ON REGULARITY AND STABILITY FOR A CLASS OF NONLOCAL EVOLUTION EQUATIONS WITH NONLINEAR PERTURBATIONS

TRAN DINH KE *, NGUYEN NHU THANG

Abstract. We study a class of nonlocal partial differential equations with nonlinear perturbations, which can be seen as an interpolation between the Basset equation and nonclassical diffusion one. Our aim is to analyze some sufficient conditions ensuring the global solvability, regularity and stability of solutions. Our analysis is based on the theory of completely positive kernel functions, local estimates and a new Gronwall type inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following problem

$$\partial_t u + \partial_t(m * (-\Delta)^\gamma u) - \Delta u = f(u) \text{ in } \Omega, t > 0, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(\cdot, 0) = \xi \text{ in } \Omega, \tag{1.3}$$

where $\partial_t = \frac{\partial}{\partial t}$, $m \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function, $\gamma \in [0, 1]$, and the notation '$*$' stands for the Laplace convolution with respect to the time t, i.e.,

$$(m * v)(t) = \int_0^t m(t-s)v(s)ds.$$

In our model, $(-\Delta)^\gamma$ denotes the fractional power of the Laplacian, f is a nonlinear function and $\xi \in L^2(\Omega)$ is given.

We first mention some special cases of (1.1). If $\gamma = 1$ and m is a nonnegative constant then (1.1) is the classical reaction-diffusion equation with nonlinear sources. In the case $\gamma = 1$ and $m(t) = m_0 g_{1-\alpha}(t) = \dfrac{m_0 t^{-\alpha}}{\Gamma(1-\alpha)}, m_0 > 0$ and $\alpha \in (0, 1)$, our equation reads

$$\partial_t u - (1 + m_0\partial_t^\alpha)\Delta u = f(u),$$

which is the generalized Rayleigh-Stokes equation (see, e.g. [2]), here ∂_t^α denotes the fractional derivative of order α in the sense of Riemann-Liouville. This equation is employed to describe the behavior of non-Newtonian fluids. In the case $\gamma = 0$ and $m(t) = m_0 g_{1-\alpha}(t)$, we get

$$\partial_t u + m_0\partial_t^\alpha u - \Delta u = f(u),$$

that is the Basset equation mentioned in [1, 12, 17].

2010 *Mathematics Subject Classification.* 35B40,35B65,35C15,45K05.
Key words and phrases. nonlocal PDE; regularity; stability.
* Corresponding author. Email: ketd@hnue.edu.vn (T.D.Ke).

In addition, if $\gamma = 1$ and m is a regular function, e.g. $m \in C^1(\mathbb{R}^+)$, then (1.1) is a nonclassical diffusion equation, namely

$$\partial_t u - (1 + m(0))\Delta u - \int_0^t m'(t-s)\Delta u(s)ds = f(u),$$

which has been a topic of an extensive study, see e.g., [3, 4, 6, 7, 9, 14, 15].

In this work, we are interested in the case m is possibly singular (e.g., the case of Rayleigh-Stokes or Basset equation). Up to our knowledge, the regularity and stability analysis for (1.1)-(1.3) have not been investigated in literature, and we aim at filling this gap. Based on the relaxation integral equation with completely positive kernel, we give an explicit representation of the resolvent operator, which enjoys some properties such as smoothness, decaying estimate, etc. In addition, a new Gronwall type inequality related to the relaxation function will be established and utilized in stability analysis. This inequality is also employed to prove the convergence of solution to equilibrium point, i.e. the solution of the elliptic problem

$$-\Delta v = f(v) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

The construction of the resolvent operator and its properties are represented in the next section. It should be noted that, in our case the regularity of the resolvent family cannot be obtained by using the resolvent theory given by Prüss [16] as in the recent work [11]. Fortunately, we get the differentiablity of the resolvent family in a particular case of kernel function m, namely, m is nonincreasing and $m(0^+) = \infty$. This property is enable us to prove the regularity of solutions. Section 3 is devoted to the results on global solvability and regularity. More precisely, our problem is globally solvable if the nonlinearity f gets the behavior like $\|f(v)\| = \ell\|v\| + o(\|v\|)$ as $\|v\| \to 0$, provided $\ell \in [0, \lambda_1)$ with λ_1 being the first eigenvalue of $-\Delta$. In the case when f is locally Lipschitzian, we show that the mild solution to (1.1)-(1.3) is classical, provided that m is nonincreasing. In Section 4, we prove some results on stability of solution to (1.1)-(1.3) such as the dissipativity, the asymptotic stability and the convergence to equilibrium. It should be mentioned that, the obtained results can be applied to the problem governed by the semilinear Rayleigh-Stokes equation or the Basset equation.

2. Preliminaries

Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ subject to the homogeneous boundary condition, i.e.,

$$-\Delta\varphi_n = \lambda_n\varphi_n \text{ in } \Omega,\ \varphi_n = 0 \text{ on } \partial\Omega,$$

where one can assume that $0 < \lambda_1 \le \lambda_2 \le ...,\ \lambda_n \to \infty$ as $n \to \infty$. For $\beta \in \mathbb{R}$, the fractional power operator $(-\Delta)^\beta$ is defined as follows

$$(-\Delta)^\beta v = \sum_{n=1}^\infty \lambda_n^\beta (v, e_n)e_n,$$

$$D((-\Delta)^\beta) = \{v \in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\beta}(v, e_n)^2 < \infty\},$$

here the notation $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.

We find a representation for solution of the linear problem

$$\partial_t u + \partial_t(m * (-\Delta)^\gamma u) - \Delta u = F \ \text{ in } \Omega, t \in (0,T], \tag{2.1}$$

$$u = 0 \ \text{ on } \partial\Omega,\ t \in [0,T], \tag{2.2}$$

$$u(\cdot,0) = \xi \ \text{ in } \Omega, \tag{2.3}$$

where $F \in C([0,T]; L^2(\Omega))$.

Assume that

$$u(\cdot,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n,\ F(\cdot,t) = \sum_{n=1}^{\infty} F_n(t)\varphi_n.$$

Substituting into (2.1), we get

$$u_n'(t) + \lambda_n u_n(t) + \lambda_n^\gamma (m * u_n)'(t) = F_n(t), \tag{2.4}$$

$$u_n(0) = \xi_n := (\xi, \varphi_n). \tag{2.5}$$

In order to find a representation of u_n, we consider *the relaxation equation*

$$\omega'(t) + \mu\omega(t) + \nu(m * \omega)'(t) = 0, \text{ for } t > 0,\ \omega(0) = 1,$$

where μ and ν are positive numbers. This equation can be rewritten as

$$\omega(t) + \mu(1 + \nu\mu^{-1} m) * \omega(t) = 1. \tag{2.6}$$

We make the following standing assumption:

(M) *$m \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function such that $a_\eta(t) := 1 + \eta m(t)$ is completely positive for any $\eta > 0$.*

Recall that, the complete positivity of a function $\ell \in L^1_{loc}(\mathbb{R}^+)$ means that the solutions of the following equations

$$s(t) + \theta(\ell * s)(t) = 1, \tag{2.7}$$

$$r(t) + \theta(\ell * r)(t) = \ell(t), \tag{2.8}$$

take nonnegative values for all $\theta > 0$. This is equivalent to that, there exist $\varepsilon \geq 0$ and a nonnegative and nonincreasing function $k \in L^1_{loc}(\mathbb{R}^+)$ such that $\varepsilon\ell + \ell * k = 1$ on $(0,\infty)$ (see [13]). It is easily seen that, if $\ell(0^+) = \infty$ then $\varepsilon = 0$. In this case, it holds that (see [5])

$$\int_0^t r(\tau)d\tau = \theta^{-1}(1 - s(t)),\ \forall t \geq 0. \tag{2.9}$$

Following [13], if a_η is completely monotonic, i.e. $(-1)^n a_\eta^{(n)}(t) \geq 0$ for all $n \in \mathbb{N}$ and $t \in (0,\infty)$, then it is completely positive. Noting that, if m is completely monotonic, so is a_η. A weaker condition ensuring the complete positivity of a_η is that, $a_\eta \in C^1(0,\infty)$ is positive and $\log a_\eta$ is a convex function (log-convex). It should be mentioned that, if $m \in C^1(0,\infty)$ is positive and log-convex, so is a_η. Indeed, if m is log-convex then it is convex. That means $\dfrac{m'}{m}$ and m' are increasing simultaneously, which implies that $\dfrac{d}{dt}\log a_\eta = \dfrac{\eta m'}{1 + \eta m}$ is increasing as well. That is, a_η is log-convex for any $\eta > 0$.

Under the assumption (**M**), the solution of (2.6) is positive on $\mathbb{R}^+$, since ω is the solution of (2.7) with $\theta = \mu$ and $\ell = 1 + \mu^{\gamma-1} m$. It should be noted that, the positivity of ω plays an important role in our analysis. We have the following additional properties of ω.

