Widths of Embeddings of Weighted Wiener Classes

Van Dung Nguyen[a], Van Kien Nguyen *[a], and Winfried Sickel[b]

[a]Faculty of Basic Sciences, University of Transport and Communications
No.3 Cau Giay Street, Lang Thuong Ward, Dong Da District, Hanoi, Vietnam
Email: dungnv@utc.edu.vn, kiennv@utc.edu.vn
[b]Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07737 Jena, Germany
Email: winfried.sickel@uni-jena.de

December 7, 2020

Abstract

In this paper we will study the asymptotic behaviour of certain widths of the embeddings $\mathcal{A}_\omega(\mathbb{T}^d) \to L_p(\mathbb{T}^d)$, $2 \le p \le \infty$, and $\mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)$, where $\mathcal{A}_\omega(\mathbb{T}^d)$ is the weighted Wiener class and $\mathcal{A}(\mathbb{T}^d)$ is the Wiener algebra on the d-dimensional torus $\mathbb{T}^d$. Our main interest will consist in the calculation of the associated asymptotic constant. As one of the consequences we also obtain the asymptotic constant related to the embedding of $C^m_{\text{mix}}(\mathbb{T}^d)$ into $L_2(\mathbb{T}^d)$ for Weyl and Bernstein numbers.

Keywords and Phrases: Mixed smoothness, Wiener algebra, compact embeddings, widths, asymptotic constant, dimensional dependence

1 Introduction

In this paper we are concerned with the behaviour of certain widths (s-numbers) with respect to the embedding of a weighted Wiener class $\mathcal{A}_\omega(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d$ into either the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ itself or into $L_p(\mathbb{T}^d)$, $2 \le p \le \infty$. Most interesting for us will be the case of dominating mixed smoothness. In this case the weight $\omega = \omega_{s,r}$ has a tensor product structure and the related spaces $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ are counterparts of the well-known Sobolev spaces $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ of dominating mixed smoothness $s > 0$. The parameter $r \in (0, \infty]$ does not influence the set of functions but the norm. Our motivation comes from some recent articles in high-dimensional approximation, see, e.g., [3], [10] and [11], which are dealing either with approximation of functions from $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ or with approximation in the norm of $\mathcal{A}(\mathbb{T}^d)$.

In specific situations we shall not only investigate the optimal order of the decay of the widths (as classically done) but we shall determine the asymptotic constant as well. This sheds some light not only on the dependence on n, but also on the dependence on s, r and in particular

*Corresponding author

on d. Our main results are the following. For $0 < s < \infty$, $0 < r \le \infty$ and $d \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{s_n\big(id:\, \mathcal{A}^{s,r}_{\rm mix}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} = \left[\frac{2^d}{(d-1)!}\right]^s,$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n\big(id:\mathcal{A}^{s,r}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} &= \lim_{n\to\infty} \frac{d_n\big(id:\mathcal{A}^{s,r}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} \\ &= \left(\frac{2s}{2s+1}\right)^s \left(\frac{2^d}{(d-1)!}\right)^s, \end{aligned} \tag{1.1}$$

and

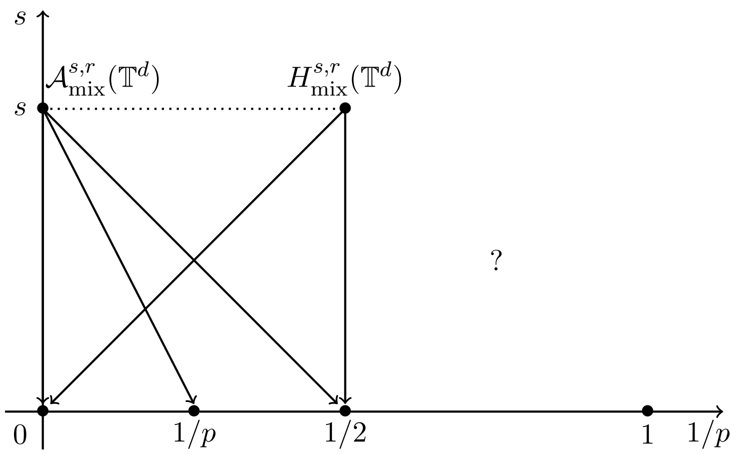
$$\begin{aligned} \lim_{n\to\infty} \frac{b_n\big(id:\mathcal{A}^{s,r}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s-1/2}(\ln n)^{s(d-1)}} &= \lim_{n\to\infty} \frac{x_n\big(id:\mathcal{A}^{s,r}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s-1/2}(\ln n)^{s(d-1)}} \\ &= \sqrt{2s+1}\left(\frac{2^d}{(d-1)!}\right)^s. \end{aligned}$$

Here $s_n \in \{a_n, b_n, d_n, x_n\}$. By a_n we denote the approximation numbers (linear widths), by b_n the Bernstein numbers, d_n refers to the Kolmogorov numbers and x_n to the Weyl numbers. From these relations it is immediate that in the first two cases the optimal order of the decay (asymptotic behaviour) is given by $n^{-s}\,(\ln n)^{s(d-1)}$, $n \in \mathbb{N}$, whereas in the third case we have $n^{-s-1/2}\,(\ln n)^{s(d-1)}$, $n \in \mathbb{N}$. With other words, for approximation numbers there is not much difference in the approximation with respect to the norm of the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$ or with respect to the norm in $L_2(\mathbb{T}^d)$, but for Bernstein and Weyl numbers it is. The value of these limits above we shall call asymptotic constant. In case of the target space $L_p(\mathbb{T}^d)$, $2 < p \le \infty$, we are not able to determine the asymptotic constant, we do not even no whether the limit exists. However, we will prove two-sided estimates for the approximation numbers (and sometimes also other widths) with constants independent of n and d.

Our method consists in a reduction of our original problem to the investigation of the widths of certain diagonal operators between sequence spaces. This is essentially standard and has been employed at many places.

The original motivation to consider the d-dependence of the behaviour of approximation numbers comes from the needs of numerical analysis in high dimensions. We refer to Bungartz, Griebel [1, Theorem 3.8] as well as to Schwab, Süli, and Todor [27]. In both papers the non-periodic situation is considered. These papers have found a lot of followers. Let us refer to Dinh Dũng, Ullrich [7], Chernov, Dinh Dũng [4], Krieg [14], Kühn, Mayer, Ullrich [15] Kühn, Sickel, Ullrich [16, 17, 18] and Cobos, Kühn, Sickel [5, 6]. In all these quoted papers the asymptotic behaviour as well as the preasymptotic behaviour of the approximation numbers of the embedding of a weighted Hilbert space $F_\omega(\mathbb{T}^d)$ either into $L_2(\mathbb{T}^d)$ or into $L_\infty(\mathbb{T}^d)$ have been investigated.

Let us give a short overview what is known about the d-dependence of certain n-widths at this moment. For simplicity we concentrate on approximation numbers.



Each arrow has to be interpreted as the existence of a sharp two-sided estimate for approximation numbers of a corresponding embedding with constants independent of n and d. Of course, it would be desirable to replace $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ by the classical (mixed-type) Hölder-Zygmund space $C^s_{\text{mix}}(\mathbb{T}^d)$ (or by the Nikol'skij-Besov space $S^s_{\infty,\infty}B(\mathbb{T}^d)$ of dominating mixed smoothness). However, with this respect we do not have a complete answer. Only for Bernstein and Weyl numbers and $m \in \mathbb{N}$ we know

$$\begin{aligned}\lim_{n\to\infty} \frac{b_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-m-\frac{1}{2}}(\ln n)^{m(d-1)}} &= \lim_{n\to\infty} \frac{x_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-m-\frac{1}{2}}(\ln n)^{m(d-1)}} \\ &= \sqrt{2m+1}\bigg(\frac{2^d}{(d-1)!}\bigg)^m .\end{aligned}$$

In addition, since we have sharp estimates for embeddings of $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d)$ into $L_2(\mathbb{T}^d)$ and into $L_\infty(\mathbb{T}^d)$, it is not a surprise that we are able to derive also sharp two-sided estimates for the L_p-case, $2 < p < \infty$. But for $1 \le p < 2$ we do not know so much, we refer to [6] for some results, mainly based on duality. Finally, we shall deal with the behaviour of s-numbers with respect to the embedding $\mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$. As a preparation for this we shall supplement the knowledge about the asymptotic behaviour of s-numbers of the embedding $H^{s,2}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$, see, e.g., [1], [7] and [18], by determining the asymptotic constant. Clearly, $H^1_{\text{mix}}(\mathbb{T}^d) \neq H^1(\mathbb{T}^d)$, in fact the embedding $H^1_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$ is proper. It holds

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+1}} &= \lim_{n\to\infty} \frac{d_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+1}} \\ &= \bigg(\frac{2s}{2s+1}\bigg)^s (2d)^{s-1}(2S+1)^{(s-1)(d-1)} ,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{b_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+1/2}} &= \lim_{n\to\infty} \frac{x_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+1/2}} \\ &= \sqrt{2s+1}\,(2d)^{s-1}(2S+1)^{(s-1)(d-1)} ,\end{aligned} \tag{1.2}$$

where $d \in \mathbb{N}$, $s > 1$ and

$$S := \sum_{k=1}^{+\infty} \frac{1}{(k^2+1)^{\frac{s}{2(s-1)}}} .$$

Those embeddings are of particular importance with respect to the numerical solution of the Poisson equation, we refer to [1]. Having a look onto the asymptotic constants in (1.1) and (1.2) then it becomes clear that there is no simple lifting argument behind. Indeed, this can not be expected because $H^1_{\text{mix}}(\mathbb{T}^d)$ belongs to a different scale of spaces than $H^1(\mathbb{T}^d)$. The latter space is not a tensor product space. So in the embedding $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$ we have a break in the scale.

It would be also of interest to study the isotropic situation. In this case the related spaces $\mathcal{A}^{s,r}(\mathbb{T}^d)$ are counterparts of the classical periodic Sobolev spaces $H^{s,r}(\mathbb{T}^d)$ on the d-dimensional torus. The behaviour of the s-numbers $(s_n)_n$ in dependence on n for the embedding $H^{s,r}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$ has been investigated at various places, but for investigations in dependence of n and d we refer to [16].

The paper is organized as follows. In Section 2 we collect the needed material about the functions spaces and the widths under consideration. The next Section 3 is devoted to the study of the widths $s_n(id: \ \mathcal{A}_\omega(\mathbb{T}^d) \to X(\mathbb{T}^d))$ for general weights ω and $X \in \{\mathcal{A}, L_2, L_\infty\}$. These results will be used in Section 4, where we deal with the particular family of weights associated to the dominating mixed smoothness. In the final Section 5 we shall determine the asymptotic constant for the embedding $\mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the non-negative integers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. By $\mathbb{T}$ we denote the torus, represented by the interval $[0, 2\pi]$, where the end points of the interval are identified. For a real number a we denote by $\lfloor a \rfloor$ the greatest integer not larger than a. The letter d is always reserved for the dimension in $\mathbb{N}^d$, $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$. For $0 < p \le \infty$ and $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote $|\boldsymbol{x}|_p = \big(\sum_{i=1}^d |x_i|^p\big)^{1/p}$ with the usual modification for $p = \infty$. If $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $\boldsymbol{x} \in \mathbb{C}^d$ we use $\boldsymbol{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^d x_i^{\alpha_i}$ with the convention $0^0 := 1$. Let Ω be a discrete set. Then the symbol $|\Omega|$ stands for the cardinality of the set Ω. If X and Y are two Banach spaces, the norm of an element x in X will be denoted by $\|x|X\|$ and the norm of an operator $A : X \to Y$ by $\|A : X \to Y\|$. The symbol $X \hookrightarrow Y$ indicates that there is a continuous embedding from X into Y. By id we denote always identity operators.

The equivalence $a_n \asymp b_n$ means that there are constants $0 < c_1 \le c_2 < \infty$ such that $c_1\, a_n \le b_n \le c_2\, a_n$ for all $n \in \mathbb{N}$. Sometimes we write $a_n \asymp_{s,q} b_n$, then we mean that the constants c_1 and c_2 depend on the parameters s and q only.

2 Preparations - widths and function spaces

First we recall the definition of various widths which will play a role in our investigations.

