we get

$$\begin{cases}
\big\langle \overline{\lambda}(T)+\alpha^2 A_1\overline{\lambda}(T), \phi_1(T)\big\rangle_{D(A_1)'} - \langle \partial_t(\overline{\lambda}(t)+\alpha^2 A_1\overline{\lambda}(t)), \phi_1\rangle_{L^2(0,T;D(A_1)')}\\
\qquad\qquad = \big\langle G_{u^*}(t), \phi_1\big\rangle_{L^2(0,T;D(A_1)')} + \alpha_T(u^*(T)-u_T, \phi_1(T)),\\
\big\langle \overline{\omega}(T), \phi_2(T)\big\rangle_{V_2'} - \langle \partial_t\overline{\omega}, \phi_2\rangle_{L^2(0,T;V_2')}\\
\qquad\qquad = \big\langle G_{B^*}(t), \phi_2\big\rangle_{L^2(0,T;V_2')} + \alpha_T(B^*(T)-B_T, \phi_2(T)).
\end{cases} \tag{4.14}$$

Taking $v=\phi_1(t)$ and $\varphi=\phi_2(t)$ in (4.13), then integrating from 0 to T and using (4.14), we get

$$\begin{aligned}
\big\langle \overline{\lambda}(T)+\alpha^2 A_1\overline{\lambda}(T), \phi_1(T)\big\rangle_{D(A_1)'} &= \alpha_T(u^*(T)-u_T, \phi_1(T)),\\
\big\langle \overline{\omega}(T), \phi_2(T)\big\rangle_{V_2'} &= \alpha_T(B^*(T)-B_T, \phi_2(T)),
\end{aligned}$$

for all $\phi=(\phi_1,\phi_2)\in W_0$. Since $\phi(T)$ is arbitrary in $D(A_1)\times V_2$, $(\overline{\lambda},\overline{\omega})$ satisfies the last equation in (4.2). This and (4.13) imply that $(\overline{\lambda},\overline{\omega})$ is a weak solutions of the equations (4.2).

Finally, we will establish a necessary optimality condition. By hypothesis, $J(z,h)-J(z^*,h^*)\geq 0$. On the other hand,

$$\begin{aligned}
J(z,h)-J(z^*,h^*) =&\beta\bigg(\alpha_T\int_\Omega s(x,T)\cdot\big(z^*(x,T)-z_T(x)\big)dx\\
&+\alpha_Q\iint_Q s(x,t)\cdot\big(z^*(x,t)-z_Q(x,t)\big)dxdt\\
&+\gamma\iint_Q h^*(x,t)\cdot\overline{h}(x,t)dxdt\bigg)+\beta Z_\beta,
\end{aligned}$$

where

$$\begin{aligned} Z_\beta =&\beta\left(\frac{\alpha_T}{2}\int_\Omega |s(x,T)+s_\beta(x,T)|^2dx + \frac{\alpha_Q}{2}\iint_Q |s(x,t)+s_\beta(x,t)|^2dxdt \right.\\ &\left. +\frac{\gamma}{2}\iint_Q |\overline{h}(x,t)|^2dxdt\right) + \alpha_T\int_\Omega s_\beta(x,T)\cdot\big(z^*(x,T)-z_T(x)\big)dx \\ &+\alpha_Q\iint_Q s_\beta(x,t)\cdot\big(z^*(x,t)-z_Q(x,t)\big)dxdt. \end{aligned} \tag{4.15}$$

Since $s, s_\beta \in C([0,T];V_1\times H_2)$ and $\overline{h}\in L^2(0,T;\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$, we have the first three integral terms of Z_β tend to 0 as $\beta\to 0^+$. Moreover, since the boundedness of z^* in $C([0,T];H_1\times H_2)$ and $s_\beta\to(0,0)$ in $C([0,T];V_1\times H_2)$ as $\beta\to 0^+$, we have the last two integral terms of (4.15) also tend to 0 as $\beta\to0^+$. As a result, we have $Z_\beta\to 0$ as $\beta\to 0^+$.

