Spanning trees with few peripheral branch vertices

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Abstract

Let T be a tree, a vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. Let T be a tree with $B(T) \neq \emptyset$, for each a vertex $x \in L(T)$, set $y_x \in B(T)$ such that $(V(P_T[x, y_x]) \setminus \{y_x\}) \cap B(T) = \emptyset$, where $P_T[u, v]$ is the unique path in T connecting u and v. We delete $V(P_T[x, y_x]) \setminus \{y_x\}$ from T for all $x \in Leaf(T)$. The resulting graph is a subtree of T and denoted by $R(Stem(T))$. It is called the *reducible stem* of T. A leaf of $R(Stem(T))$ is called a *peripheral branch vertex* of T. In this paper, we give some sharp sufficient conditions on the independence number and the degree sum to show that a graph G to have a few peripheral branch vertices.

Keywords: spanning tree; leaf; peripheral branch vertex; independence number; degree sum
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1 Introduction

In this paper, we only consider finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ (or $N(v)$ and $\deg(v)$ if there is no ambiguity) to denote the set of neighbors of v and the degree of v in G, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of X. Sometime, we denote by $|G|$ instead of $|V(G)|$. We define $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $\deg_G(X) = \sum_{x \in X} \deg_G(x)$. For $k \geq 1$, we let $N_k(X) = \{x \in V(G) \mid |N(x) \cap X| = k\}$. We use $G - X$ to denote the graph obtained from G by deleting the vertices in X together with their incident edges. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G. For two vertices u and v of G, the distance between u and v in G is denoted by $d_G(u, v)$. We use K_n to denote the complete graph on n vertices. We write $A := B$ to rename B as A.

For an integer $m \geqslant 2$, let $\alpha^m(G)$ denote the number defined by

$$\alpha^m(G) = \max\{|S| : S \subseteq V(G), d_G(x, y) \geqslant m \text{ for all distinct vertices } x, y \in S\}.$$

For an integer $p \geqslant 2$, we define

$$\sigma_p^m(G) = \min\left\{\sum_{a \in S} \deg_G(a) : S \subseteq V(G), |S| = p, d_G(x, y) \geqslant m \text{ for all distinct vertices } x, y \in S\right\}.$$

For convenience, we define $\sigma_p^m(G) = +\infty$ if $\alpha^m(G) < p$. We note that, $\alpha^2(G)$ is often written $\alpha(G)$, which is the independence number of G, and $\sigma_p^2(G)$ is often written $\sigma_p(G)$, which is the minimum degree sum of p independent vertices.

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. There are several well-known conditions (such as the independence number conditions and the degree sum conditions) ensuring that a graph G contains a spanning tree with a bounded number of leaves (see the survey paper [14] and the references cited therein for details). Win [16] obtained a sufficient condition related to the independence number for l-connected graphs, which confirms a conjecture of Las Vergnas [11]. Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with at most k leaves.

Theorem 1.1 (Win [16]) *Let $l \geq 1$ and $k \geq 2$ be integers and let G be a l-connected graph. If $\alpha(G) \leq k + l - 1$, then G has a spanning tree with at most k leaves.*

Theorem 1.2 (Broerma and Tuinstra [1]) *Let G be a connected graph and let $k \geq 2$ be an integer. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning tree with at most k leaves.*

The subtree $T - L(T)$ of a tree T is called the *stem* of T and is denoted by $Stem(T)$. Recently, many researches are studied on spanning trees in connected graphs whose stems have a bounded number of leaves (see [7], [8] and [15] for more details). We introduce here some results on spanning trees whose stems have a few leaves.

Theorem 1.3 (Tsugaki and Zhang [15]) *Let G be a connected graph and let $k \geqslant 2$ be an integer. If $\sigma_3(G) \geqslant |G| - 2k + 1$, then G have a spanning tree whose stem has at most k leaves.*

Theorem 1.4 (Kano and Yan [7]) *Let G be a connected graph and let $k \geqslant 2$ be an integer. If either $\alpha^4(G) \leq k$ or $\sigma_{k+1}(G) \geqslant |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves.*

Furthermore, by considering the graph G to be restricted in some special graph classes, many analogue researches have been introduced (see [2], [3], [5], [6], [9], [10] and [12] for examples).

