ON STABILITY FOR SEMILINEAR GENERALIZED RAYLEIGH-STOKES EQUATION INVOLVING DELAYS

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Abstract. We consider a functional semilinear Rayleigh-Stokes equation involving fractional derivative. Our aim is to analyze some circumstances, in those the global solvability and some results on asymptotic behavior of solutions take place. By establishing a new Halanay type inequality, we show the dissipativity and asymptotic stability of solutions to our problem. In addition, we prove the existence of a compact set of decay solutions by using local estimates and fixed point arguments.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following problem

$$\partial_t u - (1+\gamma\partial_t^\alpha)\Delta u = f(t, u_\rho) \ \text{ in } \Omega, t > 0, \tag{1.1}$$

$$u = 0 \ \text{ on } \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(x,s) = \xi(x,s),\ x \in \Omega, s \in [-\tau, 0], \tag{1.3}$$

where $\gamma > 0$, $\alpha \in (0,1)$, $\partial_t = \frac{\partial}{\partial t}$, ∂_t^α stands for the Riemann-Liouville derivative of order α defined by

$$\partial_t^\alpha v(t) = \frac{d}{dt}\int_0^t g_{1-\alpha}(t-s)v(s)ds,$$

where $g_\beta(t) = \dfrac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta > 0, t > 0$. In this model, u_ρ is defined by $u_\rho(x,t) = u(x, t-\rho(t))$ with ρ being a continuous function on $\mathbb{R}^+$ such that $-\tau \leq t - \rho(t) \leq t$ and $\lim\limits_{t\to\infty}(t-\rho(t)) = \infty$, $f : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$ is a nonlinear map and $\xi \in C([-\tau, 0]; L^2(\Omega))$ is given.

In the study of fluid dynamics, the behavior of non-Newtonian fluids possessing both elasticity and viscosity has received much attention due to its practical applications in industry and engineering (rheology, geophysics, chemical and petroleum industries, etc). In this direction, models of generalized second grade fluids have been investigated by many researchers. Equation (1.1) arose in a generalized Rayleigh-Stokes problem, where its constitutive relation was given in [9, 16], in order to describe the flow of generalized second grade fluids occupied in cylinders. This is also regarded as a special case of modeling equation for the generalized Oldroyd-B fluid [9]. It should be noted that, the term of fractional derivative gives a significant description for the viscoelasticity of fluids under examination.

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As a matter of fact, there have been lots of works devoted to finding numerical methods for solving Rayleigh-Stokes problem, see e.g. [3, 4, 5, 6, 15, 20]. Regarding analytic representation for solution of this problem in linear form, we refer to [9, 12, 16, 19, 21]. Recently, the final value problem involving (1.1) has been addressed in [13, 17], as an interesting supplement to qualitative investigation for this equation.

In the present work, we concern with the nonlinear model, where the nonlinearity f contains a delay term and it stands for an external force that depends on history state of the system. A typical example for the delay term is that, $\rho(t) = (1-q)t+\tau$, $u_\rho(x,t) = u(x, qt-\tau)$, for some $q \in (0,1]$, which is a proportional delay. For this model, the long-time behavior of solutions is an issue that has not been addressed in literature, and we aim at closing this gap. We first prove that, the problem is globally solvable in both cases when f has a linear or superlinear growth. Then we analyze some sufficient conditions ensuring the dissipativity and asymptotic stability for our system. Finally, we show a result on existence of decay solutions in the case f is of superlinear.

Our work is as follows. In the next section, we show typical properties of relaxation function and prove a Halanay type equality for using in stability analysis. We also prove the compactness of the Cauchy operator, which plays an important role in fixed point arguments. Section 3 is devoted to proving the solvability and stability results. In the last section, we present the existence of a compact set of decay solutions to our problem.

2. Preliminaries

We first consider the relaxation problem

$$\omega'(t) + \mu(1+\gamma\partial_t^\alpha)\omega(t) = 0,\ t > 0, \tag{2.1}$$

$$\omega(0) = 1, \tag{2.2}$$

where the unknown ω is a scalar function, μ and γ are positive parameters. We collect some properties of ω in the following proposition.

Proposition 2.1. *Let ω be the solution of (2.1)-(2.2). Then*

(1) $0 < \omega(t) \le 1$ for all $t \ge 0$.

(2) The function ω is completely monotone for $t \ge 0$, i.e. $(-1)^n\omega^{(n)}(t) \ge 0$ for $t \ge 0$ and $n \in \mathbb{N}$. Consequently, ω is a nonincreasing function.

(3) $\mu\omega(t) \le (t + g_{2-\alpha}(t))^{-1} \le \min\{t^{-1}, t^{\alpha-1}\}$, for all $t > 0$.

(4) $\int_0^t \omega(s)ds \le \mu^{-1}(1-\omega(t))$, for any $t > 0$.

Proof. The proof for (1)-(2) can be found in [3, Theorem 2.2]. For the last statement, taking integration of (2.1) yields

$$\omega(t) + \mu\int_0^t \omega(s)ds + \mu\int_0^t g_{1-\alpha}(t-s)\omega(s)ds = 1.$$

Then

$$\mu\omega(t)\left(\frac{1}{\mu} + t + \int_0^t g_{1-\alpha}(t-s)ds\right) \le 1, \tag{2.3}$$

$$\omega(t) + \mu\int_0^t \omega(s)ds \le 1, \tag{2.4}$$

thanks to the monotonicity and positivity of $\omega(\cdot)$. So (3) follows from (2.3) and (4) is deduced from (2.4). The proof is complete. $\square$

Denote by $\omega(\cdot,\mu)$ the solution of (2.1)-(2.2), respecting to parameter μ. In what follows, we use the notation $u*v$ to express the Laplace convolution of u and v, i.e.

$$(u*v)(t) = \int_0^t u(t-s)v(s)ds,\ u,v\in L^1_{loc}(\mathbb{R}^+).$$

We have the following result on the function ω.

Proposition 2.2. *For fixed $t\geq 0$ and $\gamma>0$, the function $\mu\mapsto\omega(t,\mu)$ is nonincreasing on $[0,\infty)$.*

Proof. Note that, the Laplace transform of ω is determined by

$$\hat{\omega}(\lambda,\mu) := \mathcal{L}[\omega](\lambda) = \frac{1}{\lambda+\gamma\mu\lambda^\alpha+\mu}.$$

It follows that

$$\frac{\partial\hat{\omega}}{\partial\mu} = -\frac{1+\gamma\lambda^\alpha}{(\lambda+\gamma\mu\lambda^\alpha+\mu)^2} = -[(1+\gamma\lambda^\alpha)\hat{\omega}]\hat{\omega}.$$

Taking into account the fact that

$$(1+\gamma\lambda^\alpha)\hat{\omega} = \mathcal{L}[(1+\gamma\partial_t^\alpha)\omega],$$

we see that

$$\frac{\partial\hat{\omega}}{\partial\mu} = -\mathcal{L}[(1+\gamma\partial_t^\alpha)\omega]\mathcal{L}[\omega].$$

Applying the inverse transform to the last equation, we obtain

$$\frac{\partial\omega}{\partial\mu} = -(\omega+\gamma\partial_t^\alpha\omega)*\omega,$$

thanks to the convolution rule of Laplace transform. Now using (2.1), one gets

$$\frac{\partial\omega}{\partial\mu} = \frac{1}{\mu}(\omega'*\omega)\leq 0,$$

according to the complete monotonicity of ω. The proof is complete. $\square$

We now concern with the inhomogeneous problem

$$v'(t)+\mu(1+\gamma\partial_t^\alpha)v(t) = g(t), t>0, \tag{2.5}$$
$$v(0) = v_0, \tag{2.6}$$

where $\mu>0$, $\gamma>0$ and g is a continuous function.

