NULL CONTROLLABILITY WITH CONSTRAINTS ON THE STATE FOR A SEMILINEAR NONLOCAL PARABOLIC SYSTEM

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ABSTRACT. This paper is addressed to study a null controllability problem with finite number of constraints on the state for a parabolic system with local and nonlocal nonlinearities. We first transform the linearized problem into an equivalent problem of null controllability problem with constraint on the control. Then, by using a suitable Carleman inequality adapted to the constraint we get the null controllability with constraint on the control. Then, the main result is proved by an application of a fixed-point method.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N, (N \in \mathbb{N}_+)$ be a bounded open set with boundary $\partial\Omega$ of class C^2. For a time $T > 0$, we set $Q = \Omega \times (0,T), \Sigma = \partial\Omega \times (0,T)$.

We are concerned with the following initial boundary valued problem for a parabolic coupled system with nonlinear terms of local and nonlocal kinds

$$\begin{cases} y_t - a(\int_\Omega y dx, \int_\Omega z dx)\Delta y + f(y,z) = v1_\omega & \text{in} \quad Q, \\ z_t - b(\int_\Omega y dx, \int_\Omega z dx)\Delta z + g(y,z) = 0 & \text{in} \quad Q, \\ y = z = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y_0(x), \quad z(x,0) = z_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where v is the control, (y,z) is the state and 1_ω is the characteristic function of a non-empty open subset ω of Ω. Moreover, $a(r,s)$, $b(r,s)$, $f(r,s)$ and $g(r,s)$ are C^1 functions with bounded derivatives and satisfy

$$0 < L_1 \le a(r,s); b(r,s) \le L_2 < +\infty, \ \ \forall (r,s) \in \mathbb{R} \times \mathbb{R}, \tag{1.2}$$

and

$$f(0,0) = g(0,0) = 0, \tag{1.3}$$

and there exist positive constants c_0 and T_0 such that

$$\left|\frac{\partial g}{\partial y}\right| \ge c_0 > 0 \text{ in } \omega \times (0,T_0). \tag{1.4}$$

This condition implies (2.5) which needs to get Lemma 2.5.

As in [11], if $y_0,\ z_0 \in L^2(\Omega)$, $v \in L^2(Q)$, the functions a, b, f and g satisfy (1.2)-(1.3), then the system (1.1) admits a unique weak solution (y,z) satisfying

$$(y,z) \in \Big(L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))\Big)^2, (y_t,z_t) \in (L^2(0,T;H^{-1}(\Omega)))^2.$$

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The system (1.1) is said to be nonlocal in view of structure of the diffusion coefficients. It appears in some phenomena which nonlocal terms appear naturally. To this respect, let us list several examples of real physical models as follows.

In the case of migration of populations such as the bacteria in a container, we can find in practice that, see [4, 18],

$$a = a\Big(\int_\Omega y(x,t)dx, \int_\Omega z(x,t)dx, \int_\Omega \nabla y(x,t)dx, \int_\Omega \nabla z(x,t)dx\Big).$$

In the case of reaction-diffusion equations, see [3], the nonlocal coefficient can be found by form

$$a = a(\langle h, y(.,t)\rangle, \langle k, z(.,t)\rangle),$$

where h, k are continuous linear form on $L^2(\Omega)$.

In the case of hyperbolic equations, we also find the terms of this kind, see [12] for instance

$$a = a\Big(\int_\Omega |\nabla y|^2 dx, \int_\Omega |\nabla z|^2 dx\Big).$$

We can find more details from the above references. In addition, we refer [5, 8] and the references therein. In last decade, the control of reaction-diffusion system has attracted the interest of the control community. The controllability of nonlinear parabolic systems have been studied by many authors in recent years. In case of $a = b = 1$, the null controllability of some reaction-diffusion systems are studied (see [1, 9, 10, 17]). In [5], the authors obtained the local null controllability and numerical experiments of (1.1). In [8], for $a = b = 1$, the author has proved the null controllability with constraints on the state of the reaction-diffusion system. In this paper, we focus on the null controllability problem with a finite number of constraints on the state that we describe now.

For any fixed $M \in \mathbb{N}_+$, we consider $e_i \in L^2(Q)$, $i = 1, \ldots, M$ are such that

$$\{e_i 1_\omega\}_{i=1}^M \quad \text{are linearly independent.} \tag{1.5}$$

It will be said that the problem (1.1) is null controllable with a finite number of constraints on the state at time T if given $e_i \in L^2(Q), i = 1, \ldots, M$, satisfy (1.5), and y_0, $z_0 \in L^2(\Omega)$, there exists control $v \in L^2(Q)$ such that the associated states (y, z) satisfy

$$y(x,T) = z(x,T) = 0 \quad \text{in} \quad \Omega, \tag{1.6}$$

and

$$\iint_Q z e_i dxdt = 0, \qquad \text{for } i = 1, \ldots, M. \tag{1.7}$$

Our approach is based on earlier works on the local and global null controllability of parabolic equations and systems (see [5, 6, 8, 13, 14, 16]). The main tool to establish such a result is a Carleman estimate. Let us now give the main result and explain the methods used in the paper. We will prove following theorem.

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 and $e_i \in L^2(Q), i = 1, \ldots, M$ verifying* (1.5). *Assume that $y_0, z_0 \in L^2(\Omega)$, one can find a control $v \in L^2(Q)$ such that the solution (y, z) of* (1.1) *satisfies* (1.6) *and* (1.7).

This paper is in keeping with the idea of the framework can be found in [8, 13] and it is organized as follows. In Section 2, we prove the null controllability with constraints on the state of the linearized system by reducing to the null controllability with constraints on the control. The null controllability with constraint on the control of the linearized system is based on a new adapted Carleman estimate for an adjoint system respectively. The last section gives the null controllability with constraints on the state of the nonlinear system by using the null controllability of the linearized system and a fixed-point method.

Throughout this paper we use C to denote various constants, which may change from line to line. When the dependence of the constant on some index is important we highlight it in the notation.

2. Null controllability of the linearized system at zero with constraints on the state

Since we prove the main result by using a fixed-point argument, we need to analyze first the controllability of the linearized system. For any fixed $(\widehat{y},\widehat{z})\in (L^2(Q))^2$, using the condition (1.2), we have the following linearized system of (1.1) at $(\widehat{y},\widehat{z})$,

$$\begin{cases} y_t - a\Delta y + A_1 y + B_1 z = v1_\omega & \text{in} \quad Q,\\ z_t - b\Delta z + A_2 y + B_2 z = 0 & \text{in} \quad Q,\\ y = z = 0 & \text{on} \quad \Sigma,\\ y(x,0) = y_0(x),\ z(x,0) = z_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where

$$a = a\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big),\ b = b\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big),$$
$$A_1 = \int_0^1 \frac{\partial f(\lambda\widehat{y},\lambda\widehat{z})}{\partial y}d\lambda,\ B_1 = \int_0^1 \frac{\partial f(\lambda\widehat{y},\lambda\widehat{z})}{\partial z}d\lambda, \tag{2.2}$$

and

$$A_2 = \int_0^1 \frac{\partial g(\lambda\widehat{y},\lambda\widehat{z})}{\partial y}d\lambda,\ B_2 = \int_0^1 \frac{\partial g(\lambda\widehat{y},\lambda\widehat{z})}{\partial z}d\lambda, \tag{2.3}$$

Due to the condition (1.2) we have

$$a, b \in [L_1, L_2].$$

Since $f(r,s)$ and $g(r,s)$ are real C^1 function with bounded derivatives, then A_1, A_2, B_1, B_2 satisfy

$$\begin{aligned} \|A_i(\widehat{y},\widehat{z})\|_{L^\infty(Q)} \le K, \quad i = 1,2,\\ \|B_i(\widehat{y},\widehat{z})\|_{L^\infty(Q)} \le K, \quad i = 1,2, \end{aligned} \tag{2.4}$$

with some positive constant K, and it follows from (1.4) that

$$|A_2| \ge c_0 > 0 \text{ in } \omega \times (0, T_0). \tag{2.5}$$

If $(y_0, z_0) \in (L^2(\Omega))^2$ and $v \in L^2(Q)$, then it is well-know that the problem (2.1) also has a unique solution $(y,z) \in \Big(C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))\Big)^2$.

The adjoint system of (2.1) is given by

$$\begin{cases} -\varrho_t - a\,\Delta\varrho + A_1\varrho + A_2\psi = 0 & \text{in} \quad Q,\\ -\psi_t - b\,\Delta\psi + B_1\varrho + B_2\psi = 0 & \text{in} \quad Q,\\ \varrho = \psi = 0 & \text{on} \quad \Sigma,\\ \varrho(x,T) = \varrho_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in} \quad \Omega. \end{cases} \tag{2.6}$$

We consider the null controllability with a finite number of constraints on the state as follows: *Given* $y_0, z_0 \in L^2(\Omega)$ *and* $e_i \in L^2(Q), i = 1, \dots, M$ *verifying* (1.5). *Find* $v \in L^2(Q)$ *such that the solution of* (2.1) *satisfies* (1.6) *and* (1.7).