In this paper, we would like to introduce a new situation on spanning tree. For two distinct vertices u, v of T, we denote by $P_T[u, v]$ the unique path in T connecting u and v. We define the *orientation* of $P_T[u, v]$ is from u to v. We refer to [4] for terminology and notation not defined here. Let T be a tree with $B(T) \neq \emptyset$. For every $x \in L(T)$, set $y_x \in B(T)$ such that $(V(P_T[x, y_x]) \setminus \{y_x\}) \cap B(T) = \emptyset$. We delete $V(P_T[x, y_x]) \setminus \{y_x\}$ from T for all $x \in L(T)$. The resulting graph is denoted by $R(Stem(T))$. It is called the *reducible stem* of T. The path with vertex set $V(P_T[x, y_x]) \setminus \{y_x\}$ is called a *branch* of T, usually denoted by B_x. B_x is called to be incident to x. Then $R(Stem(T)) = T - B$ with $B = \bigcup_{x \in L(T)} V(B_x)$ (see Figure 1 for a picture of T and $R(Stem(T))$). A leaf of $R(Stem(T))$ is also called a *peripheral branch vertex* of T (also see [13, page 234]). We denote by $P(B(T))$ the peripheral branch vertex set of T. We study

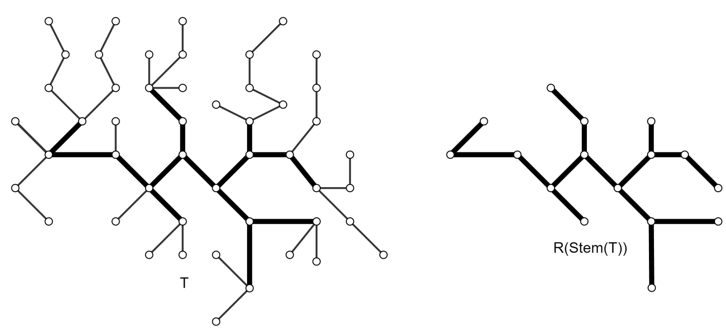

Figure 1: Tree T and R(Stem(T))

sufficient conditions to show that a graph to have a spanning tree T with few peripheral branch vertices, i.e., $R(Stem(T))$ has a few leaves. In particular, we state the following theorem.

Theorem 1.5 *Let G be a connected graph and let $k \geq 2$ be an integer. If one of the following conditions holds, then G has a spanning tree with at most k peripheral branch vertices.*

(i) $\alpha(G) \leq 2k + 2$,

(ii) $\sigma^4_{k+1}(G) \geq \lfloor \frac{|G|-k}{2} \rfloor$.

Here, the notation $\lfloor r \rfloor$ stands for the biggest integer not exceed the real number r.

To end this section, we give an example to see that our main results are sharp. Let $k \geq 2$ and $m \geq 1$ be integers, and let $D_1, D_2, ..., D_{k+1}$ and $H_1, H_2, ..., H_{k+1}$ be $2k+2$ disjoint copies of the complete graph K_m of order m. Let $w, x_1, x_2, ..., x_{k+1}$ be $k+2$ vertices not contained in $D_1 \cup D_2 \cup ... \cup D_{k+1} \cup H_1 \cup H_2 \cup ... \cup H_{k+1}$. Join w to all vertices of $\{x_1, x_2, ..., x_{k+1}\}$ and join x_i to all the vertices of $D_i \cup H_i$ for every $1 \leq i \leq k+1$. Let G denote the resulting graph (see Figure 2). Then $\alpha(G) = 2k+3$. Moreover, we also obtain

$$\sigma^4_{k+1}(G) = \sum_{i=1}^{k+1} \deg_G(y_i) = (k+1)m = \lfloor \frac{|G|-k}{2} \rfloor - 1,$$

with y_i is any vertex of D_i for every $1 \leq i \leq k+1$.