2.1 Widths

Let X, Y be Banach spaces and T be a continuous linear operator from X to Y, i.e., $T \in \mathcal{L}(X, Y)$. Then the nth approximation number of T is defined as

$$a_n(T) := \inf\big\{\|T - A : X \to Y\| : \ A \in \mathcal{L}(X,Y), \ \ \text{rank}(A) < n\big\}, \qquad n \in \mathbb{N}.$$

The nth Kolmogorov number of the linear operator T is defined as

$$d_n(T) = \inf_{L_{n-1}} \sup_{\|x|X\| \le 1} \inf_{y \in L_{n-1}} \|Tx - y|Y\|,$$

where the left-most infimum is taken over all $(n-1)$-dimensional subspaces L_{n-1} in Y.
The nth Weyl number of T is defined as

$$x_n(T) := \sup\big\{a_n(TA) : \ A \in \mathcal{L}(\ell_2, X), \ \|A\| \le 1\big\}, \qquad n \in \mathbb{N}.$$

The nth Bernstein number of T is defined as

$$b_n(T) = \sup_{L_n} \inf_{\substack{x \in L_n \\ \|x | X\| \le 1}} \|Tx \,|\, Y\|,$$

where the supremum is taken over all n-dimensional subspaces L_n in X.

There is a huge amount of literature about these numbers, their different roles in approximation and operator theory and applications. Approximation, Kolmogorov, Bernstein and Weyl numbers are s-numbers in the sense of Pietsch [21], [22, Section 11.1]. Later, in [23, Section 2.2], [25] Pietsch modified the definition of s-numbers and Bernstein numbers do not satisfy these new conditions, but the other still do. All these numbers have its own history. Good references are the monographs of Pietsch [22, 23] and of Pinkus [26].

Some properties

Many times we shall employ the ideal property, i.e., if $s = (s_n)_{n\in\mathbb{N}}$ is one of the above sequences, then

$$s_n(RST) \le \|R\| \, s_n(S) \, \|T\| \,, \qquad n \in \mathbb{N}\,, \tag{2.1}$$

holds, see [21], [22, Section 11.1]. Here R, S, T are linear operators satisfying $T \in \mathcal{L}(X_0, X)$, $S \in \mathcal{L}(X, Y)$, and $R \in \mathcal{L}(Y, Y_0)$ for certain Banach spaces X_0, X, Y_0, Y. Furthermore, the approximation numbers represent the largest s-numbers. For compact operators T it holds

$$b_n(T) \le d_n(T) \le a_n(T)\,, \qquad n \in \mathbb{N}\,.$$

We refer, e.g., to [21, Theorems 8.1, 8.2]. There is no general inequality relating Weyl and Bernstein numbers, Weyl and Kolmogorov numbers, see [25]. However, if $T: \ X \to Y$, where Y is a Hilbert space, then there is a relation between them. For any $A : \ell_2 \to X$ with $\|A\| \le 1$ we have $TA: \ \ell_2 \to Y$ and by Theorem 11.3.4 in [22] we conclude that

$$a_n(TA) = b_n(TA) \le \|A\| \cdot b_n(T : X \to Y) \le b_n(T : X \to Y)\,.$$

It follows immediately from the definition of the Weyl numbers that

$$x_n(T) \le b_n(T) \tag{2.2}$$

in this special situation. We also have if Y is a Hilbert space that

$$a_n(T) = d_n(T),$$

see [22, Proposition 11.6.2].

As it will be seen below, Bernstein numbers and Weyl numbers will behave similarly in our context and in the same way Approximation numbers and Kolmogorov numbers. Therefore we will use the following convention.

Convention: The notation $v_n(T)$ will be used instead of $x_n(T)$ and $b_n(T)$. The notation $u_n(T)$ will be used instead of $a_n(T)$ and $d_n(T)$. This means if we say v_n has properties x, y, z then both, Bernstein numbers and Weyl numbers have these properties etc..

2.2 Function spaces. I

Let $\mathbb{T} = [0, 2\pi]$ denote the torus and $\mathbb{T}^d$ be the $d-$dimensional torus. We equip $\mathbb{T}^d$ with the probability measure $(2\pi)^{-d}\mathrm{d}\boldsymbol{x}$. This is for convenience. It implies $|\mathbb{T}^d| = 1$ and this is needed for the theory developed in [5]. Furthermore it has the advantage that several related embeddings will have operator norm equal to 1. For a function $f \in L_1(\mathbb{T}^d)$, its Fourier coefficients are defined as

$$\hat{f}(\boldsymbol{k}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x}) e^{-\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{k} \in \mathbb{Z}^d.$$

Note, that the system $\{e^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}} : \boldsymbol{k} \in \mathbb{Z}^d\}$ is an orthonormal basis in $L_2(\mathbb{T}^d)$. Hence, it holds for any $f \in L_2(\mathbb{T}^d)$ that

$$\|f|L_2(\mathbb{T}^d)\|^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{k})|^2 .$$

Let $\omega = (\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a sequence of positive real numbers. Those sequences we will call a weight in what follows. We introduce the weighted Wiener class $\mathcal{A}_\omega(\mathbb{T}^d)$ as the collection of all functions $f \in L_1(\mathbb{T}^d)$ such that

$$\|f|\mathcal{A}_\omega(\mathbb{T}^d)\| := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega(\boldsymbol{k})|\hat{f}(\boldsymbol{k})| < \infty .$$

In case $\omega(\boldsymbol{k}) = 1$ for all $\boldsymbol{k} \in \mathbb{Z}^d$ we get back the classical Wiener algebra $\mathcal{A}(\mathbb{T}^d)$.

As a second scale of function spaces we introduce some weighted Hilbert spaces built on $L_2(\mathbb{T}^d)$. The class $F_\omega(\mathbb{T}^d)$ which is the collection of integrable functions $f \in L_1(\mathbb{T}^d)$ such that

$$\|f|F_\omega(\mathbb{T}^d)\| := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega(\boldsymbol{k})^2|\hat{f}(\boldsymbol{k})|^2 \right)^{1/2} < \infty .$$

For later use we remark that

$$\| \, f \,|F_\omega(\mathbb{T}^d)\| \le \| \, f \,|\mathcal{A}_\omega(\mathbb{T}^d)\| \,, \tag{2.3}$$

or with other words, the norm of $id : \mathcal{A}_\omega(\mathbb{T}^d) \to F_\omega(\mathbb{T}^d)$ is equal to 1. In what follows we suppose

$$\lim_{|\boldsymbol{k}|\to\infty} |\omega(\boldsymbol{k})| = \infty . \tag{2.4}$$

Then an important role will be played by the non-increasing rearrangement of the sequence $(1/\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$, denoted by $(\sigma_n)_{n\in\mathbb{N}}$. Observe that (2.4) implies on the one side the compactness of the embedding $F_\omega(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d))$ and on the other side $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 2.1. *Let $(\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a sequence of positive real numbers such that (2.4) is satisfied. Let $s_n \in \{x_n, b_n, d_n, a_n\}$. Then we have*

$$s_n\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) = \sigma_n, \qquad n \in \mathbb{N} .$$

This lemma represents nothing but a convenient reformulation of a classical result on the behaviour of approximation numbers of certain diagonal operators, see e.g. Pietsch [22, Theorem 11.3.2.], König [13, Section 1.b], Pinkus [26, Theorem IV.2.2], or Novak and Woźniakowski [19, Corollary 4.12]. Comments on the history may be found in Pietsch [24, 6.2.1.3]. The lemma has been used already in several of our earlier papers devoted to this subject, see, e.g. [16, 17, 5]. There it is formulated for approximation numbers only. However, we are in a Hilbert space case. There all theses numbers coincide, see [21] and [22, Theorem 11.3.4].

3 Widths of weighted Wiener classes - general results

Essentially we will consider two different cases, namely $s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$ and $s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)$, where $s_n \in \{a_n, b_n, d_n, x_n\}$. In addition a supplement to the case $id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big)$ will be given.

3.1 Widths of $id: \ \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)$

The main result in this subsection will be an surprising identity, see (3.5) below, very easy to prove. But before we have to calculate $s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$.

Theorem 3.1. *Let $\omega = (\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a weight satisfying (2.4). Let $s_n \in \{x_n, b_n, d_n, a_n\}$.*
(i) *Then we have*

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) = \sigma_n, \qquad n \in \mathbb{N}\,. \tag{3.1}$$

(ii) *Let $n \in \mathbb{N}$. Let $\Lambda \subset \mathbb{Z}^d$ be a set with the two properties*

- *The cardinality $|\Lambda|$ of Λ is $n-1$.*
- *For any $\boldsymbol{k} \in \Lambda$ and any $\boldsymbol{l} \notin \Lambda$ it holds $\omega(\boldsymbol{k}) \le \omega(\boldsymbol{l})$.*

For $f \in L_1(\mathbb{T}^d)$ we define

$$S_\Lambda f(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\Lambda} \hat{f}(\boldsymbol{k})\, e^{i\boldsymbol{k}\boldsymbol{x}}\,, \qquad \boldsymbol{x} \in \mathbb{T}^d\,.$$

Then

$$a_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) = \sup_{\|f\,|\mathcal{A}_\omega(\mathbb{T}^d)\|\le 1} \|\, f - S_\Lambda f\,|\mathcal{A}(\mathbb{T}^d)\|\,.$$

Proof. Step 1. Proof of (i). We consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{\ id\ } & \mathcal{A}(\mathbb{T}^d) \\ \big\downarrow A & & \big\uparrow B \\ \ell_1(\mathbb{Z}^d) & \xrightarrow{\ D_\omega\ } & \ell_1(\mathbb{Z}^d)\,, \end{array}$$

where the linear operators $A: \ \mathcal{A}_\omega(\mathbb{T}^d) \to \ell_1(\mathbb{Z}^d)$, $B: \ \ell_1(\mathbb{Z}^d) \to \mathcal{A}(\mathbb{T}^d)$ and $D_\omega: \ \ell_1(\mathbb{Z}^d) \to \ell_1(\mathbb{Z}^d)$ are defined as follows:

$$Af \ := \ (\omega(\boldsymbol{k})\hat{f}(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}\,, \tag{3.2}$$

$$D_\omega\xi \ := \ (\xi_{\boldsymbol{k}}/\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}\,, \qquad \xi = (\xi_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d} \tag{3.3}$$

$$(B\xi)(\boldsymbol{x}) \ := \ \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \xi_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\boldsymbol{x}}\,, \qquad \boldsymbol{x} \in \mathbb{T}^d\,. \tag{3.4}$$

It is obvious that $\|A\| = \|B\| = 1$. By the ideal property (2.1) and the identity $id = B\, D_\omega\, A$ it follows

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) \le s_n\big(D_\omega : \ell_1(\mathbb{Z}^d) \to \ell_1(\mathbb{Z}^d)\big)\,.$$

Theorem 7.1 in [21], see also Proposition 2.9.5 in [23], yields

$$s_n\big(D_\omega : \ell_1(\mathbb{Z}^d) \to \ell_1(\mathbb{Z}^d)\big) = \sigma_n\,.$$

This proves the estimate from above. Now we employ the same type of arguments with respect to the diagram

$$\begin{array}{ccc} \ell_1(\mathbb{Z}^d) & \xrightarrow{D_\omega} & \ell_1(\mathbb{Z}^d) \\ \Big\downarrow A^{-1} & & \Big\uparrow B^{-1} \\ \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{id} & \mathcal{A}(\mathbb{T}^d)\,. \end{array}$$

It is easy to see that the operators A and B are invertible and that $\|A^{-1}\| = \|B^{-1}\| = 1$. As above we conclude

$$\sigma_n = s_n\big(D_\omega : \ell_1(\mathbb{Z}^d) \to \ell_1(\mathbb{Z}^d)\big) \le s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)\,,$$

which completes the proof of (i).
Step 2. Proof of (ii). By definition of the sequence $(\sigma_n)_n$ we conclude

$$\begin{aligned} \|\, f - S_\Lambda f \,|\mathcal{A}(\mathbb{T}^d)\| &= \sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus\Lambda} |\hat{f}(\boldsymbol{k})| \le \sup_{\boldsymbol{l}\in\mathbb{Z}^d\setminus\Lambda} \frac{1}{\omega(\boldsymbol{l})} \Bigg(\sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus\Lambda} |\omega(\boldsymbol{k})\,\hat{f}(\boldsymbol{k})|\Bigg) \\ &\le \sigma_n \,\|\, f\,|A_\omega(\mathbb{T}^d)\|\,. \end{aligned}$$

This proves

$$\|\, id - S_\Lambda \,|\mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\| \le \sigma_n = a_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)\,.$$

But by definition of a_n we know $a_n \le \|\, id - S_\Lambda \,|\mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\|$. This proves (ii). □

Corollary 3.2. *Let $\omega = (\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a weight satisfying* (2.4). *Let $s_n \in \{x_n, b_n, d_n, a_n\}$. Then, for all $n \in \mathbb{N}$, we have*

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) = s_n\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)\,. \tag{3.5}$$

Furthermore, in both cases the operator S_Λ, defined in Theorem 3.1(ii), realizes the optimal approximation, i.e.,

$$\|\, id - S_\Lambda \,|\mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\| = a_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$$

and

$$\|\, id - S_\Lambda \,|F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\| = a_n\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)\,.$$

Proof. Lemma 2.1 and Theorem 3.1 yield the first part of the claim. The second part is a consequence of Theorem 3.1(ii) and Remark 4.2 in [16]. □

3.2 Widths of $id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$

Now we turn to the L_2-case. Here we need our convention. Weyl- and Bernstein numbers have a different behaviour than approximation and Kolmogorov numbers.