In case of $a = b = 1$, [10] has considered the null controllability of (2.1) and [16] has considered its null controllability with constraint on the control. The null controllability with a finite number of constraints on the sate has studied by Gao [8]. However, the term $A_2 \geq c_0 > 0$ was required. To our situation, we will relax restriction, i.e., $|A_2| \geq c_0 > 0$. Moreover, the diffusion coefficients still depends on time. By exploiting wisely some estimates used by authors (see [7, 8, 10]), we still achieve the same results. To make it clear, let us give the main ideas.

We will need some well-known results from Fursikov [7] (see also [6]) that will be used subsequently. Let ω' be a subdomain of ω such that $\omega' \Subset \omega \Subset \Omega$, there exists a function $\beta \in C^2(\overline{\Omega})$ without critical points in $\overline{\Omega \setminus \omega'}$ satisfying

$$\frac{\partial \beta}{\partial \nu} \leq 0, \quad \text{on } \partial\Omega,$$

and

$$\min_{x \in \overline{\Omega \setminus \omega'}} |\nabla \beta(x)| > 0, \quad \min_{x \in \overline{\Omega}} \beta(x) \geq \max \left\{ \frac{3}{4} \|\beta\|_{L^\infty(\Omega)}, \ln 3 \right\},$$

where ν denotes the outward unit normal to $\partial\Omega$. We adopt the following notations with parameters $\lambda > 0$ and $\tau > 0$:

$$\varphi(x,t) = \frac{e^{\lambda \beta(x)}}{t(T-t)}, \quad \alpha(x,t) = \tau \frac{e^{\frac{4}{3}\lambda \|\beta\|_{L^\infty(\Omega)}} - e^{\lambda \beta(x)}}{t(T-t)},$$

$$\theta = e^{\alpha}, \;\; \gamma^2 = e^{(2-r)\alpha}, \;\; E^2 = \sum_{i=1}^{M} \|e_i\|^2_{L^\infty(Q)}.$$

$$L^2 = \|A_1\|^2_{L^\infty(Q)} + \|A_2\|^2_{L^\infty(Q)} + \|B_1\|^2_{L^\infty(Q)} + \|B_2\|^2_{L^\infty(Q)}.$$

The main result of this section is the following theorem.

Theorem 2.1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with boundary* $\partial\Omega$ *of class* C^2 *and* $e_i \in L^2(Q), i = 1, \dots, M$ *verifying* (1.5). *For every* y_0, $z_0 \in L^2(\Omega)$, *one can find a control* $v \in L^2(Q)$ *such that the solution* (y, z) *of* (2.1) *satisfies* (1.6) *and* (1.7).

2.1. **Equivalence to the controllability problem with constraint on the control.** In this section, we transform the linearized problem into an equivalent linear problem of null controllability with constraint on the control. The argument of the proof is inspired the work by Gao [8]. Therefore, by following step by step (see [8, Lemmas 2.3-2.6, Proposition 4.1]), we will obtain the same results, so we omit the proofs here and state the main results as follows.

Lemma 2.1. *The functions* $\{p_i 1_\omega\}_{i=1,\dots,M}$ *are linearly independent, where* (p_i, q_i) *are the solutions of the following systems.*

$$\begin{cases} -p_{it} - a\,\Delta p_i + A_1 p_i + A_2 q_i = 0 & \text{in} \quad Q, \\ -q_{it} - b\,\Delta q_i + B_1 p_i + B_2 q_i = e_i & \text{in} \quad Q, \\ p_i(x,t) = q_i(x,t) = 0 & \text{on} \quad \Sigma, \\ p_i(x,T) = q_i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.7}$$

Moreover, the functions $\{\theta^{-1} p_i 1_\omega\}_{i=1,\dots,M}$ *are also linearly independent.*

We define

$$\begin{aligned}&\mathcal{U}=\operatorname{span}\{p_1 1_\omega, p_2 1_\omega, \dots, p_M 1_\omega\},\\ &\mathcal{U}_\theta=\theta^{-1}\mathcal{U},\end{aligned} \tag{2.8}$$

where (p_i, q_i) is the solution to (2.7).

Lemma 2.2. *If (p, q) is the solution of the following system*

$$\begin{cases}-p_t - a\,\Delta p + A_1 p + A_2 q = 0 & in \quad \omega\times(0,T),\\ -q_t - b\,\Delta q + B_1 p + B_2 q = 0 & in \quad \omega\times(0,T),\\ p|_\omega \in \mathcal{U}.\end{cases}$$

Then p, q are identically zero in $\omega \times (0,T)$.

Lemma 2.3. *Let $y_0, z_0 \in L^2(\Omega)$. There then exists a unique $v_0 \in \mathcal{U}_\theta$ such that*

$$-\int_\Omega y_0 p_i(0)dx - \int_\Omega z_0 q_i(0)dx = \iint_{\omega\times(0,T)} v_0 p_i dxdt, \tag{2.9}$$

for $i = 1, \dots, M$, and where (p_i, q_i) are solution to (2.7). Moreover, there exists a constant $C = C(\Omega, T, L, E)$ such that

$$\begin{aligned}\|v_0\|_{L^2(\omega\times(0,T))} &\le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),\\ \|\theta v_0\|_{L^2(\omega\times(0,T))} &\le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}).\end{aligned}$$

Proposition 2.1. *Let y_0, $z_0 \in L^2(\Omega)$, $\mathcal{U}$ and $\mathcal{U}_\theta$ be defined in (2.8). The null controllability problem with constraints on the state (2.1), (1.6), (1.7) is equivalent to the following null controllability problem with constraint on the control: Given $v_0 \in \mathcal{U}_\theta$ verifying (2.9), find $v_\theta \in L^2(\omega \times (0,T))$ such that*

$$v_\theta \in \mathcal{U}^\perp, \tag{2.10}$$

and the solution to

$$\begin{cases}y_t - a\,\Delta y + A_1 y + B_1 z = (v_0 + v_\theta)1_\omega & in\ Q,\\ z_t - b\,\Delta z + A_2 y + B_2 z = 0 & in\ Q,\\ y(x,t) = z(x,t) = 0 & on\ \Sigma,\\ y(x,0) = y_0(x),\ z(x,0) = z_0(x) & in\ \Omega,\end{cases} \tag{2.11}$$

satisfies (1.6).

2.2. **Null controllability of the linearized system with constraint on the control.** Thanks to Proposition 2.1, we only focus on solving the null controllability problem with constraint on the control (2.10), (2.11) and (1.6). To do this, we use a Carleman inequality adapted to the constraint (2.10).

We begin with the following backward parabolic equation

$$\begin{cases}\xi_t + \mu(t)\Delta\xi = h(x,t) & \text{in } Q,\\ \xi = 0 & \text{on } \Sigma,\end{cases} \tag{2.12}$$

where $\mu(t)$ is a positive, bounded and continuously differentiable function, and $h \in L^2(Q)$.

Denoting

$$I(\lambda, s, \tau, \xi) = \iint_Q \varphi^{2s-1} e^{-2\alpha}\Big[\lambda^{-1}(|\xi_t|^2 + |\Delta\xi|^2) + \lambda\tau^2\varphi^2|\nabla\xi|^2 + \lambda^4\tau^4\varphi^4|\xi|^2\Big]dxdt.$$

for positive constants λ, s, τ. Because of the properties of $\mu(t)$, we imitate step by step of the proof of Theorem 7.1 in [7, p. 288], we get the following result.

Lemma 2.4. *Let ξ and h satisfy the problem* (2.12)*, and let $s \geq -3$. There exist $\lambda_0 > 0, \tau_0 > 0$ and a positive constant C such that, for $\lambda \geq \lambda_0$, $\tau \geq \tau_0$, we have*

$$I(\lambda, s, \tau, \xi) \leq C\Big[\tau \iint_Q \varphi^{2s} e^{-2\alpha}|h|^2 dxdt + \lambda^4\tau^4 \iint_{\omega'\times(0,T)} \varphi^{2s+3}e^{-2\alpha}|\xi|^2 dxdt\Big].$$

Furthermore, similarly to [10, Lemma 2, Theorem 3] (see aslo [16, Lemma 2.3, Theorem 2.4, Corollary 2.5]), we also have the following result.

Lemma 2.5. *Let (ϱ, ψ) be the solution to* (2.6) *and C is determined in Lemma 2.4. For any $\lambda \geq \lambda_0$, $\tau \geq \tau_1 := \frac{T^2}{4}\Big(\frac{C}{\lambda_0^4}\Big)^{1/3} L^{2/3}$ and $s \geq -3$, the following inequality*

$$\iint_Q \Big(|\varrho|^2 + |\psi|^2\Big)\zeta^{2s+3}e^{-2\alpha}dxdt \leq C\lambda^4 \iint_{\omega'\times(0,T)} \Big(|\varrho|^2 + |\psi|^2\Big)\zeta^{2s+3}e^{-2\alpha}dxdt$$

holds. Moreover, for all $r \in [0, 2)$, there exists a constant $C = C(T, L, c_0, r)$ such that

$$\iint_Q \Big(|\varrho|^2 + |\psi|^2\Big)e^{-2\alpha}dxdt \leq C \iint_{\omega\times(0,T)} |\varrho|^2 e^{-r\alpha}dxdt. \tag{2.13}$$

We now prove the following Carleman inequality.