Proposition 2.3. *The solution of (2.5)-(2.6) is given by*

$$v(t) = \omega(t,\mu)v_0 + \omega(\cdot,\mu)*g(t),\ t\geq 0,$$

where ω is defined by (2.1)-(2.2).

Proof. Taking the Laplace transform of (2.5), we have

$$\lambda\hat{v} - v_0 + \mu(1+\gamma\lambda^\alpha)\hat{v} = \hat{g}.$$

Then

$$\hat{v} = \frac{v_0}{\lambda+\gamma\mu\lambda^\alpha+\mu} + \frac{\hat{g}}{\lambda+\gamma\mu\lambda^\alpha+\mu} = \hat{\omega}v_0 + \hat{\omega}\hat{g}.$$

Applying the inverse transform yields

$$v(t) = \omega(t)v_0 + \int_0^t \omega(t-s)g(s)ds. \tag{2.7}$$

Conversely, one can show that the function v given by (2.7) is the solution of (2.5)-(2.6). Indeed, let $L[v] = v' + \mu(1+\gamma\partial_t^\alpha)v$, then

$$L[v] = L[\omega]v_0 + L[\omega * g] = L[\omega * g].$$

It suffices to prove that $L[\omega * g] = g$. By computation, we see that

$$\begin{aligned}
(\omega * g)' + \mu(\omega * g) &= g + \omega' * g + \mu(\omega * g) \\
&= g + (\omega' + \mu\omega) * g, \\
\partial_t^\alpha(\omega * g) &= \frac{d}{dt}[h_{1-\alpha} * (\omega * g)] \\
&= \frac{d}{dt}(h_{1-\alpha} * \omega) * g \\
&= (\partial_t^\alpha \omega) * g.
\end{aligned}$$

Hence

$$\begin{aligned}
L[\omega * g] &= (\omega * g)' + \mu(\omega * g) + \mu\gamma\partial_t^\alpha(\omega * g) \\
&= g + (\omega' + \mu\omega + \mu\gamma\partial_t^\alpha\omega) * g \\
&= g + L[\omega] * g = g.
\end{aligned}$$

The proof is complete. □

Let $\{\varphi_n\}_{n=1}^\infty$ be the orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplacian $-\Delta$ subject to homogeneous Dirichlet boundary condition, that is

$$-\Delta\varphi_n = \lambda_n\varphi_n \text{ in } \Omega,\ \varphi_n = 0 \text{ on } \partial\Omega,$$

where we can assume that $\{\lambda_n\}_{n=1}^\infty$ is an increasing sequence, $\lambda_n > 0$ and $\lambda_n \to +\infty$ as $n \to \infty$. Then one can give a representation of solution to the linear problem

$$\partial_t u - (1+\gamma\partial_t^\alpha)\Delta u = F \quad \text{in } \Omega, t > 0, \tag{2.8}$$

$$u = 0 \quad \text{on } \partial\Omega, t \geq 0, \tag{2.9}$$

$$u(\cdot, 0) = \xi \quad \text{in } \Omega, \tag{2.10}$$

where $F \in L^1_{loc}(\mathbb{R}^+; L^2(\Omega))$ and $\xi \in L^2(\Omega)$. Indeed, let

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^\infty u_n(t)\varphi_n(x), \\
F(x,t) &= \sum_{n=1}^\infty F_n(t)\varphi_n(x),\ \xi(x) = \sum_{n=1}^\infty \xi_n\varphi_n(x).
\end{aligned}$$

Then

$$u_n'(t) + \lambda_n(1+\gamma\partial_t^\alpha)u_n(t) = F_n(t),\ u_n(0) = \xi_n.$$

Employing Proposition 2.3, we get

$$u_n(t) = \omega(t,\lambda_n)\xi_n + \int_0^t \omega(t-s,\lambda_n)F_n(s)ds.$$

This implies

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-s)F(\cdot,s)ds, \tag{2.11}$$

where $S(t) : L^2(\Omega) \to L^2(\Omega)$ is the resolvent operator defined by

$$S(t)\xi = \sum_{n=1}^{\infty} \omega(t,\lambda_n)\xi_n. \tag{2.12}$$

In the sequel, the notation $\|\cdot\|$ is understood as the standard norm in $L^2(\Omega)$, and $\|\cdot\|_{\mathcal{L}}$ stands for the operator norm of bounded linear operators on $L^2(\Omega)$. We collect some properties of the resolvent operator in the following lemma.

Lemma 2.4. *For any $v \in L^2(\Omega)$, $T > 0$, we have:*

(1) *$S(\cdot)v \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$.*
(2) *$\|S(t)v\| \le \omega(t,\lambda_1)\|v\|$, for all $t \ge 0$. In particular, $\|S(t)\| \le 1$ for all $t \ge 0$.*
(3) *$S(\cdot)v \in C^{(m)}((0,T]; L^2(\Omega))$ for all $m \in \mathbb{N}$, and $\|S^{(m)}(t)v\| \le Ct^{-m}\|v\|$, where C is a positive constant.*
(4) *$\|\Delta S^{(m)}(t)v\| \le Ct^{-m-1+\alpha}\|v\|$ for all $t > 0$ and $m \in \mathbb{N}$.*

Proof. The statements (1), (3) and (4) were stated in [3, Theorem 2.1]. We prove (2) as follows.

$$\begin{aligned}\|S(t)\xi\|^2 &= \sum_{n=1}^{\infty} \omega(t,\lambda_n)^2\xi_n^2 \\ &\le \omega(t,\lambda_1)^2 \sum_{n=1}^{\infty} \xi_n^2 = \omega(t,\lambda_1)^2\|\xi\|^2,\end{aligned}$$

thanks to Proposition 2.2. Since $\omega(t,\lambda_1) \le 1$, one gets $\|S(t)\|_{\mathcal{L}} \le 1$ for all $t \ge 0$. The proof is complete. □

Remark 2.1. *a) It should be mentioned that, the resolvent $S(t)$ can be defined as the solution operator of the following integral equation (see* [3]*)*

$$u(x,t) = \xi(x) + \int_0^t k(t-s)\Delta u(x,s)ds, \tag{2.13}$$

$$k(t) = 1 + \gamma g_{1-\alpha}(t). \tag{2.14}$$

More precisely, let

$$\begin{aligned}&\Gamma_{\delta,\theta} = \{re^{-i\theta} : r \ge \delta\} \cup \{\delta e^{i\psi} : |\psi| < \theta\} \cup \{re^{i\theta} : r \ge \delta\}, \\ &H(z) = \frac{g(z)}{z}(g(z)I - \Delta)^{-1},\ g(z) = \frac{1}{\hat{k}(z)} = \frac{z}{1+\gamma z^{\alpha}}.\end{aligned} \tag{2.15}$$

Then

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_{\delta,\pi-\theta}} e^{zt}H(z)dz,\ \delta > 0, \theta \in (0, \frac{\pi}{2}).$$

b) The first statement of Lemma 2.4 ensures that $S(t) : L^2(\Omega) \to L^2(\Omega)$ is a compact operator for any $t > 0$, due to the compactness of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Consider the *Cauchy operator* $\mathcal{Q} : C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$ given by

$$\mathcal{Q}(g)(t) = \int_0^t S(t-s)g(s)ds. \tag{2.16}$$

Denote by $\|\cdot\|_\infty$ the sup norm in $C([0,T]; L^2(\Omega))$, i.e. $\|g\|_\infty = \sup_{t\in[0,T]} \|g(t)\|$. The following lemma shows the compactness of $\mathcal{Q}$.