Theorem 3.3. *Let $\omega = (\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a weight satisfying* (2.4).
(i) *In case of Weyl and Bernstein numbers we have*

$$v_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) = \left(\sum_{k=1}^{n} \sigma_n^{-2}\right)^{-1/2}, \qquad n \in \mathbb{N}\,.$$

(ii) *For approximation and Kolmogorov numbers it holds*

$$u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) = \sup_{h\ge n} \left(\frac{h-n+1}{\sum_{k=1}^{h} \sigma_k^{-2}} \right)^{1/2}, \qquad n \in \mathbb{N}.$$

Proof. The proof is quite similar to the proof of Theorem 3.1. Again let $s_n \in \{x_n, b_n, d_n, a_n\}$. We consider the following diagram

$$\begin{array}{ccc} \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{\ id\ } & L_2(\mathbb{T}^d) \\ \Big\downarrow A & & \Big\uparrow B \\ \ell_1(\mathbb{Z}^d) & \xrightarrow{\ D_\omega\ } & \ell_2(\mathbb{Z}^d) \end{array}$$

where A, B and D_ω are defined as in (3.2)-(3.4). Again we have $\|A\| = \|B\| = 1$. By the ideal property of these numbers, see (2.1), we obtain

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \le\ s_n\big(D_\omega : \ell_1(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)\big)\,.$$

Let D_σ be the diagonal operator from $\ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})$ defined by $D_\sigma \xi := (\sigma_k\, \xi_k)_{k\in\mathbb{N}}$. Since $(\sigma_n)_{n\in\mathbb{N}}$ is the non-increasing rearrangement of $(1/\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$ we obtain

$$s_n\big(D_\omega : \ell_1(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)\big) =\ s_n\big(D_\sigma : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})\big)\,,$$

which leads to

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \le s_n\big(D_\sigma : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})\big). \tag{3.6}$$

To prove the reverse direction we consider the modified diagram

$$\begin{array}{ccc} \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{\ id\ } & L_2(\mathbb{T}^d) \\ \Big\uparrow A^{-1} & & \Big\downarrow B^{-1} \\ \ell_1(\mathbb{Z}^d) & \xrightarrow{\ D_\omega\ } & \ell_2(\mathbb{Z}^d)\,. \end{array}$$

By the same argument as used above we get the reverse inequality of (3.6). Consequently we find that

$$s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) = s_n\big(D_\sigma : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})\big).$$

Finally, from

$$v_n\big(D_\sigma : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})\big) = \left(\sum_{k=1}^{n} \sigma_k^{-2} \right)^{-1/2}$$

see [21, 9] and

$$u_n\big(D_\sigma : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})\big) = \sup_{h\ge n} \left(\frac{h-n+1}{\sum_{k=1}^{h} \sigma_k^{-2}} \right)^{1/2},$$

[22, Theorem 11.11.7], we obtain the desired results. □

Theorem 3.4. *Let $s > 0$, $d \in \mathbb{N}$, $\beta \ge 0$ and let ω be a weight satisfying (2.4). Assume that there exists a real number C such that*

$$\lim_{n\to\infty} \frac{s_n\big(id :\ F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^\beta} = C\,. \tag{3.7}$$

(i) *In case of Bernstein or Weyl numbers we have*

$$\lim_{n\to\infty} \frac{v_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s-\frac{1}{2}}(\ln n)^\beta} = \sqrt{2s+1}\, C\,.$$

(ii) *For approximation or Kolmogorov numbers it holds*

$$\lim_{n\to\infty} \frac{u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^\beta} = \Big(\frac{2s}{2s+1}\Big)^s C\,.$$

Proof. Step 1. Proof of (i). From Theorem 3.3 we have

$$v_n^{-2} := v_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)^{-2} = \sum_{k=1}^n \sigma_k^{-2}\,,$$

where $(\sigma_k)_{k\in\mathbb{N}}$ is a non-increasing rearrangement of $(1/\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$. By Lemma 2.1 it follows $\sigma_k = s_k\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)$. Assumption (3.7) indicates that for any $\varepsilon > 0$ there exists $n_0 := n_0(\varepsilon) \in \mathbb{N}$ such that for all $n > n_0$ we have

$$\Big|\frac{\sigma_n}{n^{-s}(\ln n)^\beta} - C\Big| \le \varepsilon\,,$$

or equivalently

$$C - \varepsilon \le \frac{\sigma_n}{n^{-s}(\ln n)^\beta} \le \varepsilon + C \tag{3.8}$$

which implies

$$v_n^{-2} \le \sum_{k=1}^{n_0} \sigma_k^{-2} + \big(C-\varepsilon\big)^{-2} \sum_{k=n_0+1}^{n} k^{2s}(\ln k)^{-2\beta}\,.$$

Observe that $f(t) = t^{2s}(\ln t)^{-2\beta}$ is an increasing function if $t \ge t_0 > 0$ for some appropriate $t_0 = t_0(s)$. Estimating the sum by an integral and afterwards changing variable we find

$$\begin{aligned}\sum_{k=n_0+1}^{n} k^{2s}(\ln k)^{-2\beta} &\le \int_{n_0+1}^{n+1} t^{2s}(\ln t)^{-2\beta}\mathrm{d}t\\ &= \frac{(n+1)^{2s+1}}{(\ln(n+1))^{2\beta}} \int_{n_0+1}^{n+1} \Big(\frac{t}{n+1}\Big)^{2s+1}\Big(\frac{\ln(n+1)}{\ln t}\Big)^{2\beta}\frac{\mathrm{d}t}{t}\\ &= \frac{(n+1)^{2s+1}}{(\ln(n+1))^{2\beta}} \int_{\frac{n_0+1}{n+1}}^{1} y^{2s}\Big(\frac{\ln(n+1)}{\ln(y(n+1))}\Big)^{2\beta}\mathrm{d}y\,.\end{aligned}$$

We claim that

$$\lim_{n\to\infty} \int_{\frac{n_0+1}{n+1}}^{1} y^{2s}\Big(\frac{\ln(n+1)}{\ln(yn+y)}\Big)^{2\beta}\mathrm{d}y = \frac{1}{2s+1}\,.$$

This is just a particular case of Lemma 3.5 below. Choosing $n \ge n_1$ large enough such that

$$\int_{\frac{n_0+1}{n+1}}^{1} y^{2s}\Big(\frac{\ln(n+1)}{\ln(yn+y)}\Big)^{2\beta}\mathrm{d}y \le \frac{1+\varepsilon}{2s+1} \qquad \text{and} \qquad \frac{(\ln(n+1))^{2\beta}}{(n+1)^{2s+1}}\sum_{k=1}^{n_0}\sigma_k^{-2} \le \varepsilon$$

we finally obtain that

$$v_n^{-2} = \sum_{k=1}^n \sigma_k^{-2} \le \frac{(n+1)^{2s+1}}{(\ln(n+1))^{2\beta}}\Big[\varepsilon + \big(C-\varepsilon\big)^{-2}\Big(\frac{1+\varepsilon}{2s+1}\Big)\Big] \tag{3.9}$$

for all $n \geq n_1$. The other direction is carried out similarly.
Step 2. Proof of (ii). We obtain from Theorem 3.3 that

$$u_n^2 := u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)^2 = \sup_{h \geq n} \left(\frac{h-n+1}{\sum_{k=1}^h \sigma_k^{-2}} \right), \qquad n \in \mathbb{N}.$$

Substep 2.1. Estimate from below. As a consequence of (3.9) we have for any $h \geq n \geq n_1$

$$u_n^2 \geq \frac{h-n+1}{\sum_{k=1}^h \sigma_k^{-2}} \geq (h-n+1)\frac{(\ln(h+1))^{2\beta}}{(h+1)^{2s+1}} \left[\varepsilon + (C-\varepsilon)^{-2}\left(\frac{1+\varepsilon}{2s+1}\right)\right]^{-1}.$$

Choosing $h \in \mathbb{N}$ and $\theta \in [0,1)$ such that $h+1 = (1+\frac{1}{2s})n + \theta$ we conclude

$$u_n^2 \geq \left(\frac{n}{2s} + \theta\right) \frac{\left[\ln\left((1+\frac{1}{2s})n+\theta\right)\right]^{2\beta}}{\left[(1+\frac{1}{2s})n+\theta\right]^{2s+1}} \left[\varepsilon + (C-\varepsilon)^{-2}\left(\frac{1+\varepsilon}{2s+1}\right)\right]^{-1}$$

or equivalently

$$\frac{u_n^2}{n^{-2s}(\ln n)^{2\beta}} \geq \frac{n^{2s}(\frac{n}{2s}+\theta)}{(\ln n)^{2\beta}} \frac{\left[\ln\left((1+\frac{1}{2s})n+\theta\right)\right]^{2\beta}}{\left[(1+\frac{1}{2s})n+\theta\right]^{2s+1}} \left[\varepsilon + (C-\varepsilon)^{-2}\left(\frac{1+\varepsilon}{2s+1}\right)\right]^{-1}.$$

Note that

$$\lim_{n\to\infty} \left[\frac{n^{2s}(\frac{n}{2s}+\theta)}{(\ln n)^{2\beta}} \frac{\left[\ln\left((1+\frac{1}{2s})n+\theta\right)\right]^{2\beta}}{\left[(1+\frac{1}{2s})n+\theta\right]^{2s+1}} \right] = \frac{1}{2s}\left(\frac{2s}{1+2s}\right)^{2s+1}$$

which leads to

$$\frac{u_n^2}{n^{-2s}(\ln n)^{2\beta}} \geq \left[\frac{1}{2s}\left(\frac{2s}{1+2s}\right)^{2s+1} - \varepsilon\right]\left[\varepsilon + (C-\varepsilon)^{-2}\left(\frac{1+\varepsilon}{2s+1}\right)\right]^{-1} \tag{3.10}$$

for all $n \geq n_2$.
Substep 2.2. The estimate from above. From (3.8) we derive

$$\frac{h-n+1}{\sum_{k=1}^h \sigma_k^{-2}} \leq \frac{h-n+1}{\sum_{k=n_0+1}^h \sigma_k^{-2}} \leq \frac{h-n+1}{\sum_{k=n_0+1}^h k^{2s}(\ln k)^{-2\beta}} (C+\varepsilon)^2.$$

Similarly as in Step 1 we obtain

$$\sum_{k=n_0+1}^h k^{2s}(\ln k)^{-2\beta} \geq \int_{n_0}^h \frac{t^{2s}}{(\ln t)^{2\beta}} \mathrm{d}t \geq \frac{h^{2s+1}}{(\ln h)^{2\beta}} \int_{\frac{n_0}{h}}^1 t^{2s}\left(\frac{\ln h}{\ln(th)}\right)^{2\beta} \mathrm{d}t \geq \frac{h^{2s+1}}{(\ln h)^{2\beta}} \frac{1-\varepsilon}{2s+1},$$

where we used Lemma 3.5 below in the last step. As a consequence we have

$$\frac{h-n+1}{\sum_{k=1}^h \sigma_k^{-2}} \leq \frac{(h-n+1)(\ln h)^{2\beta}}{h^{2s+1}} (C+\varepsilon)^2 \left(\frac{1-\varepsilon}{2s+1}\right)^{-1}.$$