Lemma 2.6. *(Adapted Carleman inequality). There exists a positive constant C such that for all the (ϱ, ψ) which is the solution to* (2.6)

$$\int_\Omega \Big(|\varrho(0)|^2+|\psi(0)|^2\Big)dx+\iint_Q \theta^{-2}\Big(|\varrho|^2+|\psi|^2\Big)dxdt \leq C\iint_Q \gamma^2|\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt,$$

where P is the orthogonal projection operator from $L^2(\omega\times(0,T))$ into $\mathcal{U}$.

Proof. It follows from (2.13) that

$$\iint_Q \theta^{-2}\Big(|\varrho|^2 + |\psi|^2\Big)dxdt \leq C \iint_{\omega\times(0,T)} \frac{e^{(2-r)\alpha}}{\theta^2}|\varrho|^2 dxdt,$$

where $\theta^{-2} = e^{-2\alpha}$ and $r \in [0, 2)$. Then replacing $\gamma^2 = e^{(2-r)\alpha}$ and using the boundedness of θ^{-2}, we obtain that

$$\iint_Q \theta^{-2}\Big(|\varrho|^2 + |\psi|^2\Big)dxdt \leq C \iint_Q \gamma^2|\varrho 1_\omega|^2 dxdt. \tag{2.14}$$

We first prove that there exists a positive constant C such that

$$\iint_Q \theta^{-2}\Big(|\varrho|^2 + |\psi|^2\Big)dxdt \leq C \iint_Q \gamma^2|\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt \tag{2.15}$$

We argue by contradiction, i.e., suppose that (2.15) does not hold. Then for any n, there exists $\varrho_T^n(x), \psi_T^n(x) \in L^2(\Omega)$ satisfying

$$\iint_Q \theta^{-2}\Big(|\varrho_n|^2 + |\psi_n|^2\Big)dxdt = 1, \tag{2.16}$$

$$\iint_Q \gamma^2|\varrho_n 1_\omega - P(\varrho_n 1_\omega)|^2 dxdt < \frac{1}{n}, \tag{2.17}$$

where (ϱ_n, ψ_n) is the solution to

$$\begin{cases} L_a^*(\varrho_n, \psi_n) = 0 & \text{in } Q, \\ L_b^*(\varrho_n, \psi_n) = 0 & \text{in } Q, \\ \varrho_n(x,t) = \psi_n(x,t) = 0 & \text{on } \Sigma, \\ \varrho_n(x,T) = \varrho_T^n(x),\ \psi_n(x,T) = \psi_T^n(x) & \text{in } \Omega, \end{cases} \tag{2.18}$$

where

$$L_a^*(\varrho_n, \psi_n) = -\varrho_{nt} - a\,\Delta\varrho_n + A_1\varrho_n + A_2\psi_n,$$
$$L_b^*(\varrho_n, \psi_n) = -\psi_{nt} - b\,\Delta\psi_n + B_1\varrho_n + B_2\psi_n.$$

Thanks to the Cauchy inequality, for all $n \in \mathbb{N}^*$, we have

$$\iint_Q \theta^{-2}|P(\varrho_n 1_\omega)|^2 dxdt \le 2\Big(\iint_Q \theta^{-2}|\varrho_n 1_\omega|^2 dxdt + \iint_Q \theta^{-2}|\varrho_n 1_\omega - P(\varrho_n 1_\omega)|^2 dxdt\Big).$$

Since θ^{-2} and γ^{-2} are bounded then it follows from (2.16), and (2.17) that there exists a positive constant C satisfying

$$\iint_Q \theta^{-2}|P(\varrho_n 1_\omega)|^2 dxdt \le C.$$

Since $P\varrho_n \in \mathcal{U}$ and $\mathcal{U}$ is finite dimensional subspace of $L^2(\omega \times (0,T))$, we deduce that

$$\iint_Q |P(\varrho_n 1_\omega)|^2 dxdt \le C. \tag{2.19}$$

On the other hand,

$$\iint_Q |\varrho_n 1_\omega|^2 dxdt \le 2\Big(\iint_Q |P(\varrho_n 1_\omega)|^2 dxdt + \iint_Q |\varrho_n 1_\omega - P(\varrho_n 1_\omega)|^2 dxdt\Big).$$

According to (2.17) and (2.19), we infer from the last inequality that there exists a positive constant C satisfying

$$\iint_Q |\varrho_n 1_\omega|^2 dxdt \le C.$$

Consequently, we can extract a subsequence of $\{\varrho_n 1_\omega\}$ (still denoted by $\{\varrho_n 1_\omega\}$) and $\varrho 1_\omega \in L^2(\omega \times (0,T))$ such that

$$\varrho_n 1_\omega \rightharpoonup \varrho 1_\omega \quad \text{in} \quad L^2(\omega \times (0,T)). \tag{2.20}$$

Since $P(\varrho_n 1_\omega)$ belongs to $\mathcal{U}$, an application of Lemma 2.3 in [13] with $H = L^2(\omega \times (0,T))$, $p_i^n = p_i 1_\omega$ and $h^n = P(\varrho_n 1_\omega)$ leads to

$$P(\varrho_n 1_\omega) \to P(\varrho 1_\omega) \quad \text{in} \quad L^2(\omega \times (0,T)). \tag{2.21}$$

On the other hand, it follows from (2.17) that

$$\gamma(\varrho_n 1_\omega - P(\varrho_n 1_\omega)) \to 0 \quad \text{in} \quad L^2(\omega \times (0,T)),$$

and therefore,

$$\varrho_n 1_\omega - P(\varrho_n 1_\omega) \to 0 \quad \text{in} \quad L^2(\omega \times (0,T)). \tag{2.22}$$

Putting (2.22) together with (2.21), we obtain

$$\varrho_n 1_\omega \to P(\varrho 1_\omega) \quad \text{in} \quad L^2(\omega \times (0,T)). \tag{2.23}$$

From (2.20) and (2.23) we deduce $P(\varrho 1_\omega) = \varrho 1_\omega$. This means that $\varrho 1_\omega \in \mathcal{U}$ and

$$\varrho_n 1_\omega \to \varrho 1_\omega \quad \text{in} \quad L^2(\omega \times (0,T)). \tag{2.24}$$

Let us go back (2.16), as a result

$$\iint_Q \theta^{-2}|\varrho_n|^2 dxdt \leq 1, \quad \iint_Q \theta^{-2}|\psi_n|^2 dxdt \leq 1.$$

Thanks to the boundedness of θ^{-2}, we can extrapolate that $\{\varrho_n\}$, $\{\psi_n\}$ are also bounded in $L^2(\Omega \times (\varepsilon, T-\varepsilon))$ for every $\varepsilon \in (0,T)$. Therefore, we can extract subsequences of $\{\varrho_n\}$ and $\{\psi_n\}$ (still denoted by $\{\varrho_n\}$, $\{\psi_n\}$) such that

$$\varrho_n \rightharpoonup \varrho \quad \text{in} \quad L^2(\Omega \times (\varepsilon, T-\varepsilon)).$$

$$\psi_n \rightharpoonup \psi \quad \text{in} \quad L^2(\Omega \times (\varepsilon, T-\varepsilon)).$$

We deduce that

$$\varrho_n \to \varrho, \quad \psi_n \to \psi \quad \text{in} \quad \mathcal{D}'(Q).$$

Moreover, $L_a^*(\varrho_n, \psi_n)$, $L_b^*(\varrho_n, \psi_n)$ are also weakly continuous in $\mathcal{D}(Q)$. It means

$$L_a^*(\varrho_n, \psi_n) \to L_a^*(\varrho, \psi) \quad \text{in} \quad \mathcal{D}'(Q).$$

$$L_b^*(\varrho_n, \psi_n) \to L_b^*(\varrho, \psi) \quad \text{in} \quad \mathcal{D}'(Q).$$

where

$$L_a^*(\varrho, \psi) = -\varrho_t - a\,\Delta\varrho + A_1\varrho + A_2\psi,$$
$$L_b^*(\varrho, \psi) = -\psi_t - b\,\Delta\psi + B_1\varrho + B_2\psi.$$

It follows from (2.18) that

$$\begin{cases} L_a^*(\varrho, \psi) = 0 & \text{in} \quad \omega \times (0,T), \\ L_b^*(\varrho, \psi) = 0 & \text{in} \quad \omega \times (0,T), \\ \varrho 1_\omega \in \mathcal{U}. \end{cases}$$

Thanks to Lemma 2.2, we obtain that $\varrho = \psi = 0$ in $\omega \times (0,T)$. Therefore, it follows from (2.24) that

$$\varrho_n 1_\omega \to 0 \quad \text{in} \quad L^2(\omega \times (0,T)).$$

Because of (2.14), we deduce that

$$\iint_Q \theta^{-2}\Big(|\varrho_n|^2 + |\psi_n|^2\Big) dxdt \to 0,$$

which is a contradiction with (2.16).

