State bounding for positive singular discrete-time systems with unbounded delay and bounded disturbances

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Abstract

This paper considers the state-bounding problem for positive singular discrete-time systems with unbounded delay and bounded disturbances. Based on conditions given in terms of the Linear programming / spectral radius, and by using the suitable transformation, we get the smallest componentwise estimate for the singular discrete-time system with unbounded delay and bounded disturbances. Some illustrative examples are given.

Keywords: State bounding, Positive systems, Singular systems, Unbounded time delays, Discrete-time systems.

1. Introduction

Singular system class is more suitable to describe the state of some practical systems such as mechanical systems, biological systems, economic systems, and chemical systems (see, [1–4] and the references therein). In general, a singular system consists of differential parts and algebraic constraints, thus a generalized representation of the state-space system. It is well known that the study of singular systems is much more difficult and complicated than standard systems because we have to consider regularity and causality (discrete-time systems) or non-impulsiveness (continuous-time) at the same time.

Positive systems are dynamical systems whose state and output trajectories are always non-negative whenever the inputs and initial conditions are non-negative. Positive singular systems are both positive systems and singular systems. Therefore, positive singular systems better describe physical systems than regular dynamical systems. As well known, time delays, such as discrete delays [5], distributed delays [6], neutral delays [7], leakage delay, probabilistic time-varying delays [8], and mixed delays [9, 10], often occur in positive singular systems and it is a

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source of instability and poor performance. Therefore, investigation of positive singular systems with time delays is not only of theoretical importance but also of practical significance. In recent years, many significant research developments have been devoted to the problems of stability analysis and $l_\infty-$gain analysis for positive continuous-time singular systems [11–13] subject to time-varying delays or positive discrete-time singular systems with time delays [14, 15]. The problem of exponential stability for linear singular positive with a constant time delay was considered in [13]. In this work, the authors first presented necessary and sufficient conditions for the positivity of the considered system by using the singular value decomposition method, and then they derived a sufficient condition for exponential stability of the system. However, this work deal with constant time-delays and required the existence of a monomial matrix to obtain the positivity and stability conditions for linear singular time-delay systems. By analyzing the monotonic property of the system trajectory, the authors in [11] extend the results in [13] to linear positive singular systems with a bounded time-varying delay. They proved that the stability of the considered systems is not sensitive to the magnitude of delays and is fully determined by the system matrices. The authors of the work [15] addressed the stability and $l_\infty-$gain analysis for discrete-time positive singular systems with an unbounded time-varying delay by using mathematical induction method. Based on upper bounding of the state vector by a decreasing function, the authors in [16] derived a criterion which ensures asymptotic stability of positive coupled differential-difference equations with an unbounded time-varying delay. The problems of stability and $L_\infty-$gain analysis for linear positive differential-algebraic equations with an unbounded time-varying delay were studied in [17]. It should be mentioned here that the stability and $l_\infty-$gain analysis conditions in [15–17] is independent of the magnitude of delays and fully determined by the systems matrices.

On the other hand, external disturbances are usually unavoidable in practical engineering systems due to many reasons such as linear approximation, modelling inaccuracies, external noises, measurement errors, and so on. The asymptotic stability for the systems cannot be achieved, and therefore the problem of state bounding for perturbed dynamical systems has attracted considerable attention during the past decades [18, 20–25]. In addition, with the rapid development of computer-based computational techniques, singular discrete-time systems are more suitable for computer-based simulation, experiment, and computation. To the best of our knowledge, there are two common approaches to study the state bounding problem for discrete-time systems with time-varying delays. The first approach is based on the like-Lyapunov functional method combining with linear matrix inequalities techniques and the second one is based on the properties of Metzler–Schur matrices combining with the solution comparison method and linear programming technique. The first method is widely used for classes of linear or nonlinear discrete-time systems whose matrices are constant, while, the second method is very useful for classes of positive linear systems and classes of nonlinear/time-varying systems which are bounded by positive linear systems. There are some interesting works on the problem of state bounding for discrete-time singular systems based on the first approach have been done [26, 27]. However, the research on state bounding for discrete-time singular systems based on the second approach is very limited. Very recently, Sau and Thuan [28] considered the problem of state bounding for positive singular discrete-time systems with a bounded time-varying delay and bounded disturbances. Note that the results derived in [28] were for singular discrete-time systems with bounded time-varying delays. To the best of our knowledge, the problem of state bounding for positive singular discrete-time systems with an unbounded delay and bounded disturbances has not yet been investigated in the literature.

In this paper, we provide sufficient conditions in terms of the spectral radius/ Linear programming to solve the state bounding problems of the positive singular discrete-time systems with a unbounded time-varying delay and bounded disturbances. Based on a new lemma, we provide sufficient conditions for the singular discrete-time system without disturbances as regular, causal, and positive. Then we estimate componentwise ultimate bound of the state vector of the singular discrete-time positive system with unbounded delay and without disturbances. Using the suitable transform, we present the sufficient conditions given in terms of the Linear programming / spectral radius to obtain the smallest componentwise estimate for the singular discrete-time system with unbounded delay and bounded disturbances.

2. Problem formulation and preliminaries

Notation: $\mathbb{R}^n_+$ ($\mathbb{R}^n_{0,+}$) denotes the set of all positive (nonnegative) vectors in $\mathbb{R}^n$; The set of real matrices of size $r\times h$ is denoted as $\mathbb{R}^{r\times h}$. The identity matrix of size $q\times q$ is denoted by I_q. $\mathbb{N}$ (respectively, $\mathbb{N}_+$) denotes the set of nonnegative integers (respectively, positive integers). $\mathbb{Z}$ stands for the set of all the integers. A vector $x=(\mathtt{x}_1,\mathtt{x}_2,...,\mathtt{x}_n)\in R^n$ is called positive ($x\succ 0$) if $\mathtt{x}_i>0, i=1,2,...,n$. $H=(h_{ij})\in\mathbb{R}^{m\times n}$, $H\succeq 0(\succ 0)$ if $h_{ij}\geq 0(>0)$, $\forall i,j$. $H\succeq K(H\succ K)$. For $M=(m_{ij})\in\mathbb{R}^{k\times k}$, M is the matrix Metzler if $m_{ij}\geq 0$ for all $i\neq j$; $i,j=1,2,\ldots,k$. $\lceil a\rceil$ is the smallest integer greater than or equal to real number a. The symbols $s(R)$ and $\rho(R)$ denote the spectral abscissa and the spectral radius of matrix R, respectively, that is, $s(R)=\max\{Re\eta : \eta\in\sigma(R)\}$, $\rho(R)=\max\{|\eta| : \eta\in\sigma(R)\}$ where $\sigma(R)$ is the spectrum of R.