Proposition 2.1. *Let $\omega(\cdot,\mu,\nu)$ be the solution of (2.6). Then*

(a) *The function $t\mapsto\omega(t,\mu,\nu)$ is nonincreasing on $\mathbb{R}^+$ and obeys the inequality*

$$0<\omega(t,\mu,\nu)\le\frac{1}{1+\mu\int_0^t(1+\nu\mu^{-1}m(\tau))d\tau},\ \forall t\ge 0.$$

Consequently, $\lim_{t\to\infty}\omega(t,\mu,\nu)=0$.

(b) *We have*

$$\int_0^t\omega(\tau,\mu,\nu)d\tau\le\mu^{-1}(1-\omega(t,\mu,\nu)),\ \forall t\ge 0.$$

(c) *The function $\mu\mapsto\omega(t,\mu,\mu^\gamma)$ is nonincreasing on $(0,\infty)$ for each $t\ge 0$.*

Proof. (a) As pointed out in [5], the solution of (2.7) is nonincreasing, so is $\omega(\cdot,\mu)$. Then it follows that

$$\begin{aligned}1&=\omega(t,\mu,\nu)+\mu\int_0^t(1+\nu\mu^{-1}m(t-\tau))\omega(\tau,\mu)d\tau\\&\ge\omega(t,\mu,\nu)+\mu\omega(t,\mu,\nu)\int_0^t(1+\nu\mu^{-1}m(t-\tau))d\tau,\end{aligned}$$

which implies

$$\omega(t,\mu,\nu)\le\frac{1}{1+\mu\int_0^t(1+\nu\mu^{-1}m(\tau))d\tau},\ \forall t\ge 0.$$

(b) Thanks to the fact that m is nonnegative, we have

$$\begin{aligned}1&=\omega(t,\mu,\nu)+\mu\int_0^t(1+\nu\mu^{-1}m(t-\tau))\omega(\tau,\mu,\nu)d\tau\\&\ge\omega(t,\mu,\nu)+\mu\int_0^t\omega(\tau,\mu,\nu)d\tau,\end{aligned}$$

which ensures that

$$\int_0^t\omega(\tau,\mu,\nu)d\tau\le\mu^{-1}(1-\omega(t,\mu,\nu)).$$

(c) Let $\nu=\mu^\gamma$. Taking differentiation of (2.6) in μ, one gets

$$\partial_\mu\omega+\mu(1+\mu^{\gamma-1}m)*\partial_\mu\omega=-(1+\gamma\mu^{\gamma-1}m)*\omega.\tag{2.10}$$

Let $\tilde\ell=1+\mu^{\gamma-1}m$, then by (**M**), $\tilde\ell$ is completely positive and there exist $\varepsilon\ge 0$ and a function $\tilde k\in L^1_{loc}(\mathbb{R}^+)$ which is nonnegative and nonincreasing such that $\varepsilon\tilde\ell+\tilde\ell*\tilde k=1$. Noting that

$$\begin{aligned}1+\gamma\mu^{\gamma-1}m&=\gamma(1+\mu^{\gamma-1}m)+1-\gamma\\&=\gamma\tilde\ell+(1-\gamma)(\varepsilon\tilde\ell+\tilde\ell*\tilde k)\\&=\big(\gamma+(1-\gamma)\varepsilon\big)\tilde\ell+(1-\gamma)\tilde\ell*\tilde k,\end{aligned}$$

we can rewrite (2.10) as

$$\partial_\mu\omega+\mu\tilde\ell*\partial_\mu\omega=-[\big(\gamma+(1-\gamma)\varepsilon\big)\omega+\omega*\tilde k]*\tilde\ell.\tag{2.11}$$

Let $\tilde{r}$ be the solution of $\tilde{r} + \mu\tilde{\ell} * \tilde{r} = \tilde{\ell}$. Then we see that, the solution of (2.11) is given by

$$\partial_\mu \omega = -[(\gamma + (1-\gamma)\varepsilon)\omega + \omega * \tilde{k}] * \tilde{r}.$$

The last relation guarantees that $\partial_\mu \omega(t, \mu, \mu^\gamma) \le 0$ for each $t \ge 0$ and $\mu > 0$. The proof is complete. ☐

We are now in a position to consider the inhomogeneous equation

$$v'(t) + \mu v(t) + \nu(m * v)'(t) = g(t), \text{ for } t > 0, \; v(0) = v_0, \tag{2.12}$$

where $g \in C(\mathbb{R}^+)$ is a given function, μ and ν are positive numbers.

Proposition 2.2. *The solution of* (2.12) *is unique and given by*

$$v(t) = \omega(t, \mu, \nu)v_0 + \int_0^t \omega(t-\tau, \mu, \nu) g(\tau) d\tau, \; t \ge 0. \tag{2.13}$$

Proof. Put $L[y] = y' + \mu y + \nu(m * y)'$, $y \in C^1(\mathbb{R}^+)$. Then we have $L[\omega] = 0$. In addition, we see that

$$L[v] = L[\omega]v_0 + L[\omega * g] = L[\omega * g].$$

We will prove that $L[\omega * g] = g$. Indeed, one gets

$$\begin{aligned}
(\omega * g)' + \mu\omega * g + \nu(m * \omega * g)' &= g + \omega' * g + \mu\omega * g + \nu(m * \omega)' * g \\
&= g + [\omega' + \mu\omega + \nu(m * \omega)'] * g \\
&= g + L[\omega] * g = g.
\end{aligned}$$

Conversely, if v is a solution of (2.12), we have

$$z\hat{v}(z) + \mu\hat{v}(z) + \nu z\hat{m}(z)\hat{v}(z) = v_0 + \hat{g}(z),$$

where $\hat{v}$ is the Laplace transform of v. Then

$$\begin{aligned}
\hat{v}(z) &= (z + \mu + \nu z\hat{m}(z))^{-1} v_0 + (z + \mu + \nu z\hat{m}(z))^{-1}\hat{g}(z) \\
&= \hat{\omega}(z)v_0 + \hat{\omega}(z)\hat{g}(z),
\end{aligned}$$

where $\hat{\omega}$ is the Laplace transform of ω with respect to t. Taking the inverse Laplace transform yields $v = \omega v_0 + \omega * g$, which is (2.13). The proof is complete. ☐

In what follows, in the case $\nu = \mu^\gamma$, we write $\omega(t, \mu)$ instead of $\omega(t, \mu, \mu^\gamma)$. Using the last proposition, we have the following formula for the solution of (2.4)-(2.5):

$$u_n(t) = \omega(t, \lambda_n)\xi_n + \int_0^t \omega(t-\tau, \lambda_n) F_n(\tau) d\tau.$$

Thus one has the following representation for the solution of (2.1)-(2.3):

$$\begin{aligned}
u(\cdot, t) &= \sum_{n=1}^\infty \omega(t, \lambda_n)\xi_n\varphi_n + \sum_{n=1}^\infty \int_0^t \omega(t-\tau, \lambda_n) F_n(\tau) d\tau \varphi_n \\
&= S(t)\xi + \int_0^t S(t-\tau) F(\cdot, \tau) d\tau,
\end{aligned} \tag{2.14}$$

where $S(t)$ is defined by

$$S(t)\xi = \sum_{n=1}^\infty \omega(t, \lambda_n)\xi_n\varphi_n, \; \xi \in L^2(\Omega). \tag{2.15}$$

In the sequel, the notation $\|\cdot\|$ stands for the standard norm in $L^2(\Omega)$ and $\|\cdot\|_{\mathcal{L}}$ represents the operator norm of bounded linear operator acting on $L^2(\Omega)$.

Lemma 2.3. *Let $\{S(t)\}_{t\geq 0}$ be the resolvent operators defined by (2.15), $\xi \in L^2(\Omega)$ and $T > 0$. Then*

(a) $S(\cdot)v \in C([0,T];L^2(\Omega))$ and $\|S(t)\|_{\mathcal{L}} \leq \omega(t,\lambda_1)$ for all $t \geq 0$.
(b) For $g \in C([0,T];L^2(\Omega))$, $(-\Delta)^{\frac{1}{2}}S * g \in C([0,T];L^2(\Omega))$ and it holds that

$$\|(-\Delta)^{\frac{1}{2}}S * g(t)\| \leq \left(\int_0^t \omega(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}}, \ \forall t \geq 0. \tag{2.16}$$

(c) If m is nonincreasing and $m(0^+) = \infty$, then $S(\cdot)v \in C^1((0,T];L^2(\Omega))$ and it holds that

$$\|S'(t)\|_{\mathcal{L}} \leq t^{-1} \text{ for all } t > 0.$$

(d) $\Delta S(\cdot)\xi \in C((0,T];L^2(\Omega)) \cap L^1(0,T;L^2(\Omega))$ and we have the estimates

$$\begin{aligned}
&\|\Delta S(t)\xi\| \leq t^{-1}\|\xi\|, \quad \text{for all } t > 0,\\
&\|\int_0^t \Delta S(\tau)\xi d\tau\| \leq \|\xi\|, \quad \text{for all } t \geq 0.
\end{aligned}$$