Dividing $J(z,h)-J(z^*,h^*)$ by β and taking limits as $\beta\to0^+$, we obtain

$$\begin{aligned} \alpha_T\int_\Omega s(x,T)\cdot(z^*(x,T)-z_T(x))dx + \alpha_Q\iint_Q s(x,t)\cdot\big(z^*(x,t)-z_Q(x,t)\big)dxdt \\ +\gamma\iint_Q h^*(x,t)\cdot\overline{h}(x,t)dxdt \ge 0. \end{aligned} \tag{4.16}$$

Multiplying the first equation of (4.2) and (4.5) by w and $\overline{\lambda}$, the second equation of (4.2) and (4.5) by θ and $\overline{\omega}$, respectively, then integrating over Q and using integration by parts yield the following identity

$$\begin{aligned} \alpha_T\int_\Omega s(x,T)\cdot(z^*(x,T)-z_T(x))dx + \alpha_Q\iint_Q s(x,t)\cdot(z^*(x,t)-z_Q(x,t))dxdt \\ =\iint_Q \overline{y}(x,t)\cdot\overline{h}(x,t)dxdt. \end{aligned}$$

This together with (4.16) give the inequality

$$\iint_Q (\overline{y}+\gamma h^*)\cdot\overline{h}dxdt \ge 0. \tag{4.17}$$

The inequality (4.17) must hold for any feasible direction $\overline{h}$ at h^*. For any $\zeta\in U_{ad}$, $\zeta - h^*$ is a feasible direction, so we can take $\overline{h}=\zeta-h^*$ and get (4.4). From (4.1), (4.17) imply (4.3) and we have completed the proof. □

5. Second-order sufficient optimality conditions

In this section, we will show the second-order sufficient optimality conditions for **Problem P**.

Theorem 5.1. *Let (z^*,h^*) be the admissible pair and suppose that (z^*,h^*) satisfies, together with the adjoint state $\overline{y}=(\overline{\lambda},\overline{\omega})$, the first-order necessary optimality conditions, i.e., the equations* (4.2) *and the inequality* (4.3). *Furthermore, we assume that the pair (z^*,h^*) satisfies the following assumption, in the sequel called* *the second-order sufficient condition. Then, it holds*

$$\begin{aligned}\frac{\alpha_T}{2}\int_\Omega |s(x,T)|^2dx + \frac{\alpha_Q}{2}\iint_Q |s(x,t)|^2dxdt + \frac{\gamma}{2}\iint_Q |\overline{h}(x,t)|^2dxdt\\ + \int_0^T \big(b(\theta(t),\theta(t),\overline{\lambda}(t)) - \overline{b}(w(t), w(t)+\alpha^2A_1w(t),\overline{\lambda}(t))\big)dt\\ + \int_0^T \overline{b}(w(t),\overline{\omega}(t),\theta(t))dt > 0,\end{aligned} \tag{5.1}$$

for all $\overline{h}\in\mathcal{T}_{U_{ad}}(h^*)\cap\mathcal{C}(h^*)$*, where*

$$\mathcal{C}(h^*) = \left\{\overline{h}\in\big(L^2(Q)\big)^3\times\big(L^2(Q)\big)^3 : \iint_Q(\overline{y}+\gamma h^*)\cdot\overline{h}dxdt = 0\right\}$$

and $s=(w,\theta)$ *is the unique solution of the following problem*

$$\begin{cases}\partial_t(w+\alpha^2A_1w)+\nu A_1(w+\alpha^2A_1w)+\widetilde{\mathcal{B}}(u^*,w+\alpha^2A_1w)\\ \qquad +\widetilde{\mathcal{B}}(w,u^*+\alpha^2A_1u^*) = \mathcal{B}(B^*,\theta)+\mathcal{B}(\theta,B^*)+\overline{h}_1, & x\in\Omega, t>0,\\ \partial_t\theta+\eta A_2\theta+\mathcal{B}(u^*,\theta)-\mathcal{B}(\theta,u^*)+\mathcal{B}(w,B^*)-\mathcal{B}(B^*,w)=\overline{h}_2, & x\in\Omega, t>0,\\ \nabla\cdot w=\nabla\cdot\theta=0, & x\in\Omega,t>0,\\ w=\Delta w=0,\quad \theta\cdot n=0, & x\in\partial\Omega, t>0,\\ w(x,0)=0,\quad \theta(x,0)=0, & x\in\Omega.\end{cases} \tag{5.2}$$