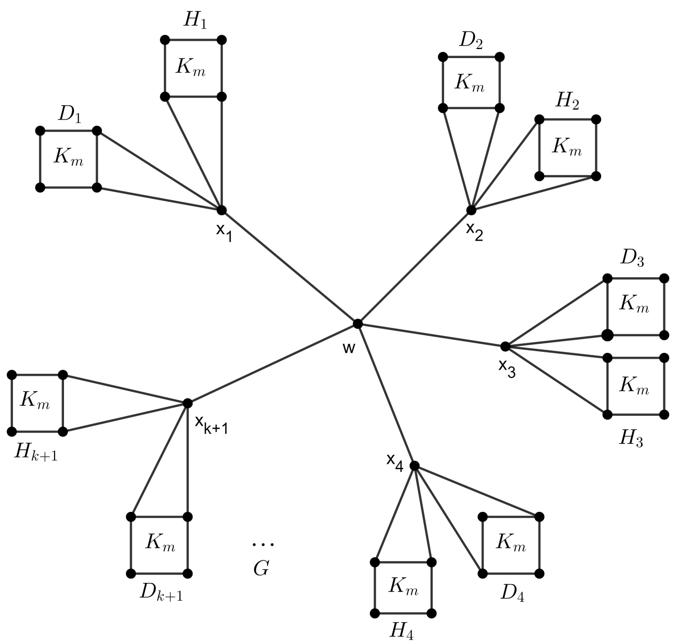

Figure 2: Graph G

But G has no a spanning tree with at most k peripheral branch vertices. Then, the conditions of Theorem 1.5 are sharp.

2 Proof of the main results

Proof. Suppose for a contradiction that there does not exist a spanning tree T of G such that $|P(B(T))| \leq k$. Then every spanning tree T of G satisfies $|P(B(T))| \geq k+1$.

Choose T to be a maximal tree of G such that $|P(B(T))| = k+1$ and

(C1) $|R(Stem(T))|$ is as small as possible,

(C2) The number of branches of T is as small as possible subject to $(C1)$.

Claim 2.1 *There does not exist a tree S in G such that $V(S) = V(T)$ and $|P(B(S))| \leq k$.*

Proof. Indeed, assume that there exists a tree S in G such that $V(S) = V(T)$ and $|P(B(S))| \leq k$. Since $|P(B(S))| \leq k$, S is not a spanning tree of G. Then there exists $u \in V(G) - V(S)$ such that u is adjacent to a vertex $v \in S$. Let S_1 be a tree obtained from S by adding the edge uv. Then S_1 is a tree in G such that $|V(S_1)| = |V(T)| + 1$ and $|P(B(S_1))| \leq k + 1$.

If $|P(B(S_1))| = k+1$, then S_1 contradicts the maximality of T (since $|V(S_1)| = |V(S)|+1 = |V(T)| + 1 > |V(T)|$). So we may assume that $|P(B(S_1))| \leq k$. By repeating this process, we can recursively construct a set of trees $\{S_i \mid i \geq 1\}$ in G such that S_i satisfies that $|P(B(S_i))| \leq k$ and $|V(S_{i+1})| = |V(S_i)| + 1$ for each $i \geq 1$. Since G has no spanning tree with at most k peripheral branch vertices and $|V(G)|$ is finite, the process must terminate after a finite number of steps, i.e., there exists some $h \geq 1$ such that S_{h+1} is a tree in G with $|P(B(S_{h+1}))| = k + 1$. But this contradicts the maximality of T. So the claim holds. ∎

Set $P(B(T)) = \{x_1, x_2, ..., x_{k+1}\}$. By the definition of peripheral branch vertex, we have the following claim.