Next, we prove that

$$\int_\Omega \left(|\varrho(0)|^2 + |\psi(0)|^2\right) dx \leq C \iint_Q \gamma^2 |\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt.$$

On a time interval $[T/4, 3T/4]$, we have

$$\alpha(x,t) = \tau \frac{e^{\frac{4}{3}\lambda\|\beta\|_{L^\infty(\Omega)}} - e^{\lambda\beta(x)}}{t(T-t)} \leq \tau e^{\frac{4}{3}\lambda\|\beta\|_{L^\infty(\Omega)}} (4/T)^2.$$

Thus

$$\theta^{-1} = e^{-\alpha} \geq e^{-\tau e^{\frac{4}{3}\lambda\|\beta\|_{L^\infty(\Omega)}}(4/T)^2}.$$

We deduce the existence of a positive constant C such that

$$\int_{T/4}^{3T/4} \int_\Omega \theta^{-2}(|\varrho|^2 + |\psi|^2) dxdt \geq C \int_{T/4}^{3T/4} \int_\Omega (|\varrho|^2 + |\psi|^2) dxdt.$$

Moreover, it follows from (2.15) that

$$\int_{T/4}^{3T/4}\int_\Omega \theta^{-2}\left(|\varrho|^2+|\psi|^2\right)dxdt \le C\iint_Q \gamma^2|\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt.$$

So,

$$\int_{T/4}^{3T/4}\int_\Omega \left(|\varrho|^2+|\psi|^2\right)dxdt \le C\iint_Q \gamma^2|\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt. \tag{2.25}$$

We consider the following function

$$\begin{cases}\ell(t)\in C^\infty([0,T]), \quad 0\le \ell\le 1,\\ \ell\equiv 1, \quad \forall t\in[0,T/4],\\ \ell\equiv 0, \quad \forall t\in[3T/4,T],\end{cases}$$

and with $\delta\in\mathbb{R}$, we define

$$\begin{aligned}\widehat{\varrho}(x,t) &= \ell(t)e^{-\delta t}\varrho(x,t),\\ \widehat{\psi}(x,t) &= \ell(t)e^{-\delta t}\psi(x,t).\end{aligned} \tag{2.26}$$

We see that

$$\begin{aligned}&\widehat{\varrho}|_\Sigma = \widehat{\psi}|_\Sigma = 0,\\ &\widehat{\varrho}(x,0)=\varrho(x,0), \quad \widehat{\psi}(x,0)=\psi(x,0),\\ &\widehat{\varrho}(x,T)=\widehat{\psi}(x,T)=0.\end{aligned} \tag{2.27}$$

From (2.26) one gets that

$$\begin{cases}-\frac{\partial\widehat{\varrho}}{\partial t} - a\Delta\widehat{\varrho} + (A_1-\delta)\widehat{\varrho} + A_2\widehat{\psi} &= -\ell' e^{-\delta t}\varrho,\\ -\dfrac{\partial\widehat{\psi}}{\partial t} - b\Delta\widehat{\psi} + B_1\widehat{\varrho} + (B_2-\delta)\widehat{\psi} &= -\ell' e^{-\delta t}\psi.\end{cases}$$

Taking inner product of both equations with respect to $\widehat{\varrho}$, $\widehat{\psi}$, then integrating by parts over Q, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\widehat{\varrho}(0)|^2dx + \iint_Q a|\nabla\widehat{\varrho}|^2dxdt + \iint_Q (A_1-\delta)|\widehat{\varrho}|^2dxdt&\\ + \iint_Q A_2\widehat{\varrho}\widehat{\psi}dxdt = -\iint_Q \ell'\varrho\widehat{\varrho}e^{-\delta t}dxdt,&\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\int_\Omega |\widehat{\psi}(0)|^2dx + \iint_Q b|\nabla\widehat{\psi}|^2dxdt + \iint_Q B_1\widehat{\varrho}\widehat{\psi}dxdt&\\ + \iint_Q (B_2-\delta)|\widehat{\psi}|^2dxdt = -\iint_Q \ell'\psi\widehat{\psi}e^{-\delta t}dxdt.&\end{aligned}$$

Summing these equations and using the Young inequality leads to

$$\begin{aligned}\frac{1}{2}\int_\Omega(|\widehat{\varrho}(0)|^2+|\widehat{\psi}(0)|^2)dx&+\iint_Q a|\nabla\widehat{\varrho}|^2dxdt+\iint_Q b|\nabla\widehat{\psi}|^2dxdt\\&+\iint_Q(A_1-\delta)|\widehat{\varrho}|^2dxdt+\iint_Q(B_2-\delta)|\widehat{\psi}|^2dxdt\\&=-\iint_Q\left(A_2\widehat{\varrho}\widehat{\psi}+B_1\widehat{\varrho}\widehat{\psi}\right)dxdt\\&\quad-\iint_Q\left(\ell'\ell|\varrho|^2+\ell'\ell|\psi|^2\right)e^{-2\delta t}dxdt\\&\le\frac{1}{2}\iint_Q|\widehat{\varrho}|^2dxdt+\frac{1}{2}\iint_Q|A_2\widehat{\psi}|^2dxdt\\&\quad+\frac{1}{2}\iint_Q|B_1\widehat{\varrho}|^2dxdt+\frac{1}{2}\iint_Q|\widehat{\psi}|^2dxdt\\&\quad+\iint_Q\left(|\varrho|^2+|\psi|^2\right)|\ell'\ell|e^{-2\delta t}dxdt.\end{aligned}$$

So, we get that

$$\begin{aligned}&\frac{1}{2}\int_\Omega(|\widehat{\varrho}(0)|^2+|\widehat{\psi}(0)|^2)dx+\iint_Q a|\nabla\widehat{\varrho}|^2dxdt+\iint_Q b|\nabla\widehat{\psi}|^2dxdt\\&+\iint_Q\left(A_1-\delta-\frac{1}{2}-\frac{B_1^2}{2}\right)|\widehat{\varrho}|^2dxdt+\iint_Q\left(B_2-\delta-\frac{1}{2}-\frac{A_2^2}{2}\right)|\widehat{\psi}|^2dxdt\\&\le\iint_Q\left(|\varrho|^2+|\psi|^2\right)|\ell'\ell|e^{-2\delta t}dxdt.\end{aligned}\tag{2.28}$$

We now choose

$$\delta\le-\min\left\{\|A_1\|_{L^\infty(Q)}+\frac{1}{2}+\frac{\|B_1\|^2_{L^\infty(Q)}}{2},\|B_2\|_{L^\infty(Q)}+\frac{1}{2}+\frac{\|A_2\|^2_{L^\infty(Q)}}{2}\right\},$$

then it follows from (2.28) that

$$\begin{aligned}\frac{1}{2}\int_\Omega(|\widehat{\varrho}(0)|^2+|\widehat{\psi}(0)|^2)dx&+\iint_Q a|\nabla\widehat{\varrho}|^2dxdt+\iint_Q b|\nabla\widehat{\psi}|^2dxdt\\&\le C\int_{T/4}^{3T/4}\int_\Omega\left(|\varrho|^2+|\psi|^2\right)e^{-2\delta t}dxdt.\end{aligned}$$

Taking (2.25) into account, we infer from the last inequality that

$$\int_\Omega(|\widehat{\varrho}(0)|^2+|\widehat{\psi}(0)|^2)dx\le C\iint_Q\gamma^2|\varrho 1_\omega-P(\varrho 1_\omega)|^2dxdt.$$

Thus, using (2.27) we get that

$$\int_\Omega\left(|\varrho(0)|^2+|\psi(0)|^2\right)dx\le C\iint_Q\gamma^2|\varrho 1_\omega-P(\varrho 1_\omega)|^2dxdt.\tag{2.29}$$

Combining (2.15) and (2.29) we complete the proof of Lemma 2.6. □

Thanks to Lemma 2.6, we can construct the following norm on a Hilbert space V which is the completion of $\mathcal{V} = \{(\varrho,\psi) \in (C^\infty(\overline{Q}))^2$ such that $L_a^*(\varrho,\psi) = L_b^*(\varrho,\psi) = 0$ and $\varrho = \psi = 0$ on $\Sigma\}$ with respect to the norm

$$\|(\varrho,\psi)\|_V^2 := \iint_Q \gamma^2 |\varrho 1_\omega - P(\varrho 1_\omega)|^2 dxdt.$$

Obviously, this norm is generated from the following scalar product in V which is also a symmetric bilinear form

$$\mathcal{A}((\widehat{\varrho},\widehat{\psi});(\varrho_2,\psi_2)) := \iint_Q \gamma^2 (\widehat{\varrho} 1_\omega - P(\widehat{\varrho} 1_\omega))(\varrho_2 1_\omega - P(\varrho_2 1_\omega)) dxdt.$$

According to the Cauchy-Schwarz inequality, the bilinear form $\mathcal{A}(.,.)$ is continuous on $V\times V$ and coercive on V. Let v_0 be determined by Lemma 2.3. Let us consider the linear operator $\mathcal{L}$ on V defined by

$$\mathcal{L}(\varrho,\psi) := \iint_{\omega\times(0,T)} v_0 \varrho dxdt + \int_\Omega y_0 \varrho(0) dx + \int_\Omega z_0 \psi(0) dx.$$

Using the Hölder inequality, we have the following estimate

$$\begin{aligned}|\mathcal{L}(\varrho,\psi)| \le\ & \|\theta v_0 1_\omega\|_{L^2(Q)} \|\theta^{-1}\varrho\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)} \|\varrho(0)\|_{L^2(\Omega)} \\ & + \|z_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}.\end{aligned}$$

Thanks to Lemma 2.3 and Lemma 2.6 and the Cauchy-Schwarz inequality, we infer that

$$|\mathcal{L}(\varrho,\psi)| \le C(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}) \|(\varrho,\psi)\|_V.$$