Lemma 2.5. *The Cauchy operator defined by* (2.16) *is compact.*

Proof. Let $D \subset C([0,T]; L^2(\Omega))$ be a bounded set. We first testify that $\Delta\mathcal{Q}(D)(t)$ is bounded in $L^2(\Omega)$ for each $t > 0$. Indeed, for any $g \in D$, we have

$$\Delta\mathcal{Q}(g)(t) = \int_0^t \Delta S(t-s)g(s)ds,\ t > 0.$$

By using Lemma 2.4(4) with $m = 0$, we get

$$\begin{aligned}\|\Delta\mathcal{Q}(g)(t)\| &\le \int_0^t \|\Delta S(t-s)g(s)\|ds\\ &\le C\int_0^t (t-s)^{-1+\alpha}\|g(s)\|ds \le \frac{CT^\alpha}{\alpha}\|g\|_\infty,\end{aligned}$$

which ensures the boundedness of $\Delta\mathcal{Q}(D)(t)$ in $L^2(\Omega)$. Since the embedding $D(\Delta) \hookrightarrow L^2(\Omega)$ is compact, we obtain the relative compactness of $\mathcal{Q}(D)(t)$ for each $t > 0$. Obviously, $\mathcal{Q}(D)(0) = \{0\}$ is a singleton, so $\mathcal{Q}(D)(t)$ is relatively compact for each $t \ge 0$.

Now we show that $\mathcal{Q}(D)$ is equicontinuous. Let $g \in D$, $t \in (0,T)$, $\epsilon \in (0,t)$ and $h \in (0, T-t]$, then one sees that

$$\begin{aligned}\|\mathcal{Q}(g)(t+h) - \mathcal{Q}(g)(t)\| &\le \int_0^t \|[S(t+h-s) - S(t-s)]g(s)\|ds\\ &\quad + \int_t^{t+h} \|S(t+h-s)g(s)\|ds\\ &= I_1(t) + I_2(t).\end{aligned}$$

It is easily seen that $I_2(t) \to 0$ as $h \to 0$ uniformly in $g \in D$. Regarding $I_1(t)$, we observe that

$$\begin{aligned}\|[S(t+h-s) - S(t-s)]g(s)\| &= \|\int_0^1 hS'(t-s+\theta h)g(s)d\theta\|\\ &\le h\int_0^1 \|S'(t-s+\theta h)\|_{\mathcal{L}}\|g(s)\|d\theta\\ &\le Ch\int_0^1 \frac{\|g(s)\|d\theta}{t-s+\theta h},\end{aligned}$$

thanks to the mean value formula and Lemma 2.4(3). So

$$\begin{aligned}\|[S(t+h-s) - S(t-s)]g(s)\| &\le C\|g\|_\infty \ln\left(1 + \frac{h}{t-s}\right)\\ &\le C\|g\|_\infty \frac{h^\beta}{\beta(t-s)^\beta},\ \beta \in (0,1),\end{aligned} \tag{2.17}$$

here we used the inequality $\ln(1+r) \le \dfrac{r^\beta}{\beta}$ for any $r > 0, \beta \in (0,1)$. Employing (2.17), we have

$$\begin{aligned} I_1(t) &\le \frac{C\|g\|_\infty h^\beta}{\beta} \int_0^t \frac{ds}{(t-s)^\beta} \\ &\le \frac{C\|g\|_\infty h^\beta}{\beta(1-\beta)} T^{1-\beta} \to 0 \text{ as } h \to 0 \text{ uniformly in } g \in D. \end{aligned}$$

Finally, for $h \in (0,T)$, we have

$$\|\mathcal{Q}(g)(h) - \mathcal{Q}(g)(0)\| \le \int_0^h \|S(h-s)g(s)\| ds \le h\|g\|_\infty \to 0 \text{ as } h \to 0,$$

uniformly in $g \in D$. Therefore, $\mathcal{Q}(D)$ is equicontinuous. The conclusion follows from the Arzelà-Ascoli theorem. $\square$

We are in a position to prove a Halanay type inequality for our analysis in the next section.

Lemma 2.6. *Let v be a continuous and nonnegative function satisfying*

$$v(t) \le \omega(t,\mu)v_0 + \int_0^t \omega(t-s,\mu)[a \sup_{\zeta\in[s-\rho(s),s]} v(\zeta) + b(s)]ds,\ t > 0, \tag{2.18}$$

$$v(s) = \psi(s), s \in [-\tau, 0], \tag{2.19}$$

where $0 < a < \mu$, $\psi \in C([-\tau,0];\mathbb{R}^+)$ and $b \in L^1_{loc}(\mathbb{R}^+)$ which is nondecreasing. Then

$$v(t) \le \frac{\mu}{\mu - a}\Big[v_0 + \int_0^t \omega(t-s,\mu)b(s)ds\Big] + \sup_{s\in[-\tau,0]} \psi(s),\ \forall t > 0. \tag{2.20}$$

In addition, if $\omega(\cdot,\mu) * b$ is bounded on $\mathbb{R}^+$ then

$$\limsup_{t\to\infty} v(t) \le \sup_{t\in\mathbb{R}^+} \int_0^t \omega(t-s,\mu)b(s)ds. \tag{2.21}$$

In particular, if $b = 0$ then $v(t) \to 0$ as $t \to \infty$.

Proof. We make use of the following result [18]: if $v \in C([-\tau,\infty);\mathbb{R}^+)$ is a nonnegative function satisfying

$$\begin{aligned} v(t) &\le d(t) + c \sup_{\zeta\in[-\tau,t]} v(\zeta),\ t > 0, \\ v(s) &= \psi(s),\ s \in [-\tau, 0], \end{aligned}$$

where $d(\cdot)$ is a nondecreasing function and $c \in (0,1)$, then

$$v(t) \le (1-c)^{-1} d(t) + \sup_{s\in[-\tau,0]} \psi(s),\ \forall t > 0. \tag{2.22}$$

It follows from (2.18) that

$$\begin{aligned}
v(t) &\leq v_0 + \omega(\cdot,\mu) * b(t) + a \sup_{\zeta\in[-h,t]} v(\zeta) \int_0^t \omega(t-s,\mu)ds \\
&\leq v_0 + \omega(\cdot,\mu) * b(t) + \frac{a}{\mu} \sup_{\zeta\in[-h,t]} v(\zeta)(1-\omega(t,\mu)) \\
&\leq v_0 + \omega(\cdot,\mu) * b(t) + \frac{a}{\mu} \sup_{\zeta\in[-h,t]} v(\zeta),
\end{aligned}$$

here we utilized Proposition 2.1(4). Since $b(\cdot)$ is nondecreasing, it is easily seen that the function $\omega(\cdot,\mu) * b$ is nondecreasing as well. Applying inequality (2.22) for $d(\cdot) = \omega(\cdot,\mu) * b$ and $c = a/\mu$, we get (2.20) as desired.