Considering the function

$$g(h) := \frac{h-n+1}{h^{2s+1}}(\ln h)^{2\beta}, \qquad h \in [n,\infty)$$

we have

$$g'(h) = \left(\frac{-2sh + (n-1)(2s+1)}{h^{2s+2}}\right)(\ln h)^{2\beta} + \left(\frac{h-n+1}{h^{2s+2}}\right) 2\beta (\ln h)^{2\beta-1}.$$

Hence $g'(h) = 0$ is equivalent to

$$\big[-2sh + (n-1)(2s+1)\big] \ln h + \big(h - n + 1\big) 2\beta = 0. \tag{3.11}$$

We put

$$f(h) := \big[-2sh + (n-1)(2s+1)\big] \ln h + \big(h - n + 1\big) 2\beta \, , \qquad h \in [n, \infty).$$

It follows

$$f'(h) = -2s \ln h - 2s + \frac{(n-1)(2s+1)}{h} + 2\beta < -2s \ln h + 1 + 2\beta \, .$$

This implies that $f' < 0$ if $n > e^{(1+2\beta)/(2s)}$. Observe that

$$\begin{aligned} f\bigg(\Big(1+\frac{1}{2s}\Big)(n-1)\bigg) &= \frac{2\beta\,(n-1)}{2s} \geq 0 \\ f\bigg(\Big(1+\frac{1}{s}\Big)(n-1)\bigg) &= -(n-1)\ln\bigg(\Big(1+\frac{1}{s}\Big)(n-1)\bigg) + \frac{2\beta\,(n-1)}{s} < 0 \end{aligned}$$

for $n \geq n_0$ depending on s and β. From this we conclude that the equation (3.11) has a unique solution for any $n \geq n_0$ and this solution belongs to the interval

$$I_n = \bigg[\Big(1+\frac{1}{2s}\Big)(n-1) - 1, \ \Big(1+\frac{1}{s}\Big)(n-1) + 1 \bigg].$$

Consequently, we obtain

$$\begin{aligned} \sup_{h \geq n} \left(\frac{1}{n^{-2s}(\ln n)^{2\beta}} \frac{h-n+1}{\sum_{k=1}^{h} \sigma_k^{-2}} \right) &\leq \sup_{h \in I_n} \left(\frac{(h-n+1)(\ln h)^{2\beta}}{h^{2s+1} n^{-2s} (\ln n)^{2\beta}} \right) \big(C+\varepsilon\big)^2 \left(\frac{1-\varepsilon}{2s+1} \right)^{-1} \\ &\leq \sup_{h \in \mathbb{R},\, h \geq n} \left(\frac{h-n+1}{h^{2s+1} n^{-2s}} \right) \big(1+\varepsilon\big) \big(C+\varepsilon\big)^2 \left(\frac{1-\varepsilon}{2s+1} \right)^{-1} \end{aligned}$$

as long as $n \geq n_1$. It is easy to see that the function $g_1(h) := \frac{h-n+1}{h^{2s+1}}$, $h \in [n, \infty)$, attains its maximum at $h = \big(1 + \frac{1}{2s}\big)(n-1)$. Inserting this into the above estimate we find

$$\sup_{h \geq n} \left(\frac{1}{n^{-2s}(\ln n)^{2\beta}} \frac{h-n+1}{\sum_{k=1}^{h} \sigma_k^{-2}} \right) \leq \frac{n^{2s}(1+\varepsilon)(C+\varepsilon)^2}{2s\big(1+\frac{1}{2s}\big)^{2s+1}(n-1)^{2s}} \left(\frac{1-\varepsilon}{2s+1} \right)^{-1} . \tag{3.12}$$

Taking the limits $n \to \infty$ and afterwards $\varepsilon \downarrow 0$ in (3.10) and (3.12) the claimed result for $d > 1$ follows. However, the modifications needed for $d = 1$ are obvious. □

Lemma 3.5. *Let $s > 0$, $a > 1$, and $\beta \geq 0$. Then we have*

$$\lim_{n \to \infty} \int_{\frac{a}{n}}^{1} y^s \left(\frac{\ln n}{\ln(yn)} \right)^{\beta} \mathrm{d}y = \frac{1}{s+1}.$$

Proof. We consider the sequence of functions

$$f_n(y) = y^s \left(\frac{\ln n}{\ln(yn)} \right)^{\beta} \chi_{[a/n,1]} \, , \qquad y \in (0,1) \, ,$$

where $\chi_{[a/n,1]}$ is the characteristic function of $[a/n, 1]$. It is clear that this sequence converges pointwise to $f(y) = y^s$ on $(0,1)$. Now we turn to the existence of a common majorant. The case $\beta = 0$ is obvious. So we concentrate on $\beta > 0$. Then the derivative of f_n on $(a/n, 1)$ is given by

$$f_n'(y) = y^{s-1} \left(\frac{\ln n}{\ln(yn)} \right)^{\beta} \left[s - \frac{\beta}{\ln(ny)} \right] .$$

This function has at most one sign change in $(a/n, 1)$. Hence, the maximal value of f_n with respect to the interval $[a/n, 1]$ is attained either in a/n, 1 or $e^{\beta/s}/n$. This implies

$$\max_{a/n \le y \le 1} |f_n(y)| = \max \left\{ \left(\frac{a}{n}\right)^s \left(\frac{\ln n}{\ln a}\right)^{\beta}, 1, \frac{s}{\beta}\, e^{\beta}\, \frac{\ln^{\beta} n}{n^s} \right\} .$$

Summarizing we found that there exists a constant $C_{a,s,\beta} > 0$ such that

$$\sup_{y \in (0,1)} |f_n(y)| \le C_{a,s,\beta}$$

holds. Hence, the desired result follows from Lebesgue's dominated convergence theorem. □

3.3 A supplement - the widths of $id : \ \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)$

The Hilbert space $F_\omega(\mathbb{T}^d)$ is continuously embedded into $\mathcal{A}(\mathbb{T}^d)$ if and only if $\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \omega(\boldsymbol{k})^{-2}$ is finite, see [5]. Under this restriction it has been proved in [5] that the approximation numbers of embeddings of the Hilbert spaces $F_\omega(\mathbb{T}^d)$ have the following property:

$$a_n\big(id : F_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) = a_n\big(id : F_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) , \qquad n \in \mathbb{N} . \tag{3.13}$$

Now we would like to understand whether a similar property is true for the approximation numbers of embeddings of the weighted Wiener classes $\mathcal{A}_\omega(\mathbb{T}^d)$. As a preparation for later considerations we mention the following partial result.

Lemma 3.6. *Let $\omega = (\omega(\boldsymbol{k}))_{\boldsymbol{k} \in \mathbb{Z}^d}$ be a weight satisfying (2.4). Then we have*

$$\sup_{h \ge n} \left(\frac{h - n + 1}{\sum_{k=1}^{h} \sigma_k^{-2}} \right)^{1/2} \le u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) \le \sigma_n \, , \qquad n \in \mathbb{N}.$$

Proof. We consider the following chain of embeddings

$$\mathcal{A}_\omega(\mathbb{T}^d) \hookrightarrow \mathcal{A}(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d).$$

Then the lower bound follows from Theorem 3.3 and

$$\begin{aligned} u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) &\le \big\|id : L_\infty(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big\| \cdot u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) \\ &= u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) \, . \end{aligned}$$

For the upper bound we have

$$\begin{aligned} u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) &\le \big\|id : \mathcal{A}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big\| \cdot u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) \\ &= u_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) \, . \end{aligned}$$

Finally we use (3.1) to get the desired result. □

Remark 3.7. In some special situations the quantities

$$\sup_{h\geq n}\left(\frac{h-n+1}{\sum_{k=1}^{h}\sigma_k^{-2}}\right)^{1/2} \qquad \text{and} \qquad \sigma_n$$

will be of the same order. In such a situation we will not be able to show the existence of an asymptotic constant, however, we will get information about the correct order of the decay of the $(u_n)_n$.

All the results we obtained so far have the disadvantage that the sequence $(\sigma_n)_{n\in\mathbb{N}}$ shows up. Below we shall discuss a special family of weights $\omega_{s,r}$ and the corresponding spaces $\mathcal{A}_{\omega_{s,r}}(\mathbb{T}^d)$ and $F_{\omega_{s,r}}(\mathbb{T}^d)$. Then we will be able to remove this dependence. The nonincreasing rearrangement $(\sigma_n)_n$ of our weight will be replaced by explicit quantities in n, s, d and r.

4 Spaces of dominating mixed smoothness

In this subsection we shall deal with the family of weights

$$\begin{aligned}\omega_{s,r}(\boldsymbol{k}) &:= \prod_{i=1}^{d}\big(1+|k_i|^r\big)^{s/r}, \qquad 0<r<\infty,\\ \omega_{s,r}(\boldsymbol{k}) &:= \prod_{i=1}^{d}\max(1,|k_i|)^s, \qquad r=\infty,\end{aligned}$$

$\boldsymbol{k}\in\mathbb{Z}^d$. Here the parameter s satisfies $0<s<\infty$. We shall use the notation $\mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) := \mathcal{A}_{\omega_{s,r}}(\mathbb{T}^d)$ and $H^{s,r}_{\text{mix}}(\mathbb{T}^d) := F_{\omega_{s,r}}(\mathbb{T}^d)$, respectively. In both cases, for different r, we obtain the same sets of functions. A change of the parameter r leads to a change of the quasinorm only.

The classes $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ are well-known in approximation theory. They are called periodic Sobolev spaces of dominating mixed smoothness. The space $H^{s,r}_{\text{mix}}(\mathbb{T}^d)$ is just a tensor product space of the univariate Sobolev spaces $H^{s,r}(\mathbb{T})$, i.e.,

$$H^{s,r}_{\text{mix}}(\mathbb{T}^d) = H^{s,r}(\mathbb{T})\otimes\ldots\otimes H^{s,r}(\mathbb{T})$$

(to be understood as the iterated tensor product of d Hilbert spaces). Let $s=m\in\mathbb{N}$. We define the space $H^m_{\text{mix}}(\mathbb{T}^d)$ to be the collection of all functions $f\in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^{\boldsymbol{\alpha}}f$ with $|\boldsymbol{\alpha}|_\infty\leq m$ belong to $L_2(\mathbb{T}^d)$ equipped with the norm

$$\|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\| := \left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d,\, |\boldsymbol{\alpha}|_\infty\leq m}\|\, D^{\boldsymbol{\alpha}}f\,|L_2(\mathbb{T}^d)\|^2\right)^{1/2}.$$

Then $H^m_{\text{mix}}(\mathbb{T}^d) = H^{m,r}_{\text{mix}}(\mathbb{T}^d)$ for all r in the sense of equivalent quasinorms. If $m=1$, then we have $\|\cdot|H^{1,2}_{\text{mix}}(\mathbb{T}^d)\| = \|\cdot|H^1_{\text{mix}}(\mathbb{T}^d)\|$. If $m\geq 2$, then the norm $\|\cdot|H^m_{\text{mix}}(\mathbb{T}^d)\|$ itself does not belong to the family of norms $\|\cdot|H^{m,r}_{\text{mix}}(\mathbb{T}^d)\|$, $0<r\leq\infty$. But the choice $r=2m$ leads to the following standard norm

$$\|\, f\,|H^{m,2m}_{\text{mix}}(\mathbb{T}^d)\| = \left(\sum_{\boldsymbol{\alpha}\in\{0,m\}^d}\|\, D^{\boldsymbol{\alpha}}f\,|L_2(\mathbb{T}^d)\|^2\right)^{1/2}, \tag{4.1}$$

see [17]. For a recent survey on the behaviour of widths of embeddings of these spaces into $L_p(\mathbb{T}^d)$ we refer to [8].

There is a different class of weights which would be of interest. For $0 < s < \infty$ and $0 < r \le \infty$, let

$$\begin{aligned} \widetilde{\omega}_{s,r}(\boldsymbol{k}) &:= \Big(1 + \sum_{i=1}^d |k_i|^r\Big)^{s/r}, \qquad 0 < r < \infty, \\ \widetilde{\omega}_{s,r}(\boldsymbol{k}) &:= \max\big(1, |k_1|, \ldots, |k_d|\big)^s, \qquad r = \infty, \end{aligned}$$

$\boldsymbol{k} \in \mathbb{Z}^d$. The corresponding Sobolev spaces $H^{s,r}(\mathbb{T}^d)$ are the standard periodic Sobolev spaces of fractional smoothness s, the classes $\mathcal{A}^{s,r}(\mathbb{T}^d)$ would be the natural counterparts. The investigation of the behaviour of the s-numbers for the embeddings $\mathcal{A}^{s,r}(\mathbb{T}^d) \hookrightarrow \mathcal{A}(\mathbb{T}^d)$ and $\mathcal{A}^{s,r}(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d)$ will be postponed.