We deduce that the linear operator $\mathcal{L}$ is continuous on V. In view of the Lax-Milgram theorem, there exists a unique $(\varrho_\theta,\psi_\theta)$ of the following equation.

$$\begin{aligned}&\iint_Q \gamma^2 (\varrho_\theta 1_\omega - P(\varrho_\theta 1_\omega))(\varrho 1_\omega - P(\varrho 1_\omega)) dxdt \\ &= \iint_{\omega\times(0,T)} v_0 \varrho dxdt + \int_\Omega y_0 \varrho(0) dx + \int_\Omega z_0 \psi(0) dx, \text{ for every } (\varrho,\psi) \in V. \quad (2.30)\end{aligned}$$

Proposition 2.2. *Let $y_0, z_0 \in L^2(\Omega)$, $(\varrho_\theta,\psi_\theta)$ is the unique solution of* (2.30) *and*

$$v_\theta = -\gamma^2 (\varrho_\theta 1_\omega - P(\varrho_\theta 1_\omega)). \quad (2.31)$$

Then v_θ satisfies (2.10) *and the associated solution of* (2.11) *satisfies* (1.6)*. Moreover, there exists a positive constant $C(\Omega,\omega,T,L,E)$ such that*

$$\|v_\theta\|_{L^2(\omega\times(0,T))} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \quad (2.32)$$

$$\|(\varrho_\theta,\psi_\theta)\|_V \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \quad (2.33)$$

Proof. According to the construction of v_θ. We easily infer that $v_\theta \in \mathcal{U}^\perp$. We now prove that the solution (y_θ, z_θ) of (2.11) satisfies (1.6). Indeed, multiplying both sides of the first equation in (2.11) by ϱ and the second one by ψ, and then integrating in time and spaces. It follows from adding the resulting identities that

$$\int_\Omega (y(T)\varrho(T) + z(T)\psi(T))\, dx = \iint_Q (v_\theta + v_0) 1_\omega \varrho dxdt + \int_\Omega (y_0 \varrho(0) + z_0 \psi(0))\, dx,$$

for every $(\varrho,\psi) \in V$. Putting this together with (2.30), we deduce that

$$\int_\Omega (y(T)\varrho_T + z(T)\psi_T)\, dx = 0,$$

for every $(\varrho, \psi) \in V$. Therefore, $y(T) = z(T) = 0$ in Ω.

The estimation (2.32) and (2.33) are implied directly from (2.30) by replacing ϱ by ϱ_θ, using the Cauchy - Schwarz inequality, Lemma 2.3 and Lemma 2.6. $\square$

Proof of Theorem 2.1. We have just proved the null controllability of linearized system with constraint on the control (2.10), (2.11) and (1.6). Therefore, Theorem 2.1 is the result directly deduced from Propositions 2.1-2.2.

3. Null controllability of nonlinear system

We have just proved that, for any $(\widehat{y}, \widehat{z}) \in (L^2(\Omega))^2$, there exists a control $v \in L^2(Q)$ such that $v = v_\theta + v_0$ and the triple (v, y, z) satisfies the null controllability problem with constraints on the sate associated to the linearized system (2.1), (1.6) and (1.7). Thus we can construct a nonlinear map

$$\begin{aligned} S:\ (L^2(Q))^2 &\longrightarrow (L^2(Q))^2, \\ (\widehat{y}, \widehat{z}) &\longmapsto S(\widehat{y}, \widehat{z}) = (y, z), \end{aligned}$$

where (y, z) is the solution of (2.1) with $v(\widehat{y}, \widehat{z}) = v_\theta + v_0$, $v_0(\widehat{y}, \widehat{z}) \in \mathcal{U}_\theta$, and $v_\theta(\widehat{y}, \widehat{z}) \in \mathcal{U}^\perp$ are determined by Lemma 2.3 and Proposition 2.2. We now prove that S has a fixed point $(y, z) \in (L^2(Q))^2$, such that $S(y, z) = (y, z)$, since $f(y, z) = A_1(y, z)y + B_1(y, z)z$, and $g(y, z) = A_2(y, z)y + B_2(y, z)z$, will be sufficient to finish the proof of Theorem 1.1.

Proposition 3.1. *Assume that a, b, f and g be functions of class C^1 with bounded derivatives, satisfying* (1.2)-(1.4)*. Then*

(i) *S is continuous;*
(ii) *S is compact;*
(iii) *S has a bounded range, i.e., there exists a positive constant $\mathcal{C}$ such that*

$$\|S(y, z)\|_{(L^2(Q))^2} \le \mathcal{C},\ \forall (y, z) \in (L^2(Q))^2.$$

Proof. The proof is the same idea as in [13, Proposition 3.1]. However, it must be adapted to our situation. Thus, we outline it for reader's convenience.

i) Proof of the continuity of S. We proceed in five steps as follows:

Step 1. Let $(\widehat{y}_n, \widehat{z}_n) \in (L^2(Q))^2$ be such that

$$\begin{aligned} \widehat{y}_n &\to \widehat{y} \ \text{ in } L^2(Q), \\ \widehat{z}_n &\to \widehat{z} \ \text{ in } L^2(Q). \end{aligned}$$

Since a and b are C^1-functions with bounded derivatives satisfying (1.2), we have

$$a\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big) \to a\Big(\int_\Omega \widehat{y} dx, \int_\Omega \widehat{z} dx\Big) \ \text{ in } L^2(0,T), \tag{3.1}$$

$$b\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big) \to b\Big(\int_\Omega \widehat{y} dx, \int_\Omega \widehat{z} dx\Big) \ \text{ in } L^2(0,T). \tag{3.2}$$

Moreover, we can extract subsequences of $\{\widehat{y}_n\}$ and $\{\widehat{z}_n\}$ (still denoted the same) such that

$$\begin{aligned} \widehat{y}_n &\to \widehat{y} \text{ almost everywhere in } Q, \\ \widehat{z}_n &\to \widehat{z} \text{ almost everywhere in } Q. \end{aligned}$$

Since f and g are functions of C^1 class with bounded derivatives, then functions A_1, A_2, B_1 and B_2 determined by (2.2) and (2.3) are continuous. Thus

$$A_i(\widehat{y}_n, \widehat{z}_n) \to A_i(\widehat{y}, \widehat{z}) \text{ almost everywhere in } Q,$$
$$B_i(\widehat{y}_n, \widehat{z}_n) \to B_i(\widehat{y}, \widehat{z}) \text{ almost everywhere in } Q,$$

for $i = 1, 2$. We deduce from (2.4) that $|A_i(\widehat{y}_n, \widehat{z}_n)| \leq K$, and $|B_i(\widehat{y}_n, \widehat{z}_n)| \leq K$ almost everywhere in Q. It follows from the relation of modes of convergence that (see [2, Chapter 7, Theorem 7.2])

$$A_i(\widehat{y}_n, \widehat{z}_n) \to A_i(\widehat{y}, \widehat{z}) \text{ in } L^2(Q), \tag{3.3}$$

$$B_i(\widehat{y}_n, \widehat{z}_n) \to B_i(\widehat{y}, \widehat{z}) \text{ in } L^2(Q). \tag{3.4}$$

Step 2. Since Theorem 2.1 holds for every $(\widehat{y}, \widehat{z}) \in (L^2(Q))^2$, it is also true for $(\widehat{y}_n, \widehat{z}_n) \in (L^2(Q))^2$. Therefore, there exists the control $v(\widehat{y}_n, \widehat{z}_n)$ such that the solution (y_n, z_n) of

$$\begin{cases} \frac{\partial y_n}{\partial t} - a\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big)\Delta y_n & \\ \qquad\qquad + A_1(\widehat{y}_n, \widehat{z}_n) y_n + B_1(\widehat{y}_n, \widehat{z}_n) z_n = v(\widehat{y}_n, \widehat{z}_n) 1_\omega & \text{in } Q, \\ \frac{\partial z_n}{\partial t} - b\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big)\Delta z_n + A_2(\widehat{y}_n, \widehat{z}_n) y_n + B_2(\widehat{y}_n, \widehat{z}_n) z_n = 0 & \text{in } Q, \\ y_n(x,t) = z_n(x,t) = 0 & \text{on } \Sigma, \\ y_n(x,0) = y_0(x); \quad z_n(x,0) = z_0(x) & \text{in } \Omega \end{cases} \tag{3.5}$$

satisfies

$$y_n(T) = z_n(T) = 0 \text{ in } \Omega,$$
$$\int_0^T \int_\Omega z_n e_i dx dt = 0, \ i = 1, \dots, M,$$

and

$$v(\widehat{y}_n, \widehat{z}_n) = v_0(\widehat{y}_n, \widehat{z}_n) + v_\theta(\widehat{y}_n, \widehat{z}_n),$$

where, in view of (2.8),

$$v_0(\widehat{y}_n, \widehat{z}_n) \in \theta^{-1} \text{span}\{p_1(\widehat{y}_n, \widehat{z}_n)1_\omega, p_2(\widehat{y}_n, \widehat{z}_n)1_\omega, \dots, p_M(\widehat{y}_n, \widehat{z}_n)1_\omega\}$$