Now assume that $\omega(\cdot,\mu) * b$ is bounded on $\mathbb{R}^+$. Then by (2.20), $v(\cdot)$ is bounded by

$$M := \frac{\mu}{\mu-a}\Big[v_0 + \sup_{t\in\mathbb{R}^+} \int_0^t \omega(t-s,\mu)b(s)ds\Big] + \sup_{s\in[-\tau,0]} \psi(s),$$

and therefore the limit $L = \lim\limits_{t\to\infty} \sup_{\zeta\in[t,\infty)} v(\zeta)$ exists. Since $t-\rho(t)\to\infty$ as $t\to\infty$, for any $\varepsilon>0$, one can find $T_1>0$ such that

$$\sup_{\zeta\in[t-\rho(t),t]} v(\zeta) \leq \sup_{\zeta\in[t-\rho(t),\infty]} v(\zeta) \leq L+\varepsilon,\ \forall t\geq T_1.$$

Owing to the last estimate, we see that

$$\begin{aligned}
v(t) &\leq \omega(t,\mu)v_0 + \omega(\cdot,\mu) * b(t) \\
&\quad + \left(\int_0^{T_1} + \int_{T_1}^t\right) \omega(t-s,\mu) a \sup_{\zeta\in[s-\rho(s),s]} v(\zeta)ds \\
&\leq \omega(t,\mu)v_0 + \omega(\cdot,\mu) * b(t) \\
&\quad + aM \int_0^{T_1} \omega(t-s,\mu)ds + a(L+\varepsilon) \int_{T_1}^t \omega(t-s,\mu)ds \\
&\leq \varepsilon v_0 + \omega(\cdot,\mu) * b(t) \\
&\quad + aM \int_{t-T_1}^t \omega(t-s,\mu)ds + a(L+\varepsilon) \int_0^t \omega(t-s,\mu)ds \\
&\leq \varepsilon v_0 + \omega(\cdot,\mu) * b(t) + aM\varepsilon + a(L+\varepsilon)\mu^{-1},
\end{aligned} \tag{2.23}$$

provided t chosen such that

$$\omega(t,\mu) \leq \varepsilon,\ \int_{t-T_1}^t \omega(t-s,\mu)ds \leq \varepsilon,$$

which is possible since $\omega(t,\mu)\to 0$ as $t\to\infty$ and $\omega(\cdot,\mu)\in L^1(\mathbb{R}^+)$.

It follows from (2.23) that

$$L = \lim_{t\to\infty} \sup_{\zeta\in[t,\infty]} v(\zeta) \leq aL\mu^{-1} + \sup_{t\in\mathbb{R}^+} \omega(\cdot,\mu) * b(t) + (v_0 + aM + a\mu^{-1})\varepsilon,$$

which implies that

$$L \leq \frac{\mu}{\mu-a} \sup_{t\in\mathbb{R}^+} \omega(\cdot,\mu) * b(t) + \frac{\mu}{\mu-a}(v_0 + aM + a\mu^{-1})\varepsilon.$$

Hence

$$\limsup_{t\to\infty} v(t) \le L \le \frac{\mu}{\mu - a} \sup_{t\in\mathbb{R}^+} \omega(\cdot,\mu) * b(t),$$

thanks to the fact that ε is an arbitrarily positive number. $\square$

3. Solvability and stability

Based on representation (2.11), we give the following definition.

Definition 3.1. *Let $\xi \in C([-\tau,0]; L^2(\Omega))$ be given. A function $u \in C([-\tau,T]; L^2(\Omega))$ is said to be a mild solution to* (1.1)-(1.3) *on the interval $[-\tau,T]$ iff $u(\cdot,s) = \xi(\cdot,s)$ for $s \in [-\tau,0]$ and*

$$u(\cdot,t) = S(t)\xi(\cdot,0) + \int_0^t S(t-s) f(s, u_\rho(\cdot,s))ds,\ t \in [0,T].$$

For given $\xi \in C([-\tau,0]; L^2(\Omega))$, denote $C_\xi([0,T]; L^2(\Omega)) := \{u \in C([0,T]; L^2(\Omega)) : u(\cdot,0) = \xi(\cdot,0)\}$. For $u \in C_\xi([0,T]; L^2(\Omega))$, we define $u[\xi] \in C([-\tau,T]; L^2(\Omega))$ as follows

$$u[\xi](\cdot,t) = \begin{cases} u(\cdot,t) & \text{if } t \in [0,T], \\ \xi(\cdot,t) & \text{if } t \in [-\tau,0]. \end{cases}$$

Hence, we have

$$u[\xi]_\rho(\cdot,t) = \begin{cases} u(\cdot,t-\rho(t)) & \text{if } t-\rho(t) \in [0,T], \\ \xi(\cdot,t-\rho(t)) & \text{if } t-\rho(t) \in [-\tau,0]. \end{cases}$$

In what follows, we use the notation $\|\cdot\|_\infty$ for the sup norm in the spaces $C([-\tau,0]; L^2(\Omega))$, $C([-\tau,T]; L^2(\Omega))$ and $C([0,T]; L^2(\Omega))$.

Let $\Phi : C_\xi([0,T]; L^2(\Omega)) \to C_\xi([0,T]; L^2(\Omega))$ be the operator defined by

$$\Phi(u)(\cdot,t) = S(t)\xi(\cdot,0) + \int_0^t S(t-s) f(s, u[\xi]_\rho(\cdot,s))ds,$$

which will be referred to as *the solution operator*. This operator is continuous if f is a continuous map. Obviously, u is a fixed point of Φ iff $u[\xi]$ is a mild solution of (1.1)-(1.3).

In the next theorems, we show some global existence results for (1.1)-(1.3).

Theorem 3.1. *Let $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ be a continuous mapping such that*

(F1) *$\|f(t,v)\| \le p(t)G(\|v\|)$ for all $t \in [0,T]$ and $v \in L^2(\Omega)$, where $p \in L^1(0,T)$ is a nonnegative function and G is a continuous and nonnegative function obeying that*

$$\limsup_{r\to 0} \frac{G(r)}{r} \cdot \sup_{t\in[0,T]} \int_0^t \omega(t-s,\lambda_1)p(s)ds < 1.$$

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) *has at least one mild solution on $[-\tau,T]$, provided $\|\xi\|_\infty \le \delta$.*

Proof. Let

$$\ell = \limsup_{r\to 0} \frac{G(r)}{r},\ M = \sup_{t\in[0,T]} \omega(\cdot,\lambda_1) * p(t).$$