4.1 Widths of $id: \ \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)$

Here we are in a convenient situation. Based on [17, 18] and (3.5) all hard work is already done. For convenience of the reader we recall some basic facts.

Corollary 4.1. *Let $0 < s < \infty$, $0 < r \le \infty$ and $d \in \mathbb{N}$. Let $s_n \in \{x_n, b_n, d_n, a_n\}$. Then*

$$\lim_{n\to\infty} \frac{s_n\big(id: \ \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{(d-1)s}} = \left[\frac{2^d}{(d-1)!}\right]^s. \tag{4.2}$$

Proof. The proof is a direct consequence of a corresponding result for $a_n(id: \ H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see [17], [18], and Corollary 3.2. □

The asymptotic behaviour of the numbers $s_n\big(id: \ \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$ is quite different from the behaviour for small n, say $n \le 2^d$. This range is called the preasymptotic region. In case d is large this is the only region where such a result could be of practical relevance (e.g., as a benchmark).

Corollary 4.2. *Let $d \ge 3$, $0 < s < \infty$ and $1 \le r < \infty$. Let $s_n \in \{x_n, b_n, d_n, a_n\}$. For all $n \ge 2$ it holds*

$$s_n\big(id: \ \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) \le \left(\frac{C(d)}{n}\right)^{\frac{s}{r\,(1+\log_2(d-1))}},$$

where $C(d)$ is defined as

$$C(d) := \left[1 + \frac{1}{d-1}\Big(1 + \frac{2}{\log_2(d-1)}\Big)\right]^{d-1}, \qquad d \ge 3.$$

Proof. Again the proof follows from Corollary 3.2 and a corresponding result for $a_n(id: \ H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see [17, 18]. □

Obviously $\lim_{d\to\infty} C(d) = e$. That makes clear that in the preasymptotic range there is an essential change in the approximation properties. The asymptotically optimal rate $n^{-s}(\ln n)^{s(d-1)}$ is replaced by $n^{-\frac{s}{r\,(1+\log_2(d-1))}}$. As larger as d, as smaller is the convergence rate. In addition there is a tremendous influence of the special norm chosen, i.e., of the parameter r. If $r \to \infty$, then the rate is getting worse. Here is the result in the limiting situation $r = \infty$.

Corollary 4.3. *Let $d \in \mathbb{N}$ and $0 < s < \infty$. Let $s_n \in \{x_n, b_n, d_n, a_n\}$. Then*

$$s_n\big(id: \ \mathcal{A}^{s,\infty}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big) = 1, \qquad n = 1, 2, \ldots, 3^d.$$

Proof. The proof follows from Corollary 3.2 and a corresponding result for $a_n(id: \ H^{s,\infty}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see [18, Theorem 4.3]. □

4.2 Widths of $id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$

We recall a further result obtained in [17].

Proposition 4.4. *Let $0 < s < \infty$ and $0 < r \le \infty$. Let $s_n \in \{x_n, b_n, d_n, a_n\}$, $n \in \mathbb{N}$. Then it holds*

$$\lim_{n\to\infty} \frac{s_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} = \left(\frac{2^d}{(d-1)!}\right)^s .$$

In [17] this is formulated for a_n only. But we are in a Hilbert space situation, see Lemma 2.1. Theorem 3.4 and Proposition 4.4 and will be used to derive the following.

Theorem 4.5. *Let $0 < s < \infty$ and $0 < r \le \infty$.*
(i) *In case of Bernstein or Weyl numbers we have*

$$\lim_{n\to\infty} \frac{v_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s-\frac{1}{2}}(\ln n)^{s(d-1)}} = \sqrt{2s+1}\left(\frac{2^d}{(d-1)!}\right)^s .$$

(ii) *For approximation or Kolmogorov numbers it holds*

$$\lim_{n\to\infty} \frac{u_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} = \left(\frac{2s}{2s+1}\right)^s \left(\frac{2^d}{(d-1)!}\right)^s . \tag{4.3}$$

4.3 Widths of $id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_p(\mathbb{T}^d)$, $2 < p \le \infty$

Now we turn to the widths $u_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\big)$.

Theorem 4.6. *Let $0 < s < \infty$ and $0 < r \le \infty$.*
(i) *For any $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that*

$$\left(\frac{2s}{2s+1}\right)^s \left(\frac{2^d}{(d-1)!}\right)^s - \varepsilon \le \frac{u_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} \le \left(\frac{2^d}{(d-1)!}\right)^s + \varepsilon \tag{4.4}$$

holds for all $n \ge n_0$.
(ii) *Let $2 < p < \infty$. Then (4.4) remains true also for $L_\infty(\mathbb{T}^d)$ replaced by $L_p(\mathbb{T}^d)$.*

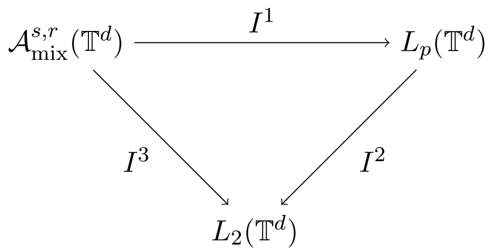
Proof. Step 1. Proof of (i). We employ Lemma 3.6, Theorem 3.3 and Theorem 4.5. This yields the following

$$\frac{u_n\big(id: \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} = \frac{\sup_{h\ge n}\left(\frac{h-n+1}{\sum_{k=1}^h \sigma_k^{-2}}\right)^{1/2}}{n^{-s}(\ln n)^{s(d-1)}} \xrightarrow[n\to\infty]{} \left(\frac{2s}{2s+1}\right)^s \left(\frac{2^d}{(d-1)!}\right)^s .$$

On the other hand we know

$$\frac{\sigma_n}{n^{-s}(\ln n)^{s(d-1)}} = \frac{u_n\big(id: H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} \xrightarrow[n\to\infty]{} \left(\frac{2^d}{(d-1)!}\right)^s ,$$

see Lemma 2.1 and Proposition 4.4. Combining these two facts yields the claim.
Step 2. Proof of (ii). In view of the commutative diagram



with $I^3 = I^2 \circ I^1$ we conclude

$$s_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \le s_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\big)\,. \tag{4.5}$$

Similarly it follows

$$s_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\big) \le s_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)\,. \tag{4.6}$$

Now we combine (4.5), (4.6) with (4.2) and (4.3). □

Remark 4.7. (i) One of the main results in [5] reads as follows. Let $d \in \mathbb{N}$, $s > 1/2$ and $0 < r \le \infty$. Then

$$\lim_{n\to\infty} \frac{a_n(I_d : H^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d))}{n^{-s+1/2}(\ln n)^{(d-1)s}} = \frac{1}{\sqrt{2s-1}}\left[\frac{2^d}{(d-1)!}\right]^s .$$

Comparing this behaviour with the result obtained in Theorem 4.6 we notice an essential difference. Indeed, we have that for any $\varepsilon > 0$ there exists some n_0 such that

$$\frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{n}}\, \frac{a_n(I_d : H^{s,r}_{\text{mix}}(\mathbb{T}^d)) \to L_\infty(\mathbb{T}^d))}{a_n(I_d : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d))} \le \left(1 + \frac{1}{2s}\right)^s \frac{1}{\sqrt{2s-1}} + \varepsilon$$

holds for all $n \ge n_0$. For the first inequality we used (2.3) and (2.1).
(ii) The left-hand side of inequality (4.4) can be simplified. Elementary analysis yields that for all $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{e}}\left(\frac{2^d}{(d-1)!}\right)^s - \varepsilon \le \frac{u_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big)}{n^{-s}(\ln n)^{s(d-1)}} \le \left(\frac{2^d}{(d-1)!}\right)^s + \varepsilon$$

is true for all $n \ge n_0$.
(iii) Above we have mentioned the identity

$$a_n\big(id : F_\omega(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) = a_n\big(id : F_\omega(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)\,, \qquad n \in \mathbb{N}\,.$$

see (3.13). Up to now it is not clear whether the counterpart

$$a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) = a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$$

holds. But at least we know that for any $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$1 \le \frac{a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)}{a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big)} \le \left(\frac{2s+1}{2s}\right)^s + \varepsilon < \sqrt{e} + 1$$

holds for all $n \ge n_0$. This follows from

$$a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\big) \le a_n\big(id : \mathcal{A}^{s,r}_{\text{mix}}(\mathbb{T}^d) \to \mathcal{A}(\mathbb{T}^d)\big)$$

and Theorem 4.6(i).

4.4 Widths of the classes $C^m_{\text{mix}}(\mathbb{T}^d)$ and $S^m_\infty W(\mathbb{T}^d)$

Let $m \in \mathbb{N}$. We define $C^m_{\text{mix}}(\mathbb{T}^d)$ as the collection of all functions f on $\mathbb{T}^d$ such that all classical derivatives $D^{\boldsymbol{\alpha}} f$ with $|\boldsymbol{\alpha}|_\infty \le m$ belong to $C(\mathbb{T}^d)$. The norm of f in $C^m_{\text{mix}}(\mathbb{T}^d)$ is given by

$$\|f|C^m_{\text{mix}}(\mathbb{T}^d)\| := \max_{\boldsymbol{\alpha}\in\mathbb{N}^d_0,\, |\boldsymbol{\alpha}|_\infty \le m} \|D^{\boldsymbol{\alpha}} f|C(\mathbb{T}^d)\| .$$

Similarly we define $S^m_\infty W(\mathbb{T}^d)$ as the collection of all functions f in $C(\mathbb{T}^d)$ such that all distributional derivatives $D^{\boldsymbol{\alpha}} f$ with $|\boldsymbol{\alpha}|_\infty \le m$ belong to $L_\infty(\mathbb{T}^d)$. The norm of f in $S^m_\infty W(\mathbb{T}^d)$ is chosen as

$$\|f|S^m_\infty W(\mathbb{T}^d)\| := \max_{\boldsymbol{\alpha}\in\mathbb{N}^d_0,\, |\boldsymbol{\alpha}|_\infty \le m} \|D^{\boldsymbol{\alpha}} f|L_\infty(\mathbb{T}^d)\| .$$

Lemma 4.8. *Let $m \in \mathbb{N}$ and $0 < r \le \infty$.*
(i) *Then we have the chain of continuous embeddings*

$$\mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow C^m_{\text{mix}}(\mathbb{T}^d) \hookrightarrow S^m_\infty W(\mathbb{T}^d) .$$

The norm of these embedding operators is always 1.
(ii) *We have the continuous embedding $S^m_\infty W(\mathbb{T}^d) \hookrightarrow H^{m,2m}_{\text{mix}}(\mathbb{T}^d)$ and*

$$\big\| id : S^m_\infty W(\mathbb{T}^d) \hookrightarrow H^{m,2m}_{\text{mix}}(\mathbb{T}^d)\big\| = 2^{d/2} .$$

Proof. Step 1. Proof of (i). Let $f \in \mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d)$ and $\boldsymbol{\alpha} \in \mathbb{N}^d_0$ with $|\boldsymbol{\alpha}|_\infty \le m$. Then $D^{\boldsymbol{\alpha}} f$ is a continuous function which coincides with its Fourier series given by $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{k})\, e^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \Big(\prod_{\ell=1}^d (\mathrm{i}k_\ell)^{\alpha_\ell}\Big)$. Furthermore, we have the obvious estimate

$$\begin{aligned}
\Big| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{k})\, e^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \Big(\prod_{\ell=1}^d (\mathrm{i}k_\ell)^{\alpha_\ell}\Big)\Big| & \le \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{k})| \prod_{\ell=1}^d |k_\ell|^{\alpha_\ell} \\
& \le \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{k})| \prod_{\ell=1}^d (1+|k_\ell|^r)^{\alpha_\ell/r} \\
& \le \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}(\boldsymbol{k})\, \omega_{m,r}(\boldsymbol{k})| \\
& = \|\, f\,|\mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d)\| .
\end{aligned}$$

Step 2. Proof of (ii). We employ the identity in (4.1). Then it follows immediately

$$\big\|\, f\,|H^{m,2m}_{\text{mix}}(\mathbb{T}^d)\big\| \le 2^{d/2}\, \big\|\, f\,|S^m_\infty W(\mathbb{T}^d)\big\| .$$

Now we turn to the estimate from below. Therefore we shall employ the test function

$$g(\boldsymbol{x}) := e^{\mathrm{i}(x_1+\ldots+x_d)} , \qquad \boldsymbol{x} \in \mathbb{T}^d .$$

Again by using (4.1) it is easy to see that $\|\, g\,|H^{m,2m}_{\text{mix}}(\mathbb{T}^d)\| = 2^{d/2}$. Moreover it is obvious that $\|\, g\,|S^m_\infty W(\mathbb{T}^d)\| = 1$. This proves the claim. □

We first consider the embedding $C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$. Here we really need to work with Bernstein numbers. For this we recall the following result in [20, Theorem 4].