Claim 2.2 *For every $i \in \{1, 2, ..., k + 1\}$, there exist at least two branchs of T which are incident to x_i.*

Claim 2.3 *For each $i \in \{1, 2, ..., k+1\}$, there exist $y_i, z_i \in L(T)$ such that B_{y_i}, B_{z_i} are incident to x_i and $N_G(y_i) \cap (V(R(Stem(T))) - \{x_i\}) = \emptyset$, $N_G(z_i) \cap (V(R(Stem(T))) - \{x_i\}) = \emptyset$.*

Proof. Let $\{a_{ij}\}_{j=1}^m$ be the subset of $L(T)$ such that $B_{a_{ij}}$ is adjacent to x_i. By Claim 2.2, we obtain $m \geq 2$.
Suppose that there are more than $m - 1$ vertices $\{a_{ij}\}_{j=1}^m$ satisfying

$$N_G(a_{ij}) \cap (V(R(Stem(T))) - \{x_i\}) \neq \emptyset.$$

Without lost of generality, we may assume that $N_G(a_{ij}) \cap (V(R(Stem(T))) - \{x_i\}) \neq \emptyset$ for all $j = 2, ..., m$. Set $b_{ij} \in N_G(a_{ij}) \cap (V(R(Stem(T))) - \{x_i\})$, for all $j \in \{2, ..., m\}$. Consider the tree

$$T' := T + \{a_{ij}b_{ij}\}_{j=2}^m - \{x_i v_{ij}\}_{j=2}^m,$$

where $v_{ij} \in N_T(x_i) \cap V(P_T[a_{ij}; x_i])$. Hence T' satisfies $|V(T')| = |V(T)|$ and $|R(Stem(T'))| < |R(Stem(T))|$, which contradicts the condition (C1). Therefore, Claim 2.3 holds. ∎

Set $U = \{y_i, z_i\}_{i=1}^{k+1}$. By the maximality of T we have $N_G(U) \subseteq V(T)$.

Claim 2.4 *U is an independent set in G.*

Proof. Suppose that there exist two vertices $u, v \in U$ such that $uv \in E(G)$. Without lost of generality, we assume that $v = y_i$ for some $i \in \{1, 2, ..., k + 1\}$. Set $v_i = N_T(x_i) \cap B_{y_i}$. Consider the tree $T' := T + uy_i - v_i x_i$. If $\deg_T(x_i) = 3$ then x_i is not a branch vertex of T'. Hence $|R(Stem(T'))| < |R(Stem(T))|$, this contradicts the condition (C1). Otherwise, $|R(Stem(T'))| = |R(Stem(T))|$ but the number of branchs of T' is smaller then ones of T. This contradicts the condition (C2). The proof of Claim 2.4 is completed. ∎

Since $k \geq 2$, then $|L(R(Stem(T)))| \geq 3$. Hence, we have $|B(R(Stem(T)))| \geq 1$. Let u be a vertex in $B(R(Stem(T)))$. By Claim 2.3 and Claim 2.4 we conclude that $U \cup \{u\}$ is an independent set in G. This implies that $\alpha(G) \geq 2k + 3$. This is a contradiction with the assumption (i) of Theorem 1.5.

Claim 2.5 *For every* $1 \leq i \neq j \leq k+1$, *then* $N_G(y_i) \cap B_{y_j} = \emptyset$ *and* $N_G(y_i) \cap B_{z_j} = \emptyset$.

Proof. By the same role of y_i and z_i, we only need to prove for the case $N_G(y_i) \cap B_{y_j} = \emptyset$. Suppose the assertion of the claim is false. Then there exists a vertex $x \in N_G(y_i) \cap B_{y_j}$. Set $T' := T + xy_i$. Then T' is a subgraph of G including a unique cycle C, which contains both x_i and x_j.