Then there exists $\varepsilon>0$ *and* $\rho>0$ *such that*

$$J(z,h)\geq J(z^*,h^*)+\varepsilon\|h-h^*\|^2_{(L^2(Q))^3\times(L^2(Q))^3} \tag{5.3}$$

holds for all admissible pairs (z,h) *with* $\|h-h^*\|^2_{(L^2(Q))^3\times(L^2(Q))^3}\leq\rho$*. In particular,* h^* *is a locally optimal control with associated state* z^**.*

Proof. Let $\overline{h}$ be in $\mathcal{F}_{U_{ad}\cap(h^*+\mathcal{C}(h^*))}$. Take $h=h^*+\beta\overline{h}$, we have $h\in U_{ad}\cap(h^*+\mathcal{C}(h^*))$ with small enough $\beta\in\mathbb{R}^+$. Let z be a state associated to h. We can write (where $s=(w,\theta)$, $\widetilde{s}=(\widetilde{w},\widetilde{\theta})$ and $s_\beta=(w_\beta,\theta_\beta)$)

$$z=z^*+\beta s+\frac{\beta^2}{2}\widetilde{s}+\beta^2 s_\beta$$

with $\widetilde{s}$ is a weak solution of the following equations

$$\begin{cases}\partial_t(\widetilde{w}+\alpha^2A_1\widetilde{w})+\nu A_1(\widetilde{w}+\alpha^2A_1\widetilde{w})\\ \qquad +\widetilde{\mathcal{B}}(u^*,\widetilde{w}+\alpha^2A_1\widetilde{w})+\widetilde{\mathcal{B}}(\widetilde{w},u^*+\alpha^2A_1u^*)\\ \qquad -\mathcal{B}(B^*,\widetilde{\theta})-\mathcal{B}(\widetilde{\theta},B^*) = -2\widetilde{\mathcal{B}}(w,w+\alpha^2A_1w)+2\mathcal{B}(\theta,\theta), & x\in\Omega,t>0,\\ \partial_t\widetilde{\theta}+\eta A_2\widetilde{\theta}+\mathcal{B}(u^*,\widetilde{\theta})-\mathcal{B}(\widetilde{\theta},u^*)\\ \qquad +\mathcal{B}(\widetilde{w},B^*)-\mathcal{B}(B^*,\widetilde{w}) = 2\mathcal{B}(\theta,w)-2\mathcal{B}(w,\theta), & x\in\Omega,t>0,\\ \nabla\cdot\widetilde{w}=\nabla\cdot\widetilde{\theta}=0, & x\in\Omega,t>0,\\ \widetilde{w}=\Delta\widetilde{w}=0,\quad \widetilde{\theta}\cdot n=0, & x\in\partial\Omega,t>0,\\ \widetilde{w}(x,0)=0,\qquad \widetilde{\theta}(x,0)=0, & x\in\Omega,\end{cases} \tag{5.4}$$