Consider the following discrete-time singular systems with unbounded time-varying delay

$$\mathtt{E}\mathtt{x}(k+1)=\mathtt{A}\mathtt{x}(k)+\mathtt{D}\mathtt{x}(k-\tau(k))+B\mathtt{w}(k), \tag{1}$$

where $\mathtt{x}(k)\in\mathbb{R}^n$ is the state vector, $\mathtt{w}(k)\in\mathbb{R}^m$ is the disturbance, satisfying the following estimate:

$$0\preceq\mathtt{w}(k)\preceq\overline{\mathtt{w}} \tag{2}$$

$\mathtt{A},\mathtt{D},B$ are known constant matrices with appropriate dimensions. The matrix $\mathtt{E}\in\mathbb{R}^{n\times n}$ is singular and $\text{rank}(\mathtt{E})=r<n$. $\tau(\cdot)\in\mathbb{N}_+$ is unknown function delay satisfies the following estimates:

$$\sup_{k\geq T}\frac{\tau(k)}{k}\leq\theta \tag{3}$$

for some $T\in\mathbb{N}_+$ and a scalar $\theta\in[0,\,1)$. From estimate (3), it is easy to see that $k-\tau(k)\geq(1-\theta)k>0,\ \forall\ k\geq T$. Let $\overline{\tau}=-\inf_{0\leq k\leq T}\{k-\tau(k)\}$. Hence, the initial condition of system (1) is given by $\mathtt{x}(s)=\varphi(s)$, $s\in\{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$ and assume that funtion $\varphi(\cdot)$ satisfies the following condition:

$$0\preceq\varphi(s)\preceq\overline{\varphi},\forall s\in\{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}. \tag{4}$$

In this paper, for simplicity, let $\mathtt{E} := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $\mathtt{A} := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $\mathtt{D} := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$, $B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $\mathtt{x}(k) = \begin{pmatrix} \mathtt{x}_1(k) \\ \mathtt{x}_2(k) \end{pmatrix}$, $\mathtt{x}_1(k) \in \mathbb{R}^r, \mathtt{x}_2(k) \in \mathbb{R}^{n-r}$, and $\mathtt{A}_\mathtt{E} := \mathtt{A} + I_n - \mathtt{E}$. Let us denote by $\mathtt{x}(k, \varphi, \mathtt{w})$ the state trajectory with the initial condition $\varphi(\cdot)$ of system (1).

Definition 1. ([2]) (i) If the pair $(\mathtt{E}, \mathtt{A})$ is regular i.e., $\det(s\mathtt{E} - \mathtt{A}) \neq 0$, then the singular system (1) is regular. (ii) If the pair $(\mathtt{E}, \mathtt{A})$ is causal i.e., $deg(det(s\mathtt{E} - \mathtt{A})) = rank(\mathtt{E})$, then the singular system (1) is causal.

Definition 2. [29] System (1) is positive if for all initial value $\varphi \succeq 0$, and for any nonnegative input $\mathtt{w}(\cdot) \succeq 0$ implies the corresponding trajectory $\mathtt{x}(k, \varphi, \mathtt{w}) \succeq 0$ for all $k \in \mathbb{N}$.

Definition 3. [19] For $\zeta \in \mathbb{R}^n_+$, ζ is called a componentwise ultimate upper bound of system (1) if for any initial condition $\varphi(s), s \in \{-\overline{\tau}, -\overline{\tau}+1, \ldots, 0\}$ and for any $\mathtt{w}(k)$ satisfying (2), we get

$$\limsup_{k \to \infty} \mathtt{x}(k, \varphi, \mathtt{w}) \preceq \zeta.$$

Lemma 1. *[29] Let D be a Metzler matrix. Then, the following statements are equivalent:*

1) $s(D) < 0$
2) $\exists\, \mu \in \mathbb{R}^n : \mu \succ 0$ *and* $D\mu \prec 0$
3) $\exists \gamma \in \mathbb{R}^n : \gamma \succ 0$ *and* $\gamma^T D \prec 0$
4) $det(D) \neq 0$ *and* $-D^{-1} \succeq 0$.

Lemma 2. *([30]) Assume that* $\mathtt{A}_\mathtt{E} \succeq 0$, $\mathtt{D} \succeq 0$, *then the following statements are equivalent.*

i) $\mathtt{A}_\mathtt{E} + \mathtt{D}$ *is a Schur matrix, i.e.,* $\rho(\mathtt{A}_\mathtt{E} + \mathtt{D}) < 1$.
ii) *There exists a vector* $q \succ 0$ *such that* $(\mathtt{A}_\mathtt{E} + \mathtt{D} - I_n)q \prec 0$.
iii) $(I - \mathtt{A}_\mathtt{E} - \mathtt{D})^{-1} \succeq 0$.

Lemma 3. *Assume that* $\mathtt{A}_\mathtt{E} \succeq 0$, $\mathtt{D} \succeq 0$, *then, the following statements are equivalent:*

i) $\rho\left(\mathtt{A}_\mathtt{E} + \mathtt{D}\right) < 1$, *i.e.,* $\mathtt{A}_\mathtt{E} + \mathtt{D}$ *is a Schur matrix.*
ii) $\rho\left(A_4 + D_4 + I_{n-r}\right) < 1$ *and* $\rho\left(A_1 + D_1 - (A_2 + D_2)(A_4 + D_4)^{-1}(A_3 + D_3)\right) < 1$.
iii) $\rho\left(A_1 + D_1\right) < 1$ *and* $\rho\left(A_4 + D_4 + I_{n-r} - (A_3 + D_3)(A_1 + D_1 - I_r)^{-1}(A_2 + D_2)\right) < 1$.
iv) $\exists \eta = (\eta_1, \eta_2) \succ 0$, $\eta_1 \in \mathbb{R}^r, \eta_2 \in \mathbb{R}^{n-r}$:

$$(\mathtt{A}_\mathtt{E} + \mathtt{D} - I_n)\,\eta = \begin{pmatrix} A_1 + D_1 - I_r & A_2 + D_2 \\ A_3 + D_3 & A_4 + D_4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \prec 0.$$

Proof. $(ii) \Rightarrow (iv)$. As $\mathtt{A}_\mathtt{E} + \mathtt{D} = \begin{pmatrix} A_1 + D_1 & A_2 + D_2 \\ A_3 + D_3 & A_4 + D_4 + I_{n-r} \end{pmatrix} \succeq 0$, then $A_1 + D_1$, $A_2 + D_2$, $A_3 + D_3$, $A_4 + D_4 + I_{n-r}$ are nonnegative matrices. This implies that $A_4 + D_4$ is a Metzler matrix. It follows from Lemma 2 and $\rho\left(A_4 + D_4 + I_{n-r}\right) < 1$ that there exists a vector $\lambda_2 \succ 0$:

$$\left(A_4 + D_4\right)\lambda_2 \prec 0. \tag{5}$$

Combining the inequality (5) and Lemma 1 implies that A_4+D_4 is a Hurwitz matrix and $-(A_4+D_4)^{-1} \succeq 0$. This implies that the $A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)$ is a nonnegative matrix. Combining this with $\rho\left(A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)\right)<1$, and Lemma 2 implies that there exists a vector $\lambda_1 \succ 0$ such that

$$\left(A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r\right)\lambda_1 \prec 0. \tag{6}$$