Since $k \geq 2$, then $|L(R(Stem(T)))| \geq 3$. Hence, we have $|B(R(Stem(T)))| \geq 1$. Then there exists a branch vertex of $R(Stem(T))$ contained in C. Let e be an edge incident to such a vertex in C and $R(Stem(T))$. By removing the edge e from T' we obtain a tree T'' of G satisfying $V(T'') = V(T)$ and $|P(B(T''))| \leq k$. This is a contradiction with Claim 2.1. Claim 2.5 is proved. ∎

Claim 2.6 *For every* $1 \leq i < j \leq k+1$, $d_G(y_i, y_j) \geq 4$ *and* $d_G(z_i, z_j) \geq 4$.

Proof. We first prove that $d_G(y_i, y_j) \geq 4$. Let $P[y_i, y_j]$ be a shortest path connecting y_i and y_j in G. Assume that all vertices of $P[y_i, y_j]$ are contained in $(V(G) - R(Stem(T))) \cup \{x_i, x_j\}$.

Let $t_i \in B_{y_i} \cup \{x_i\}, t_j \in B_{y_j} \cup \{x_j\}$ such that $t_i, t_j \in P[y_i, y_j]$ and

$$P_{P[y_i,y_j]}[t_i, t_j] \cap B_{y_i} = \{t_i\},\ P_{P[y_i,y_j]}[t_i, t_j] \cap B_{y_j} = \{t_j\}.$$

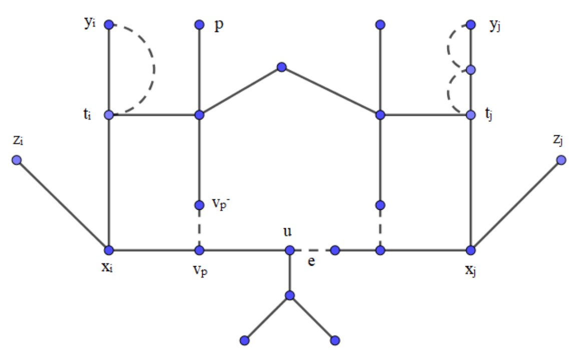

Figure 3: Tree T''

Set $P[t_i, t_j] := P_{P[y_i,y_j]}[t_i, t_j]$. For every branch B_p of T such that $B_p \cap P[t_i, t_j] \neq \emptyset$, remove all the edges of B_p in T which are incident to $R(Stem(T))$ and add $P[t_i, t_j]$. Then the resulting subgraph T' of G includes a unique cycle C, which contains two vertices x_i and x_j. Because $|B(R(Stem(T))| \geq 1$, there exists a branch vertex u of $R(Stem(T))$ to be contained in C. Let e be an edge in C which is incident to u. Denote by T'' to be a tree obtained from T' by removing the edge e (see Figure 3 for an example). Then $V(T) \subseteq V(T') = V(T'')$ and $|P(B(T''))| \leq k$. This contradicts either the maximality of T or Claim 2.1. Therefore, $P[y_i, y_j] \cap (R(Stem(T)) - \{x_i, x_j\}) \neq \emptyset$. Set $v \in P[y_i, y_j] \cap (R(Stem(T)) - \{x_i, x_j\})$. Hence, by combining with Claim 2.3, we obtain

$$d_G(y_i, y_j) = d_{P[y_i,y_j]}(y_i, y_j) \geq d_{P[y_i,y_j]}(y_i, v) + d_{P[y_i,y_j]}(v, y_j) \geq 2 + 2 = 4.$$

Now, by using the same arguments, we also obtain that $d_G(z_i, z_j) \geq 4$. These complete the proof of Claim 2.6. ∎

By Claim 2.6 we obtain that $\alpha^4(G) \geq k+1$.

Claim 2.7 $\sum_{y\in U}|N_G(y)\cap B_p|\leq|B_p|$ *for every* $p\in L(T)-U$.

Proof. Set $v_p\in B(T)$ such that $(V(P_T[p,v_p])\setminus\{v_p\})\cap B(T)=\emptyset$. Then we consider $B_p=P_T[p,v_p]-\{v_p\}$.