Theorem 4.9. *Let H be a Hilbert space, E a topological space with probability measure ν such that* $\operatorname{supp}\nu = \mathrm{E}$. *Let H_1 be a subspace of $L_2(E,\nu)$ and $X_E \subset H_1$ be a subspace of $C(E)$. Assume that a bounded linear operator $T_0 : H_1 \to H$ satisfying*

$$T_0' T_0 \varphi_k = s_k^2\, \varphi_k, \qquad k = 1, 2, \dots,$$

where $s_1 \ge s_2 \ge \ldots > 0$ and $\{\varphi_k\}$ is an orthonormal basis for the range of $T_0' T_0$ with $\varphi_k \in X_E$. Define $T : X_E \to H$ by $Tf := T_0 f$. Then

$$b_n(T) \le \left(\sum_{k=1}^{n} s_k^{-2} \right)^{-1/2}.$$

Below we shall apply the above theorem with $E := \mathbb{T}^d$, $X_E := C^m_{\text{mix}}(\mathbb{T}^d)$, $H_1 := H^{m,r}_{\text{mix}}(\mathbb{T}^d)$, $H := L_2(\mathbb{T}^d)$ and $T :\ H^{m,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ being the identity operator. Then the dual operator T_0' is given by

$$T_0' g(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{\hat{g}(\boldsymbol{k})}{\omega_{m,r}(\boldsymbol{k})^2}\, e^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}, \qquad \boldsymbol{x} \in \mathbb{T}^d,$$

i.e., $s_n = \sigma_n$ and $(\sigma_n)_{n=1}^\infty$ denotes the non-increasing rearrangement of $1/\omega_{m,r}(\boldsymbol{k})$, $\boldsymbol{k} \in \mathbb{Z}^d$.

Theorem 4.10. *Let $m, d \in \mathbb{N}$.*
(i) *Then the following identity takes place*

$$v_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) = \left(\sum_{k=1}^{n} \sigma_k^{-2} \right)^{-1/2}. \tag{4.7}$$

(ii) *In case of Weyl and Bernstein numbers the asymptotic constant is given by*

$$\lim_{n\to\infty} \frac{v_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-m-\frac{1}{2}}(\ln n)^{m(d-1)}} = \sqrt{2m+1}\left(\frac{2^d}{(d-1)!} \right)^m.$$

(iii) *For all $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that*

$$\left(\frac{2m}{2m+1}\right)^m \left(\frac{2^d}{(d-1)!} \right)^m - \varepsilon \le \frac{u_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{-m}(\ln n)^{m(d-1)}} \le 2^{d/2}\left(\frac{2^d}{(d-1)!} \right)^m + \varepsilon \tag{4.8}$$

holds for all $n \ge n_0$.

Proof. Step 1. Proof of (i). From the chain of embeddings $\mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow C^m_{\text{mix}}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$, always with operator norm 1, see Lemma 4.8, we conclude by means of (2.1) that

$$\begin{aligned} v_n\big(id : \mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) &\le \|id : \mathcal{A}^{m,r}_{\text{mix}}(\mathbb{T}^d) \to C^m_{\text{mix}}(\mathbb{T}^d)\| \cdot v_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \\ &= v_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big). \end{aligned}$$

Theorem 3.3 yields the lower estimate. Since the target space is a Hilbert space we know that $x_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \le b_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)$, see (2.2). Applying Theorem 4.9 with $E := \mathbb{T}^d$, $X_E := C^m_{\text{mix}}(\mathbb{T}^d)$, $H_1 := H^{m,r}_{\text{mix}}(\mathbb{T}^d)$, $H := L_2(\mathbb{T}^d)$ and $T = id$ we obtain (4.7). The special choice of r does not play a role.
Step 2. Proof of (ii). Part (ii) becomes a direct consequence of (i) in combination with Theorem 3.3(i) and Theorem 4.5.
Step 3. Proof of (iii). Concerning the lower bound in (4.8) we may use the same argument

as in Step 1. Concerning the upper bound we shall employ the embeddings $C^m_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{m,2m}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$. We have

$$\begin{aligned} u_n\big(id : C^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) &\le \|id : C^m_{\text{mix}}(\mathbb{T}^d) \to H^{m,2m}_{\text{mix}}(\mathbb{T}^d)\| \cdot u_n\big(id : H^{m,2m}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \\ &\le 2^{d/2}\, u_n\big(id : H^{m,2m}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)\,. \end{aligned}$$

In view of Proposition 4.4 we have finished the proof. □

Remark 4.11. It would be desirable to improve the estimate in part (iii). The correct behaviour in the dependence on d is still unclear because of the additional factor $2^{d/2}$.

Corollary 4.12. *Let $m, d \in \mathbb{N}$. Then all assertions in Theorem 4.10 remain true when replacing $C^m_{\text{mix}}(\mathbb{T}^d)$ by $S^m_\infty W(\mathbb{T}^d)$.*

5 Approximation numbers of $\mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d)$ in $H^1(\mathbb{T}^d)$

In this section, we are interested in $s_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$, $s > 1$. Here we equip $H^1(\mathbb{T}^d)$ with the norm

$$\begin{aligned} \|\, f\,|H^1(\mathbb{T}^d)\| :&= \bigg(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \Big(1 + \sum_{j=1}^d |k_j|^2\Big) |\hat{f}(\boldsymbol{k})|^2\bigg)^{1/2} \\ &= \bigg(\|\, f\,|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^d \Big\| \frac{\partial f}{\partial x_j}\,\Big|L_2(\mathbb{T}^d)\Big\|^2\bigg)^{1/2}. \end{aligned}$$

I.e., $H^1(\mathbb{T}^d)$ is the standard isotropic periodic Sobolev space with smoothness 1. As a preparation we shall investigate the asymptotic constant of $s_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$, $s > 1$. Because we are in a Hilbert space situation it will be enough to deal with the approximation numbers. We define a weight ω by

$$\omega(\boldsymbol{k}) := \frac{\prod_{j=1}^d (1+|k_j|^2)^{s/2}}{\big(1+\sum_{j=1}^d |k_j|^2\big)^{1/2}}, \qquad \boldsymbol{k} \in \mathbb{Z}^d\,. \tag{5.1}$$

In addition we need the diagonal operator $D_\omega \xi := (\xi_{\boldsymbol{k}}/\omega(\boldsymbol{k}))_{\boldsymbol{k}\in\mathbb{Z}^d}$, $\xi = (\xi_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$, see (3.3). Now we observe that

$$a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big) = a_n\big(D_\omega : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)\big) = a_n\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)\,, \tag{5.2}$$

see Lemma 2.1. Rearranging non-increasingly the sequence $\{1/\omega(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ with the outcome denoted by $(\sigma_n)_{n=1}^\infty$, we obtain $\sigma_n = a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$.

The asymptotic order of $a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$ is well-known, see [7, 2]. It holds

$$C_1(s,d) n^{-(s-1)} \le a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big) \le C_2(s,d) n^{-(s-1)}, \qquad n \in \mathbb{N}\,,$$

with constants $C_1(d,s)$ and $C_2(d,s)$ depending on s, d. Notice that there is no logarithmic factor anymore. Several preasymptotic estimates for $a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$ may be found in [18]. Our result for the asymptotic constant of $a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$ reads as follows.

Proposition 5.1. *Let $d \in \mathbb{N}$ and $s > 1$. We define*

$$S := \sum_{k=1}^{+\infty} \frac{1}{(k^2+1)^{\frac{s}{2(s-1)}}}. \tag{5.3}$$

Then we have

$$\lim_{n\to+\infty} \frac{a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{1-s}} = (2d)^{s-1}(2S+1)^{(s-1)(d-1)}.$$

The rather technical proof of this assertion is shifted to Appendix A. Our main result in this section reads as follows.

Theorem 5.2. *Let $d \in \mathbb{N}$, $s > 1$ and S be given in (5.3).*
(i) *In case of Bernstein or Weyl numbers we have*

$$\lim_{n\to\infty} \frac{v_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+\frac{1}{2}}} = \sqrt{2s+1}(2d)^{s-1}(2S+1)^{(s-1)(d-1)}.$$

(ii) *For approximation or Kolmogorov numbers it holds*

$$\lim_{n\to\infty} \frac{u_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{-s+1}} = \left(\frac{2s}{2s+1}\right)^{s} (2d)^{s-1}(2S+1)^{(s-1)(d-1)}.$$

Proof. Let ω be given in (5.1). From (5.2) and Proposition 5.1 we obtain

$$\lim_{n\to+\infty} \frac{a_n\big(id : F_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big)}{n^{1-s}} = (2d)^{s-1}(2S+1)^{(s-1)(d-1)}. \tag{5.4}$$

Next we will show that

$$s_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big) = s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big), \tag{5.5}$$

by using standard lifting arguments. We consider the diagram

$$\begin{array}{ccc} \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) & \xrightarrow{\ id\ } & H^1(\mathbb{T}^d) \\ \big\downarrow A & & \big\uparrow B \\ \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{\ id\ } & L_2(\mathbb{T}^d) \end{array}$$

where the linear operators A and B are defined for $f \in \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d)$ and $g \in L_2(\mathbb{T}^d)$ respectively by

$$\widehat{Af}(\boldsymbol{k}) := \Big(1 + \sum_{j=1}^{d} |k_j|^2\Big)^{1/2} \hat{f}(\boldsymbol{k}), \quad \widehat{Bg}(\boldsymbol{k}) := \Big(1 + \sum_{j=1}^{d} |k_j|^2\Big)^{-1/2} \hat{g}(\boldsymbol{k}), \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

It is obvious that $\|A\| = \|B\| = 1$. Now by the ideal property of s-numbers, see (2.1), we obtain

$$s_n\big(id : \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big) \le s_n\big(id : \mathcal{A}_\omega(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big).$$

The reverse inequality follows from the modified diagram

$$\begin{array}{ccc} \mathcal{A}^{s,2}_{\text{mix}}(\mathbb{T}^d) & \xrightarrow{\ id\ } & H^1(\mathbb{T}^d) \\ \big\uparrow A^{-1} & & \big\downarrow B^{-1} \\ \mathcal{A}_\omega(\mathbb{T}^d) & \xrightarrow{\ id\ } & L_2(\mathbb{T}^d)\,. \end{array}$$

Now the claims follow from Theorem 3.4, (5.4), and (5.5). □

A Appendix - the proof of Proposition 5.1

Proposition 5.1 in case $d = 1$ is almost obvious and represents an old result of Kolmogorov, see [12]. However, it can be found also in [16, Theorem. 4.14].

In the following we will assume that $d \geq 2$. For $s > 1$, $r \in \mathbb{N}$ and $\omega(\boldsymbol{k})$ as in (5.1) we define

$$C(r,d) := \left| \left\{ \boldsymbol{k} \in \mathbb{Z}^d : \ (1+r^2)^{\frac{s-1}{2}} \geq \omega(\boldsymbol{k}) \right\} \right| .$$

Later on we shall need the following observation. If $\boldsymbol{k} \in \mathbb{Z}^d$ satisfies $(1+r^2)^{\frac{s-1}{2}} \geq \omega(\boldsymbol{k})$ then

$$|k_j| \leq r, \qquad j = 1, \ldots, d \tag{A.1}$$

follows. Indeed, from

$$(1+|k_1|^2)^s \Big(1 + \sum_{j=2}^d |k_j|^2 \Big) \geq \Big(1 + \sum_{j=1}^d |k_j|^2 \Big)$$

we derive

$$\omega^2(\boldsymbol{k}) = \frac{\prod_{j=1}^d (1+|k_j|^2)^s}{1+\sum_{j=1}^d |k_j|^2} \geq \frac{\prod_{j=2}^d (1+|k_j|^2)^s}{1+\sum_{j=2}^d |k_j|^2} .$$

Iterating this argument we find $\omega^2(\boldsymbol{k}) \geq (1+|k_d|^2)^{s-1}$. This yields the observation. We shall need several further preparations.