From $-(A_2+D_2)(A_4+D_4)^{-1} \succeq 0$ and (6) we obtain estimate as follows:

$$\begin{pmatrix} A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r & -(A_2+D_2)(A_4+D_4)^{-1} \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \varepsilon\lambda_2 \end{pmatrix} \prec 0 \tag{7}$$

for sufficiently small $\varepsilon>0$. Furthermore, we have

$$\begin{aligned} &\begin{pmatrix} A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r & -(A_2+D_2)(A_4+D_4)^{-1} \\ 0 & -I_{n-r} \end{pmatrix} \\ &\qquad = \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -(A_4+D_4)^{-1}(A_3+D_3) & -(A_4+D_4)^{-1} \end{pmatrix} \end{aligned} \tag{8}$$

It follows from (7) and (8) that there exists a vector $\eta=(\eta_1,\eta_2)$, such that

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \prec 0,$$

where $\eta_1=\lambda_1 \succ 0$ and $\eta_2=-(A_4+D_4)^{-1}(A_3+D_3)\lambda_1-\varepsilon(A_4+D_4)^{-1}\lambda_2$. Note that $-(A_4+D_4)^{-1}(A_3+D_3),\ -(A_4+D_4)^{-1}$ are nonnegative matrices and $-(A_4+D_4)^{-1}$ is a nonsingular matrix, hence vector $\eta_2 \succ 0$. Therefore, (iv) holds.
$(iv) \Rightarrow (ii)$. Assume that there exists $\eta=(\eta_1,\eta_2)\succ 0$, $\eta_1\in\mathbb{R}^r, \eta_2\in\mathbb{R}^{n-r}$, such that

$$(\mathtt{A}_\mathtt{E}+\mathtt{D}-I_n)\,\eta = \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \prec 0. \tag{9}$$

From (9), we obtain

$$(A_3+D_3)\eta_1+(A_4+D_4)\eta_2 \prec 0. \tag{10}$$

This, together with $(A_3+D_3)\eta_1 \succeq 0$, we get $(A_4+D_4)\eta_2 \prec 0$. From this, we have

$$(A_4+D_4+I_{n-r}-I_{n-r})\eta_2 \prec 0. \tag{11}$$

Combining (11) with Lemma 2, we get $\rho(A_4+D_4+I_{n-r})<1$ by $A_4+D_4+I_{n-r}\succeq 0$. Moreover, we have

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix}$$

$$= \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -(A_4+D_4)^{-1}(A_3+D_3) & -(A_4+D_4)^{-1} \end{pmatrix} \times \begin{pmatrix} I_r & 0 \\ -(A_3+D_3) & -(A_4+D_4) \end{pmatrix}$$

$$= \begin{pmatrix} A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r & -(A_2+D_2)(A_4+D_4)^{-1} \\ 0 & -I_{n-r} \end{pmatrix} \times \begin{pmatrix} I_r & 0 \\ -(A_3+D_3) & -(A_4+D_4) \end{pmatrix} \tag{12}$$

It follows from (9), (12) that there exists a vector $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{pmatrix} A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r & -(A_2+D_2)(A_4+D_4)^{-1} \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \prec 0, \tag{13}$$

where $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ -(A_3+D_3) & -(A_4+D_4) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. This implies that, $\lambda_1 = \eta_1 \succ 0$ and

$$\lambda_2 = -(A_3+D_3)\eta_1 - (A_4+D_4)\eta_2.$$

From (10) implies $\lambda_2 \succ 0$. Using the inequality (13), we obtain

$$\left(A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)-I_r\right)\lambda_1 \prec 0. \tag{14}$$

Combining (14) with Lemma 2, we get $\rho\left(A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3)\right) < 1$ by $A_1+D_1-(A_2+D_2)(A_4+D_4)^{-1}(A_3+D_3) \succeq 0$. Therefore, (ii) holds.

$(iii) \Rightarrow (iv)$. Using Lemma 2, $\rho\,(A_1+D_1) < 1$ and $A_1+D_1 \succeq 0$ implies that there exists a vector $\gamma_1 \succ 0:$

$$(A_1+D_1-I_r)\,\gamma_1 \prec 0. \tag{15}$$

From (15) and Lemma 1 implies that $A_1+D_1-I_r$ is a Hurwitz matrix and $-(A_1+D_1-I_r)^{-1} \succeq 0$. This implies that the $A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)$ is a nonnegative matrix. Combining this with $\rho\left(A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)\right) < 1$, and Lemma 2 implies that there exists a vector $\gamma_2 \succ 0$ such that

$$\left(A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)-I_{n-r}\right)\gamma_2 \prec 0. \tag{16}$$

From $(A_3+D_3) \succeq 0$ and (16) we obtain

$$\begin{pmatrix} A_1+D_1-I_r & 0 \\ A_3+D_3 & A_4+D_4-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2) \end{pmatrix} \begin{pmatrix} \varepsilon_1\gamma_1 \\ \gamma_2 \end{pmatrix} \prec 0 \tag{17}$$

for sufficiently small $\varepsilon_1 > 0$. Moreover , we get

$$\begin{pmatrix} A_1+D_1-I_r & 0 \\ A_3+D_3 & A_4+D_4-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2) \end{pmatrix}$$
$$= \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} I_r & -(A_1+D_1-I_r)^{-1}(A_2+D_2) \\ 0 & I_{n-r} \end{pmatrix}. \tag{18}$$

It follows from (17) and (18) that there exists a vector $\eta = (\eta_1, \eta_2)$, such that

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \prec 0, \tag{19}$$

where $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} I_r & -(A_1+D_1-I_r)^{-1}(A_2+D_2) \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} \varepsilon_1\gamma_1 \\ \gamma_2 \end{pmatrix}$. This implies that

$$\eta_1 = \varepsilon_1\gamma_1 - (A_1+D_1-I_r)^{-1}(A_2+D_2)\gamma_2$$

and $\eta_2 = \gamma_2 \succ 0$. Note that $-(A_1+D_1-I_r)^{-1}(A_2+D_2)\gamma_2 \succeq 0$ and $\varepsilon_1\gamma_1 \succ 0$, then we obtain $\eta_1 \succ 0$. Therefore, (iv) holds.