Then by assumption, one can take $\epsilon > 0$ such that $(\ell+\epsilon)M < 1$. In addition, there is $\eta > 0$ such that

$$\frac{G(r)}{r} \leq \ell + \epsilon, \ \forall r \in [0, 2\eta].$$

Let

$$\delta_0 = \eta \inf_{t\in[0,T]} \left\{ \left[\omega(t,\lambda_1) + (\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t)\right]^{-1} \left[1 - (\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t)\right] \right\},$$

then $\delta_0 > 0$. Indeed, observing that

$$\left[\omega(t,\lambda_1) + (\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t)\right]^{-1} \geq \omega(t,\lambda_1)^{-1} \geq 1,$$

we get

$$\begin{aligned}
\delta_0 &\geq \eta \inf_{t\in[0,T]} \left[1 - (\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t)\right] \\
&\geq \eta\left[1 - (\ell+\epsilon) \sup_{t\in[0,T]} \omega(\cdot,\lambda_1) * p(t)\right] \\
&= \eta\left[1 - (\ell+\epsilon)M\right] > 0.
\end{aligned}$$

Denote by B_η the closed ball in $C_\xi([0,T];L^2(\Omega))$ centered at origin with radius η. Considering $\Phi : \mathsf{B}_\eta \to C_\xi([0,T];L^2(\Omega))$, we have

$$\|\Phi(u)(\cdot,t)\| \leq \omega(t,\lambda_1)\|\xi(\cdot,0)\| + \int_0^t \omega(t-s,\lambda_1)p(s)G(\|u[\xi]_\rho(\cdot,s)\|)ds,$$

thanks to Lemma 2.4(2). Put $\delta = \min\{\delta_0, \eta\}$. If $\xi \in C([-\tau,0];L^2(\Omega))$ such that $\|\xi\|_\infty \leq \delta$, then

$$\|u[\xi]_\rho(\cdot,s)\| \leq \|u\|_\infty + \|\xi\|_\infty \leq \eta + \delta \leq 2\eta \text{ for all } s \in [0,T].$$

So

$$\begin{aligned}
\|\Phi(u)(\cdot,t)\| &\leq \omega(t,\lambda_1)\|\xi\|_\infty + (\ell+\epsilon)\int_0^t \omega(t-s,\lambda_1)p(s)\|u[\xi]_\rho(\cdot,s)\|ds \\
&\leq \omega(t,\lambda_1)\delta + (\eta+\delta)(\ell+\epsilon)\int_0^t \omega(t-s,\lambda_1)p(s)ds \\
&\leq \left[\omega(t,\lambda_1) + (\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t)\right]\delta_0 + \eta(\ell+\epsilon)\omega(\cdot,\lambda_1) * p(t) \\
&\leq \eta, \ \forall t \in [0,T].
\end{aligned}$$

We have shown that $\Phi(\mathsf{B}_\eta) \subset \mathsf{B}_\eta$, provided $\|\xi\|_\infty \leq \delta$. Consider $\Phi : \mathsf{B}_\eta \to \mathsf{B}_\eta$. In order to apply the Schauder fixed point theorem, it remains to check that Φ is a compact operator. It should be noted that, Φ admits the representation

$$\Phi(u) = S(\cdot)\xi + \mathcal{Q} \circ N_f(u),$$

where $N_f(u)(\cdot,t) = f(t, u[\xi]_\rho(\cdot,t))$. According to the compactness of $\mathcal{Q}$ stated in Lemma 2.5, we conclude that Φ is compact. The proof is complete. □

Theorem 3.1 deals with the case that f is possibly superlinear. In the next theorem, we can relax the smallness condition on initial data, provided that f has a sublinear growth.

Theorem 3.2. *Let $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ be a continuous mapping such that*

(F2) *$\|f(t,v)\| \leq p(t)(1+\|v\|)$ for all $t \in [0,T]$ and $v \in L^2(\Omega)$, where $p \in L^1(0,T)$ is a nonnegative function.*

Then the problem (1.1)-(1.3) *has at least one mild solution on* $[-\tau, T]$.

Proof. Let $\psi \in C([0,T];\mathbb{R})$ be the unique solution of the integral equation

$$\psi(t) = \|\xi\|_\infty + (1+\|\xi\|_\infty)\int_0^t p(s)ds + \int_0^t p(s)\psi(s)ds,$$

and $D = \{u \in C_\xi([0,T];L^2(\Omega)) : \sup_{\zeta\in[0,t]}\|u(\zeta)\| \le \psi(t),\ \forall t \in [0,T]\}$. Then D is a closed and convex subset of $C_\xi([0,T];L^2(\Omega))$. Since Φ is continuous and compact, it suffices to show that $\Phi(D) \subset D$. Let $u \in D$, then

$$\begin{aligned}\|\Phi(u)(\cdot,t)\| &\le \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)(1+\|u[\xi]_\rho(\cdot,s)\|)ds\\ &\le \|\xi\|_\infty + \int_0^t p(s)(1+\|\xi\|_\infty + \sup_{\zeta\in[0,s]}\|u(\zeta)\|)ds.\end{aligned}$$

Since the last integral is nondecreasing in t, we get

$$\begin{aligned}\sup_{\zeta\in[0,t]}\|\Phi(u)(\cdot,\zeta)\| &\le \|\xi\|_\infty + \int_0^t p(s)(1+\|\xi\|_\infty + \sup_{\zeta\in[0,s]}\|u(\zeta)\|)ds\\ &\le \|\xi\|_\infty + \int_0^t p(s)(1+\|\xi\|_\infty + \psi(s))ds = \psi(t),\end{aligned}$$

which ensures that $\Phi(u) \in D$. The proof is complete. □

In the next theorem, we state an existence and uniqueness result.

Theorem 3.3. *Let* $f : [0,T]\times L^2(\Omega) \to L^2(\Omega)$ *be a continuous mapping such that*

(F3) $f(\cdot,0) = 0$ *and* $\|f(t,v_1)-f(t,v_2)\| \le p(t)\kappa(r)\|v_1-v_2\|$ *for all* $t \in [0,T]$ *and* $v_1, v_2 \in L^2(\Omega)$ *such that* $\|v_1\|, \|v_2\| \le r$, *where* $p \in L^1(0,T)$ *is a nonnegative function and* κ *is a function obeying that*

$$\limsup_{r\to 0}\kappa(r)\cdot\sup_{t\in[0,T]}\int_0^t \omega(t-s,\lambda_1)p(s)ds < 1.$$

Then there exists $\delta > 0$ *such that the problem* (1.1)-(1.3) *has a unique mild solution on* $[-\tau,T]$, *provided* $\|\xi\|_\infty \le \delta$.

Proof. The existence result can be obtained by applying Theorem 3.1 with $G(r) = \kappa(r)r$. It remains to prove the uniqueness. Assume that $u_1[\xi], u_2[\xi]$ are solutions of (1.1)-(1.3). Let $R = \max\{\|u_1[\xi]\|_\infty, \|u_2[\xi]\|_\infty\}$, then

$$\begin{aligned}\|u_1(\cdot,t)-u_2(\cdot,t)\| &\le \int_0^t p(s)\kappa(R)\|u_1[\xi]_\rho(\cdot,s)-u_2[\xi]_\rho(\cdot,s)\|ds\\ &\le \int_0^t p(s)\kappa(R)\sup_{\zeta\in[0,s]}\|u_1(\cdot,\zeta)-u_2(\cdot,\zeta)\|ds,\end{aligned}$$

due to the fact that $u_1(\cdot,s) = u_2(\cdot,s)$ for $s \in [-\tau,0]$. Observing that, the last integral is nondecreasing in t, we have

$$\sup_{\zeta\in[0,t]}\|u_1(\cdot,t)-u_2(\cdot,t)\| \le \int_0^t p(s)\kappa(R)\sup_{\zeta\in[0,s]}\|u_1(\cdot,\zeta)-u_2(\cdot,\zeta)\|ds,\ t\in[0,T],$$

which implies that $u_1 = u_2$, by means of the Gronwall inequality. The proof is complete. □

We are now in a position to show the dissipativity of our system.