satisfies

$$-\int_\Omega y_0 p_i(\widehat{y}_n, \widehat{z}_n)(0) dx - \int_\Omega z_0 q_i(\widehat{y}_n, \widehat{z}_n)(0) dx = \iint_{\omega \times (0,T)} v_0(\widehat{y}_n, \widehat{z}_n) p_i(\widehat{y}_n, \widehat{z}_n) dx dt, \tag{3.6}$$

for $i = 1, \dots, M$, and $(p_i(\widehat{y}_n, \widehat{z}_n), q_i(\widehat{y}_n, \widehat{z}_n))$ is the solution of

$$\begin{cases} -\frac{\partial p_i(\widehat{y}_n, \widehat{z}_n)}{\partial t} - a\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big) \Delta p_i(\widehat{y}_n, \widehat{z}_n) & & \\ \qquad + A_1(\widehat{y}_n, \widehat{z}_n) p_i(\widehat{y}_n, \widehat{z}_n) + A_2(\widehat{y}_n, \widehat{z}_n) q_i(\widehat{y}_n, \widehat{z}_n) & = 0 & \text{in } Q, \\ -\frac{\partial q_i(\widehat{y}_n, \widehat{z}_n)}{\partial t} - b\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big) \Delta q_i(\widehat{y}_n, \widehat{z}_n) & & \\ \qquad + B_1(\widehat{y}_n, \widehat{z}_n) p_i(\widehat{y}_n, \widehat{z}_n) + B_2(\widehat{y}_n, \widehat{z}_n) q_i(\widehat{y}_n, \widehat{z}_n) & = e_i & \text{in } Q, \\ p_i(\widehat{y}_n, \widehat{z}_n)(x,t) = q_i(\widehat{y}_n, \widehat{z}_n)(x,t) & = 0 & \text{on } \Sigma, \\ p_i(\widehat{y}_n, \widehat{z}_n)(x,T) = q_i(\widehat{y}_n, \widehat{z}_n)(x,T) & = 0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Let $P_n = P(\widehat{y}_n, \widehat{z}_n)$ be the orthogonal projection operator from $L^2(\omega \times (0,T))$ into

$$\mathcal{U}(\widehat{y}_n, \widehat{z}_n) = \text{span}\{p_1(\widehat{y}_n, \widehat{z}_n)1_\omega, p_2(\widehat{y}_n, \widehat{z}_n)1_\omega, \dots, p_M(\widehat{y}_n, \widehat{z}_n)1_\omega\}.$$

In view of (2.31),

$$v_\theta(\widehat{y}_n,\widehat{z}_n) = -\gamma^2(\varrho_\theta(\widehat{y}_n,\widehat{z}_n)1_\omega - P_n(\varrho_\theta(\widehat{y}_n,\widehat{z}_n)1_\omega)), \tag{3.8}$$

where $(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n)) \in V$ satisfies

$$\begin{cases} L_a^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n)) = 0 & \text{in} \quad Q, \\ L_b^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n)) = 0 & \text{in} \quad Q, \\ \varrho_\theta(\widehat{y}_n,\widehat{z}_n)(x,t) = \psi_\theta(\widehat{y}_n,\widehat{z}_n)(x,t) = 0 & \text{on} \quad \Sigma, \\ \varrho_\theta(\widehat{y}_n,\widehat{z}_n)(x,T) = \varrho_T(x),\ \psi_\theta(\widehat{y}_n,\widehat{z}_n)(x,T) = \psi_T(x) & \text{in} \quad \Omega. \end{cases} \tag{3.9}$$

Furthermore, according to Proposition 2.2, Lemmas 2.3 and 2.6, there exists a positive constant $C(\Omega,\omega,T,L,E,K)$ such that

$$\|(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))\|_V \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \tag{3.10}$$

$$\|(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))\|_{(L^2(\omega\times(0,T)))^2} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \tag{3.11}$$

$$\|v_0(\widehat{y}_n,\widehat{z}_n)\|_{L^2(\omega\times(0,T))} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),$$

$$\|\theta v_0(\widehat{y}_n,\widehat{z}_n)\|_{L^2(\omega\times(0,T))} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \tag{3.12}$$

$$\|v_\theta(\widehat{y}_n,\widehat{z}_n)\|_{L^2(\omega\times(0,T))} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),$$

$$\|v(\widehat{y}_n,\widehat{z}_n)\|_{L^2(\omega\times(0,T))} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \tag{3.13}$$

Therefore, we can extract subsequences (see [15, Theorem 4.18, Corollary 4.19]) (still denoted the same) such that

$$(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n)) \rightharpoonup (\widetilde{\varrho}_\theta,\widetilde{\psi}_\theta) \ \text{ in } \ V,$$

$$(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n)) \rightharpoonup (\widetilde{\varrho}_\theta,\widetilde{\psi}_\theta) \ \text{ in } \ (L^2(\omega\times(0,T)))^2, \tag{3.14}$$

$$v_0(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{v}_0 \ \text{ in } \ L^2(\omega\times(0,T)), \tag{3.15}$$

$$\theta v_0(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{v}_1 \ \text{ in } \ L^2(\omega\times(0,T)), \tag{3.16}$$

$$v_\theta(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{v}_\theta \ \text{ in } \ L^2(\omega\times(0,T)). \tag{3.17}$$

Therefore,

$$v(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{v} = \widetilde{v}_0 + \widetilde{v}_\theta \ \text{ in } \ L^2(\omega\times(0,T)). \tag{3.18}$$

Step 3. Since (y_n, z_n) is the solution of (3.5), by using (3.13), we have

$$\|y_n\|_{W(0,T)} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),$$

$$\|z_n\|_{W(0,T)} \le C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),$$

where

$$W(0,T) = L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)).$$

Consequently, there exists $(\widetilde{y},\widetilde{z}) \in (W(0,T))^2$ such that

$$y_n \rightharpoonup \widetilde{y} \ \text{ in } \ W(0,T), \tag{3.19}$$

$$z_n \rightharpoonup \widetilde{z} \ \text{ in } \ W(0,T). \tag{3.20}$$

In addition, the embedding of $W(0,T)$ into $L^2(Q)$ is compact, we deduce that

$$y_n \to \widetilde{y} \ \text{ in } L^2(Q), \tag{3.21}$$

$$z_n \to \widetilde{z} \ \text{ in } L^2(Q). \tag{3.22}$$

Now, passing to the limits in (3.5), while using (3.1), (3.2), (3.3), (3.4), (3.18), (3.19), (3.20), (3.21) and (3.22), we deduce that the triple $(\widetilde{v}, \widetilde{y}(\widetilde{v}), \widetilde{z}(\widetilde{v}))$ verifies

$$\begin{cases} \frac{\partial \widetilde{y}}{\partial t} - a\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big)\ \Delta\widetilde{y} + A_1(\widehat{y},\widehat{z})\widetilde{y} + B_1(\widehat{y},\widehat{z})\widetilde{z} = \widetilde{v}1_\omega & \text{in } Q, \\ \frac{\partial \widetilde{z}}{\partial t} - b\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big)\ \Delta\widetilde{z} + A_2(\widehat{y},\widehat{z})\widetilde{y} + B_2(\widehat{y},\widehat{z})\widetilde{z} = 0 & \text{in } Q, \\ \widetilde{y}(x,t) = \widetilde{z}(x,t) = 0 & \text{on } \Sigma, \\ \widetilde{y}(x,0) = y_0(x); \quad \widetilde{z}(x,0) = z_0(x) & \text{in } \Omega \end{cases}$$

satisfies

$$\widetilde{y}(T) = \widetilde{z}(T) = 0 \text{ in } \Omega,$$

and

$$\int_0^T \int_\Omega \widetilde{z}e_i dxdt = 0,\ 1 \le i \le M.$$

Step 4. Since $(p_i(\widehat{y}_n,\widehat{z}_n), q_i(\widehat{y}_n,\widehat{z}_n))$ is the solution of (3.7), we deduce that

$$\|p_i(\widehat{y}_n,\widehat{z}_n)\|_{L^2(0,T;H_0^1(\Omega))} \le C\|e_i\|_{L^2(Q)}, \tag{3.23}$$

$$\|q_i(\widehat{y}_n,\widehat{z}_n)\|_{L^2(0,T;H_0^1(\Omega))} \le C\|e_i\|_{L^2(Q)}. \tag{3.24}$$