Lemma A.1. *For $r \geq 1$, we have*

$$C(r,d) = 1 + \sum_{\ell=1}^d 2^\ell \binom{d}{\ell} A(r,\ell), \tag{A.2}$$

where $A(r,\ell) = |\mathscr{M}(r,\ell)|$ and

$$\mathscr{M}(r,\ell) = \left\{ \boldsymbol{k} \in \mathbb{N}^\ell : \ (1+r^2)^{\frac{s-1}{2}} \geq \omega(\boldsymbol{k}) \right\}.$$

Proof. We consider $\boldsymbol{k} \in \mathbb{Z}^d$ such that $\operatorname{supp}(\boldsymbol{k}) = \ell$, $\ell = 0, \ldots, d$, where

$$\operatorname{supp} \boldsymbol{k} := \left| \{ k_j : \ k_j \neq 0, j = 1, \ldots, d \} \right| .$$

It is clear that

$$\left| \left\{ \boldsymbol{k} \in \mathbb{Z}^d, \operatorname{supp}(\boldsymbol{k}) = \ell : \ (1+r^2)^{\frac{s-1}{2}} \geq \omega(\boldsymbol{k}) \right\} \right| = 2^\ell \binom{d}{\ell} \left| \left\{ \boldsymbol{k} \in \mathbb{N}^\ell : \ (1+r^2)^{\frac{s-1}{2}} \geq \omega(\boldsymbol{k}) \right\} \right| .$$

From this the assertion follows. □

We claim and we shall prove below in Lemma A.3 that for $\ell \in \mathbb{N}$, $\ell \geq 2$, $s > 1$, it holds

$$A(r,\ell) = \ell r S^{\ell-1} + o(r) \qquad (r \to +\infty). \tag{A.3}$$

From this property Proposition 5.1 can be derived as follows.

Proof of Proposition 5.1. For $n \in \mathbb{N}$, there exists some $r \in \mathbb{N}$ such that

$$C(r-1,d) < n \le C(r,d).$$

Then by the definition of $C(r,d)$ we get

$$\frac{1}{(r^2+1)^{\frac{s-1}{2}}} \le a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big) \le \frac{1}{((r-1)^2+1)^{\frac{s-1}{2}}}$$

(recall $\sigma_n = a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)$, where $(\sigma_n)_n$ denotes the nonincreasing rearrangement of the sequence $\{1/\omega(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$) which leads to

$$\frac{C(r-1,d)^{s-1}}{(r^2+1)^{\frac{s-1}{2}}} \le \frac{a_n\big(id : H^{s,2}_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)\big)}{n^{1-s}} \le \frac{C(r,d)^{s-1}}{((r-1)^2+1)^{\frac{s-1}{2}}}. \tag{A.4}$$

From (A.2) and (A.3), we have:

$$\begin{aligned} C(r,d) &= 1 + \sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell} \ell S^{\ell-1} r + o(r) \\ &= 1 + 2\sum_{\ell=1}^{d} (2S)^{\ell-1} \ell \binom{d}{\ell} r + o(r) = 1 + 2d(2S+1)^{d-1} r + o(r). \end{aligned}$$

Consequently we find

$$\lim_{r\to+\infty} \frac{C(r,d)}{r} = 2d(2S+1)^{d-1}.$$

Now (A.4) implies the desired result. □

The next preparation will be used in the proof of (A.3).

Lemma A.2. *For $r \ge 1$, $2 \le \ell \le d$, and arbitrary $r_\ell \in \mathbb{N}$, $r_\ell \le r$ we have*

$$A(r,\ell) = \sum_{j=0}^{\ell} \binom{\ell}{j} A(r,\ell,j), \tag{A.5}$$

where $A(r,\ell,j) = |\mathscr{M}(r,\ell,j)|$ and

$$\mathscr{M}(r,\ell,j) = \left\{\boldsymbol{k} \in \mathscr{M}(r,\ell);\ k_1,k_2,\ldots,k_j \le r_\ell;\ k_{j+1},\ldots,k_\ell > r_\ell\right\}.$$

Proof. Let $0 \le j \le \ell$. For any $\Lambda \subset \{1,\ldots,\ell\}$ such that $|\Lambda| = j$ we put

$$\mathscr{M}_\Lambda := \big\{\boldsymbol{k} \in \mathscr{M}(r,\ell);\ k_i \le r_\ell \text{ if } i \in \Lambda \text{ and } k_i > r_\ell \text{ if } i \in \{1,\ldots,\ell\}\backslash\Lambda\big\}.$$

That is, $\mathscr{M}_\Lambda$ has exactly j components less than or equal to r_ℓ with index in Λ. It is obvious that $|\mathscr{M}_\Lambda| = |\mathscr{M}(r,\ell,j)|$. Since for each j there are exactly $\binom{\ell}{j}$ different sets Λ with $|\Lambda| = j$, the claim follows. □

Note that all numbers $A(r,\ell,j)$ depend on r_ℓ, but $A(r,\ell)$ not. In our notation we just ignored this dependence (to keep the notation as simple as possible). In what follows we will show that by choosing appropriate r_ℓ, depending on r and ℓ, $A(r,\ell,\ell-1)$ has order $r\, S^{\ell-1}$ when $r \to +\infty$. All other summands $A(r,\ell,j)$, $j \ne \ell-1$, see (A.5), are much smaller and as a consequence, (A.3) will follow.

Lemma A.3. *Let $2 \le \ell \le d$ and $r_\ell := \lfloor r^{\lambda_\ell} \rfloor$, where $0 < \lambda_\ell < \frac{s-1}{s\ell}$. Then we have*
(i) $A(r,\ell,\ell) = o(r), \quad r \to +\infty,$
(ii) $A(r,\ell,j) = o(r), \quad r \to +\infty$ *if* $0 \le j \le \ell - 2,$
(iii) $\lim_{r\to+\infty} \frac{A(r,\ell,\ell-1)}{r} = S^{\ell-1}$.

Therefore, it holds

$$A(r,\ell) = \ell r S^{\ell-1} + o(r), \quad r \to +\infty.$$

Proof. Step 1. Proof of (i). From the definition of $A(r,\ell,\ell)$ we derive

$$A(r,\ell,\ell) \le r_\ell^\ell \le r^{\ell\lambda_\ell} \le r^{\frac{\ell(s-1)}{\ell s}} = r^{\frac{s-1}{s}}.$$

Since $\frac{s-1}{s} < 1$ we get immediately that $A(r,\ell,\ell) = o(r),\ r \to +\infty$.
Step 2. Proof of (ii). For $\boldsymbol{k} \in \mathscr{M}(r,\ell,j)$, we have

$$\frac{\left(1+\sum_{i=1}^\ell k_i^2\right)^{1/2}}{\prod_{i=1}^\ell (1+k_i^2)^{s/2}} \ge \frac{1}{(1+r^2)^{\frac{s-1}{2}}} \quad \Longleftrightarrow \quad \prod_{i=1}^{\ell}(1+k_i^2)^s \le (1+r^2)^{s-1}\left(1+\sum_{i=1}^{\ell} k_i^2\right)$$

which implies

$$(1+k_\ell^2)^{s-1} \le \frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} + \frac{(1+r^2)^{s-1}(\sum_{i=1}^{\ell-1} k_i^2)}{(1+k_\ell^2)\prod_{i=1}^{\ell-1}(1+k_i^2)^s}. \tag{A.6}$$

Since $k_\ell > r^{\lambda_\ell}$ and

$$\frac{\sum_{i=1}^{\ell-1} k_i^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} \le \frac{\prod_{i=1}^{\ell-1} k_i^2}{\prod_{i=1}^{\ell-1} k_i^{2s}}, \quad s > 1, \tag{A.7}$$

we have

$$\frac{(1+r^2)^{s-1}(\sum_{i=1}^{\ell-1} k_i^2)}{(1+k_\ell^2)\prod_{i=1}^{\ell-1}(1+k_i^2)^s} \le \frac{(2r^2)^{s-1}\prod_{i=1}^{\ell-1} k_i^2}{r^{2\lambda_\ell}\prod_{i=1}^{\ell-1} k_i^{2s}} = \frac{2^{s-1} r^{2(s-1-\lambda_\ell)}}{\prod_{i=1}^{\ell-1} k_i^{2(s-1)}}.$$

From (A.6) and $j = \ell - 1$ (i.e., $k_i < r_\ell,\ i = 1,\dots,\ell-1$), we obtain

$$k_\ell^{2(s-1)} < \frac{2^{s-1} r^{2(s-1)}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} + \frac{2^{s-1} r^{2(s-1-\lambda_\ell)}}{\prod_{i=1}^{\ell-1} k_i^{2(s-1)}},$$

which leads to

$$k_\ell < c\left(\frac{r}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + \frac{r^{1-\frac{\lambda_\ell}{s-1}}}{\prod_{i=1}^{\ell-1} k_i}\right) \tag{A.8}$$

for some $c = c(s) > 0$. The definition of $A(r,\ell,j)$ yields

$$A(r,\ell,j) = \sum_{\boldsymbol{k}\in\mathscr{M}(r,\ell,j)} 1 = \sum_{\substack{k_1,\dots,k_j \le r_\ell \\ k_{j+1},\dots,k_{\ell-1} > r_\ell}} \left|\left\{k_\ell \in \mathbb{N} :\ (1+r^2)^{\frac{s-1}{2}} \ge \omega(\boldsymbol{k})\right\}\right|. \tag{A.9}$$

This together with (A.8) and $0 \le k_i \le r$, $i = 1, \dots, \ell$, if $\boldsymbol{k} \in \mathscr{M}(r, \ell, j)$, see (A.1), implies

$$\begin{aligned}
\frac{A(r,\ell,j)}{r} &< c \sum_{\substack{k_1,\dots,k_j \le r_\ell \\ k_{j+1},\dots,k_{\ell-1} \in (r_\ell, r]}} \left[\frac{1}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + \frac{r^{-\frac{\lambda_\ell}{s-1}}}{\prod_{i=1}^{\ell-1} k_i} \right] \\
&< c \sum_{\substack{k_1,\dots,k_j \in \mathbb{N} \\ k_{j+1},\dots,k_{\ell-1} > r_\ell}} \frac{1}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + c \sum_{k_1,\dots,k_{\ell-1} \in [1,r]} \frac{r^{-\frac{\lambda_\ell}{s-1}}}{\prod_{i=1}^{\ell-1} k_i} \\
&= c \left(\sum_{m=1}^{\infty} \frac{1}{(1+m^2)^{\frac{s}{2(s-1)}}} \right)^j \left(\sum_{m=r_\ell+1}^{\infty} \frac{1}{(1+m^2)^{\frac{s}{2(s-1)}}} \right)^{\ell-j-1} + \frac{c}{r^{\frac{\lambda_\ell}{s-1}}} \left(\sum_{m=1}^{r} \frac{1}{m} \right)^{\ell-1} \\
&\le c' \left[\left(\sum_{m=r_\ell+1}^{\infty} \frac{1}{(1+m^2)^{\frac{s}{2(s-1)}}} \right)^{\ell-j-1} + \frac{(\ln r)^{\ell-1}}{r^{\frac{\lambda_\ell}{s-1}}} \right]
\end{aligned}$$

with $c' > 0$ independent of r. This estimate and $\ell - j - 1 \ge 1$ yields that $\frac{A(r,\ell,j)}{r}$ tends to 0 when $r \to +\infty$.