and s_β is a weak solution of the following equations

$$\begin{cases}
\partial_t(w_\beta+\alpha^2A_1w_\beta)+\nu A_1(w_\beta+\alpha^2A_1w_\beta)\\
\qquad +\widetilde{\mathcal{B}}(u^*,w_\beta+\alpha^2A_1w_\beta)+\widetilde{\mathcal{B}}(w_\beta,u^*+\alpha^2A_1u^*)\\
\qquad +\beta\widetilde{\mathcal{B}}(w,w_\beta+\alpha^2A_1w_\beta)+\beta\widetilde{\mathcal{B}}(w_\beta,w+\alpha^2A_1w)\\
\qquad +\frac{\beta}{2}\widetilde{\mathcal{B}}(w,\widetilde{w}+\alpha^2A_1\widetilde{w})+\frac{\beta}{2}\widetilde{\mathcal{B}}(\widetilde{w},w+\alpha^2A_1w)\\
\qquad +\frac{\beta^2}{2}\widetilde{\mathcal{B}}(w_\beta,\widetilde{w}+\alpha^2A_1\widetilde{w})+\frac{\beta^2}{2}\widetilde{\mathcal{B}}(\widetilde{w},w_\beta+\alpha^2A_1w_\beta)\\
\qquad +\beta^2\widetilde{\mathcal{B}}(w_\beta,w_\beta+\alpha^2A_1w_\beta)+\frac{\beta^2}{4}\widetilde{\mathcal{B}}(\widetilde{w},\widetilde{w}+\alpha^2A_1\widetilde{w})\\
\qquad =\mathcal{B}(B^*,\theta_\beta)+\mathcal{B}(\theta_\beta,B^*)+\beta\mathcal{B}(\theta,\theta_\beta)+\beta\mathcal{B}(\theta_\beta,\theta)\\
\qquad +\frac{\beta}{2}\mathcal{B}(\theta,\widetilde{\theta})+\frac{\beta}{2}\mathcal{B}(\widetilde{\theta},\theta)+\frac{\beta^2}{2}\mathcal{B}(\widetilde{\theta},\theta_\beta)+\frac{\beta^2}{2}\mathcal{B}(\theta_\beta,\widetilde{\theta})\\
\qquad +\beta^2\mathcal{B}(\theta_\beta,\theta_\beta)+\frac{\beta^2}{4}\mathcal{B}(\widetilde{\theta},\widetilde{\theta}), & x\in\Omega,t>0,\\
\partial_t\theta_\beta+\eta A_2\theta_\beta+\mathcal{B}(u^*,\theta_\beta)-\mathcal{B}(\theta_\beta,u^*)+\mathcal{B}(w_\beta,B^*)-\mathcal{B}(B^*,w_\beta)\\
\qquad +\beta\big(\mathcal{B}(w,\theta_\beta)-\mathcal{B}(\theta_\beta,w)+\mathcal{B}(w_\beta,\theta)-\mathcal{B}(\theta,w_\beta)\big)\\
\qquad +\frac{\beta}{2}\big(\mathcal{B}(w,\widetilde{\theta})-\mathcal{B}(\widetilde{\theta},w)+\mathcal{B}(\widetilde{w},\theta)-\mathcal{B}(\theta,\widetilde{w})\big)\\
\qquad +\frac{\beta^2}{2}\big(\mathcal{B}(\widetilde{w},\theta_\beta)-\mathcal{B}(\theta_\beta,\widetilde{w})+\mathcal{B}(w_\beta,\widetilde{\theta})-\mathcal{B}(\widetilde{\theta},w_\beta)\big)\\
\qquad +\beta^2\big(\mathcal{B}(w_\beta,\theta_\beta)-\mathcal{B}(\theta_\beta,w_\beta)\big)+\frac{\beta^2}{4}\big(\mathcal{B}(\widetilde{w},\widetilde{\theta})-\mathcal{B}(\widetilde{\theta},\widetilde{w})\big)=0, & x\in\Omega,t>0,\\
\nabla\cdot w_\beta=\nabla\cdot\theta_\beta=0, & x\in\Omega,t>0,\\
w_\beta=\Delta w_\beta=0,\quad \theta_\beta\cdot n=0, & x\in\partial\Omega,t>0,\\
w_\beta(x,0)=0,\quad \theta_\beta(x,0)=0, & x\in\Omega.
\end{cases} \tag{5.5}$$

By using similar arguments as in the proof of Theorem 3.1 in [20], we can show that (5.4) possesses exactly one weak solution $\widetilde{s}\in C([0,T];V_1\times H_2)\cap L^2(0,T;D(A_1)\times V_2)$ with $\partial_t\widetilde{s}\in L^2(0,T;H_1\times V_2')$, and that for any $\beta>0$, (5.5) also has a unique weak solution $s_\beta\in C([0,T];V_1\times H_2)\cap L^2(0,T;D(A_1)\times V_2)$ and $\partial_t s_\beta\in L^2(0,T;H_1\times V_2')$.