Theorem 3.4. *Let the hypotheses of Theorem 3.2 hold for all $T > 0$ and $\|p\|_\infty = \text{esssup}_{t\geq 0} p(t) < \lambda_1$. Then there exists a bounded absorbing set for solution of* (1.1)-(1.3) *with arbitrary initial data.*

Proof. Let u be a solution of (1.1)-(1.3). Then

$$\begin{aligned}\|u(\cdot,t)\| &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)(1+\|u_\rho(\cdot,s)\|)ds\\ &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)\|p\|_\infty(1+\sup_{\zeta\in[s-\rho(s),s]}\|u(\cdot,\zeta)\|)ds.\end{aligned}$$

Applying the Halanay type inequality formulated in Lemma 2.6, we get

$$\begin{aligned}\limsup_{t\to\infty}\|u(\cdot,t)\| &\leq \sup_{t\in\mathbb{R}^+}\int_0^t \omega(t-s,\lambda_1)\|p\|_\infty ds\\ &= \|p\|_\infty\lambda_1^{-1}\sup_{t\in\mathbb{R}^+}(1-\omega(t,\lambda_1)) = \|p\|_\infty\lambda_1^{-1}.\end{aligned}$$

This implies that the ball $B(0,R)\subset L^2(\Omega)$ with $R = \|p\|_\infty\lambda_1^{-1}+1$ turns out to be an absorbing set for solution of (1.1)-(1.3) with arbitrary initial data. □

The next theorem shows the asymptotic stability of zero solution to (1.1).

Theorem 3.5. *Let $f : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$ be a continuous mapping such that*

(**F**4) *$f(\cdot,0) = 0$ and $\|f(t,v_1) - f(t,v_2)\| \leq p(t)\kappa(r)\|v_1 - v_2\|$ for all $t\in\mathbb{R}^+$ and $v_1, v_2 \in L^2(\Omega)$ such that $\|v_1\|, \|v_2\| \leq r$, where $p \in L^\infty(\mathbb{R}^+)$ is a nonnegative function and κ is a continuous function satisfying that*

$$\|p\|_\infty \cdot \limsup_{r\to 0}\kappa(r) < \lambda_1.$$

Then the zero solution of (1.1) *is asymptotically stable.*

Proof. Let $\ell = \limsup_{r\to 0}\kappa(r)$. Choosing $\theta > 0$ such that $\|p\|_\infty(\ell+\theta) < \lambda_1$, we can find $\eta > 0$ such that $\kappa(r) \leq \ell+\theta$ for all $r \in [0, 2\eta]$. Reasoning as in the proof of Theorem 3.1 and 3.3, there exists $\delta > 0$ such that the problem (1.1)-(1.3) has a unique mild solution $u \in \mathsf{B}_\eta$ as long as $\|\xi\|_\infty \leq \delta$, which is defined on $[-\tau, T]$ for all $T > 0$. Moreover, one has the following estimate

$$\begin{aligned}\|u(\cdot,t)\| &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)(\ell+\theta)\|u[\xi]_\rho(\cdot,s)\|ds\\ &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)\|p\|_\infty(\ell+\theta)\sup_{\zeta\in[s-\rho(s),s]}\|u(\cdot,\zeta)\|ds.\end{aligned}$$

Employing Lemma 2.6 with $b(\cdot) = 0, a = \|p\|_\infty(\ell+\theta)$, we obtain

$$\begin{aligned}\|u(\cdot,t)\| &\leq \left(\frac{\lambda_1}{\lambda_1 - \|p\|_\infty(\ell+\theta)} + 1\right)\|\xi\|_\infty,\ \forall t\geq 0,\\ \lim_{t\to\infty}\|u(\cdot,t)\| &= 0,\end{aligned}$$

which imply the asymptotic stability of the zero solution of (1.1). The proof is complete. □

4. Existence of decay solutions

Our goal of this section is to prove the existence of decay solutions to the problem (1.1)-(1.3) under the assumption that, the nonlinearity is non-Lipschitzian and possibly superlinear. Specifically, assume that

(**F**5) $f : \mathbb{R}^+ \times L^2(\Omega)) \to L^2(\Omega)$ is a continuous mapping such that

$$\|f(t,v)\| \le p(t)G(\|v\|),\ \forall t \in \mathbb{R}^+, v \in L^2(\Omega),$$

where $p \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function and $G \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{G(r)}{r} \cdot \sup_{t\ge 0} \int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau < 1, \tag{4.1}$$

and

$$\lim_{T\to\infty} \sup_{t\ge T} \int_0^{\frac{t}{2}} \omega(t-\tau,\lambda_1)p(\tau)d\tau = 0. \tag{4.2}$$

In order to study the existence of decay solutions to (1.1)-(1.3), we make use of the fixed point theory for condensing maps.

Definition 4.1. [10] *Let E be a Banach space and $\mathcal{P}_b(E)$ the collection of all nonempty and bounded subsets of E. A function $\mu : \mathcal{P}_b(E) \to \mathbb{R}^+$ is said to be a measure of noncompactness (MNC) if $\mu(\overline{\text{co}}\, D) = \mu(D)$ for all $D \in \mathcal{P}_b(E)$, here the notation $\overline{\text{co}}$ denote the closure of convex hull of subsets in E. An MNC is called*

- *nonsingular if $\mu(D \cup \{x\}) = \mu(D)$ for all $D \in \mathcal{P}_b(E)$, $x \in E$.*
- *monotone if $\mu(D_1) \le \mu(D_2)$ provided that $D_1 \subset D_2$.*

The MNC defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ admits a finite } \varepsilon-\text{net}\}$$

is called the Hausdorff measure of noncompactness.

Definition 4.2. [10] *Let E be a Banach space and $D \in \mathcal{P}_b(E)$. A continuous map $\mathcal{F} : D \to E$ is said to be condensing with respect to MNC μ (μ-condensing) iff the relation $\mu(B) \le \mu(\mathcal{F}(B)), B \subset D$, implies that B is relatively compact.*

The following theorem states a fixed point principle for condensing maps.

Theorem 4.1. [10] *Let μ be a monotone and nonsingular MNC on E. Assume that $D \subset E$ is a closed convex set and $\mathcal{F} : D \to D$ is μ-condensing. Then $\mathcal{F}$ admits a fixed point.*

Let $BC_0(\mathbb{R}^+; L^2(\Omega))$ be the space of continuous functions on $\mathbb{R}^+$, taking values in $L^2(\Omega)$ and decaying as $t \to \infty$. Given $\xi \in C([-\tau, 0]; L^2(\Omega))$, put $\mathcal{BC}_0^\xi = \{u \in BC_0(\mathbb{R}^+; L^2(\Omega)) : u(\cdot, 0) = \xi(\cdot, 0)\}$. Then $\mathcal{BC}_0^\xi$ with the supremum norm $\|\cdot\|_\infty$ is a closed subspace of $BC_0(\mathbb{R}^+; L^2(\Omega))$.