On the other hand, we can rewrite (3.7) as follows

$$\begin{cases} -\frac{\partial p_i(\widehat{y}_n,\widehat{z}_n)}{\partial t} - a\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big)\ \Delta p_i(\widehat{y}_n,\widehat{z}_n) = c_i(\widehat{y}_n,\widehat{z}_n) & \text{in } Q, \\ -\frac{\partial q_i(\widehat{y}_n,\widehat{z}_n)}{\partial t} - b\Big(\int_\Omega \widehat{y}_n dx, \int_\Omega \widehat{z}_n dx\Big)\ \Delta q_i(\widehat{y}_n,\widehat{z}_n) = d_i(\widehat{y}_n,\widehat{z}_n) & \text{in } Q, \\ \quad p_i(\widehat{y}_n,\widehat{z}_n)(x,t) = q_i(\widehat{y}_n,\widehat{z}_n)(x,t) = 0 & \text{on } \Sigma, \\ \quad p_i(\widehat{y}_n,\widehat{z}_n)(x,T) = q_i(\widehat{y}_n,\widehat{z}_n)(x,T) = 0 & \text{in } \Omega, \end{cases}$$

where

$$c_i(\widehat{y}_n,\widehat{z}_n) = -A_1(\widehat{y}_n,\widehat{z}_n)p_i(\widehat{y}_n,\widehat{z}_n) - A_2(\widehat{y}_n,\widehat{z}_n)q_i(\widehat{y}_n,\widehat{z}_n),$$
$$d_i(\widehat{y}_n,\widehat{z}_n) = e_i - B_1(\widehat{y}_n,\widehat{z}_n)p_i(\widehat{y}_n,\widehat{z}_n) - B_2(\widehat{y}_n,\widehat{z}_n)q_i(\widehat{y}_n,\widehat{z}_n)$$

are uniformly bounded in $L^2(Q)$ according to (3.23), (3.24), and (2.4). Then we get that $(p_i(\widehat{y}_n,\widehat{z}_n), q_i(\widehat{y}_n,\widehat{z}_n))$ is bounded uniformly in n in $(L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)))^2 =: X(Q)^2$. Therefore, we can extract a subsequences of $(p_i(\widehat{y}_n,\widehat{z}_n), q_i(\widehat{y}_n,\widehat{z}_n))$ (still denoted the same) such that

$$p_i(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{p}_i \ \text{ in } \ X(Q), \tag{3.25}$$

$$q_i(\widehat{y}_n,\widehat{z}_n) \rightharpoonup \widetilde{q}_i \ \text{ in } \ X(Q). \tag{3.26}$$

Thanks to the compactness of the embedding $X(Q)$ into $L^2(0,T;H_0^1(\Omega))$, we have

$$p_i(\widehat{y}_n,\widehat{z}_n) \to \widetilde{p}_i \ \text{ in } L^2(0,T;H_0^1(\Omega)), \tag{3.27}$$

$$q_i(\widehat{y}_n,\widehat{z}_n) \to \widetilde{q}_i \ \text{ in } L^2(0,T;H_0^1(\Omega)). \tag{3.28}$$

From the energy inequality for $p_i(\widehat{y}_n,\widehat{z}_n)$, $q_i(\widehat{y}_n,\widehat{z}_n)$, (3.23) and (3.24), it follows that

$$\|p_i(\widehat{y}_n,\widehat{z}_n)(0)\|_{L^2(\Omega)} \le C\|e_i\|_{L^2(Q)}, \tag{3.29}$$

$$\|q_i(\widehat{y}_n,\widehat{z}_n)(0)\|_{L^2(\Omega)} \le C\|e_i\|_{L^2(Q)}. \tag{3.30}$$

Therefore, passing to the limits in (3.7), while using (3.1), (3.2), (3.3), (3.4), (3.25), (3.26), (3.27) and (3.28), we get

$$\begin{cases} -\frac{\partial \widetilde{p}_i}{\partial t} - a\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big) \, \Delta\widetilde{p}_i + A_1(\widehat{y},\widehat{z})\widetilde{p}_i + A_2(\widehat{y},\widehat{z})\widetilde{q}_i = 0 & \text{in } Q, \\ -\frac{\partial \widetilde{q}_i}{\partial t} - b\Big(\int_\Omega \widehat{y}dx, \int_\Omega \widehat{z}dx\Big) \, \Delta\widetilde{q}_i + B_1(\widehat{y},\widehat{z})\widetilde{p}_i(\widehat{y},\widehat{z}) + B_2(\widehat{y},\widehat{z})\widetilde{q}_i = e_i & \text{in } Q, \\ \widetilde{p}_i(x,t) = \widetilde{q}_i(x,t) = 0 & \text{on } \Sigma, \\ \widetilde{p}_i(x,T) = \widetilde{q}_i(x,T) = 0 & \text{in } \Omega. \end{cases}$$

It follows from (3.29) and (3.30) that

$$p_i(\widehat{y}_n,\widehat{z}_n)(0) \rightharpoonup \widetilde{p}_i(0) \text{ in } L^2(\Omega), \tag{3.31}$$

$$q_i(\widehat{y}_n,\widehat{z}_n)(0) \rightharpoonup \widetilde{q}_i(0) \text{ in } L^2(\Omega). \tag{3.32}$$

Therefore, for each e_i, $i = 1,\dots,M$, $(\widetilde{p}_i,\widetilde{q}_i)$ is the solution of (2.7). Hence, thanks to the uniqueness of the solution of (2.7), one gets

$$p_i(\widehat{y},\widehat{z}) = \widetilde{p}_i(\widehat{y},\widehat{z}), \ i = 1,\dots,M, \tag{3.33}$$

$$q_i(\widehat{y},\widehat{z}) = \widetilde{q}_i(\widehat{y},\widehat{z}), \ i = 1,\dots,M. \tag{3.34}$$

Step 5. Since $\theta v_0(\widehat{y}_n,\widehat{z}_n) \in \text{span}\{p_1(\widehat{y}_n,\widehat{z}_n)1_\omega, p_2(\widehat{y}_n,\widehat{z}_n)1_\omega,\dots,p_M(\widehat{y}_n,\widehat{z}_n)1_\omega\}$ and satisfies (3.12), by using [13, Lemma 2.3] with $H = L^2(\omega\times(0,T)), h^n = \theta v_0(\widehat{y}_n,\widehat{z}_n)$, $p_i^n = p_i(\widehat{y}_n,\widehat{z}_n)1_\omega$ while taking into account (3.16), (3.27), (3.28), (3.33) and (3.34), we deduce that there exists $\widetilde{\alpha}_j \in \mathbb{R}, i = 1,\dots,M$, such that

$$\theta v_0(\widehat{y}_n,\widehat{z}_n) \to \sum_{j=1}^M \widetilde{\alpha}_j p_j(\widehat{y},\widehat{z})1_\omega \text{ in } L^2(\omega\times(0,T)).$$

Since θ^{-1} is bounded in $L^\infty(Q)$ and $v_0(\widehat{y}_n,\widehat{z}_n)$ satisfying (3.15), we infer that

$$v_0(\widehat{y}_n,\widehat{z}_n) \to \widetilde{v}_0 = \sum_{j=1}^M \widetilde{\alpha}_j \frac{1}{\theta} p_j(\widehat{y},\widehat{z})1_\omega \text{ in } L^2(\omega\times(0,T)).$$

Passing to the limits in (3.6), while using (3.15), (3.27), (3.28), (3.31), (3.32), (3.33) and (3.34) we get

$$-\int_\Omega y_0 p_i(\widehat{y},\widehat{z})(0)dx - \int_\Omega z_0 q_i(\widehat{y},\widehat{z})(0)dx = \iint_{\omega\times(0,T)} \widetilde{v}_0 p_i(\widehat{y},\widehat{z})dxdt,$$

for $i = 1,\dots,M$. Thanks to the uniqueness of $v_0 \in \mathcal{U}_\theta$ which verifies (2.9), we conclude that $v_0(\widehat{y},\widehat{z}) = \widetilde{v}_0$.

On the other hand, since

$$v_\theta(\widehat{y}_n,\widehat{z}_n) \in \mathcal{U}^\perp(\widehat{y}_n,\widehat{z}_n) = \text{span}\{p_1(\widehat{y}_n,\widehat{z}_n)1_\omega, p_2(\widehat{y}_n,\widehat{z}_n)1_\omega,\dots,p_M(\widehat{y}_n,\widehat{z}_n)1_\omega\}^\perp,$$

then we have

$$\iint_{\omega\times(0,T)} v_\theta(\widehat{y}_n,\widehat{z}_n)p_i(\widehat{y}_n,\widehat{z}_n)dxdt = 0, \ i = 1,\dots,M.$$

Taking the limits in this identity while using (3.17), (3.27), (3.28), (3.33) and (3.34), we obtain

$$\iint_{\omega\times(0,T)} \widetilde{v}_\theta p_i(\widehat{y},\widehat{z})dxdt = 0, \ i = 1,\dots,M.$$

This means that $\widetilde{v}_\theta \in \mathcal{U}^\perp = \text{span}\{p_1(\widehat{y},\widehat{z})1_\omega, p_2(\widehat{y},\widehat{z})1_\omega,\dots,p_M(\widehat{y},\widehat{z})1_\omega\}^\perp$.