Analogously as the proof of Theorem 4.4, we obtain $s_\beta\to(0,0)$ in $C([0,T];V_1\times H_2)$ as $\beta\to0^+$. On the other hand

$$\begin{aligned}
J(z,h)-J(z^*,h^*)=&\beta\bigg(\alpha_T\int_\Omega s(x,T)\cdot\big(z^*(x,T)-z_T(x)\big)dx\\
&+\alpha_Q\iint_Q s(x,t)\cdot\big(z^*(x,t)-z_Q(x,t)\big)dxdt\\
&+\gamma\iint_Q h^*(x,t)\cdot\overline{h}(x,t)dxdt\bigg)\\
&+\beta^2\bigg(\frac{\alpha_T}{2}\int_\Omega\Big(|s(x,T)|^2+\widetilde{s}(x,T)\cdot\big(z^*(x,T)-z_T(x)\big)\Big)dx\\
&+\frac{\alpha_Q}{2}\iint_Q\Big(|s(x,t)|^2+\widetilde{s}(x,t)\cdot\big(z^*(x,t)-z_Q(x,t)\big)\Big)dxdt\\
&+\frac{\gamma}{2}\iint_Q|\overline{h}(x,t)|^2dxdt\bigg)+\beta^2Z_\beta
\end{aligned}$$

$$
\begin{aligned}
&= \beta \iint_Q (\overline{y} + \gamma h^*) \cdot \overline{h} dxdt \\
&+ \beta^2 \left(\frac{\alpha_T}{2} \int_\Omega \Big(|s(x,T)|^2 + \widetilde{s}(x,T) \cdot \big(z^*(x,T) - z_T(x) \big) \Big) dx \right. \\
&+ \frac{\alpha_Q}{2} \iint_Q \Big(|s(x,t)|^2 + \widetilde{s}(x,t) \cdot \big(z^*(x,t) - z_Q(x,t) \big) \Big) dxdt \\
&+ \left. \frac{\gamma}{2} \iint_Q |\overline{h}(x,t)|^2 dxdt \right) + \beta^2 Z_\beta.
\end{aligned}
$$

Multiplying the first and second equations of (4.2) by $\widetilde{w}$ and $\widetilde{\theta}$ pointwise with respect to time, respectively, and integrating from 0 to T, then integrating by parts we obtain

$$
\begin{aligned}
&\int_0^T \Big((\overline{\lambda}, \partial_t \widetilde{w}) + \alpha^2 ((\overline{\lambda}, \partial_t \widetilde{w}))_1 \Big) dt + \nu \int_0^T \Big(((\overline{\lambda}, \widetilde{w}))_1 + \alpha^2 (A_1 \overline{\lambda}, A_1 \widetilde{w}) \Big) dt \\
&+ \int_0^T \big(\overline{b}(u^*, \widetilde{w} + \alpha^2 A_1 \widetilde{w}, \overline{\lambda}) + \overline{b}(\widetilde{w}, u^* + \alpha^2 A_1 u^*, \overline{\lambda}) \big) dt - \int_0^T \big(b(B^*, \widetilde{\theta}, \overline{\lambda}) + b(\widetilde{\theta}, B^*, \overline{\lambda}) \big) dt \\
&= \iint_Q \alpha_Q (u^* - u_Q) \cdot \widetilde{w} dxdt + \big(\overline{\lambda}(T), \widetilde{w}(T) \big) + \alpha^2 \big(\big(\overline{\lambda}(T), \widetilde{w}(T) \big) \big)_1, \\
&\int_0^T (\overline{\omega}, \partial_t \widetilde{\theta}) dt + \eta \int_0^T ((\overline{\omega}, \widetilde{\theta}))_2 dt + \int_0^T \big(b(u^*, \widetilde{\theta}, \overline{\omega}) - b(\widetilde{\theta}, u^*, \overline{\omega}) \big) dt \\
&+ \int_0^T \big(b(\widetilde{w}, B^*, \overline{\omega}) - b(B^*, \widetilde{w}, \overline{\omega}) \big) dt = \iint_Q \alpha_Q (B^* - B_Q) \cdot \widetilde{\theta} dxdt + \big(\overline{\omega}(T), \widetilde{\theta}(T) \big).
\end{aligned}
\tag{5.6}
$$