Let D be a bounded set in $\mathcal{BC}_0^\xi$ and $\pi_T : \mathcal{BC}_0^\xi \to C([0,T]; L^2(\Omega))$ the restriction operator on $\mathcal{BC}_0^\xi$, i.e. $\pi_T(u)$ is the restriction of $u \in \mathcal{BC}_0^\xi$ to the interval $[0,T]$. Define

$$d_\infty(D) = \lim_{T\to\infty} \sup_{u\in D} \sup_{t\ge T} \|u(\cdot,t)\|,$$

$$\chi_\infty(D) = \sup_{T>0} \chi_T(\pi_T(D)),$$

where $\chi_T(\cdot)$ is the Hausdorff MNC in $C([0,T];L^2(\Omega))$. Then the following MNC defined in [1],

$$\chi^*(D) = d_\infty(D) + \chi_\infty(D), \tag{4.3}$$

possesses all properties stated in Definition 4.1. In addition, if $\chi^*(D)=0$ then D is relatively compact in $BC_0(\mathbb{R}^+;L^2(\Omega))$.

Lemma 4.2. *Let (**F**5) hold. Then there exist positive numbers δ and η such that for $\|\xi\|_\infty \le \delta$, the solution operator Φ obeys $\Phi(\mathsf{B}_\eta)\subset \mathsf{B}_\eta$, where B_η is the closed ball in $\mathcal{BC}_0^\xi$ centered at origin with radius η.*

Proof. Denote

$$\ell = \limsup_{r\to 0}\frac{G(r)}{r},\ M=\sup_{t\ge 0}\int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau.$$

Then by (4.1), one can take $\zeta>0$ such that

$$(\ell+\zeta)M<1. \tag{4.4}$$

Moreover, there exists $\eta>0$ such that $\frac{G(r)}{r}\le \ell+\zeta$ for all $r\in(0,2\eta]$. Recall that the solution operator Φ is defined by

$$\Phi(u)(\cdot,t) = S(t)\xi(\cdot,0)+\int_0^t S(t-s)f(s,u[\xi]_\rho(\cdot,s))ds,\ u\in\mathcal{BC}_0^\xi.$$

Considering the operator Φ on B_η with $\|\xi\|_\infty\le\eta$, we have

$$\|\Phi(u)(\cdot,t)\| \le \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-\tau,\lambda_1)p(\tau)G(\|u[\xi]_\rho(\cdot,s)\|)ds. \tag{4.5}$$

We first check that $\Phi(u)\in\mathcal{BC}_0^\xi$, provided $u\in\mathcal{BC}_0^\xi$. It suffices to prove that $\Phi(u)(\cdot,t)\to 0$ as $t\to\infty$ in $L^2(\Omega)$. According to (4.5), one has to testify that

$$I(t) := \int_0^t \omega(t-s,\lambda_1)p(s)G(\|u[\xi]_\rho(\cdot,s)\|)ds \to 0 \text{ as } t\to\infty.$$

Since $t-\rho(t)\to\infty$ as $t\to\infty$, we get $\|u[\xi](\cdot,t-\rho(t))\|\to 0$ as $t\to\infty$. So for any $\varepsilon>0$, there exists $T>0$ such that $G(\|u[\xi](\cdot,s-\rho(s))\|)\le\varepsilon$ for all $s\ge T$, thanks to the fact that G is continuous and $G(0)=0$. Hence for $t>T$, we get

$$\begin{aligned}
I(t) &= \left(\int_0^T + \int_T^t\right)\omega(t-s,\lambda_1)p(s)G(\|u[\xi](\cdot,s-\rho(s))\|)d\tau\\
&\le G(2\eta)\int_0^T \omega(t-s,\lambda_1)p(s)ds + \varepsilon\int_T^t \omega(t-s,\lambda_1)p(s)ds\\
&\le G(2\eta)\omega(t-T,\lambda_1)\int_0^T p(s)ds + \varepsilon M\\
&\le [G(2\eta)+M]\varepsilon,
\end{aligned}$$

for all t chosen so that

$$\omega(t-T,\lambda_1)\int_0^T p(s)ds < \varepsilon,$$

which is possible since $\omega(t,\lambda_1)\to 0$ as $t\to\infty$. We have proved that $\Phi(u)\in\mathcal{BC}_0^\xi$. Let

$$\delta_0 = \eta\inf_{t\ge 0}\left[\big(\omega(t,\lambda_1)+(\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\big)^{-1}\big(1-(\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\big)\right], \tag{4.6}$$

then $\delta_0 > 0$. Indeed, one has

$$\big(\omega(t,\lambda_1) + (\ell+\zeta)\omega(\cdot,\lambda_1) * p(t)\big)^{-1} \geq \omega(t,\lambda_1)^{-1} \geq 1,\ \forall t \geq 0,$$

then

$$\begin{aligned}\delta_0 &\geq \eta \inf_{t\geq 0}\left(1 - (\ell+\zeta)\int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau\right)\\ &\geq \eta\left(1-(\ell+\zeta)\sup_{t\geq 0}\int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau\right) > 0,\end{aligned}$$

thanks to (4.4). Choosing $\delta = \min\{\eta,\delta_0\}$, we show that $\Phi(u) \in \mathsf{B}_\eta$ provided $\|\xi\|_\infty \leq \delta$. For $\|\xi\| \leq \delta$, $u \in \mathsf{B}_\eta$, we get $\|u[\xi](\cdot,s)\| \leq 2\eta$ for any $s \geq -\tau$. In addition,

$$\begin{aligned}\|\Phi(u)(\cdot,t)\| &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)G(\|u[\xi](\cdot,s-\rho(s))\|)ds\\ &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)(\ell+\zeta)\|u[\xi](\cdot,s-\rho(s))\|ds\\ &\leq \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)(\ell+\zeta)(\|u\|_\infty + \|\xi\|_\infty)ds\\ &\leq \omega(t,\lambda_1)\|\xi\|_\infty + (\eta+\|\xi\|_\infty)(\ell+\zeta)\int_0^t \omega(t-s,\lambda_1)p(s)ds\\ &\leq \big[\omega(t,\lambda_1) + (\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\big]\|\xi\|_\infty + \eta(\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\\ &\leq \big[\omega(t,\lambda_1) + (\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\big]\delta_0 + \eta(\ell+\zeta)\omega(\cdot,\lambda_1)*p(t)\\ &\leq \eta,\ \forall t\geq 0,\end{aligned}$$

due to the formulation of δ_0 in (4.6). Therefore $\Phi(\mathsf{B}_\eta) \subset \mathsf{B}_\eta$. The proof is complete. □

The following theorem represents the main result of this section.