$(iv) \Rightarrow (iii)$. Assume that there exists $\eta = (\eta_1, \eta_2) \succ 0$, $\eta_1 \in \mathbb{R}^r, \eta_2 \in \mathbb{R}^{n-r}$, such that

$$(\mathtt{A}_\mathtt{E} + \mathtt{D} - I_n)\,\eta = \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \prec 0. \tag{20}$$

From (20), we obtain

$$(A_1+D_1-I_r)\eta_1 + (A_2+D_2)\eta_2 \prec 0. \tag{21}$$

Combine this with $(A_2+D_2)\eta_2 \succeq 0$, we get $(A_1+D_1-I_r)\eta_1 \prec 0$. This, together with Lemma 2, we get $\rho(A_1+D_1) < 1$ by $A_1+D_1 \succeq 0$. Moreover, we have

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix}$$
$$= \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} I_r & -(A_1+D_1-I_r)^{-1}(A_2+D_2) \\ 0 & I_{n-r} \end{pmatrix}$$
$$\times \begin{pmatrix} I_r & (A_1+D_1-I_r)^{-1}(A_2+D_2) \\ 0 & I_{n-r} \end{pmatrix}$$
$$= \begin{pmatrix} A_1+D_1-I_r & 0 \\ A_3+D_3 & A_4+D_4-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2) \end{pmatrix}$$
$$\times \begin{pmatrix} I_r & (A_1+D_1-I_r)^{-1}(A_2+D_2) \\ 0 & I_{n-r} \end{pmatrix}. \tag{22}$$

It follows from (20), (22) that there exists a vector $\lambda = (\lambda_1, \lambda_2)$ such that

$$\begin{pmatrix} A_1+D_1-I_r & 0 \\ A_3+D_3 & A_4+D_4-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \prec 0, \tag{23}$$

where $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} I_r & (A_1+D_1-I_r)^{-1}(A_2+D_2)\\ 0 & I_{n-r}\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$. This implies that, $\lambda_2 = \eta_1 2 \succ 0$ and

$$\lambda_2 = \eta_1 + (A_1+D_1-I_r)^{-1}(A_2+D_2)\eta_2.$$

It follows from $-(A_1+D_1-I_r)^{-1} \succeq 0$, and (21), we obtain

$$\lambda_2 = \eta_1 + (A_1+D_1-I_r)^{-1}(A_2+D_2)\eta_2 \succ 0.$$

Using the inequality (23) and $(A_3+D_3)\lambda_1 \succeq 0$, we obtain

$$\begin{aligned}&\big(A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)-I_{n-r}\big)\lambda_2\\ &\qquad = \big(A_4+D_4-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)\big)\,\lambda_2 \prec 0.\end{aligned} \tag{24}$$

Combining (24) with Lemma 2, we get

$$\rho\left(A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2)\right) < 1$$

by $A_4+D_4+I_{n-r}-(A_3+D_3)(A_1+D_1-I_r)^{-1}(A_2+D_2) \succeq 0$. Therefore, (iii) holds.
$(i) \Leftrightarrow (iv)$. Using Lemma 2, we have (i) and (iv) are equivalent.

□

Lemma 4. *Suppose that* $\mathtt{A_E} \succeq 0$, $\mathtt{D} \succeq 0$ *and there exists a vector* $\mu = (\mu_1, \mu_2)$ *satisfying* $(\mathtt{A_E}+\mathtt{D}-I_n)\,\mu \prec 0$. *Then, the systems* (1) *is regular, causal, and positive.*

Proof. We have

$$(\mathtt{A_E}+\mathtt{D}-I_n)\,\mu = \begin{pmatrix}A_1+D_1-I_r & A_2+D_2\\ A_3+D_3 & A_4+D_4\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \prec 0. \tag{25}$$

Using the inequalities (25) and $\mathtt{D} \succeq 0$, $\mu \succ 0$, we get $(-\mathtt{E}+\mathtt{A})\,\mu \prec 0$. This implies that

$$A_3\mu_1 + A_4\mu_2 \prec 0\,. \tag{26}$$

From $\mathtt{A}+(I_n-\mathtt{E}) = \begin{pmatrix}A_1 & A_2\\ A_3 & A_4+I_{n-r}\end{pmatrix} \succeq 0$ implies that $A_1, A_2, A_3, A_4+I_{n-r}$ are non-negative matrices. From this, A_4 is a Metzler matrix and $A_3\mu_1 \succeq 0$. Combine this with the estimate (26) we get $A_4\mu_2 \prec 0$. From this, the matrix A_4 is Hurwitz and $det(A_4) \neq 0$ by Lemma 1. This implies that the system (1) is regular and causal (see [3]). Let us denote

$$\begin{aligned}\overline{A}_1 &:= A_1 - A_2A_4^{-1}A_3,\ \overline{D}_1 := D_1 - A_2A_4^{-1}D_3,\quad \overline{D}_3 = -A_4^{-1}D_3,\ \overline{D}_4 := -A_4^{-1}D_4,\\ \overline{A}_3 &:= -A_4^{-1}A_3,\qquad \overline{D}_2 := D_2 - A_2A_4^{-1}D_4,\quad \overline{B}_1 = B_1 - A_2A_4^{-1}B_2,\ \overline{B}_2 := -A_4^{-1}B_2.\end{aligned}$$

Then, system (1) rewrite the following:

$$\begin{aligned}\mathtt{x}_1(k+1) &= \overline{A}_1\mathtt{x}_1(k) + \overline{D}_1\mathtt{x}_1(k-\tau(k)) + \overline{D}_2\mathtt{x}_2(k-\tau(k)) + \overline{B}_1\mathtt{w}(k),\\ \mathtt{x}_2(k) &= \overline{A}_3\mathtt{x}_1(k) + \overline{D}_3\mathtt{x}_1(k-\tau(k)) + \overline{D}_4\mathtt{x}_2(k-\tau(k)) + \overline{B}_2\mathtt{w}(k).\end{aligned} \tag{27}$$

It is easy to show that the matrices $\overline{A}_1, \overline{A}_3$, $\overline{D}_1, \overline{D}_2, \overline{D}_3$, $\overline{D}_4, \overline{B}_1, \overline{B}_2$ are nonnegative. Since $0 < \tau(k) \in \mathbb{N}$, there exists $h_1 \in \mathbb{N}$ such that $0 < h_1 \leq \tau(k), k \in \mathbb{N}$. We can easily show that the solution $\mathtt{x}(k)$ of the system (27) is positive on $[0, h_1]$. Using the step method, we can extend the consideration for the intervals $[h_1, 2h_1], [2h_1, 3h_1]$, etc. Then, the system (1) is positive. □

3. MAIN RESULTS

We consider the singular system:

$$\begin{cases} \mathtt{E}\bar{\mathtt{x}}(k+1) = \mathtt{A}\bar{\mathtt{x}}(k) + \mathtt{D}\bar{\mathtt{x}}(k-\tau(k)) + B\bar{\mathtt{w}},\ k \geq 0, \\ \bar{\mathtt{x}}(s) = \vartheta(s),\ s \in \{-\overline{\tau}, -\overline{\tau}+1, \ldots, 0\}. \end{cases} \tag{28}$$

The following lemma provides a relationship between the state trajectory of the system (1) and the state trajectory of the system (28).