Subclaim 2.7.1. For every $i\in\{1,2,...,k+1\}$, if $x\in N_G(y_i)\cap B_p$ then $x^+\notin N_G(U-\{y_i\})\cap B_p$.

Suppose that there exists $x^+\in N_G(z)\cap B_p$ with $z\in U-\{y_i\}$. Let $T':=T+xy_i+x^+z-xx^+-v_pv_p^-$. Then T' is a tree in G satisfying $V(T')=V(T)$ and the number of branches of T' is smaller than number of branches of T. Hence this contradicts the condition (C2).

Subclaim 2.7.2. For every $x\in B_p$ then x is adjacent to at most 2 vertices in U.

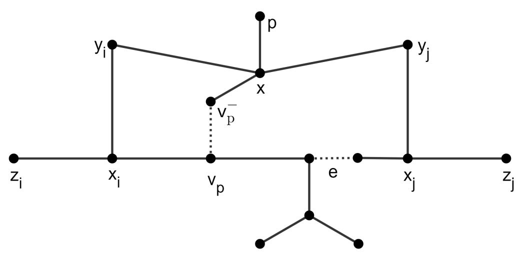

Figure 4: Tree T''

Indeed, we first prove that if $x\in N_G(y_i)\cap B_p$ then $x\notin N_G(y_j)\cap B_p$ and $x\notin N_G(z_j)\cap B_p$ for all $1\leq i\neq j\leq k+1$. To the contrary, without loss of generality, assume that there exist $1\leq i\neq j\leq k+1$ such that $x\in N_G(y_i)\cap B_p$ and $x\in N_G(y_j)$. Set $T':=T+xy_i+xy_j-v_pv_p^-$. Then T' is subgraph of G including a unique cycle C, which contains two vertices x_i and x_j. Since $|B(R(Stem(T)))|\geq 1$, there exists a branch vertex in the $R(Stem(T))$ contained in C. Let e be an edge which is incident to such vertex in C. By removing the edge e we obtain a tree T'' of G (see Figure 4 for an example). Then $|V(T'')|\geq|V(T)|$ and $|P(B(T)|\leq k$. This contradicts either the maximality of T or Claim 2.1. Therefore, we have $|U\cap N_G(x)|\leq 2$.

Subclaim 2.7.3. $p\notin N_G(U)$.

Indeed, to the contrary, without loss of generality, assume that $p\in N_G(y_i)$ for some $y_i\in U$. We consider the tree $T':=T+y_ip-v_pv_p^-$. Hence, T' is a tree with $|V(T')|=|V(T)|$, $|R(Stem(T'))|\leq|R(Stem(T))|$ and the number of branchs of T' is smaller then the ones of T. This contradicts either the condition (C1) or (C2). Therefore $p\notin N_G(U)$.

Now, by Subclaims *2.7.1*-*2.7.2* we conclude that $\{p\}, N_G(y_i)\cap B_p$, $(N_G(U-\{y_i\})\cap B_p)^+$ and $(N_2(U)-N(y_i))\cap B_p$ are pairwise disjoint subsets in B_p for every $1\leq i\leq k+1$. Recall that $N_3(U)\cap B_p=\emptyset$ by Subclaim *2.7.2*. Then by combining with Subclaim *2.7.3* we obtain

$$\begin{aligned}|B_p|&\geq|N(y_i)\cap B_p|+|(N(U-\{y_i\})\cap B_p)^+|+|(N_2(U)-N(y_i))\cap B_p|\\&=|N(y_i)\cap B_p|+|N(U-\{y_i\})\cap B_p|+|(N_2(U)-N(y_i))\cap B_p|\\&=\sum_{y\in U}|N_G(y)\cap B_p|.\end{aligned}$$

Claim 2.7 is proved. ∎

Claim 2.8 *For every* $1 \le i \le k+1$, *then* $\sum_{y\in U} |N_G(y)\cap B_{y_i}| \le |B_{y_i}|-1$ *and* $\sum_{y\in U} |N_G(y)\cap B_{z_i}| \le |B_{z_i}|-1$.