Proof. (a) It follows from (2.15) that

$$\begin{aligned}
\|S(t)v\|^2 &= \sum_{n=1}^{\infty} \omega(t,\lambda_n)^2\xi_n^2\\
&\leq \omega(t,\lambda_1)^2 \sum_{n=1}^{\infty} \xi_n^2 = \omega(t,\lambda_1)^2\|\xi\|^2, \ \xi_n = (\xi,\varphi_n),
\end{aligned}$$

thanks to Proposition 2.1(c), which implies the uniform convergence of series (2.15) on $[0,T]$ and the estimate $\|S(t)\|_{\mathcal{L}} \leq \omega(t,\lambda_1)$ for all $t \geq 0$.
(b) We observe that

$$(-\Delta)^{\frac{1}{2}}S * g(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \int_0^t \omega(t-\tau,\lambda_n)g_n(\tau)d\tau\varphi_n, \tag{2.17}$$

where $g_n(t) = (g(t),\varphi_n)$. Using the Hölder inequality, we get

$$\begin{aligned}
\lambda_n\left(\int_0^t \omega(t-\tau,\lambda_n)g_n(\tau)d\tau\right)^2 &\leq \lambda_n \int_0^t \omega(t-\tau,\lambda_n)d\tau \int_0^t \omega(t-\tau,\lambda_n)|g_n(\tau)|^2 d\tau\\
&\leq (1-\omega(t,\lambda_n))\int_0^t \omega(t-\tau,\lambda_n)|g_n(\tau)|^2 d\tau\\
&\leq \int_0^t \omega(t-\tau,\lambda_1)|g_n(\tau)|^2 d\tau,
\end{aligned}$$

thanks to Proposition 2.1(b). So

$$\begin{aligned}\|(-\Delta)^{\frac{1}{2}}S*g(t)\|^2 &= \sum_{n=1}^{\infty}\lambda_n\left(\int_0^t \omega(t-\tau,\lambda_n)g_n(\tau)d\tau\right)^2\\ &\le \sum_{n=1}^{\infty}\int_0^t \omega(t-\tau,\lambda_1)|g_n(\tau)|^2 d\tau\\ &= \int_0^t \omega(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau,\end{aligned}$$

which implies (2.16). In order to show $(-\Delta)^{\frac{1}{2}}S*g \in C([0,T];L^2(\Omega))$, it suffices to check that series (2.17) is uniformly convergent on $[0,T]$. Since g is continuous, the series $\sum_{n=1}^{\infty}|g_n(\tau)|^2$ is uniformly convergent on $[0,T]$. That means, for any $\epsilon>0$, there exists $N_\epsilon\in\mathbb{N}$ such that $\sum_{n=N}^{N+p}|g_n(\tau)|^2<\epsilon$ for all $N\ge N_\epsilon$, $p\in\mathbb{N}$ and $\tau\in[0,T]$. It follows that

$$\begin{aligned}\sum_{n=N}^{N+p}\lambda_n\left(\int_0^t \omega(t-\tau,\lambda_n)g_n(\tau)d\tau\right)^2 &\le \sum_{n=N}^{N+p}\int_0^t \omega(t-\tau,\lambda_1)|g_n(\tau)|^2 d\tau\\ &= \int_0^t \omega(t-\tau,\lambda_1)\sum_{n=N}^{N+p}|g_n(\tau)|^2 d\tau \le \lambda_1^{-1}\epsilon,\end{aligned}$$

for all $t\in[0,T]$, which guarantees the uniform convergence of (2.17) on $[0,T]$.

(c) Let $r(\cdot,\lambda)$ be the solution of (2.8) with $\theta=\lambda$ and $\ell(t)=1+\lambda^{\gamma-1}m(t)$. Then, due to the assumption that m is nonincreasing, we have

$$r(t,\lambda)+\lambda(1+\lambda^{\gamma-1}m(t))\int_0^t r(\tau,\lambda)d\tau \le 1+\lambda^{\gamma-1}m(t).$$

In addition, it follows from (2.9) that

$$\int_0^t r(\tau,\lambda)d\tau = \lambda^{-1}(1-\omega(t,\lambda)) \ge \frac{t+\lambda^{\gamma-1}(1*m)(t)}{1+\lambda t+\lambda^{\gamma}(1*m)(t)},$$

in accordance with Proposition 2.1(a). Hence

$$r(t,\lambda)\le[1+\lambda^{\gamma-1}m(t)]\left[1-\frac{\lambda t+\lambda^{\gamma}(1*m)(t)}{1+\lambda t+\lambda^{\gamma}(1*m)(t)}\right] = \frac{1+\lambda^{\gamma-1}m(t)}{1+\lambda t+\lambda^{\gamma}(1*m)(t)}. \tag{2.18}$$

Considering the series

$$\sum_{n=1}^{\infty}\omega'(t,\lambda_n)\xi_n\varphi_n,\ t>0, \xi_n=(\xi,\varphi_n),\ \xi\in L^2(\Omega), \tag{2.19}$$

we see that

$$\begin{aligned}|\omega'(t,\lambda_n)| &= \lambda_n r(t,\lambda_n)\\ &\le \frac{\lambda_n+\lambda_n^{\gamma}m(t)}{1+\lambda_n t+\lambda_n^{\gamma}(1*m)(t)} \le \frac{\lambda_n+\lambda_n^{\gamma}m(t)}{\lambda_n t+\lambda_n^{\gamma}tm(t)} = t^{-1},\ \forall t>0.\end{aligned}$$

thanks to (2.18) and the fact that $1 * m(t) \geq tm(t)$ for $t > 0$. This ensures the uniform convergence of series (2.19) on $[\epsilon, T]$ and it holds that

$$S'(t)\xi = \sum_{n=1}^{\infty} \omega'(t, \lambda_n)\xi_n\varphi_n, \ \|S'(t)\xi\| \leq t^{-1}\|\xi\|, \ \forall t > 0.$$

(d) It follows from Proposition 2.1 that $\lambda_n\omega(t, \lambda_n) \leq t^{-1}$ for all $t > 0$. Then

$$\|\Delta S(t)\xi\|^2 = \sum_{n=1}^{\infty}[\lambda_n\omega(t, \lambda_n)]^2\xi_n^2 \leq t^{-2}\|\xi\|^2,$$

which infers that the series

$$\Delta S(t)\xi = \sum_{n=1}^{\infty} \lambda_n\omega(t, \lambda_n)\xi_n\varphi_n$$

is uniformly convergent on $[\epsilon, T]$ for any $\epsilon \in (0, T)$. Thus $\Delta S(\cdot)\xi \in C((0, T]; L^2(\Omega))$.

On the other hand, we see that

$$\int_0^t \Delta S(\tau)\xi d\tau = -\sum_{n=1}^{\infty}\int_0^t \lambda_n\omega(\tau, \lambda_n)\xi_n d\tau\varphi_n. \tag{2.20}$$

Then

$$\begin{aligned}
\|\int_0^t \Delta S(\tau)\xi d\tau\|^2 &= \sum_{n=1}^{\infty}\left(\int_0^t \lambda_n\omega(\tau, \lambda_n)d\tau\right)^2\xi_n^2 \\
&\leq \sum_{n=1}^{\infty}(1 - \omega(t, \lambda_n))^2\xi_n^2 \leq \|\xi\|^2,
\end{aligned}$$

due to Proposition 2.1(b), which implies that $\Delta S(\cdot)\xi \in L^1(0, T; L^2(\Omega))$. The proof is complete. $\square$

Based on the properties of $S(t)$ stated in Lemma 2.3, we will show that the Cauchy operator

$$\begin{aligned}
&\mathcal{Q} : C([0, T]; L^2(\Omega)) \to C([0, T]; L^2(\Omega)), \\
&\mathcal{Q}(g)(t) = \int_0^t S(t - \tau)g(\tau)d\tau, \qquad (2.21)
\end{aligned}$$

is compact in the next lemma.

Lemma 2.4. *Let (**M**) hold. If the function m is nonincreasing and $m(0^+) = \infty$, then the operator $\mathcal{Q}$ defined by* (2.21) *is compact.*

Proof. Let $D \subset C([0, T]; L^2(\Omega))$ be a bounded set. We first testify that $(-\Delta)^{\frac{1}{2}}\mathcal{Q}(D)(t)$ is bounded in $L^2(\Omega)$ for each $t \geq 0$. Indeed, by using Lemma 2.3(b), we get

$$\|(-\Delta)^{\frac{1}{2}}\mathcal{Q}(g)(t)\| \leq \int_0^t \omega(t - \tau, \lambda_1)\|g(\tau)\|^2 d\tau, \ \forall t \geq 0,$$

which ensures the boundedness of $(-\Delta)^{\frac{1}{2}}\mathcal{Q}(D)(t)$ in $L^2(\Omega)$ for all $t \geq 0$. Since the embedding $D((-\Delta)^{\frac{1}{2}}) \hookrightarrow L^2(\Omega)$ is compact, we obtain the relative compactness of $\mathcal{Q}(D)(t)$ for each $t \geq 0$.