Lemma 5. *Assume that A_4 is Hurwitz matrix. The following statements hold:*

(i) If $\varphi(s) \preceq \vartheta_1(s)\ \ \forall s \in \{-\overline{\tau}, -\overline{\tau}+1, \ldots, 0\}$ we get that $\mathtt{x}(k,\varphi,\mathtt{w}) \preceq \bar{\mathtt{x}}(k,\vartheta_1,\bar{\mathtt{w}}), \forall k \geq 0$.

(ii) If $\vartheta_1(s) \preceq \vartheta_2(s)\ \ \forall s \in \{-\overline{\tau}, -\overline{\tau}+1, \ldots, 0\}$, we obtain $\bar{\mathtt{x}}(k,\vartheta_1,\bar{\mathtt{w}}) \preceq \bar{\mathtt{x}}(k,\vartheta_2,\bar{\mathtt{w}}), \forall k \geq 0$.

Proof. (i) We consider the system

$$\begin{aligned} &\mathtt{E}u(k+1) = \mathtt{A}u(k) + \mathtt{D}u(k-\tau(k)) + Bp(k),\ k \geq 0, \\ &u(s) = \vartheta_1(s) - \varphi(s),\ s \in \{-\overline{\tau}, -\overline{\tau}+1, \ldots, 0\}, \end{aligned} \tag{29}$$

where $u(k) := \bar{\mathtt{x}}(k) - \mathtt{x}(k), p(k) := \bar{\mathtt{w}} - \mathtt{w}(k)$. In virtue of Lemma 1, and A_4 is Hurwitz, we obtain $det(A_4) \neq 0$ and $-{A_4}^{-1} \succeq 0$. Apply Lemma 4; we can immediately deduce that $u(k,\vartheta_1 - \varphi,\ p(k)) \succeq 0, \forall k \geq 0$, it follows that $\bar{\mathtt{x}}(k,\vartheta_1,\ \bar{\mathtt{w}}) - \mathtt{x}(k,\varphi,\mathtt{w}) \succeq 0, \forall k \geq 0$.
(ii) By the same method as in the proof of part (i) and Lemma 4, we get that (ii). □

Now, we give some estimate for the singular positive system without disturbances.

Lemma 6. *Assume that the conditions in Lemma 3 are satisfied. Then, the system (1) with $\mathtt{w}(k) = 0$ is regular, causal, positive and $\exists\ \alpha \in (0,1), \exists \mu \in \mathbb{R}^n_+$ and a sequence $0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots < +\infty$ such that*

$$\mathtt{x}(k,\mu,0) \preceq \alpha^{n+1}\mu,\ \forall k \in \{T_n+1, \ldots, T_{n+1}\}. \tag{30}$$

Proof. Using condition (iv) in Lemma 3, i.e., there exists a vector $\mu \succ 0$ such that

$$(\mathtt{A}+\mathtt{D}-\mathtt{E})\,\mu \prec 0. \tag{31}$$

Similar to Lemma 4, we get $det(A_4) \neq 0,\ -A_4^{-1} \succeq 0$, and the system (1) with $\mathtt{w}(k) = 0$ is regular, causal, positive. We show that there exists $\alpha \in (0,1)$ such that

$$\left(A_1+D_1\right)\mu_1 + \left(A_2+D_2\right)\mu_2 \prec \alpha\mu_1, \tag{32}$$

$$-A_4^{-1}\left(A_3+D_3\right)\mu_1 - A_4^{-1}D_4\mu_2 \prec \alpha\mu_2. \tag{33}$$

Indeed, we have the matrix $\begin{pmatrix} I_r & 0 \\ 0 & -A_4^{-1} \end{pmatrix} \succeq 0$ and nonsingular. Left multiplying (31) by $\begin{pmatrix} I_r & 0 \\ 0 & -A_4^{-1} \end{pmatrix}$, we obtain:

$$\begin{pmatrix} I_r & 0 \\ 0 & -A_4^{-1} \end{pmatrix} \begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ A_3+D_3 & A_4+D_4 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \prec 0,$$

which is equivalent to

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ -A_4^{-1}(A_3+D_3) & -A_4^{-1}D_4-I_{n-r} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \prec 0. \tag{34}$$

By (34), there exists a small enough scalar $\nu \in (0,1)$ satisfying

$$\begin{pmatrix} A_1+D_1-I_r & A_2+D_2 \\ -A_4^{-1}(A_3+D_3) & -A_4^{-1}D_4-I_{n-r} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \nu \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \prec 0,$$

which is equivalent to

$$\begin{pmatrix} A_1+D_1 & A_2+D_2 \\ -A_4^{-1}(A_3+D_3) & -A_4^{-1}D_4 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \prec (1-\nu) \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

From this, we obtain (32) and (33) with $\alpha = 1-\nu \in (0,1)$. We rewrite the system (1) ($\mathtt{w}(k)=0$) as:

$$\begin{cases} \mathtt{x}_1(k+1) = A_1\mathtt{x}_1(k) + A_2\mathtt{x}_2(k) + D_1\mathtt{x}_1(k-\tau(k)) + D_2\mathtt{x}_2(k-\tau(k)), \\ \mathtt{x}_2(k) = -A_4^{-1}A_3\mathtt{x}_1(k) - A_4^{-1}D_3\mathtt{x}_1(k-\tau(k)) - A_4^{-1}D_4\mathtt{x}_2(k-\tau(k)). \end{cases} \tag{35}$$

Using the inequalities (31), (32), (33) and by [15], we obtain (30) with $T_0 = 0, T_1 = T \geq 2$ (given in (3)), $T_{n+1} = \left\lceil \frac{T_n+1}{1-\theta} \right\rceil, n \in \mathbb{N}$.

□

Remark 1. Note that, if $\mathtt{A_E}$, $\mathtt{D}$ satisfy the conditions in Lemma 3, then the system (1)($\mathtt{w}(k)=0$) is regular, causal, positive and asymptotically stable. Indeed, similar to Lemma 6, the system (1) with $\mathtt{w}(k)=0$ is regular, causal, positive. For any $\varepsilon > 0$, let $M = \|\mu\|_\infty$, choose $\alpha_1 = \frac{\varepsilon}{M^2}$. Then, for all $\varphi(\cdot)$ satisfying $\max_{s\in\{-\overline{\tau},\cdots,-1\}} \|\varphi(s)\|_\infty < \alpha_1$, we obtain $\max_{s\in\{-\overline{\tau},\cdots,-1\}} \varphi(s) < \frac{\varepsilon}{M}\mu$. By linearity of system (1) (with $\mathtt{w}(k)=0$) and (30) we get $\mathtt{x}(k,\varphi,0) \preceq \frac{\varepsilon}{M}\mathtt{x}(k,\mu,0) \preceq \frac{\varepsilon}{M}\mu$, $\forall k \in \mathbb{N}$. This implies that $\|\mathtt{x}(k,\varphi,0)\|_\infty \leq \varepsilon$. On the other hand, (30) implies that $\lim_{t\to\infty} \mathtt{x}(k,\mu,0) = 0$. Then, we get $\lim_{t\to\infty} \mathtt{x}(k,\varphi,0) = 0$.