Theorem 4.3. *Let (**F**5) hold. Then there exists $\delta > 0$ such that, the problem* (1.1)-(1.3) *has a compact set of decay solutions, provided $\|\xi\|_\infty \leq \delta$.*

Proof. Taking δ and B_η from Lemma 4.2, we consider the solution map $\Phi : \mathsf{B}_\eta \to \mathsf{B}_\eta$. By standard reasoning, we get that Φ is continuous. We will show that Φ is χ^*-condensing. Let $D \subset \mathsf{B}_\rho$. Then arguing as in the proof of Theorem 3.1, one has $\pi_T \circ \Phi$ is a compact mapping, i.e., $\pi_T(\Phi(D))$ is relatively compact in $C([0,T];L^2(\Omega))$. This implies $\chi_T(\pi_T(\Phi(D))) = 0$ and then $\chi_\infty(\Phi(D)) = 0$. We are now in a position to estimate $d_\infty(\Phi(D))$.

Let $z \in \Phi(D)$ and $u \in D$ be such that $z = \Phi(u)$. Then

$$\begin{aligned}
\|z(\cdot,t)\| &\le \omega(t,\lambda_1)\|\xi\|_\infty + \int_0^t \omega(t-s,\lambda_1)p(s)G(\|u[\xi](\cdot,s-\rho(s))\|)ds \\
&\le \omega(t,\lambda_1)\|\xi\|_\infty + (\ell+\zeta)\int_0^t \omega(t-s,\lambda_1)p(s)\|u[\xi](\cdot,s-\rho(s))\|ds \\
&\le \omega(t,\lambda_1)\|\xi\|_\infty + (\ell+\zeta)\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\omega(t-s,\lambda_1)p(s)\|u[\xi](\cdot,s-\rho(s))\|ds \\
&\le \omega(t,\lambda_1)\|\xi\|_\infty + 2(\ell+\zeta)\eta\int_0^{\frac{t}{2}} \omega(t-s,\lambda_1)p(s)ds \\
&\quad + \sup_{s\ge\frac{t}{2}}\|u[\xi](\cdot,s-\rho(s))\|(\ell+\zeta)\int_{\frac{t}{2}}^t \omega(t-s,\lambda_1)p(s)ds.
\end{aligned}$$

Noting that, for given $T > 0$, one can find $T_1 > T$ such that $t - \rho(t) \ge T$ for all $t \ge T_1$. So for $t \ge 2T_1$, we have

$$\begin{aligned}
\|z(\cdot,t)\| &\le \omega(t,\lambda_1)\|\xi\|_\infty + 2(\ell+\zeta)\eta\int_0^{\frac{t}{2}} \omega(t-s,\lambda_1)p(s)ds \\
&\quad + \sup_{s\ge T}\|u(\cdot,s)\|(\ell+\zeta)\int_0^t \omega(t-s,\lambda_1)p(s)ds \\
&\le \omega(t,\lambda_1)\|\xi\|_\infty + 2(\ell+\zeta)\eta\int_0^{\frac{t}{2}} \omega(t-s,\lambda_1)p(s)ds \\
&\quad + \sup_{u\in D}\sup_{s\ge T}\|u(\cdot,s)\|(\ell+\zeta)\int_0^t \omega(t-s,\lambda_1)p(s)ds.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\sup_{t\ge 2T_1}\|z(\cdot,t)\| &\le \omega(2T_1,\lambda_1)\|\xi\|_\infty + 2(\ell+\zeta)\eta\sup_{t\ge 2T_1}\int_0^{\frac{t}{2}} \omega(t-s,\lambda_1)p(s)ds \\
&\quad + \sup_{u\in D}\sup_{s\ge T}\|u(\cdot,s)\|(\ell+\zeta)M,
\end{aligned}$$

where

$$M = \sup_{t\ge 0}\int_0^t \omega(t-s,\lambda_1)p(s)ds.$$

Since $z \in \Phi(D)$ is taken arbitrarily, we get

$$\begin{aligned}
\sup_{z\in\Phi(D)}\sup_{t\ge 2T_1}\|z(\cdot,t)\| &\le \omega(2T_1,\lambda_1)\|\xi\|_\infty + 2(\ell+\zeta)\eta\sup_{t\ge 2T_1}\int_0^{\frac{t}{2}} \omega(t-s,\lambda_1)p(s)ds \\
&\quad + \sup_{u\in D}\sup_{s\ge T}\|u(\cdot,s)\|(\ell+\zeta)M,
\end{aligned}$$

which ensures that

$$d_\infty(\Phi(D)) \le (\ell+\zeta)Md_\infty(D),$$

thanks to (4.2) and the fact that $T_1 \to \infty$ as $T \to \infty$. Therefore,

$$\begin{aligned}
\chi^*(\Phi(D)) = \chi_\infty(\Phi(D)) + d_\infty(\Phi(D)) &= d_\infty(\Phi(D)) \le (\ell+\zeta)Md_\infty(D) \\
&\le (\ell+\zeta)M[d_\infty(D) + \chi_\infty(D)] = (\ell+\zeta)M\chi^*(D).
\end{aligned}$$

Now if $\chi^*(D) \le \chi^*(\Phi(D))$ then $\chi^*(D) \le (\ell+\zeta)M\chi^*(D)$ which implies $\chi^*(D) = 0$, thanks to the fact that $(\ell+\zeta)M < 1$. Thus Φ is χ^*-condensing and it admits a fixed point, according to Theorem 4.1. Denote by $\mathcal{D}$ the fixed point set of Φ in B_η. Then $\mathcal{D}$ is closed and $\mathcal{D} \subset \Phi(\mathcal{D})$. Hence,

$$\chi^*(\mathcal{D}) \le \chi^*(\Phi(\mathcal{D})) \le (\ell+\zeta)M\chi^*(\mathcal{D}),$$

which ensures $\chi^*(\mathcal{D}) = 0$ and $\mathcal{D}$ is a compact set. The proof is complete. □

Remark 4.1. *Let us give a notice on the condition* (4.1) *and* (4.2). *Let* $p \in L^\infty(\mathbb{R}^+)$ *and* $\|p\|_\infty = \operatorname{ess\,sup}_{t\ge 0}|p(t)|$. *Then* (4.2) *is satisfied. Indeed, we see that*

$$\begin{aligned}\sup_{t\ge T}\int_0^{\frac{t}{2}} \omega(t-\tau,\lambda_1)p(\tau)d\tau &\le \|p\|_\infty \sup_{t\ge T}\int_0^{\frac{t}{2}} \omega(t-\tau,\lambda_1)d\tau\\ &\le \|p\|_\infty \sup_{t\ge T}\int_{\frac{t}{2}}^{t} \omega(\tau,\lambda_1)d\tau\\ &\le \|p\|_\infty \int_{\frac{T}{2}}^{\infty} \omega(\tau,\lambda_1)d\tau \to 0 \ \text{as}\ T\to\infty,\end{aligned}$$

thanks to the fact that $\omega(\cdot,\lambda_1) \in L^1(\mathbb{R}^+)$.

On the other hand, if f *is superlinear, e.g.* $G(r) = r^q$ *for some* $q > 1$, *then* (4.1) *is testified obviously. If* f *as a sublinear growth, e.g.* $G(r) = r$, *then* (4.1) *becomes*

$$\sup_{t\ge 0}\int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau < 1. \tag{4.7}$$

Noting that

$$\int_0^t \omega(t-\tau,\lambda_1)p(\tau)d\tau \le \|p\|_\infty \int_0^t \omega(\tau,\lambda_1)d\tau \le \|p\|_\infty \lambda_1^{-1},$$

we get that (4.7) *is fulfilled provided* $\|p\|_\infty < \lambda_1$.

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