Now we show that $\mathcal{Q}(D)$ is equicontinuous. Let $g \in D$, $t \in (0,T)$, and $h \in (0, T-t]$, then one sees that

$$\begin{aligned}\|\mathcal{Q}(g)(t+h) - \mathcal{Q}(g)(t)\| &\le \int_0^t \|[S(t+h-\tau) - S(t-\tau)]g(\tau)\|d\tau \\ &\quad + \int_t^{t+h} \|S(t+h-\tau)g(\tau)\|d\tau \\ &= I_1(t) + I_2(t).\end{aligned}$$

It is easily seen that $I_2(t) \to 0$ as $h \to 0$ uniformly in $g \in D$. Regarding $I_1(t)$, we observe that

$$\begin{aligned}\|[S(t+h-\tau) - S(t-\tau)]g(\tau)\| &= \|\int_0^1 hS'(t-\tau+\theta h)g(\tau)d\theta\| \\ &\le h\int_0^1 \|S'(t-\tau+\theta h)\|_{\mathcal{L}}\|g(\tau)\|d\theta \\ &\le h\int_0^1 \frac{\|g(\tau)\|d\theta}{t-\tau+\theta h},\end{aligned}$$

thanks to the mean value formula and Lemma 2.3(c). So

$$\begin{aligned}\|[S(t+h-\tau) - S(t-\tau)]g(\tau)\| &\le C\|g\|_\infty \ln\left(1 + \frac{h}{t-\tau}\right) \\ &\le C\|g\|_\infty \frac{h^\beta}{\beta(t-\tau)^\beta},\ \beta \in (0,1), \qquad (2.22)\end{aligned}$$

here $\|g\|_\infty = \sup\limits_{t\in[0,T]} \|g(t)\|$, and we used the inequality $\ln(1+r) \le \dfrac{r^\beta}{\beta}$ for any $r > 0, \beta \in (0,1)$. Employing (2.22), we have

$$\begin{aligned}I_1(t) &\le \frac{\|g\|_\infty h^\beta}{\beta}\int_0^t \frac{ds}{(t-\tau)^\beta} \\ &\le \frac{\|g\|_\infty h^\beta}{\beta(1-\beta)}T^{1-\beta} \to 0 \text{ as } h \to 0 \text{ uniformly in } g \in D.\end{aligned}$$

Finally, for $h \in (0,T)$, we have

$$\|\mathcal{Q}(g)(h) - \mathcal{Q}(g)(0)\| \le \int_0^h \|S(h-\tau)g(\tau)\|d\tau \le h\|g\|_\infty \to 0 \text{ as } h \to 0,$$

uniformly in $g \in D$. Therefore, $\mathcal{Q}(D)$ is equicontinuous. We have the conclusion due to the Arzelà-Ascoli theorem. □

We end this section by proving a Gronwall type inequality, which will be used in our stability analysis.

Proposition 2.5. *Let z be a nonnegative function obeying the inequality*

$$z(t) \le \omega(t,\mu,\nu)z_0 + \int_0^t \omega(t-\tau,\mu,\nu)[az(\tau) + b(\tau)]d\tau,\ t \ge 0, \qquad (2.23)$$

where $a \in [0,\mu)$, $\nu > 0$, $b \in L^1_{loc}(\mathbb{R}^+)$. Then

$$z(t) \le \omega(t,\mu-a,\nu)z_0 + \int_0^t \omega(t-\tau,\mu-a,\nu)b(\tau)d\tau.$$

Proof. Let $y(t)$ be the right hand side of (2.23). Then $z(t) \le y(t)$ and y solves the equation

$$y'(t) + \mu y(t) + \nu(m * y)'(t) = az(t) + b(t),\ t > 0, y(0) = z_0,$$

as stated by Proposition 2.2. It follows that

$$y'(t) + (\mu - a)y(t) + \nu(m * y)'(t) = a[z(t) - y(t)] + b(t),\ t > 0, y(0) = z_0,$$

and then y admits the representation

$$\begin{aligned} y(t) &= \omega\Big(t, \mu - a, \nu\Big) z_0 \\ &\quad + \int_0^t \omega(t-\tau, \mu - a, \nu)(a[z(\tau) - y(\tau)] + b(\tau))d\tau \\ &\le \omega(t, \mu - a, \nu)z_0 + \int_0^t \omega(t-\tau, \mu - a, \nu) b(\tau) d\tau, \end{aligned}$$

thanks to the positivity of ω and the fact that $z(\tau) - y(\tau) \le 0$ for $\tau \ge 0$. So we get the conclusion as desired. □

3. Solvability and regularity

Based on representation (2.14), we give the following definition of mild solution for (1.1)-(1.3).

Definition 3.1. *A function $u \in C([0,T]; L^2(\Omega))$ is said to be a mild solution to the problem (1.1)-(1.3) on $[0,T]$ iff*

$$u(\cdot, t) = S(t)\xi + \int_0^t S(t-\tau) f(u(\cdot,\tau)) d\tau \text{ for any } t \in [0,T].$$

We first prove a global solvability result for (1.1)-(1.3).

Theorem 3.1. *Let (**M**) hold. Assume that*

(**F**1) *The function $f : L^2(\Omega) \to L^2(\Omega)$ satisfies $f(0) = 0$ and is locally Lipschitzian, i.e.*

$$\|f(v_1) - f(v_2)\| \le \kappa(r)\|v_1 - v_2\|,\ \forall v_1, v_2 \in B_r,$$

where B_r is the closed ball in $L^2(\Omega)$ with radius r and center at origin, $\kappa(\cdot)$ is a nonnegative function such that $\limsup\limits_{r\to 0} \kappa(r) = \ell \in [0, \lambda_1)$.

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) has a unique mild solution on $[0,T]$, provided $\|\xi\| \le \delta$.

Proof. Let $\Phi : C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$ be the mapping defined by

$$\Phi(u)(t) = S(t)\xi + \int_0^t S(t-\tau) f(u(\cdot,\tau)) d\tau \text{ for } t \in [0,T].$$

We first look for $\rho > 0$ such that $\Phi(\mathsf{B}_\rho) \subset \mathsf{B}_\rho$, where B_ρ is the closed ball in $C([0,T]; L^2(\Omega))$ centered at origin with radius ρ. Taking $\epsilon \in (0, \lambda_1 - \ell)$, we can find

$\rho > 0$ such that $\kappa(r) \le \ell + \epsilon$ for any $r \le \rho$. Considering $\Phi : \mathsf{B}_\rho \to C([0,T]; L^2(\Omega))$, we have

$$\begin{aligned}\|\Phi(u)(\cdot,t)\| &\le \|S(t)\xi\| + \int_0^t \|S(t-\tau)\|_{\mathcal{L}} \|f(u(\cdot,\tau))\| d\tau \\ &\le \omega(t,\lambda_1)\|\xi\| + \int_0^t \omega(t-\tau,\lambda_1)\kappa(\rho)\|u(\cdot,\tau)\| d\tau \\ &\le \omega(t,\lambda_1)\|\xi\| + (\ell+\epsilon)\rho \int_0^t \omega(t-\tau,\lambda_1) d\tau \\ &\le \omega(t,\lambda_1)\|\xi\| + (\ell+\epsilon)\rho\lambda_1^{-1}(1-\omega(t,\lambda_1)) \\ &= \omega(t,\lambda_1)[\|\xi\| - (\ell+\epsilon)\rho\lambda_1^{-1}] + (\ell+\epsilon)\rho\lambda_1^{-1},\ \forall u \in \mathsf{B}_\rho, t \in [0,T],\end{aligned}$$

here we used Lemma 2.3(1) and Proposition 2.1(2). Choosing $\|\xi\| \le \delta := \ell\rho\lambda_1^{-1}$, we see that

$$\|\Phi(u)(\cdot,t)\| \le (\ell+\epsilon)\rho\lambda_1^{-1} \le \rho,\ \forall u \in \mathsf{B}_\rho, t \in [0,T],$$

which implies $\Phi(\mathsf{B}_\rho) \subset \mathsf{B}_\rho$. We now prove that Φ is a contraction mapping on B_ρ. For $u_1, u_2 \in \mathsf{B}_\rho$, one gets

$$\begin{aligned}\|\Phi(u_1)(\cdot,t) - \Phi(u_2)(\cdot,t)\| &\le \int_0^t \omega(t-\tau,\lambda_1)\|f(u_1(\cdot,\tau)) - f(u_2(\cdot,\tau))\| d\tau \\ &\le \kappa(\rho) \int_0^t \omega(t-\tau,\lambda_1)\|u_1(\cdot,\tau) - u_2(\cdot,\tau)\| d\tau \\ &\le (\ell+\epsilon)\|u_1 - u_2\|_\infty \int_0^t \omega(t-\tau,\lambda_1) d\tau \\ &\le (\ell+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\|u_1 - u_2\|_\infty,\ \forall t \in [0,T],\end{aligned}$$

which ensures that

$$\|\Phi(u_1) - \Phi(u_2)\|_\infty \le (\ell+\epsilon)\lambda_1^{-1}\|u_1 - u_2\|_\infty.$$

Hence Φ is a contraction mapping and it admits a fixed point in B_ρ, which is a mild solution to (1.1)-(1.3). In order to testify the uniqueness, we observe that, if $u, v \in C([0,T]; L^2(\Omega))$ are solutions of (1.1)-(1.3), then one can assume that $u, v \in \mathsf{B}_R$ for some $R > 0$. So

$$\begin{aligned}\|u(\cdot,t) - v(\cdot,t)\| &\le \int_0^t \omega(t-\tau,\lambda_1)\kappa(R)\|u(\cdot,\tau) - v(\cdot,\tau)\| d\tau \\ &\le \kappa(R) \int_0^t \|u(\cdot,\tau) - v(\cdot,\tau)\| d\tau,\ \forall t \in [0,T],\end{aligned}$$

according to the fact that $\omega(t,\lambda_1) \le 1$ for all $t \ge 0$. By using the Gronwall inequality, we get $\|u(\cdot,t) - v(\cdot,t)\| = 0$ for all $t \in [0,T]$, which implies $u = v$. The proof is complete. □

In the next theorem, we show an existence result without Lipschitz condition.