The following theorem provides a condition sufficient to ensure that system (1) is regular, causal, positive and the existence of an componentwise ultimate bound for the system.

Let us denote: $\zeta_{\overline{\mathtt{w}}} = -\left(\mathtt{A_E} + \mathtt{D} - I_n\right)^{-1} B\overline{\mathtt{w}}$.

Theorem 1. *Assume that the conditions in Lemma 3 are satisfied. Then, the system (1) is regular, causal, positive and*
(i) There exist $\alpha \in (0,1)$, $\eta \in \mathbb{R}^n_+$ and a sequence $0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots < +\infty$ and $\zeta_{\overline{\mathtt{w}}}$ such that

$$\mathtt{x}(k,\varphi,\mathtt{w}) \preceq \zeta_{\overline{\mathtt{w}}} + \alpha^{n+1}\eta,\ \forall k \in [T_n, T_{n+1}]. \tag{36}$$

(ii) $\zeta_{\overline{\mathtt{w}}} \in \mathbb{R}^n_{0,+}$ is the smallest vector such that

$$\limsup_{k\to\infty} \mathtt{x}(k,\varphi,\mathtt{w}) \preceq \zeta_{\overline{\mathtt{w}}}. \tag{37}$$

Proof. In case (i), similar to i) of Lemma 3, implies that $\exists \mu \in \mathbb{R}^n_+$ such that

$$(\mathtt{A}+\mathtt{D}-\mathtt{E})\,\mu \prec 0. \tag{38}$$

Similar to Lemma 6, from the condition (38) implies that A_4 is Hurwitz matrix and $-{A_4}^{-1} \succeq 0$. By $\det(A_4) \neq 0$, we can conclude that system (1) is regular, causal. Moreover, the system (1) can be rewritten as follows

$$\begin{cases} x_1(k+1) = A_1x_1(k) + A_2x_2(k) + D_1x_1(k-\tau(k)) + D_2x_2(k-\tau(k)) + B_1\mathtt{w}(k) \\ x_2(k) = -{A_4}^{-1}\left(A_3x_1(k) + D_3x_1(k-\tau(k)) + D_4x_2(k-\tau(k)) + B_2\mathtt{w}(k)\right). \end{cases}$$

It follows from this and Lemma 4 that the system (1) is positive. Let $\alpha_1 = \max\left\{\frac{\overline{\varphi}_1}{a_1},\ \cdots,\ \frac{\overline{\varphi}_n}{a_n}\right\}$, where $\mu = (a_1,a_2,\ldots,a_n) \in \mathbb{R}^n_+$, $\overline{\varphi} = (\overline{\varphi}_1,\overline{\varphi}_2,\ldots,\overline{\varphi}_n) \in \mathbb{R}^n_{0,+}$ and choose $\xi = \alpha_1\,\mu$. Then, we get $\xi \succeq \overline{\varphi}$ and

$$(\mathtt{A}+\mathtt{D}-\mathtt{E})\,\xi \prec 0. \tag{39}$$

Let $\vartheta_\xi(s) = \xi,\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$. Using Lemma 5 and $\varphi(s) \preceq \overline{\varphi} \preceq \xi,\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$, we obtain that

$$\mathtt{x}(k,\varphi,\mathtt{w}) \preceq \bar{\mathtt{x}}(k,\vartheta_\xi,\overline{\mathtt{w}}),\ k \geq 0. \tag{40}$$

By $\xi \succ 0$, it is easy to choose $\rho > 1 : \rho\xi \succeq \zeta_{\overline{\mathtt{w}}}$. Set $\eta := \rho\xi$ this together with (39) we have that

$$(\mathtt{A}+\mathtt{D}-\mathtt{E})\,\eta \prec 0. \tag{41}$$

Setting $\vartheta_\eta(s) = \eta,\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$. Since $\rho > 1$ we have $\xi \prec \eta$, it follows from Lemma 5, we obtain that

$$\bar{\mathtt{x}}(k,\vartheta_\xi,\overline{\mathtt{w}}) \preceq \bar{\mathtt{x}}(k,\vartheta_\eta,\overline{\mathtt{w}}),\ k \geq 0. \tag{42}$$

We can easily get $\zeta_{\overline{\mathtt{w}}} \succeq 0$. Let $\vartheta_{\zeta_{\overline{\mathtt{w}}}}(s) = \zeta_{\overline{\mathtt{w}}},\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$, it follows from $\zeta_{\overline{\mathtt{w}}} \preceq \eta$ then, we obtain that

$$\vartheta_{\zeta_{\overline{\mathtt{w}}}}(s) \preceq \vartheta_\eta(s),\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}. \tag{43}$$

Setting $\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}}(s) := \vartheta_\eta(s) - \vartheta_{\zeta_{\overline{\mathtt{w}}}}(s),\ s \in \{-\overline{\tau},-\overline{\tau}+1,\ldots,0\}$, using (43) we get $\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}}(s) \succeq 0$. Under coordinate transformation

$$\bar{\mathtt{x}}(k) = z(k) + \zeta_{\overline{\mathtt{w}}}, \tag{44}$$

then, from system (28), we get

$$\mathtt{E}z(k+1)=\mathtt{A}z(k)+\mathtt{D}z(k-\tau(k)), \tag{45}$$

and

$$\bar{\mathtt{x}}(k,\vartheta_\eta,\overline{\mathtt{w}})=\zeta_{\overline{\mathtt{w}}}+z(k,\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}}), \tag{46}$$

where $z(k,\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}})$ is the solution of system (45) with the initial function $\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}}(\cdot)$. It follows from $\eta-\zeta_{\overline{\mathtt{w}}}\preceq\eta$, implies that $\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}}(s)\preceq\vartheta_\eta(s),\ s\in\{-\overline{\tau},-\overline{\tau}+1,\dots,0\}$, then using Lemma 5, we get

$$z(k,\vartheta_{\eta-\zeta_{\overline{\mathtt{w}}}})\preceq z(k,\vartheta_\eta),\ k\geq 0. \tag{47}$$

It follows from Lemma 6 for system (45) and (41) imply $\exists\ \alpha\in(0,1)$ and a sequence $0=T_0<T_1<T_2<\cdots<T_n<\cdots<+\infty$ such that

$$z(k,\vartheta_\eta)\preceq\alpha^{n+1}\eta,\ \forall k\in[T_n,T_{n+1}]. \tag{48}$$

Combining (40), (42), (46), (47) and (48), we get that (36).