Multiplying the first and second equations of (5.4) by $\overline{\lambda}$ and $\overline{\omega}$ pointwise with respect to time, respectively, and then integrating from 0 to T, by using (2.2), (2.5) and from (5.6), we deduce

$$
\begin{aligned}
&2 \int_0^T \Big(b(\theta(t), \theta(t), \overline{\lambda}(t)) - \overline{b}(w(t), w(t) + \alpha^2 A_1 w(t), \overline{\lambda}(t)) \Big) dt \\
&\qquad = \iint_Q \alpha_Q (u^* - u_Q) \cdot \widetilde{w} dxdt + \big(\overline{\lambda}(T), \widetilde{w}(T) \big) + \alpha^2 \big(\big(\overline{\lambda}(T), \widetilde{w}(T) \big) \big)_1, \\
&2 \int_0^T \overline{b}(w(t), \overline{\omega}(t), \theta(t)) dt = \iint_Q \alpha_Q (B^* - B_Q) \cdot \widetilde{\theta} dxdt + \big(\overline{\omega}(T), \widetilde{\theta}(T) \big).
\end{aligned}
$$

From this and the last equation of (4.2), we get

$$
\begin{aligned}
&\alpha_T \int_\Omega \widetilde{s}(x,T) \cdot \big(z^*(x,T) - z_T(x) \big) dx + \alpha_Q \iint_Q \widetilde{s}(x,t) \cdot \big(z^*(x,t) - z_Q(x,t) \big) dxdt \\
&= 2 \int_0^T \big(b(\theta(t), \theta(t), \overline{\lambda}(t)) - \overline{b}(w(t), w(t) + \alpha^2 A_1 w(t), \overline{\lambda}(t)) \big) dt \\
&\quad + 2 \int_0^T \overline{b}(w(t), \overline{\omega}(t), \theta(t)) dt.
\end{aligned}
$$

Thus, we deduce

$$\begin{aligned}
&J(z,h)-J(z^*,h^*)=\beta\iint_Q(\overline{y}+\gamma h^*)\cdot\overline{h}dxdt\\
&+\beta^2\left(\frac{\alpha_T}{2}\int_\Omega|s(x,T)|^2dx+\frac{\alpha_Q}{2}\iint_Q|s(x,t)|^2dxdt+\frac{\gamma}{2}\iint_Q|\overline{h}(x,t)|^2dxdt\right.\\
&+\int_0^T\big(b(\theta(t),\theta(t),\overline{\lambda}(t))-\overline{b}(w(t),w(t)+\alpha^2A_1w(t),\overline{\lambda}(t))\big)dt\\
&\left.+\int_0^T\overline{b}(w(t),\overline{\omega}(t),\theta(t))dt\right)+\beta^2Z_\beta.
\end{aligned}\tag{5.7}$$

Next, we set

$$\begin{aligned}
q(s,\overline{h})=&\frac{\alpha_T}{2}\int_\Omega|s(x,T)|^2dx+\frac{\alpha_Q}{2}\iint_Q|s(x,t)|^2dxdt+\frac{\gamma}{2}\iint_Q|\overline{h}(x,t)|^2dxdt\\
&+\int_0^T\big(b(\theta(t),\theta(t),\overline{\lambda}(t))-\overline{b}(w(t),w(t)+\alpha^2A_1w(t),\overline{\lambda}(t))\big)dt\\
&+\int_0^T\overline{b}(w(t),\overline{\omega}(t),\theta(t))dt,
\end{aligned}$$

where $\overline{h}\in\big(L^2(Q)\big)^3\times\big(L^2(Q)\big)^3$ and s is the unique solution to the problem (5.2).

Let us suppose that the first-order necessary and the second-order sufficient conditions are satisfied, whereas (5.3) does not hold. Then for all $\varepsilon>0$ and $\rho>0$ there exists $h_{\varepsilon,\rho}\in U_{ad}$ with $\|h_{\varepsilon,\rho}-h^*\|_{(L^2(Q))^3\times(L^2(Q))^3}\le\rho$ and

$$J(z_{\varepsilon,\rho},h_{\varepsilon,\rho})<J(z^*,h^*)+\varepsilon\|h_{\varepsilon,\rho}-h^*\|^2_{\left(L^2(Q)\right)^3\times\left(L^2(Q)\right)^3},$$

where $z_{\varepsilon,\rho}$ is the state associated with $h_{\varepsilon,\rho}$. Hence, for any $k\in\mathbb{Z}^+$, let us choose $\varepsilon_k=\rho_k=1/k$ and $z_k=z_{\varepsilon,\rho}$, $h_k=h_{\varepsilon,\rho}$, then we have

$$J(z_k,h_k)<J(z^*,h^*)+\frac{1}{k}\|h_k-h^*\|^2_{(L^2(Q))^3\times(L^2(Q))^3}\tag{5.8}$$

and $\|h_k-h^*\|_{(L^2(Q))^3\times(L^2(Q))^3}<\dfrac{1}{k}$.