Now, it follows from Lemma 2.4 that

$$\begin{aligned}&I(\lambda,s,\tau,\varrho_\theta(\widehat{y}_n,\widehat{z}_n))+I(\lambda,s,\tau,\psi_\theta(\widehat{y}_n,\widehat{z}_n))\\&\le C\Big[\tau\iint_Q\varphi^{2s}e^{-2\alpha}\big(|A_1\varrho_\theta(\widehat{y}_n,\widehat{z}_n)+A_2\psi_\theta(\widehat{y}_n,\widehat{z}_n)|^2\\&\qquad+|B_1\varrho_\theta(\widehat{y}_n,\widehat{z}_n)+B_2\psi_\theta(\widehat{y}_n,\widehat{z}_n)|^2\big)dxdt\\&\quad+\lambda^4\tau^4\iint_{\omega'\times(0,T)}\varphi^{2s+3}e^{-2\alpha}\left(|\varrho_\theta(\widehat{y}_n,\widehat{z}_n)|^2+|\psi_\theta(\widehat{y}_n,\widehat{z}_n)|^2\right)dxdt\Big].\end{aligned}\tag{3.35}$$

Using the Cauchy-Schwarz inequality in the right hand side of (3.35), one gets

$$\begin{aligned}&I(\lambda,s,\tau,\varrho_\theta(\widehat{y}_n,\widehat{z}_n))+I(\lambda,s,\tau,\psi_\theta(\widehat{y}_n,\widehat{z}_n))\\&\le C\Big[L^2\tau\iint_Q\varphi^{2s}e^{-2\alpha}\Big(|\varrho_\theta(\widehat{y}_n,\widehat{z}_n)|^2+|\psi_\theta(\widehat{y}_n,\widehat{z}_n)|^2\Big)dxdt\\&\quad+\lambda^4\tau^4\iint_{\omega\times(0,T)}\varphi^{2s+3}e^{-2\alpha}\Big(|\varrho_\theta(\widehat{y}_n,\widehat{z}_n)|^2+|\psi_\theta(\widehat{y}_n,\widehat{z}_n)|^2\Big)dxdt\Big].\end{aligned}\tag{3.36}$$

On the other hand, since $(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))$ verifies (3.9) and (3.11), then it follows from (3.36) that $(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))$ is bounded uniformly in n in $(L^2(\varepsilon,T-\varepsilon;H^2(\Omega)))^2$, for all $T>\varepsilon>0$. This implies that

$$\begin{aligned}(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ (L^2(\Omega\times(\varepsilon,T-\varepsilon)))^2,\\(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ (L^2(\partial\Omega\times(\varepsilon,T-\varepsilon)))^2.\end{aligned}$$

So

$$\begin{aligned}(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ (\mathcal{D}'(Q))^2,\\(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ (\mathcal{D}'(\Sigma))^2.\end{aligned}$$

Setting

$$\begin{aligned}L_a^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))=&-\frac{\partial\varrho_\theta(\widehat{y}_n,\widehat{z}_n)}{\partial t}-a\Big(\int_\Omega\widehat{y}_ndx,\int_\Omega\widehat{z}_ndx\Big)\,\Delta\varrho_\theta(\widehat{y}_n,\widehat{z}_n)\\&+A_1(\widehat{y}_n,\widehat{z}_n)\varrho_\theta(\widehat{y}_n,\widehat{z}_n)+A_2(\widehat{y}_n,\widehat{z}_n)\psi_\theta(\widehat{y}_n,\widehat{z}_n),\\L_b^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))=&-\frac{\partial\psi_\theta(\widehat{y}_n,\widehat{z}_n)}{\partial t}-b\Big(\int_\Omega\widehat{y}_ndx,\int_\Omega\widehat{z}_ndx\Big)\,\Delta\psi_\theta(\widehat{y}_n,\widehat{z}_n)\\&+B_1(\widehat{y}_n,\widehat{z}_n)\varrho_\theta(\widehat{y}_n,\widehat{z}_n)+B_2(\widehat{y}_n,\widehat{z}_n)\psi_\theta(\widehat{y}_n,\widehat{z}_n).\end{aligned}$$

It follows from (3.1), (3.2), (3.3), and (3.4) that

$$\begin{aligned}L_a^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup L_a^*(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ \mathcal{D}'(Q),\\L_b^*(\varrho_\theta(\widehat{y}_n,\widehat{z}_n),\psi_\theta(\widehat{y}_n,\widehat{z}_n))&\rightharpoonup L_b^*(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))\ \text{ in }\ \mathcal{D}'(Q).\end{aligned}$$

Combining with (3.9), we deduce that

$$\begin{cases}L_a^*(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))=0 & \text{in}\quad Q,\\L_b^*(\widetilde{\varrho}_\theta(\widehat{y},\widehat{z}),\widetilde{\psi}_\theta(\widehat{y},\widehat{z}))=0 & \text{in}\quad Q,\\\widetilde{\varrho}_\theta(\widehat{y},\widehat{z})=\widetilde{\psi}_\theta(\widehat{y},\widehat{z})=0 & \text{on}\quad\Sigma,\\\widetilde{\varrho}_\theta(\widehat{y},\widehat{z})(x,T)=\varrho_T(x),\ \ \widetilde{\psi}_\theta(\widehat{y},\widehat{z})(x,T)=\psi_T(x) & \text{in}\quad\Omega.\end{cases}$$

Thanks to uniqueness of the solution of (2.30) and $v_0 = \widetilde{v}_0$, we have $\varrho_\theta = \widetilde{\varrho}_\theta, \psi_\theta = \widetilde{\psi}_\theta$. According to (3.10) and the definition of the norm on V, we have

$$\|P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega) - \varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega\|_{L^2(Q)} \leq C(\Omega, \omega, T, L, E, K)(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \tag{3.37}$$

Applying inequality (2.13) to $(\varrho_\theta(\widehat{y}_n, \widehat{z}_n), \psi_\theta(\widehat{y}_n, \widehat{z}_n))$ and taking into account (3.9) and (3.11), we obtain

$$\|\theta^{-1}\varrho_\theta(\widehat{y}_n, \widehat{z}_n)\|_{L^2(Q)} \leq C(\Omega, \omega, T, L, E, K)(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \tag{3.38}$$

$$\|\theta^{-1}\psi_\theta(\hat{y}_n, \widehat{z}_n)\|_{L^2(Q)} \leq C(\Omega, \omega, T, L, E, K)(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \tag{3.39}$$

Then, proceeding as in the proof of Lemma 2.6 while using (3.37), (3.38) and (3.39), we get

$$\|P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega)\|_{L^2(Q)} \leq C(\Omega, \omega, T, L, E, K)(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}).$$

Therefore, $P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega)$ being in $\mathcal{U}(\widehat{y}_n, \widehat{z}_n)$, using [13, Lemma 2.3] with $H = L^2(\omega \times (0,T))$, $h^n = P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega)$, $p_i^n = p_i(\widehat{y}_n, \widehat{z}_n)1_\omega$ while putting together with (3.27), (3.28), (3.33), and (3.34), one gets

$$P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega) \rightharpoonup \chi \in \operatorname{span}\{p_1(\widehat{y}, \widehat{z})1_\omega, p_2(\widehat{y}, \widehat{z})1_\omega, \ldots, p_M(\widehat{y}, \widehat{z})1_\omega\}.$$

This means that $\chi \in \mathcal{U}$.

By using (3.8), (3.14) and (3.17), we deduce that

$$\begin{aligned} v_\theta(\widehat{y}_n, \widehat{z}_n) &= -\gamma^2(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega - P_n(\varrho_\theta(\widehat{y}_n, \widehat{z}_n)1_\omega)) \\ &\rightharpoonup -\gamma^2(\varrho_\theta(\widehat{y}, \widehat{z})1_\omega - \chi) = \widetilde{v}_\theta \ \text{ in } \ L^2(\omega \times (0,T)). \end{aligned}$$

Observing that $P(\varrho_\theta(\widehat{y}, \widehat{z})1_\omega - \chi) = 0$ and $P(\chi) = \chi$ because $\varrho_\theta(\widehat{y}, \widehat{z})1_\omega - \chi \in \mathcal{U}^\perp$ and $\chi \in \mathcal{U}$. We derive $P(\varrho_\theta(\widehat{y}, \widehat{z})1_\omega) = \chi$. This leads to

$$\widetilde{v}_\theta = v_\theta = -\gamma^2(\varrho_\theta(\widehat{y}, \widehat{z})1_\omega - P(\varrho_\theta(\widehat{y}, \widehat{z})1_\omega)).$$

It follows from the relation (3.18) that $\widetilde{v} = v_0(\widehat{y}, \widehat{z}) + v_\theta(\widehat{y}, \widehat{z}) = v$. Therefore, the triple (v, y, z) verifies (2.1), (1.6) and (1.7).

ii) Proof of the compactness of S***.*** We deduce from the argument above that when $(\widehat{y}, \widehat{z})$ lies in bounded subset B of $(L^2(Q))^2$, $S(\widehat{y}, \widehat{z}) = (y, z)$ lies in bounded set of $(W(0,T))^2$. Since the compactness of the embedding $(W(0,T))^2 \hookrightarrow (L^2(Q))^2$, we deduce that $S(B)$ is relatively compact in $(L^2(Q))^2$. Thus, S is a compact operator.

iii) Proof of the boundedness of the range of S***.*** Let $(\widehat{y}, \widehat{z}) \in (L^2(Q))^2$. Since (y, z) is solution of (2.1) with $v(\widehat{y}, \widehat{z})$ satisfying

$$\|v\|_{L^2(Q)} \leq C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}),$$

we deduce that

$$\begin{aligned} \|y\|_{L^2(0,T;H_0^1(\Omega))} &\leq C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \\ \|z\|_{L^2(0,T;H_0^1(\Omega))} &\leq C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \end{aligned}$$

Thanks to the continuous embedding of $L^2(0,T;H_0^1(\Omega))$ into $L^2(Q)$, it follows that

$$\begin{aligned} \|y\|_{L^2(Q)} &\leq C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}), \\ \|z\|_{L^2(Q)} &\leq C(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}). \end{aligned}$$

□

Proof of Theorem 1.1. Due to the Proposition 3.1, all hypotheses of the Schauder fixed-point theorem are satisfied. Consequently, the mapping S has a fixed point (y, v), and then we complete the proof of Theorem 1.1.

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