(ii) For $k\to\infty$, it follows from (36), we get (37). Then, $\zeta_{\overline{\mathtt{w}}}$ is a componentwise ultimate bound of system (1). We now show that $\lim_{k\to\infty}\bar{\mathtt{x}}(k,\ \vartheta_0,\ \overline{\mathtt{w}})=\zeta_{\overline{\mathtt{w}}}$, where $\vartheta_0(s)=0, s\in\{-\overline{\tau},-\overline{\tau}+1,\dots,0\}$. Using coordinate transformation

$$y(k)=\zeta_{\overline{\mathtt{w}}}-\bar{\mathtt{x}}(k). \tag{49}$$

This together with (28), imply that

$$\mathtt{E}y(k+1)=\mathtt{A}y(k)+\mathtt{D}y(k-\tau(k)),\ k\geq 0, \tag{50}$$

and

$$y(k,\vartheta_{0\zeta_{\overline{\mathtt{w}}}})=\zeta_{\overline{\mathtt{w}}}-\bar{\mathtt{x}}(k,\vartheta_0,\ \overline{\mathtt{w}}),\forall k\geq 0, \tag{51}$$

where $\vartheta_{0\zeta_{\overline{\mathtt{w}}}}(s)=\zeta_{\overline{\mathtt{w}}}, s\in\{-\overline{\tau},-\overline{\tau}+1,\dots,0\}$. It follows from Lemma 4 that $y(k,\vartheta_{0\zeta_{\overline{\mathtt{w}}}})\succeq 0, k\geq 0$. Note that $\zeta_{\overline{\mathtt{w}}}\preceq\eta$, then $\vartheta_{0\zeta_{\overline{\mathtt{w}}}}(s)\preceq\vartheta_\eta(s), s\in\{-\overline{\tau},-\overline{\tau}+1,\dots,0\}$. By Lemma 5, we get

$$y(k,\vartheta_{0\zeta_{\overline{\mathtt{w}}}})\preceq y(k,\vartheta_\eta),k\geq 0. \tag{52}$$

Applying Lemma 6 to system (50), implies $\exists\ \alpha\in(0,1)$ and a sequence $0=T_0<T_1<T_2<\cdots<T_n<\cdots<+\infty$ such that

$$y(k,\vartheta_\eta)\preceq\alpha^{n+1}\eta,\ \forall k\in[T_n,T_{n+1}]. \tag{53}$$

Combining (51), (52) and (53) we get $\zeta_{\overline{\mathtt{w}}}-\alpha^{n+1}\eta\preceq\bar{\mathtt{x}}(k,\vartheta_0,\overline{\mathtt{w}})\preceq\zeta_{\overline{\mathtt{w}}}, \forall k\geq 0$. This implies that $\lim_{k\to\infty}\bar{\mathtt{x}}(k,\vartheta_0,\overline{\mathtt{w}})=\zeta_{\overline{\mathtt{w}}}$. Then we have $\zeta_{\overline{\mathtt{w}}}$ is the smallest componentwise ultimate bound of system (1). □

4. Numerical example

Example 1. Let us consider system (1) where

$$\mathtt{E}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \mathtt{A}=\begin{bmatrix}0.4&0.25&0\\0.25&0.2&0\\0.15&0.15&-0.99\end{bmatrix},$$

$$\mathtt{D}=\begin{bmatrix}0.25&0.1&0\\0.2&0.2&0\\0.3&0.11&0\end{bmatrix}B=\begin{bmatrix}0.15\\0.2\\0.1\end{bmatrix},$$

and $0\le \mathtt{w}(k)\le 2, k\in\mathbb{N}$, the delay is given by $\tau(k)=\left[\frac{k}{5}\right]+1, k\in\mathbb{N}$, where $\left[\cdot\right]$ is the integer function. It is easy to show that condition (3) is satisfied with $T=2$, $\theta=0.9$, and then $\overline{\tau}=-\inf_{0\le k\le T}\{k-\tau(k)\}=1$. Moreover, we have

$$\mathtt{A}_\mathtt{E}=\mathtt{A}+I_3-\mathtt{E}=\begin{pmatrix}0.4&0.25&0\\0.25&0.2&0\\0.15&0.15&0.01\end{pmatrix}\succeq 0$$

and $\mathtt{D}, B\succeq 0$. It can be readily verified that $\rho\left(\mathtt{A}_\mathtt{E}+\mathtt{D}\right)=0.9411<1$. Therefore, by Remark 1, system (1) with $\mathtt{w}(k)=0, k\ge 0$, is asymptotically stable. Moreover, using Theorem 1, we can compute the componentwise ultimate bound of (1)

$$\zeta_{\overline{\mathtt{w}}}=-\left(\mathtt{A}_\mathtt{E}+\mathtt{D}-I_3\right)^{-1}B\overline{\mathtt{w}}=\begin{pmatrix}6.0952\\5.2381\\4.3482\end{pmatrix}.$$

For a visual simulation, we choose $\mathtt{w}(k)=b\left[2sin^2(0.1k)\right]$, where $b\in\{0.5;\ 1\}$, and the initial condition $\varphi(s)=[1.5\ \ 1\ \ 1]^T,\ s\in\{-1,0\}$. Figure 1 shows the trajectories of $x_1(k),\ x_2(k)$ and $x_3(k)$ of the systems (1) with $\mathtt{w}(k)=0;\ k\ge 0$. Figure 2-4 shows trajectories of $x_1(k),\ x_2(k)$ $x_3(k)$ and its bound, respectively.

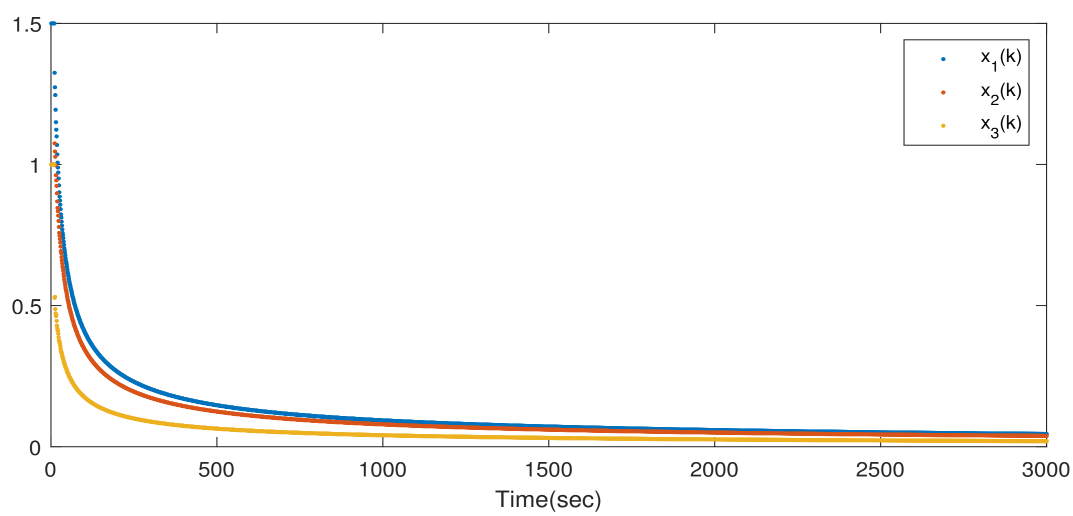

Figure 1: Responses of state trajectory of system (1) with $\mathtt{w}(k) = 0, k \geq 0$.

