STABILITY OF ASSOCIATED PRIME IDEALS AND DEPTHS OF INTEGRAL CLOSURES OF POWERS OF EDGE IDEALS

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ABSTRACT. Let G be a graph on the vertex set $[r]$ and $R = k[x_1, \ldots, x_r]$ a polynomial ring over a field k. In this paper, we explicitly determine two numbers n_0 and n_1 in terms of G such that $\operatorname{Ass}(R/\overline{I(G)^n}) = \operatorname{Ass}(R/\overline{I(G)^{n_0}}$ for all $n \geqslant n_0$; and $\operatorname{depth}(R/\overline{I(G)^n}) = \operatorname{depth}(R/\overline{I(G)^{n_1}}$ for all $n \geqslant n_1$. Furthermore, our n_0 and n_1 are sharp.

INTRODUCTION

Let $R = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k and I an ideal in R. Brodmann [4] showed that the sequence $\{\operatorname{Ass} R/I^n\}_{n\geqslant 1}$ is constant for n big enough. We call the smallest number n_0 for which this sequence becomes constant to be *the index of stability* of I (see [24] and [12]) and denote by:

$$\operatorname{astab}(I) := \min\{n_0 \mid \operatorname{Ass} R/I^n = \operatorname{Ass} R/I^{n_0} \text{ for all } n \geqslant n_0\}.$$

Find a reasonable bound of $\operatorname{astab}(I)$ when I is a monomial ideal is studied recently (see [12],[13], [24], [14]). In general, L. T. Hoa [14] gave a bound of $\operatorname{dstab}(I)$ (see [14, Theorem 2.12]). This bound is extremely large. However, if we restrict to special classes of monomial ideals, much smaller bounds can be found, for example, edge ideals [6], polymatroidal ideals [13].

For integral closures, S. McAdam and P. Eakin [19] showed that the sequence $\{\operatorname{Ass}(R/\overline{I^n})\}_{n\geqslant 1}$ stabilizes. Hence we are also interested in bounding the number:

$$\overline{\operatorname{astab}}(I) := \min\{n_0 \mid \operatorname{Ass} R/\overline{I^n} = \operatorname{Ass} R/\overline{I^{n_0}} \text{ for all } n \geqslant n_0\}.$$

If I is a monomial ideal and $d(I)$ is the maximal degree of a generator. Then

$$\overline{\operatorname{astab}}(I) \leqslant r2^{r-1}d(I)^{r-2}$$

whenever $r \geqslant 3$; and this bound is optimal in the sense that the exponent in $d(I)^{r-2}$ cannot reduce. But for the case I is an edge ideal, this bound becomes quite large and far from the true value of $\overline{\operatorname{astab}}(I)$. The first main result of the paper is to get a better bound for $\overline{\operatorname{astab}}(I)$ when I is an edge ideal. Before stating our result we recall some definitions.

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Throughout this paper, every graph G is assumed to be simple (i.e., a finite, undirected, loopless and without multiple edges) without isolated vertices on the vertex set $V(G) = [r] := \{1, \ldots, r\}$ and the edge set $E(G)$ unless otherwise indicated. We associate to G the quadratic squarefree monomial ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq R = k[x_1, \ldots, x_r]$$

which is called the edge ideal of G.

In a graph, a *leaf* is a vertex of degree one and a *leaf edge* is an edge incident with a leaf. A connected graph is called a *tree* if it contains no cycles, and it is called a *unicyclic* graph if it contains exactly one cycle. We use the symbols $\upsilon(G)$, $\varepsilon(G)$ and $\varepsilon_0(G)$ to denote the number of vertices, edges and leaf edges of G, respectively.

Theorem 2.4. *Let G be a graph. Let $G_1, \ldots, G_s$ be all connected nonbipartite components of G. Let $2m-1$ be the minimum length of odd cycles of G. Let*

$$\overline{n}_0(G) := \begin{cases} 1 & \text{if } s = 0, \text{ i.e. } G \text{ is bipartite,} \\ \sum_{i=1}^{s}(\upsilon(G_i) - \varepsilon_0(G_i)) - m + 1 & \text{if } s \geqslant 1. \end{cases}$$

Then, $\overline{\operatorname{astab}}(I(G)) \leqslant \overline{n}_0(I(G))$. In particular, $\overline{\operatorname{astab}}(I(G)) < r$.