Step 3. We will show that

$$\limsup_{r \to +\infty} \frac{A(r, \ell, \ell-1)}{r} \le S^{\ell-1}. \tag{A.10}$$

First we need an elementary inequality. We have

$$\frac{\sum_{i=1}^{\ell-1} k_i^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} \le \frac{1}{\prod_{i=1}^{\ell-1} k_i^2 (s-1)} \tag{A.11}$$

for all $\ell \ge 2$ and arbitrary $\boldsymbol{k} \in \mathbb{N}^\ell$. This can be proved by induction. From (A.6) and (A.11) we conclude

$$k_\ell^{2(s-1)} < (1+k_\ell^2)^{s-1} \le \frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} + \frac{2^{s-1} r^{2(s-1-\lambda_\ell)}}{\prod_{i=1}^{\ell-1} k_i^{2(s-1)}}.$$

If $s \ge \frac{3}{2}$ (i.e. $\frac{1}{2(s-1)} \le 1$) we get

$$k_\ell < \frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + \frac{\sqrt{2} r^{1-\frac{\lambda_\ell}{s-1}}}{\prod_{i=1}^{\ell-1} k_i},$$

where we applied $(a+b)^p \le a^p + b^p$, $a, b > 0$, and $0 < p \le 1$. Consequently, by using the identity (A.9) with $j = \ell - 1$,

$$\begin{aligned}
A(r,\ell,\ell-1) &< \sum_{k_1,\dots,k_{\ell-1} \le r_\ell} \left\{ \frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + \frac{\sqrt{2} r^{1-\frac{\lambda_\ell}{s-1}}}{\prod_{i=1}^{\ell-1} k_i} \right\} \\
&< \sqrt{1+r^2} \left(\sum_{m=1}^{\infty} \frac{1}{(1+m^2)^{\frac{2}{2(s-1)}}} \right)^{\ell-1} + \sqrt{2} r^{1-\frac{\lambda_\ell}{s-1}} \left(\sum_{m=1}^{r} \frac{1}{m} \right)^{\ell-1} \\
&< \sqrt{1+r^2} S^{\ell-1} + c\, r^{1-\frac{\lambda_\ell}{s-1}} (\ln r)^{\ell-1}
\end{aligned} \tag{A.12}$$

for some constant $c > 0$ depending on ℓ. This yields (A.10) in case $s \ge \frac{3}{2}$.

If $1 < s < \frac{3}{2}$ we use the inequality: for $a, b > 0$, $b \le ma$ and $p \ge 1$ it holds

$$(a+b)^p \le a^p + p(1+m)^{p-1} a^{p-1} b$$

which can be derived from $(1+x)^p \le 1 + p(1+m)^{p-1}x$ for $0 \le x \le m$. Observe that if $\boldsymbol{k} \in \mathscr{M}(r,\ell,\ell-1)$ we have

$$b = \frac{(1+r^2)^{s-1}(\sum_{i=1}^{\ell-1} k_i^2)}{(1+k_\ell^2)\prod_{i=1}^{\ell-1}(1+k_i^2)^s} \le (\ell-1)\frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} = m \cdot a\,. \tag{A.13}$$

From (A.13), (A.6) and the above inequality with $p := \frac{1}{2(s-1)}$, $m := \ell - 1$ we can estimate

$$\begin{aligned} k_\ell &\le \frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + C(s,\ell)\left(\frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s}\right)^{\frac{1}{2(s-1)}-1}\left(\frac{(1+r^2)^{s-1}(\sum_{i=1}^{\ell-1} k_i^2)}{(1+k_\ell^2)\prod_{i=1}^{\ell-1}(1+k_i^2)^s}\right) \\ &\le \frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + C(s,\ell)\left(\frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1} k_i^{2s}}\right)^{\frac{1}{2(s-1)}-1}\left(\frac{(1+r^2)^{s-1}(\prod_{i=1}^{\ell-1} k_i^2)}{(1+k_\ell^2)\prod_{i=1}^{\ell-1} k_i^{2s}}\right) \\ &\le \frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} + C(s,\ell)\frac{\sqrt{1+r^2}}{r^{2\lambda_\ell}\prod_{i=1}^{\ell-1} k_i^{\frac{2-s}{s-1}}}\,, \end{aligned}$$

where in the second line of the estimate we used (A.7). Here $C(s,\ell)$ denotes a constant which depends on s and ℓ only. Since $\frac{2-s}{s-1} > 1$, a similar argument to (A.12) yields (A.10) in case $s < \frac{3}{2}$.

We next prove the reverse direction, i.e.,

$$\liminf_{r\to+\infty} \frac{A(r,\ell,\ell-1)}{r} \ge S^{\ell-1}\,. \tag{A.14}$$

Let $\boldsymbol{k} = (k_1,\dots,k_\ell)$ be such that $(1+r^2)^{\frac{s-1}{2}} \ge \omega(\boldsymbol{k})$. For $k_1,\dots,k_{\ell-1}$ fixed we have $k_\ell \le K$, where $K := K(k_1,\dots,k_{\ell-1})$ is the positive solution of the equation

$$(1+X^2)^{s-1} = \frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} + \frac{(1+r^2)^{s-1}(\sum_{i=1}^{\ell-1} k_i^2)}{(1+X^2)\prod_{i=1}^{\ell-1}(1+k_i^2)^s},$$

see (A.6). From this we get

$$(1+K^2)^{s-1} > \frac{(1+r^2)^{s-1}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^s} \qquad \text{as well as} \qquad K > \left(\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} - 1\right)^{1/2}.$$

Observe that $1 \le k_i \le r_\ell$, $i = 1,\dots,\ell-1$, implies

$$\begin{aligned} \left\lfloor\left(\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} - 1\right)^{1/2}\right\rfloor - r_\ell &\ge \left\lfloor\left(\frac{1+r^2}{(1+r_\ell^2)^{\frac{s(\ell-1)}{s-1}}} - 1\right)^{1/2}\right\rfloor - r_\ell \\ &\ge \left\lfloor\left(\frac{1+r^2}{\left(1+r^{\frac{2(s-1)}{\ell s}}\right)^{\frac{s(\ell-1)}{s-1}}} - 1\right)^{1/2}\right\rfloor - \lfloor r^{\frac{s-1}{\ell s}}\rfloor. \end{aligned} \tag{A.15}$$

As a consequence of this estimate we obtain that the left-hand side of (A.15) is positive when r is large enough. Hence, the set

$$\left\{\boldsymbol{k} = (k_1,\dots,k_\ell) : 1 \le k_i \le r_\ell,\ i = 1,\dots,\ell-1;\ r_\ell < k_\ell \le \left\lfloor\left(\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} - 1\right)^{1/2}\right\rfloor\right\}$$

is a subset of $\mathscr{M}(r,\ell,\ell-1)$ when r is large enough. Therefore, for those large r, we obtain

$$A(r,\ell,\ell-1) \geq \sum_{k_1,\ldots,k_{\ell-1}\leq r_\ell} \left(\left\lfloor \left(\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} - 1 \right)^{1/2} \right\rfloor - r_\ell \right)$$
$$= \sum_{k_1,\ldots,k_{\ell-1}\leq r_\ell} \left\lfloor \left(\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} - 1 \right)^{1/2} \right\rfloor - r_\ell^\ell.$$

The estimate (A.15) also implies that

$$\frac{1+r^2}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{s-1}}} \geq 1$$

when r is large enough. From this and the inequalities

$$\lfloor \sqrt{a-1} \rfloor > \sqrt{a-1} - 1 \geq \sqrt{a} - 2$$

for $a \geq 1$ we derive

$$A(r,\ell,\ell-1) \geq \sum_{k_1,\ldots,k_{\ell-1}\leq r_\ell} \left(\frac{\sqrt{1+r^2}}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} - 2 \right) - r_\ell^\ell$$
$$\geq \sum_{k_1,\ldots,k_{\ell-1}\leq r_\ell} \frac{r}{\prod_{i=1}^{\ell-1}(1+k_i^2)^{\frac{s}{2(s-1)}}} - 2r_\ell^{\ell-1} - r_\ell^\ell$$
$$\geq r \left(\sum_{m=1}^{r_\ell} \frac{1}{(1+m^2)^{\frac{s}{2(s-1)}}} \right)^{\ell-1} - 2r^{\frac{(\ell-1)(s-1)}{\ell s}} - r^{\frac{s-1}{s}},$$

which implies (A.14). The proof is completed. □

Acknowledgments. The work of V. K. Nguyen is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 102.01-2020.03. A part of this paper was done when V. K. Nguyen was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for providing a fruitful research environment and working condition.

References

[1] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13: 147–269, 2004.

[2] G. Byrenheid, D. Dũng, W. Sickel, and T. Ullrich. Sampling on energy-norm based sparse grids for the optimal recovery of Sobolev type functions in H^γ. *J. Approx. Theory*, 207: 207—231, 2016.

[3] G. Byrenheid, L. Kaemmerer, T. Ullrich and T. Volkmer. Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness. *Numer. Math.*, 136: 993–1034, 2017.

[4] A. Chernov and D. Dũng. New explicit-in-dimension estimates for the cardinality of high-dimensional hyperbolic crosses and approximation of functions having mixed smoothness. *J. Complexity*, 32: 92–121, 2016.

[5] F. Cobos, T. Kühn, and W. Sickel. Optimal approximation of Sobolev functions in the sup-norm. *J. Funct. Anal.*, 270: 4196–4212, 2016.

[6] F. Cobos, T. Kühn, and W. Sickel. Optimal approximation of Sobolev functions in the sup-norm. *J. Math. Anal. Appl.*, 474: 1441–1462, 2019.

[7] D. Dũng and T. Ullrich. N-Widths and ε-dimensions for high-dimensional approximations. *Found. Comp. Math.*, 13: 965–1003, 2013.

[8] D. Dũng, V. Temlyakov and T. Ullrich. *Hyperbolic Cross Approximation.* Birkhäuser, Cham, 2018.

[9] E.M. Galeev. Bernstein diameters of the classes of periodic function of several variables. *Math. Balk.(N.S.)*, 5: 229–244, 1991.

[10] L. Kämmerer, D. Potts and T. Volkmer. Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling. *J. Complexity*, 31: 543–576, 2015.

[11] L. Kämmerer and T. Volkmer. Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices. *J. Approx. Theory*, 246: 1–27, 2019.

[12] A.N. Kolmogorov. Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse. *Ann. Math.*, 37: 107–110, 1936.

[13] H. König. *Eigenvalue Distribution of Compact Operators.* Birkhäuser, Basel, 1986.

[14] D. Krieg. Tensor power sequences and the approximation of tensor product operators. *J. Complexity*, 44: 30–51, 2018.

[15] T. Kühn, S. Mayer and T. Ullrich. Counting via entropy - new preasymptotics for the approximation numbers of Sobolev embeddings. *SIAM J. Numer. Anal.*, 54: 3625–3647, 2016.

[16] T. Kühn, W. Sickel, and T. Ullrich. Approximation numbers of Sobolev embeddings - sharp constants and tractability. *J. Complexity*, 30: 95–116, 2014.

[17] T. Kühn, W. Sickel, and T. Ullrich. Approximation of mixed order Sobolev functions on the d-torus – asymptotics, preasymptotics and d-dependence. *Constr. Approx.*, 42: 353–398, 2015.

[18] T. Kühn, W. Sickel, and T. Ullrich. How anisotropic mixed smoothness affects the decay of singular numbers of Sobolev embeddings. *J. Complexity* (online). https://doi.org/10.1016/j.jco.2020.101523.

[19] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems. Volume I: Linear Information.* EMS, Zürich, 2008.

[20] K. Yu. Osipenko and O. G. Parfenov. Ismagilov type theorems for linear, Gelfand and Bernstein n-widths. *J. Complexity*, 11: 474–492, 1995.

[21] A. Pietsch. s-Numbers of operators in Banach spaces. *Studia Math.*, 51: 201–223, 1974.

[22] A. Pietsch. *Operator Ideals.* North-Holland, Amsterdam, 1980.

[23] A. Pietsch. *Eigenvalues and s-Numbers.* Cambridge University Press, Cambridge, 1987.

[24] A. Pietsch, *History of Banach Spaces and Linear Operators.* Birkhäuser, Basel, 2007.

[25] A. Pietsch, Bad properties of Bernstein numbers. *Studia Math.* 184: 263–269, 2008.

[26] A. Pinkus. *n-Widths in Approximation Theory.* Springer, Berlin, 1985.

[27] Ch. Schwab, E. Süli and R.A. Todor. Sparse finite element approximation of high-dimensional transport-dominated diffusion problems. *ESAIM–Math. Model Num.* 42: 777–819, 2008.