Theorem 3.2. *Let **(M)** hold. Assume that*

(**F**2) *The function $f : L^2(\Omega) \to L^2(\Omega)$ is a continuous such that*

$$\limsup_{\|v\|\to 0} \frac{\|f(v)\|}{\|v\|} = \ell \in [0, \lambda_1).$$

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) *has a unique mild solution on* $[0, T]$*, provided* $\|\xi\| \le \delta$.

Proof. Choosing $\epsilon \in (0, \lambda_1 - \ell)$, we can find $\rho > 0$ such that $\|f(v)\| \le (\ell + \epsilon)\|v\|$ for any $v \in \mathsf{B}_\rho$. Arguing as in the proof of Theorem 3.1, we have $\Phi(\mathsf{B}_\rho) \subset \mathsf{B}_\rho$, provided $\|\xi\| \le \delta = \ell\rho\lambda^{-1}$. Considering $\Phi : \mathsf{B}_\rho \to \mathsf{B}_\rho$, we see that Φ is a continuous mapping, thanks to the continuity of f. In addition, one can represent Φ as

$$\Phi(u) = S(\cdot)\xi + \mathcal{Q} \circ N_f(u), \tag{3.1}$$

where N_f is defined by $N_f(u)(\cdot, t) = f(u(\cdot, t))$, which is continuous as mapping from $C([0, T]; L^2(\Omega))$ into itself. Due to the compactness of the operator $\mathcal{Q}$ stated in Lemma 2.4, we get that Φ is compact. Therefore, Φ admits a fixed point in B_ρ, according to the Schauder fixed point theorem. The proof is complete. □

Remark 3.1. *In the case f is globally Lipschitzian, i.e.*

$$\|f(v_1) - f(v_2)\| \le \kappa_0\|v_1 - v_2\|, \ \forall v_1, v_2 \in L^2(\Omega),$$

for some $\kappa_0 > 0$, one can prove the existence and uniqueness of mild solution to (1.1)-(1.3) *by using the contraction mapping principle, regardless the assumption $f(0) = 0$ and the smallness of initial data (see, e.g.* [11, Theorem 4.2]*).*

We are in a position to show the first result on regularity of mild solution to (1.1)-(1.3).

Theorem 3.3. *Assume that m satisfies **(M)** with additional conditions that, m is nonincreasing and $m(0^+) = \infty$. If $f : L^2(\Omega) \to L^2(\Omega)$ is locally Lipschitzian, i.e.*

$$\|f(v_1) - f(v_2)\| \le \kappa(r)\|v_1 - v_2\|, \ \forall v_1, v_2 \in B_r,$$

where $\kappa(\cdot)$ is a nonnegative function, and u is a mild solution to (1.1)-(1.3) *on* $[0, T]$*, then $u \in C^1((0, T]; L^2(\Omega))$.*

Proof. The proof consists of two steps.
Step 1. We show that u is Hölder continuous on $(0, T]$. By assumption, the resolvent family $S(\cdot)$ is continuously differentiable on $(0, \infty)$ and

$$\|S'(t)\|_{\mathcal{L}} \le t^{-1}, \ \text{ for all } t > 0,$$

thanks to Lemma 2.3.

For $t \in (0, T]$ and $h \in (0, T - t)$, we have

$$\begin{aligned}
\|u(\cdot, t + h) - u(\cdot, t)\| &\le \|[S(t + h) - S(t)]\xi\| \\
&\quad + \int_t^{t+h} \|S(t + h - \tau)f(u(\cdot, \tau))\|d\tau \\
&\quad + \int_0^t \|[S(t + h - \tau) - S(t - \tau)]f(u(\cdot, \tau))\|d\tau \\
&= E_1(t) + E_2(t) + E_3(t).
\end{aligned}$$

Using the mean value formula, we have

$$[S(t + h) - S(t)]\xi = h\int_0^1 S'(t + \zeta h)\xi d\zeta.$$

Then

$$E_1(t) = \|[S(t+h) - S(t)]\xi\| \le h\|\xi\| \int_0^1 \frac{d\zeta}{t+\zeta h}$$
$$= M\|\xi\| \ln\left(1 + \frac{h}{t}\right) \le \|\xi\|\beta^{-1}\left(\frac{h}{t}\right)^\beta, \text{ for any } \beta \in (0,1),$$

where we used the inequality $\ln(1+r) \le \dfrac{r^\beta}{\beta}$ for all $r > 0$, $\beta \in (0,1)$.

Dealing with $E_2(t)$, we put $R = \|u\|_\infty$ and make use of the inequality

$$\|f(u(\cdot,t))\| \le \kappa(R)\|u(\cdot,t)\| + \|f(0)\| \le \kappa(R)R + \|f(0)\|.$$

So

$$E_2(t) \le \int_t^{t+h} \|f(u(\cdot,\tau))\| d\tau \le [\kappa(R)R + \|f(0)\|]h$$
$$\le [\kappa(R)R + \|f(0)\|]T^{1-\beta}h^\beta.$$

Regarding $E_3(t)$, we note that

$$[S(t+h-\tau) - S(t-\tau)]f(u(\cdot,\tau)) = h\int_0^1 S'(t-\tau+\zeta h)f(u(\cdot,\tau))d\zeta.$$

Then by the same argument used to estimate $E_1(t)$, we obtain

$$\|[S(t+h-\tau) - S(t-\tau)]f(u(\cdot,\tau))\| \le [\kappa(R)R + \|f(0)\|]\beta^{-1}\left(\frac{h}{t-\tau}\right)^\beta.$$

Therefore,

$$E_3(t) \le [\kappa(R)R + \|f(0)\|]\beta^{-1}h^\beta \int_0^t \frac{d\tau}{(t-\tau)^\beta}$$
$$= [\kappa(R)R + \|f(0)\|]\beta^{-1}(1-\beta)^{-1}T^{1-\beta}h^\beta.$$

Summing up, we get

$$E_1(t) + E_2(t) + E_3(t) \le (C_1 t^{-\beta} + C_2)h^\beta,$$

where

$$C_1 = \|\xi\|\beta^{-1},$$
$$C_2 = [\kappa(R)R + \|f(0)\|]T^{1-\beta}(1 + \beta^{-1}(1-\beta)^{-1}).$$

Step 2. We now prove that u is continuously differentiable on $(0,T]$. Writing

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)f(u(\cdot,\tau))d\tau = u_1(t) + u_2(t),$$

we see that $u_1 = S(\cdot)\xi \in C^1((0,T]; L^2(\Omega))$ as stated in Lemma 2.3(c). In addition, we have

$$\partial_t u_2(t) = f(u(\cdot,t)) + \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau,$$

where the second term makes sense by the following reasoning

$$\begin{aligned}
\|\int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau\| &\le \|\int_0^t S'(t-\tau)[f(u(\cdot,\tau))-f(u(\cdot,t))]d\tau\| \\
&\quad + \|\int_0^t S'(t-\tau)f(u(\cdot,t))d\tau\| \\
&\le \int_0^t \|S'(t-\tau)\|_{\mathcal{L}}\|f(u(\cdot,\tau))-f(u(\cdot,t))\|d\tau \\
&\quad + \|(S(t)-I)f(u(\cdot,t))\| \\
&\le \int_0^t (t-\tau)^{-1}(C_1\tau^{-\beta}+C_2)(t-\tau)^{\beta}d\tau \\
&\quad + \|(S(t)-I)f(u(\cdot,t))\| \\
&\le \int_0^t (t-\tau)^{\beta-1}(C_1\tau^{-\beta}+C_2)d\tau + \|(S(t)-I)f(u(\cdot,t))\| \\
&< \infty, \quad \text{for all } t \ge 0,
\end{aligned}$$

thanks to the result of Step 1. It remains to show that the mapping

$$F(t) = \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau$$

is continuous on $(0,T]$. Let $t>0$ and $h \in (0, T-t)$, then

$$\begin{aligned}
\|F(t+h)-F(t)\| &\le \|\int_t^{t+h} S'(t+h-\tau)f(u(\cdot,\tau))d\tau\| \\
&\quad + \|\int_0^t [S'(t+h-\tau)-S'(t-\tau)]f(u(\cdot,t))d\tau\| \\
&\quad + \|\int_0^t [S'(t+h-\tau)-S'(t-\tau)][f(u(\cdot,\tau))-f(u(\cdot,t))]d\tau\| \\
&= F_1(t)+F_2(t)+F_3(t).
\end{aligned}$$