For any ideal I of R, Brodmann [3] showed that $\operatorname{depth} R/I^n$ is a constant for sufficiently large n, moreover

$$\lim_{n\to\infty} \operatorname{depth} R/I^n \leqslant \dim R - \ell(I)$$

where $\ell(I)$ is the analytic spread of I. As introduced in [12] we define *the index of depth stability* of I to be the number:

$$\operatorname{dstab}(I) := \min\{n_0 \mid \operatorname{depth} R/I^n = \operatorname{depth} R/I^{n_0} \text{ for all } n \geqslant n_0\}.$$

It is natural to find a bound for $\operatorname{dstab}(I)$. As until now we only knew the index of depth stability of few special classes of monomial ideals (see [7]), [11]), [13],[23] [29]).

As in the case of the stability of associated primes we also interested in bounding of the number:

$$\overline{\operatorname{dstab}}(I) := \min\{n_0 \mid \operatorname{depth} R/\overline{I^n} = \operatorname{depth} R/\overline{I^{n_0}} \text{ for all } n \geqslant n_0\}.$$

In [1] we obtain an upper bound for $\overline{\operatorname{dstab}}(I)$, but this bound is extremely large. In this paper we find a better bound for edge ideals.

Theorem 3.2. *Let G be a graph. Let $G_1, \ldots, G_s$ be all connected bipartite components of G and let $G_{s+1}, \ldots, G_{s+t}$ be all connected nonbipartite components of G. Let $2k_i$ be the maximum length of cycles of G_i ($k_i = 1$ if G_i is a tree) for all $i = 1, \ldots, s$; and let $2k_i - 1$ be the maximum length of odd cycles of G_i for every $i = s+1, \ldots, s+t$; and let $2m-1$ be the minimum length of odd cycles of G. Let*

$$\overline{n}(G) := \begin{cases} \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1 & \text{if } t = 0, \text{ i.e. } G \text{ is bipartite,} \\ \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + m & \text{if } t = 2j \text{ for some } j \geqslant 1, \\ \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + 1 & \text{if } t = 2j+1 \text{ for some } j \geqslant 0. \end{cases}$$

Then, $\overline{\operatorname{dstab}}(I(G)) \leqslant \overline{n}(I(G))$. *In particular,* $\overline{\operatorname{dstab}}(I(G)) < r$.

Furthermore, these bounds are always achieved if G is a unicyclic nonbipartite graph.

Our approach is based on a generalized Hochster formula for computing local cohomology modules of arbitrary monomial ideals formulated by Takayama [27]. The efficiency of this formula was shown in recent papers (see [9], [15], [21], [22]). Using this formula and the explicit description of that formula for symbolic powers of Stanley-Reisner ideals which were developed in [21], we are able to study the behavior of depths of powers of edge ideals.

The paper is organized as follows. In Section 1, we give the generalized Hochster formula to compute local cohomological modules of monomial ideals and some preliminary results on integral closures of monomial ideals. In section 2 we get an upper bound of $\overline{\operatorname{astab}}(I(G))$. In the last section, we get an upper bound for $\overline{\operatorname{dstab}}(I(G))$.

1. Preliminary

We recall some standard notation and terminology from graph theory here. Let G be a graph. The ends of an edge of G are said to be incident with the edge, and vice versa. Two vertices which are incident with a common edge are adjacent, and two distinct adjacent vertices are neighbors. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ and the degree of a vertex v in G, denoted by $\deg_G(v)$, is the number of neighbours of v in G. If there is no ambiguity in the context, we write $\deg v$ instead $\deg_G(v)$.

The graph G is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of G. It is well-known that G is bipartite if and only if G contains no odd cycle (see [2, Theorem 4.7]). We also have a nice algebraic characterization of bipartite graphs.

Lemma 1.1. ([25]) *Let G be a graph. Then $I(G)^n = I(G)^{(n)}$ for all $n \geqslant 1$ if and only if G is bipartite.*

In general case, our main tool to study $\overline{\operatorname{dstab}}(I(G))$ is a generalized version of a Hochster's formula (see [26, Theorem 4.1]) to compute local cohomology modules of monomial ideals given in [27].