By the construction, it follows that $h_k\to h^*$ in $\big(L^2(Q)\big)^3\times\big(L^2(Q)\big)^3$ as $k\to\infty$. Hence, we can write $h_k=h^*+t_k\overline{h}_k$, where $t_k>0$, $\overline{h}_k\in\mathcal{F}_{U_{ad}}(h^*)$ and $\|\overline{h}_k\|_{(L^2(Q))^3\times(L^2(Q))^3}=1$ and $t_k\to0$ as $k\to\infty$. Because of the boundedness of the set of these $\{\overline{h}_k\}$ in $(L^2(Q))^3\times(L^2(Q))^3$, we can extract a subsequence denoted again by $\{\overline{h}_k\}$ converging weakly in $\mathcal{T}_{U_{ad}}(h^*)\subset(L^2(Q))^3\times(L^2(Q))^3$ to some limit $\hat{h}$. The set $\mathcal{T}_{U_{ad}}(h^*)$ is convex and closed, so it is weakly closed, therefore $\hat{h}\in\mathcal{T}_{U_{ad}}(h^*)$. Moreover, by Theorem 2.4, there exists a unique solution s_k to the problem (5.2) with the right-hand side of the first two equations $\overline{h}_k$. And we obtain that the set $\{s_k\}$ is bounded in $L^2(0,T;D(A_1)\times V_2)$ and $\{\partial_ts_k\}\in L^2(0,T;H_1\times V_2')$. Thus, we can extract subsequence (again indexed with k) $\{s_k\}$ converging weakly to $\hat{s}\in L^2(0,T;D(A_1)\times V_2)$, $\partial_t\hat{s}\in L^2(0,T;H_1\times V_2')$. By the Compactness Lemma (see [24]) we deduce $s_k\to\hat{s}$ in $\big(L^2(Q)\big)^3\times\big(L^2(Q)\big)^3$ as $k\to\infty$. It follows that $\hat{s}$ is the unique solution to the problem (5.2) with the right-hand side of the first two equations $\hat{h}$. We will show that $\hat{h}\in\mathcal{C}(h^*)$ and $q(\hat{s},\hat{h})\le0$, which contradicts (5.1) and so we get the claim.

Using the first-order necessary optimality condition (4.3), we have that

$$\iint_Q n \cdot \hat{h} dxdt \geq 0, \tag{5.9}$$

where $n = \overline{y} + \gamma h^*$. As in (5.7), we can write

$$J(z_k, h_k) = J(z^*, h^*) + t_k \iint_Q n \cdot \overline{h}_k dxdt + t_k^2 q(s_k, \overline{h}_k) + t_k^2 Z_k,$$

where $Z_k \to 0$ as $k \to \infty$. From (5.8) and $\|\overline{h}_k\|_{(L^2(Q))^3 \times (L^2(Q))^3} = 1$, we obtain

$$t_k \iint_Q n \cdot \overline{h}_k dxdt + t_k^2 q(s_k, \overline{h}_k) + t_k^2 Z_k < \frac{t_k^2}{k}. \tag{5.10}$$

This implies

$$\iint_Q n \cdot \overline{h}_k dxdt < \frac{t_k}{k} - t_k q(s_k, \overline{h}_k) - t_k Z_k. \tag{5.11}$$

From the boundedness of $\{\overline{h}_k\}$ and $\{s_k\}$ in $\big(L^2(Q)\big)^3 \times \big(L^2(Q)\big)^3$ and $C([0,T]; V_1 \times H_2) \cap L^2(0,T; D(A_1) \times V_2)$, respectively, there exists a constant $M > 0$ such that $|q(s_k, \overline{h}_k)| \leq M$ for all k. Passing limit in (5.11) we obtain

$$\iint_Q n \cdot \hat{h} dxdt \leq 0.$$

This and (5.9) imply that $\hat{h} \in \mathcal{C}(h^*)$. Since $\hat{h} \in \mathcal{T}_{U_{ad}}(h^*)$, we conclude that $\hat{h} \in \mathcal{T}_{U_{ad}}(h^*) \cap \mathcal{C}(h^*)$. Finally, we have to show that $q(\hat{s}, \hat{h}) \leq 0$.