Obviously, $F_1(t) \to 0$ as $h \to 0$ due to the fact that $\tau \mapsto S'(t-\tau)f(u(\cdot,\tau))$ is integrable on $(0,t)$ for any $t \in (0,T)$. Regarding $F_2(t)$, we see that

$$F_2(t) = \|[S(t+h)-S(t)+I-S(h)]f(u(\cdot,t))\| \to 0 \text{ as } h \to 0.$$

Dealing with $F_3(t)$, we get

$$\begin{aligned}
F_3(t) &\le \kappa(R)\int_0^t \|S'(t+h-\tau)-S'(t-\tau)\|_{\mathcal{L}}\|u(\cdot,t)-u(\cdot,\tau)\|d\tau \\
&\le \kappa(R)\int_0^t \|S'(t+h-\tau)-S'(t-\tau)\|_{\mathcal{L}}(C_1\tau^{-\beta}+C_2)(t-\tau)^{\beta}d\tau.
\end{aligned}$$

Let us denote $Q_h(\tau) = \|S'(t+h-\tau) - S'(t-\tau)\|_{\mathcal{L}}(C_1\tau^{-\beta} + C_2)(t-\tau)^\beta$, then $\lim\limits_{h\to 0} Q_h(\tau) = 0$ for any $\tau \in (0,t)$. Moreover, we observe that

$$\begin{aligned}
Q_h(\tau) &\le [\|S'(t+h-\tau)\|_{\mathcal{L}} + \|S'(t-\tau)\|_{\mathcal{L}}](C_1\tau^{-\beta} + C_2)(t-\tau)^\beta \\
&\le [(t+h-\tau)^{-1} + (t-\tau)^{-1}](C_1\tau^{-\beta} + C_2)(t-\tau)^\beta \\
&\le 2(t-\tau)^{-1}(C_1\tau^{-\beta} + C_2)(t-\tau)^\beta \\
&= 2(t-\tau)^{\beta-1}(C_1\tau^{-\beta} + C_2) = Q^*(\tau).
\end{aligned}$$

Since $Q^* \in L^1(0,t)$, one can utilize the Lebesgue dominated convergence theorem to claim that

$$F_3(t) \le \kappa(R)\int_0^t Q_h(\tau)d\tau \to 0 \text{ as } h \to 0.$$

Thus F is continuous on $(0,T]$ and the theorem is proved. □

Theorem 3.4. *Let the assumptions of Theorem 3.3 hold. Then the mild solution of* (1.1)-(1.3) *on* $[0,T]$ *satisfies* $\Delta u \in C((0,T];L^2(\Omega))$.

Proof. Let u be a mild solution (1.1)-(1.3), then

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)f(u(\cdot,\tau))d\tau = u_1(t) + u_2(t),\ \forall t \in [0,T].$$

By Lemma 2.3(d), we get $\Delta u_1 \in C((0,T];L^2(\Omega))$. In addition, let $R = \|u\|_\infty$ then

$$\begin{aligned}
\|\Delta u_2(t)\| &= \|\int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau\| \\
&\le \|\int_0^t \Delta S(t-\tau)[f(u(\cdot,\tau)) - f(u(\cdot,t))]d\tau\| + \|\int_0^t \Delta S(t-\tau)f(u(\cdot,t))d\tau\| \\
&\le \kappa(R)\int_0^t (t-\tau)^{-1}\|u(\cdot,\tau) - u(\cdot,t)\| + \|\int_0^t \Delta S(\tau)f(u(\cdot,t))d\tau\| \\
&\le \kappa(R)\int_0^t (t-\tau)^{\beta-1}(C_1\tau^{-\beta} + C_2)d\tau + \|f(u(\cdot,t))\|,
\end{aligned}$$

here we employed the Lipschitz condition on f, the Hölder continuity shown in the proof of Theorem 3.3, and the estimates stated in Lemma 2.3(d). This means, Δu_2 makes sense. It remains to check that the mapping $t \mapsto \Delta u_2(t)$ is continuous on $(0,T]$. Let $t > 0$ and $h \in (0, T-t]$, then

$$\begin{aligned}
\|\Delta u_2(t+h) - \Delta u_2(t)\| \le \|&\int_t^{t+h} \Delta S(t+h-\tau)f(u(\cdot,\tau))d\tau\| \\
&+ \|\int_0^t [\Delta S(t+h-\tau) - \Delta S(t-\tau)]f(u(\cdot,t))d\tau\| \\
&+ \|\int_0^t [\Delta S(t+h-\tau) - \Delta S(t-\tau)][f(u(\cdot,\tau) - f(u(\cdot,t))]d\tau\| \\
= G_1&(t) + G_2(t) + G_3(t).
\end{aligned}$$

Since the function $\tau \mapsto \Delta S(t+h-\tau)f(u(\cdot,\tau))$ is integrable, we see that $G_1(t) \to 0$ as $h \to 0$.

For $G_2(t)$, we have

$$\begin{aligned}
G_2(t) &= \| \int_h^{t+h} \Delta S(\tau) f(u(\cdot,t))d\tau - \int_0^t \Delta S(\tau) f(u(\cdot,t))d\tau \| \\
&\le \| \int_t^{t+h} \Delta S(\tau) f(u(\cdot,t))d\tau \| + \| \int_0^h \Delta S(\tau) f(u(\cdot,t))d\tau \| \\
&\to 0 \text{ as } h \to 0,
\end{aligned}$$

according to the fact that $\tau \mapsto \Delta S(\tau)\xi$ is integrable for any $\xi \in L^2(\Omega)$, as stated in Lemma 2.3(d).

Concerning $G_3(t)$, we observe that

$$G_h(\tau) := \|[\Delta S(t+h-\tau) - \Delta S(t-\tau)][f(u(\cdot,\tau) - f(u(\cdot,t))]\| \to 0 \text{ as } h \to 0,$$

for all $\tau \in (0,t)$, thanks to the fact that $\Delta S(\cdot)\xi \in C((0,T]; L^2(\Omega))$ for any $\xi \in L^2(\Omega)$. Moreover, we get

$$\begin{aligned}
G_h(\tau) &\le [(t+h-\tau)^{-1} + (t-\tau)^{-1}]\kappa(R)\|u(\cdot,t) - u(\cdot,\tau)\| \\
&\le 2\kappa(R)(t-\tau)^{\beta-1}(C_1\tau^{-\beta} + C_2),
\end{aligned}$$

according to the Hölder continuity of u and the estimate of $\Delta S(t)$ given by Lemma 2.3(d). Thus

$$G_3(t) \le \int_0^t G_h(\tau)d\tau \to 0 \text{ as } h \to 0,$$

due to the Lebesgue dominated convergence theorem. This completes the proof. □

Theorem 3.5. *Assume that the assumptions of Theorem 3.3 are satisfied. Then the mild solution of* (1.1)-(1.3) *on* $[0,T]$ *obeys* $m * (-\Delta)^\gamma u \in C^1((0,T]; L^2(\Omega))$. *Consequently, this solution is classical.*

Proof. Noting that

$$\begin{aligned}
u(\cdot,t) &= \sum_{n=1}^\infty [\omega(t,\lambda_n)\xi_n + \omega(\cdot,\lambda_n) * F_n(t)]\varphi_n, \\
&= \sum_{n=1}^\infty u_n(t)\varphi_n,
\end{aligned}$$

where $F_n(t) = (f(u(\cdot,t)), \varphi_n)$, we have

$$m * (-\Delta)^\gamma u(\cdot,t) = \sum_{n=1}^\infty \lambda_n^\gamma (m * u_n)(t)\varphi_n.$$

So it suffices to show that, the series $\sum\limits_{n=1}^\infty \lambda_n^\gamma (m*u_n)'(t)\varphi_n$ is uniformly convergent on $[\epsilon, T]$ for any $\epsilon \in (0,T)$. We first see that $\lambda_n^\gamma [m * \omega(\cdot,\lambda_n)]'(t) = -(\omega'(t,\lambda_n) + \lambda_n\omega(t,\lambda_n))$ thanks to the relaxation equation, then

$$\begin{aligned}
\lambda_n^\gamma (m*u_n)'(t) &= \lambda_n^\gamma [m * \omega(\cdot,\lambda_n)\xi_n + m * \omega(\cdot,\lambda_n) * F_n]'(t) \\
&= -[\omega'(t,\lambda_n)\xi_n + \lambda_n\omega(t,\lambda_n)\xi_n] \\
&\quad - [\omega'(\cdot,\lambda_n) * F_n + \lambda_n\omega_n(\cdot,\lambda_n) * F_n](t).
\end{aligned}$$

One can write

$$\sum_{n=1}^{\infty} \lambda_n^{\gamma}(m * u_n)'(t)\varphi_n = -(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4),$$

where

- $\sigma_1 = \sum\limits_{n=1}^{\infty} \omega'(t, \lambda_n)\xi_n\varphi_n$ is uniformly convergent on $[\epsilon, T]$ since $\sigma_1 = S'(t)\xi$ and $S(\cdot)\xi \in C^1((0,T]; L^2(\Omega))$.
- $\sigma_2 = \sum\limits_{n=1}^{\infty} \lambda_n\omega(t, \lambda_n)\xi_n\varphi_n$ is uniformly convergent on $[\epsilon, T]$ to $-\Delta S(t)\xi$ due to that $\Delta S(\cdot)\xi \in C((0,T]; L^2(\Omega))$.
- $\sigma_3 = \sum\limits_{n=1}^{\infty} \omega'(\cdot, \lambda_n) * F_n(t)\varphi_n = \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau$ which is continuous in $t \in (0, T]$ as proved in Theorem 3.3. So σ_3 is uniformly convergent on $[\epsilon, T]$.
- $\sigma_4 = \sum\limits_{n=1}^{\infty} \lambda_n\omega_n(\cdot, \lambda_n) * F_n(t)$ is uniformly convergent on $[\epsilon, T]$ to $-\int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau$, which is continuous in $t \in (0, T]$ as testified in the proof of Theorem 3.4.