Let $\mathfrak{m} := (x_1, \ldots, x_r)$ be the maximal homogeneous ideal of R and I a monomial ideal in R. Since R/I is an $\mathbb{N}^r$-graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over R/I. For every degree $\alpha \in \mathbb{Z}^r$ we denote by $H^i_{\mathfrak{m}}(R/I)_\alpha$ the α-component of $H^i_{\mathfrak{m}}(R/I)$.

Let $\Delta(I)$ denote the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$. For every $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ we set $G_\alpha := \{i \mid \alpha_i < 0\}$ and we denote by $\Delta_\alpha(I)$ the simplicial complex of all sets of the form $F \setminus G_\alpha$, where F is a face of $\Delta(I)$ containing G_α such that for every minimal generator x^β of I there exists an $i \notin F$ such that $\alpha_i < \beta_i$. To represent $\Delta_\alpha(I)$ in a more compact way, for every subset F of $[r]$ let $R_F := R[x_i^{-1} \mid i \in F \cup G_\alpha]$ and $I_F := IR_F$. This means that the ideal I_F of R_F

is generated by all monomials of I by setting $x_i = 1$ for all $i \in F \cup G_\alpha$. Then $x^\alpha \in R_F$ and by [9, Lemma 1.1] we have

$$\Delta_\alpha(I) = \{F \subseteq [r] \setminus G_\alpha \mid x^\alpha \notin I_F\}. \tag{1.1}$$

Lemma 1.2. ([27, Theorem 2.2]) $\dim_k H^i_{\mathfrak{m}}(R/I)_\alpha = \dim_k \widetilde{H}_{i-|G_\alpha|-1}(\Delta_\alpha(I); k)$.

Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. If $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$, we write $\Delta = \langle F_1, \dots, F_m\rangle$. The Stanley-Reisner ideal of Δ can be written as

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$$

where P_F is the prime ideal of R generated by variables x_i with $i \notin F$. For every integer $n \geqslant 1$, the n-th symbolic power of I_Δ is the monomial ideal

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^n.$$

Note that $\Delta(I_\Delta^{(n)}) = \Delta$. The next lemma is very useful to compute $\Delta_\alpha(I_\Delta^{(n)})$.

Lemma 1.3. ([21, Lemma 1.3]) *For all $\alpha \in \mathbb{N}^r$ and $n \geqslant 1$, we have*

$$\Delta_\alpha(I_\Delta^{(n)}) = \left\langle F \in \mathcal{F}(\Delta) \mid \sum_{i \notin F} \alpha_i \leqslant n-1 \right\rangle.$$

Throughout of this paper we let $\mathbf{e}_1, \dots, \mathbf{e}_r$ be the canonical basis of $\mathbb{R}^r$. For a monomial ideal I of R, let $G(I)$ denote the minimal system of monomial generators of I. For a vector $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, we denote by $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ a monomial of R. For a subset $\tau \subseteq [r]$ we write $\mathbf{x}^\tau$ to mean the square-free monomial $\prod_{i \in \tau} x_i$ of R. We also denote by $\mathbb{R}_+$ the set of nonnegative real numbers.

The *integral closure* of an arbitrary ideal J of R is the set of elements x in R that satisfy an integral relation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

where $a_i \in J^i$ for $i = 1, \dots, n$. It is denoted by $\overline{J}$ and it is an ideal.

The behavior of the sequence $\{\operatorname{Ass}(R/\overline{J^n})\}_{n \geqslant 1}$ is very nice, namely it is increasing.

Lemma 1.4. (See [10, Proposition 16.3]) *Let J be an arbitrary ideal of R. Then the sequence $\{\operatorname{Ass}(R/\overline{J^n})\}_{n \in \mathbb{N}}$ is increasing.*

For monomial ideals I, then $\overline{I}$ is a monomial ideal as well. We can describe the integral closure of a monomial ideal I geometrically via its Newton polyhedron.

Definition 1.5. Let I be a monomial ideal of R. We define

1. For a subset $A \subseteq R$, the exponent set of A is $E(A) := \{\alpha \mid \mathbf{x}^\alpha \in A\} \subseteq \mathbb{Z}^r$.
2. The Newton polyhedron of I is $NP(I) := \operatorname{conv}\{E(I)\}$, the convex hull of the exponent set of I in the space $\mathbb{R}^r$.