Proof. By the same role of y_i and z_i, we only need to prove for the case $\sum_{y\in U} |N_G(y)\cap B_{y_i}| \le |B_{y_i}|-1$. Now we consider $B_{y_i} = P_T[y_i, x_i] - \{x_i\}$.
By Claim 2.5, we conclude that $N_G(U)\cap B_{y_i} = N_G(\{y_i, z_i\})\cap B_{y_i}$.
Subclaim 2.8.1. $x_i^- \notin N_G(z_i)\cap B_{y_i}$.

Assume that x_i^- is adjacent to z_i in G. Consider the tree $T' = T + z_i x_i{}^- - x_i{}^- x_i$. If $\deg_T(x_i) = 3$, then T' is a tree of G such that $V(T') = V(T)$, $|R(Stem(T'))| \le |R(Stem(T))|-1$. This contradicts the condition (C1). Otherwise, $V(T') = V(T)$, and $|R(Stem(T'))| \le |R(Stem(T))|$, but the number of branches of T' is smaller than the one of T. This contradicts the condition (C2).
Subclaim 2.8.2. If $x \in N_G(y_i)\cap B_{y_i}$ then $x^- \notin N_G(z_i)\cap B_{y_i}$.

Suppose that there exists $x \in N_G(y_i)\cap B_{y_i}$ such that $x^- \in N_G(z_i)\cap B_{y_i}$. Set $T' := T + \{xy_i, z_i x^-\} - \{xx^-, x_i^- x_i\}$. Hence T' is a tree of G such that $V(T') = V(T)$, $|R(Stem(T'))| \le |R(Stem(T))|$ and the number of branches of T' is smaller than the one of T. This contradicts either the condition (C1) or the condition $(C2)$. Subclaim 2.8.2 holds.

By Subclaims 2.8.1 and 2.8.2 and Claim 2.5 we conclude that $\{y_i\}, N_G(y_i)\cap B_{y_i}$ and $(N_G(z_i)\cap B_{y_i})^-$ are pairwise disjoint subsets in B_{y_i}. Then

$$\sum_{y\in U} |N_G(y)\cap B_{y_i}| = |N_G(y_i)\cap B_{y_i}| + |N_G(z_i)\cap B_{y_i}|$$
$$= |N_G(y_i)\cap B_{y_i}| + |(N_G(z_i)\cap B_{y_i})^-| \le |B_{y_i}|-1.$$

This completes the proof of Claim 2.8. ∎

By Claim 2.7, Claim 2.8 and Claim 2.3 we obtain that

$$\begin{aligned}\deg_G(U) &= \sum_{i=1}^{k+1} (\deg_G(y_i) + \deg_G(z_i)) \\ &\le \sum_{i=1}^{k+1} (|B_{y_i}|-1) + \sum_{i=1}^{k+1} (|B_{z_i}|-1) + \sum_{p\in L(T)-U} |B_p| + 2(k+1) \\ &= |G| - |R(Stem(T))|.\end{aligned}$$

On the other hand, we note that $|R(Stem(T))| \ge k+2$. Hence

$$\sum_{i=1}^{k+1} \deg_G(y_i) + \sum_{i=1}^{k+1} \deg_G(z_i) \le |G|-k-2 \Rightarrow \min\left\{\sum_{i=1}^{k+1} \deg_G(y_i), \sum_{i=1}^{k+1} \deg_G(z_i)\right\} \le \lfloor \frac{|G|-k-2}{2} \rfloor.$$

Combining with Claim 2.6, we obtain

$$\sigma^4_{k+1}(G) \le \min\left\{\sum_{i=1}^{k+1} \deg_G(y_i), \sum_{i=1}^{k+1} \deg_G(z_i)\right\} \le \lfloor \frac{|G|-k}{2} \rfloor - 1.$$

This contradicts the assumption (ii) of Theorem 1.5.

Therefore, the proof of Theorem 1.5 is completed. ∎

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