Since $\overline{h}_k \in \mathcal{F}_{U_{ad}}(h^*) \subset \mathcal{T}_{U_{ad}}(h^*)$, by the first order necessary condition (4.3), we deduce that $\iint_Q n \cdot \overline{h}_k dxdt \geq 0$. From (5.10), we have that

$$q(s_k, \overline{h}_k) < \frac{1}{k} - Z_k.$$

Set $q^*(s, \overline{h}) = q(s, \overline{h}) - \dfrac{\gamma}{2} \iint_Q |\overline{h}(x,t)|^2 dxdt$, we will show that

$$q^*(s_k, \overline{h}_k) \to q^*(\hat{s}, \hat{h}), \quad \text{as } k \to \infty. \tag{5.12}$$

Since $s_k \to \hat{s}$ weakly in $L^2(0,T; D(A_1) \times V_2)$, follow the lines in the proof of Theorem 2.4, we have

$$\int_0^T \overline{b}\big(w_k(t), w_k(t) + \alpha^2 A_1 w_k(t), \overline{\lambda}(t)\big) dt \to \int_0^T \overline{b}\big(\hat{w}(t), \hat{w}(t) + \alpha^2 A_1 \hat{w}(t), \overline{\lambda}(t)\big) dt,$$

$$\int_0^T b\big(\theta_k(t), \theta_k(t), \overline{\lambda}(t)\big) dt \to \int_0^T b\big(\hat{\theta}(t), \hat{\theta}(t), \overline{\lambda}(t)\big) dt,$$

and

$$\int_0^T \overline{b}\big(w_k(t), \overline{\omega}(t), \theta_k(t)\big) dt = \int_0^T \big[b\big(\theta_k(t), w_k(t), \overline{\omega}\big) - b\big(w_k(t), \theta_k(t), \overline{\omega}\big)\big] dt$$

$$\to \int_0^T \big[b\big(\hat{\theta}(t), \hat{w}(t), \overline{\omega}\big) - b\big(\hat{w}(t), \hat{\theta}(t), \overline{\omega}\big)\big] dt = \int_0^T \overline{b}\big(\hat{w}(t), \overline{\omega}(t), \hat{\theta}(t)\big) dt.$$

Moreover, $s_k \to \hat{s}$ in $\big(L^2(Q)\big)^3 \times \big(L^2(Q)\big)^3$, we have

$$\frac{\alpha_Q}{2} \iint_Q |s_k(x,t)|^2 dxdt \to \frac{\alpha_Q}{2} \iint_Q |\hat{s}(x,t)|^2 dxdt.$$

Besides, with the initial $\theta_0 = 0 \in V_2$, from Theorem 2.4, we have that $\theta \in C([0,T];V_2)$, together with the boundedness of w in $C([0,T];V_1)$, we deduce $s_k \to \hat{s}$ weakly in $C([0,T];V_1 \times V_2)$ and from the fact that $V_1 \times V_2$ is compactly embebded in $H_1 \times H_2$, we get $s_k(T) \to \hat{s}(T)$ in $H_1 \times H_2$. Thus, we obtain (5.12).

Finally, we have $\overline{h}_k \to \hat{h}$ weakly in $\left(L^2(Q)\right)^3 \times \left(L^2(Q)\right)^3$ and $\|\overline{h}_k\|_{(L^2(Q))^3 \times (L^2(Q))^3} = 1$, then $\|\hat{h}\|_{(L^2(Q))^3 \times (L^2(Q))^3} \le 1$. Therefore,

$$q(\hat{s}, \hat{h}) \le \lim_{k\to\infty} q(s_k, \overline{h}_k) \le 0,$$

which contradicts (5.1) and we finish the proof. □

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