The bridge between the Newton polyhedron of I and its integral closure is given by the well-known equation:

$$E(\overline{I}) = NP(I) \cap \mathbb{Z}^r = \{\alpha \in \mathbb{N}^r \mid \mathbf{x}^{n\alpha} \in I^n \text{ for some } n \geqslant 1\}. \tag{1.2}$$

The Newton polyhedron of the power I^n for $n \geqslant 1$ is related to $NP(I)$ by

$$NP(I^n) = nNP(I) = n\,\mathrm{conv}\{E(I)\} + \mathbb{R}^r_+\,. \tag{1.3}$$

Remark 1.6. For any monomial ideal I of R, $F \subseteq [r]$ and $n \geqslant 1$, we have

$$(\overline{I^n})_F = \overline{I^n_F}.$$

We conclude this section with some remarks about operations on monomial ideals. Let $A := k[x_1, \ldots, x_s], B := k[y_1, \ldots, y_t]$ and $R := k[x_1, \ldots, x_s, y_1, \ldots, y_t]$ be polynomial rings where $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ are two disjoint sets of variables. For simplicity, for two ideals I of A and J of B we still denote IR and JR by I and J, respectively. Let $\mathfrak{m}_A = (x_1, \ldots, x_s)$ be the maximal homogeneous ideal of A.

Lemma 1.7. *$\overline{I} \cdot \overline{J} \subseteq \overline{IJ}$ for all monomial ideals I and J of R.*

Proof. Let $f \in \overline{I} \cdot \overline{J}$, so that $f = uv$ for some monomials $u \in \overline{I}$ and $v \in \overline{J}$. By Formula 1.2, one has $u^m \in I^m$ and $v^n \in J^n$ for some positive integers m and n. It follows that $u^{mn} \in I^{mn}$ and $v^{mn} \in J^{mn}$, so $f^{mn} = u^{mn}v^{mn} \in I^{mn}J^{mn} = (IJ)^{mn}$. By Formula 1.2 again, we get $f \in \overline{IJ}$. Thus, $\overline{I} \cdot \overline{J} \subset \overline{IJ}$, as required. $\square$

Lemma 1.8. *Let P be a monomial ideal of A generated by variables and J a monomial ideal of B. Then, for all $n \geqslant 1$, we have*

$$\overline{(P + J)^n} = \sum_{i=0}^{n} P^i \cdot \overline{J^{n-i}}$$

Proof. If $P = \mathbf{0}$, the lemma is obvious. Assume that $P \neq \mathbf{0}$. By Remark 1.6, we can assume that $P = \mathfrak{m}_A$.

We first prove that $\overline{(\mathfrak{m}_A + J)^n} \subseteq \sum_{i=0}^n \mathfrak{m}_A^i \cdot \overline{J^{n-i}}$. Let $\mathbf{x}^\alpha\mathbf{y}^\beta \in \overline{(\mathfrak{m}_A + J)^n}$. If $\deg(\mathbf{x}^\alpha) \geqslant n$, then $\mathbf{x}^\alpha\mathbf{y}^\beta \in \mathfrak{m}_A^n \subseteq \sum_{i=0}^n \mathfrak{m}_A^i \cdot \overline{J^{n-i}}$.

Assume that $p := \deg(\mathbf{x}^\alpha) < n$. In particular, $\mathbf{x}^\alpha \in \mathfrak{m}_A^p$. Since $\mathbf{x}^\alpha\mathbf{y}^\beta \in \overline{(\mathfrak{m}_A + J)^n}$, we have $\mathbf{x}^{m\alpha}\mathbf{y}^{m\beta} \in (\mathfrak{m}_A + J)^{mn}$ for some $m \geqslant 1$. Since $\mathbf{x}^{m\alpha} \in \mathfrak{m}_A^{pm}$ and

$$\mathbf{x}^{m\alpha}\mathbf{y}^{m\beta} \in (\mathfrak{m}_A + J)^{mn} = \sum_{j=1}^{mn} \mathfrak{m}_A^j J^{mn-j},$$

so $\mathbf{y}^{m\beta} \in J^{m(n-p)}$, and so $\mathbf{y}^\beta \in \overline{J^{n-p}}$. Therefore

$$\mathbf{x}^\alpha\mathbf{y}^\beta \in \mathfrak{m}_A^p\overline{J^{n-p}} \subseteq \sum_{i=0}^{n} \mathfrak{m}_A^i \cdot \overline{J^{n-i}}$$

and the claim follows.