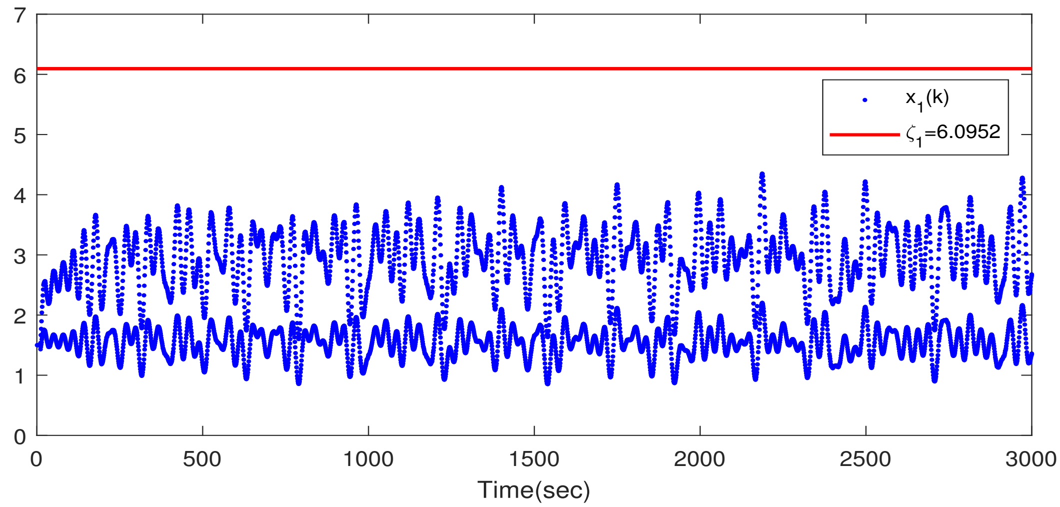

Figure 2: Responses of $x_1(k)$ and its bound $\zeta_1 = 6.0952$.

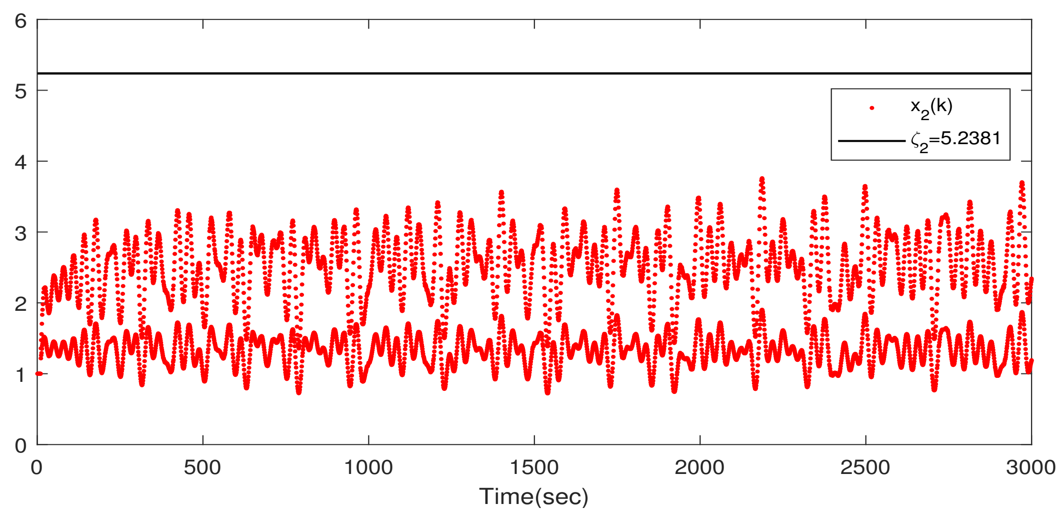

Figure 3: Responses of $x_2(k)$ and its bound $\zeta_2 = 5.2381$.

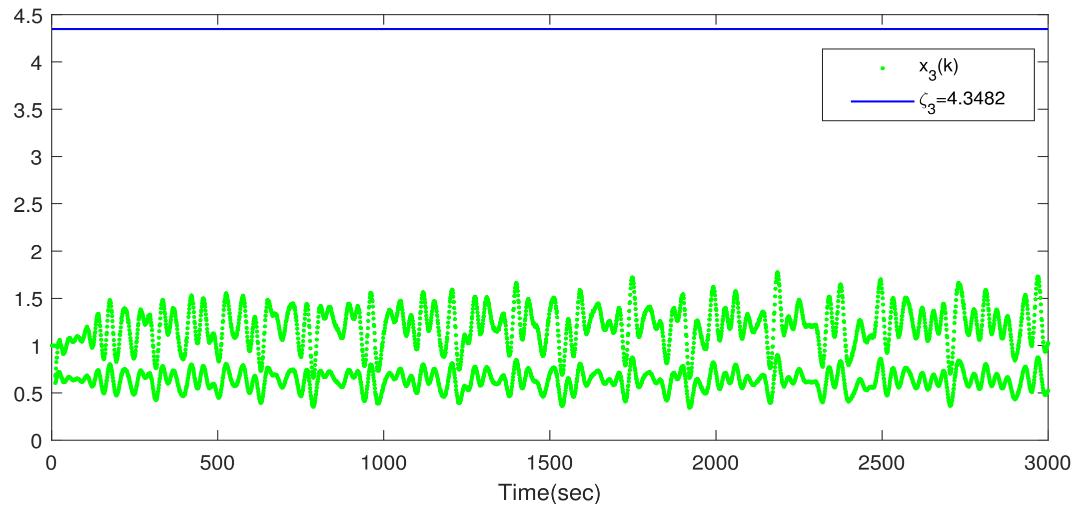

Figure 4: Responses of $x_3(k)$ and its bound $\zeta_3 = 4.3482$.

5. Conclusions

In this paper, we have presented a method to derive the smallest componentwise state for positive singular discrete-time systems with unbounded delay and bounded disturbances. Firstly, we show that the singular discrete-time systems without disturbances is regular, causal, positive, and the existence of componentwise bounds for the state vector of the system. Then, we have obtained a sufficient condition for the existence of componentwise ultimate bounds for positive singular discrete-time systems with unbounded delay and bounded disturbances. A numerical example is given to illustrate the effectiveness of the proposed results.

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