In summary, we get

$$\begin{aligned}\partial_t(m * (-\Delta)^{\gamma}u)(t) &= -(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) \\ &= -S'(t)\xi + \Delta S(t)\xi - \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau \\ &\quad + \int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau \\ &= \Delta u(\cdot,t) - \partial_t u(\cdot,t) + f(u(\cdot,t)),\ \forall t \in (0,T],\end{aligned}$$

which ensures that u satisfies (1.1). The theorem is proved. □

4. Stability results

In this section, we show some results on the asymptotic stability, dissipativity and the convergence to the solution of the elliptic equation $-\Delta v = f(v)$.

Theorem 4.1. *Let the hypotheses of Theorem 3.1 hold. Then the zero solution to* (1.1) *is asymptotically stable.*

Proof. Taking ρ, δ and $\epsilon \in (0, \lambda_1 - \ell)$ from the proof of Theorem 3.1, we see that for any initial data $\|\xi\| \le \delta$, there exists a unique solution u of (1.1)-(1.3) such that $\|u(\cdot,t)\| \le \rho$ and $\|f(u(\cdot,t))\| \le (\ell + \epsilon)\|u(\cdot,t)\|$ for any $t \ge 0$. Then

$$\begin{aligned}\|u(\cdot,t)\| &\le \|S(t)\xi\| + \int_0^t \|S(t-\tau)\|\|f(u(\cdot,\tau))\|d\tau \\ &\le \omega(t, \lambda_1, \lambda_1^{\gamma})\|\xi\| + \int_0^t \omega(t-\tau, \lambda_1, \lambda_1^{\gamma})(\ell + \epsilon)\|u(\cdot,\tau)\|d\tau,\end{aligned}$$

thanks to Proposition 2.1. Applying the Gronwall type inequality given by Proposition 2.5, we get

$$\|u(\cdot,t)\| \le \omega(t, \lambda_1 - \ell - \epsilon, \lambda_1^{\gamma})\|\xi\|,\ \forall t \ge 0,$$

which implies that the zero solution of (1.1) is asymptotically stable, according to the fact that $\lambda_1 - \ell - \epsilon > 0$ and $\omega(t,\mu) \to 0$ as $t \to \infty$ for any $\mu > 0$. $\square$

If the nonlinearity function f is globally Lipschitzian, we have a stronger result, whose proof is simply utilizing Proposition 2.5.

Theorem 4.2. *Let (**M**) hold. If f satisfies the Lipschitz condition*

$$\|f(v_1) - f(v_2)\| \le \kappa_0 \|v_1 - v_2\|,\ \forall v_1, v_2 \in L^2(\Omega),$$

where $\kappa_0 \in [0, \lambda_1)$, then every solution of (1.1)-(1.2) *is asymptotically stable.*

In the next theorem, we show a result on dissipativity of (1.1)-(1.3).

Theorem 4.3. *Let (**M**) hold. If f satisfies the condition*

$$\|f(v)\| \le a\|v\| + b,\ \forall v \in L^2(\Omega),$$

where $a \in [0, \lambda_1)$, and b is a nonnegative constant. Then there exists an absorbing set of mild solutions to (1.1)-(1.3) *for any initial data.*

Proof. Let u be a mild solution of (1.1)-(1.3). Then

$$\|u(\cdot,t)\| \le \omega(t,\lambda_1)\|\xi\| + \int_0^t \omega(t-\tau,\lambda_1)(a\|u(\cdot,\tau)\| + b)d\tau.$$

By using Proposition 2.5 again, we obtain

$$\begin{aligned}
\|u(\cdot,t)\| &\le \omega(t,\lambda_1 - a, \lambda_1^\gamma)\|\xi\| + b\int_0^t \omega(t-\tau,\lambda_1 - a, \lambda_1^\gamma)d\tau \\
&\le \omega(t,\lambda_1 - a, \lambda_1^\gamma)\|\xi\| + \frac{b}{\lambda_1 - a}(1 - \omega(t,\lambda_1 - a, \lambda_1^\gamma)) \\
&\le \omega(t,\lambda_1 - a, \lambda_1^\gamma)\|\xi\| + \frac{b}{\lambda_1 - a},
\end{aligned}$$

thanks to Proposition 2.1(b). It follows that the closed ball $B_R \subset L^2(\Omega)$ with $R = \dfrac{b}{\lambda_1 - a} + 1$ is an absorbing set of solutions to (1.1)-(1.3) for any initial data. The proof is complete. $\square$

We are in a position to show a result on the convergence of solution of (1.1)-(1.3) to equilibrium point.

Theorem 4.4. *Let (**M**) hold. Assume that $\partial\Omega \in C^2$ and $f : L^2(\Omega) \to L^2(\Omega)$ is continuous and locally bounded. If a mild solution of* (1.1)-(1.3) *obeys $\lim\limits_{t\to\infty} u(\cdot,t) = u^*$ in $L^2(\Omega)$, then u^* is a strong solution of the elliptic problem*

$$-\Delta v = f(v) \ \text{in}\ \Omega, \tag{4.1}$$

$$v = 0 \ \text{on}\ \partial\Omega, \tag{4.2}$$

provided that $z\hat{m}(z) \to 0$ as $z \to 0$ in $\mathbb{C}$, where $\hat{m}$ is the Laplace transform of m. Conversely, if f satisfies the Lipschitz condition

$$\|f(v_1) - f(v_2)\| \le \kappa_0 \|v_1 - v_2\|,\ \forall v_1, v_2 \in L^2(\Omega), \tag{4.3}$$

where $\kappa_0 \in [0, \lambda_1)$, then the solution of (1.1)-(1.3) *converges to the unique strong solution of* (4.1)-(4.2)*, provided that m is nonincreasing, $m(0^+) = \infty$ and $\lim\limits_{t\to\infty} m(t) = 0$.*

Proof. Let u be a mild solution to (1.1)-(1.3) such that $\lim\limits_{t\to\infty} u(\cdot,t) = u^*$ in $L^2(\Omega)$. Then

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]d\tau + \int_0^t S(t-\tau)f(u^*)d\tau.$$

Obviously, $\|S(t)\xi\| \le \omega(t,\lambda_1)\|\xi\| \to 0$ as $t\to\infty$. On the other hand, since $f(u(\cdot,t)) \to f(u^*)$ as $t\to\infty$ in $L^2(\Omega)$, there exists $T>0$ such that $\|f(u(\cdot,t)) - f(u^*)\| < \varepsilon$ for all $t \ge T$, here $\epsilon > 0$ is given. So for any $t\ge T$, we get

$$\begin{aligned}
\|\int_0^t S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]d\tau\| &\le \int_0^T \omega(t-\tau,\lambda_1)\|f(u(\cdot,\tau)) - f(u^*)\|d\tau \\
&\quad + \int_T^t \omega(t-\tau,\lambda_1)\|f(u(\cdot,\tau)) - f(u^*)\|d\tau \\
&\le \int_0^T \omega(t-\tau,\lambda_1)\|f(u(\cdot,\tau)) - f(u^*)\|d\tau \\
&\quad + \varepsilon\int_T^t \omega(t-\tau,\lambda_1)d\tau \\
&\le C\int_0^T \omega(t-\tau,\lambda_1)d\tau + \varepsilon\int_T^t \omega(t-\tau,\lambda_1)d\tau,
\end{aligned}$$

for some $C>0$, thanks to the local boundedness of f. Since $\omega(\cdot,\lambda_1)\in L^1(\mathbb{R}^+)$, we have

$$C\int_0^T \omega(t-\tau,\lambda_1)d\tau = C\int_{t-T}^t \omega(\tau,\lambda_1)d\tau < C\varepsilon,$$

for t large enough. In addition, we see that

$$\varepsilon\int_T^t \omega(t-\tau,\lambda_1)d\tau = \varepsilon\int_0^{t-T} \omega(\tau,\lambda_1)d\tau < \varepsilon\lambda_1^{-1}.$$

Thus

$$\int_0^t S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]d\tau \to 0 \text{ as } t\to\infty.$$

It follows that

$$\lim_{t\to\infty} u(\cdot,t) = \lim_{t\to\infty}\int_0^t S(t-\tau)f(u^*)d\tau = \int_0^\infty S(\tau)f(u^*)d\tau.$$

Equivalently, one has

$$u^* = \hat{S}(0)f(u^*), \tag{4.4}$$

where $\hat{S}$ is the Laplace transform of the resolvent family $S(\cdot)$, i.e.

$$\hat{S}(z)\xi = \int_0^\infty e^{-zt}S(t)\xi dt,\ \xi\in L^2(\Omega).$$

Since $S(t)$ is the resolvent operator for the Cauchy problem

$$\begin{aligned}
\partial_t u + \partial_t(m*(-\Delta)^\gamma u) - \Delta u = 0 \text{ in } \Omega, t>0,\\
u = 0 \text{ on } \partial\Omega, t\ge 0,\\
u(\cdot,0) = \xi \text{ in } \Omega,
\end{aligned}$$

we get $u(\cdot,t) = S(t)\xi$ and then $\hat{S}(z)\xi = \hat{u}(\cdot,z)$. Taking the Laplace transform of the last system, we have

$$z\hat{u} + z\hat{m}(z)(-\Delta)^\gamma\hat{u} - \Delta\hat{u} = \xi.$$

Thus $\hat{S}(z) = (zI + z\hat{m}(z)(-\Delta)^\gamma - \Delta)^{-1}$ and $\hat{S}(0) = (-\Delta)^{-1}$. Plugging into (4.4) yields $u^* = (-\Delta)^{-1}f(u^*)$. Employing the regularity result in [10, Sect. 6.3.2] with the assumption $\partial\Omega \in C^2$, we get $u^* \in H^2(\Omega)$. Thus u^* is a strong solution of (4.1)-(4.2).

We now prove the converse statement. Assume that (4.3) holds, m is nonincreasing, $m(0^+) = \infty$ and $m(t) \to 0$ as $t \to \infty$. By Remark 3.1 and Theorem 3.5, the problem (1.1)-(1.3) has a unique classical solution. On the other hand, due to [8, Theorem 7.4.1], the problem (4.1)-(4.2) has a unique weak solution $u^* \in H_0^1(\Omega)$ if the Lipschitz constant κ_0 satisfies $\kappa_0 < C_e^{-2}$, where C_e is the constant of embedding $H_0^1(\Omega) \subset L^2(\Omega)$. By the smoothness of $\partial\Omega$, we have

$$C_e^{-2} = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_1.$$

Observe that $f(u^*) \in L^2(\Omega)$. Then using the regularity result in [10, Sect. 6.3.2] again, we obtain $u^* \in H^2(\Omega)$ and hence u^* is a unique strong solution of (4.1)-(4.2).

We now combine (1.1)-(1.3) with (4.1)-(4.2) to obtain

$$\partial_t(u - u^*) + \partial_t[m * (-\Delta)^\gamma(u - u^*)] - \Delta(u - u^*) = f(u) - f(u^*) - m(-\Delta)^\gamma u^*$$

in Ω, with the boundary condition

$$u - u^* = 0 \text{ on } \partial\Omega,$$

and the initial condition

$$u(\cdot,0) - u^* = \xi - u^* \text{ in } \Omega.$$

Then $u - u^*$ admits the following representation

$$u(\cdot,t) - u^* = S(t)(\xi - u^*) + \int_0^t S(t-\tau)[f(\cdot,\tau) - f(u^*) - m(\tau)(-\Delta)^\gamma u^*]d\tau,$$

and hence

$$\begin{aligned}\|u(\cdot,t) - u^*\| &\le \omega(t,\lambda_1)\|\xi - u^*\| \\ &\quad + \int_0^t \omega(t-\tau,\lambda_1)[\kappa_0\|u(\cdot,\tau) - u^*\| + m(\tau)\|(-\Delta)^\gamma u^*\|]d\tau.\end{aligned}$$

Utilizing the Gronwall type inequality in Proposition 2.5, we get

$$\begin{aligned}\|u(\cdot,t) - u^*\| &\le \omega(t,\lambda_1 - \kappa_0, \lambda_1^\gamma)\|\xi - u^*\| \\ &\quad + \|(-\Delta)^\gamma u^*\| \int_0^t \omega(t-\tau,\lambda_1 - \kappa_0,\lambda_1^\gamma)m(\tau)d\tau.\end{aligned}$$

Since $\omega(t,\lambda_1 - \kappa_0,\lambda_1^\gamma) \to 0$ as $t \to \infty$, it is sufficient to testify that

$$M(t) = \int_0^t \omega(t-\tau,\lambda_1 - \kappa_0,\lambda_1^\gamma)m(\tau)d\tau \to 0 \text{ as } t \to \infty.$$

By assumption, for given $\varepsilon > 0$, there is $T > 0$ such that $m(t) < \varepsilon$ for all $t \geq T$. Then for any $t \geq T$, one gets

$$\begin{aligned} M(t) &\leq \int_0^T \omega(t-\tau, \lambda_1 - \kappa_0, \lambda_1^\gamma) m(\tau) d\tau + \varepsilon \int_T^t \omega(t-\tau, \lambda_1 - \kappa_0, \lambda_1^\gamma) d\tau \\ &\leq \omega(t-T, \lambda_1 - \kappa_0, \lambda_1^\gamma) \int_0^T m(\tau) d\tau + \varepsilon \int_0^{t-T} \omega(\tau, \lambda_1 - \kappa_0, \lambda_1^\gamma) d\tau \\ &\leq \omega(t-T, \lambda_1 - \kappa_0, \lambda_1^\gamma) \int_0^T m(\tau) d\tau + \varepsilon (\lambda_1 - \kappa_0)^{-1}, \end{aligned}$$

thanks to Proposition 2.1(b). This implies

$$M(t) \leq [1 + (\lambda_1 - \kappa_0)^{-1}]\varepsilon,$$

provided t chosen such that

$$\omega(t-T, \lambda_1 - \kappa_0, \lambda_1^\gamma) \int_0^T m(\tau) d\tau < \varepsilon,$$

which is possible since $\omega(t-T, \lambda_1 - \kappa_0, \lambda_1^\gamma) \to 0$ as $t \to \infty$. The proof is complete. $\square$

Acknowledgement. This work was supported by the Vietnam Institute for Advanced Study in Mathematics-VIASM.

References

[1] A. Allaberen, *Well-posedness of the Basset problem in spaces of smooth functions*, Appl. Math. Lett. 24 (2011), 1176-1180.

[2] E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid, *Numer. Math.* 131 (2015), no. 1, 1-31.

[3] P. Cannarsa, H. Frankowska, E. M. Marchini, *Optimal control for evolution equations with memory*, J. Evol. Equ. 13 (2013), no. 1, 197-227.

[4] J. R. Cannon, Y.P. Lin, *A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory*, SIAM J. Numer. Anal. 27 (1990), no. 3, 595-607.

[5] Ph. Clément, J. A. Nohel, *Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels*, SIAM J. Math. Anal. 12 (1981), 514-535.

[6] M. Conti, Elsa M. Marchini, V. Pata, *Reaction-diffusion with memory in the minimal state framework*, Trans. Amer. Math. Soc. 366 (2014), no. 9, 4969-4986.

[7] G. Di Blasio, *Parabolic Volterra integrodifferential equations of convolution type*, J. Integral Equations Appl. 6 (1994), no. 4, 479-508.

[8] P. Drábek, J. Milota, Methods of nonlinear analysis. Applications to differential equations. Birkhäuser Verlag, Basel, 2007.

[9] K.Ezzinbi, S.Ghnimi, M.-A. Taoudi, *Existence results for some nonlocal partial integrodifferential equations without compactness or equicontinuity*, J. Fixed Point Theory Appl. 21 (2019), no. 2, Art. 53, 24 pp.

[10] L.C. Evans, Partial differential equations. Second edition. American Mathematical Society, Providence, RI, 2010.

[11] T.D. Ke, N.N. Thang, L.T.P. Thuy, *Regularity and stability analysis for a class of semilinear nonlocal differential equations in Hilbert spaces*, J. Math. Anal. Appl. 483 (2020), no. 2, 123655.

[12] S. McKee, A. Stokes, *Product integration methods for the nonlinear Basset equation*, SIAM J. Numer. Anal. 20 (1983), no. 1, 143-160.

[13] R.K. Miller, *On Volterra integral equations with nonnegative integrable resolvents*, J. Math. Anal. Appl. 22 (1968), 319-340.

[14] R. K. Miller, *An integro-differential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. 66 (1978), no. 2, 313-332.

[15] A. Mohebbi, *Crank-Nicolson and Legendre spectral collocation methods for a partial integro-differential equation with a singular kernel*, J. Comput. Appl. Math. 349 (2019), 197-206.

[16] J. Prüss, Evolutionary Integral Equations and Applications. Monographs in Mathematics 87, Birkhäuser, Basel, 1993.

[17] B. Wu, J. Yu, *Uniqueness of an inverse problem for an integro-differential equation related to the Basset problem*, Bound. Value Probl. 2014:229, 9 pp.

Tran Dinh Ke
Department of Mathematics, Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
E-mail address: ketd@hnue.edu.vn

Nguyen Nhu Thang
Department of Mathematics, Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
E-mail address: thangnn@hnue.edu.vn