In order to prove the reverse inclusion, for any $i = 0, \dots, n$, by Lemma 1.7 we have $\mathfrak{m}_A^i \overline{J^{n-i}} \subseteq \overline{\mathfrak{m}_A^i \cdot J^{n-i}} \subseteq \overline{(\mathfrak{m}_A + J)^n}$. This implies that

$$\sum_{i=0}^{n} \mathfrak{m}_A^i \overline{J^{n-i}} \subseteq \overline{\mathfrak{m}_A^i J^{n-i}} \subseteq \overline{(\mathfrak{m}_A + J)^n},$$

as required. □

Lemma 1.9. *Let I be a proper nonzero monomial ideal of R and n a positive integer. Let $P \in \operatorname{Ass} R/\overline{I^n}$ and $P = \overline{I^n} : f$ for some monomial f. Then, $f \in \overline{I^{n-1}}$.*

Proof. By assumption we have $P \neq \mathbf{0}$. Since P is generated by variables, we may assume that $x_1 \in P$. Since $fx_1 \in \overline{I^n}$, by Formula 1.2 we have $f^m x_1^m \in I^{mn}$ for some $m \geqslant 1$. Hence, $f^m x_1^m$ can be written as $f_1 \cdots f_{mn}$ where $f_1, \dots, f_{mn}$ are monomials in I. We may assume that $f_1, f_2, \dots, f_j$ are the only monomials which contain the variable x_1. Then $x_1^m \mid f_1 f_2 \cdots f_j$. Setting $p := \min\{j, m\}$, this implies $x_1^m \mid f_1 \cdots f_p$, i.e., $f_1 f_2 \cdots f_p = x_1^m g$, for some monomial g. Therefore,

$$f = g f_{p+1} f_{p+2} \cdots f_{mn} \in I^{mn-p} \subseteq I^{m(n-1)}.$$

By Formula 1.2 we get $f \in \overline{I^{n-1}}$, as claimed. □

Lemma 1.10. *Let J be a monomial ideal of B. Then, for all $n \geqslant 1$, we have*

$$\operatorname{Ass} R/\overline{(\mathfrak{m}_A + J)^n} = \{\mathfrak{m}_A + P \mid P \in \operatorname{Ass} R/\overline{J^n}\}.$$

In particular, $\overline{\operatorname{astab}}(\mathfrak{m}_A + J) = \overline{\operatorname{astab}}(J)$.

Proof. The case $J = \mathbf{0}$ is obvious, so we assume that $J \neq \mathbf{0}$.

To prove the inclusion "$\subseteq$". Let $Q \in \operatorname{Ass} R/\overline{(\mathfrak{m}_A + J)^n}$. Then, $Q = \overline{(\mathfrak{m}_A + J)^n} : \mathbf{x}^\alpha \mathbf{y}^\beta$ for some monomial $\mathbf{x}^\alpha \mathbf{y}^\beta$ of R. Note that $Q \supseteq \sqrt{\overline{(\mathfrak{m}_A + J)^n}} = \mathfrak{m}_A + \sqrt{J}$. Therefore, we can write $Q = \mathfrak{m}_A + P$ where P is generated by some variables in $y_1, \dots, y_t$. In order to prove our inclusion it suffice to show that $P \in \operatorname{Ass} R/\overline{J^n}$. Indeed, let $q := |\alpha|$. Then $q < n$ and $\mathbf{y}^\beta \notin \overline{J^{n-q}}$ by Lemma 1.8.

For any monomial $\mathbf{y}^\beta$ in R involving only variables in $y_1, \dots, y_t$, since $|\alpha| = p$, by Lemma 1.8 we imply that

$$\begin{aligned}\mathbf{y}^\gamma \in \overline{J^{n-q}} : \mathbf{y}^\beta &\Longleftrightarrow \mathbf{y}^\beta \mathbf{y}^\gamma \in \overline{J^{n-q}} \Longleftrightarrow \mathbf{x}^\alpha \mathbf{y}^\beta \mathbf{y}^\gamma \in \overline{(\mathfrak{m}_A + J)^n} \\ &\Longleftrightarrow \mathbf{y}^\gamma \in \overline{(\mathfrak{m}_A + J)^n} : \mathbf{x}^\alpha \mathbf{y}^\beta = \mathfrak{m}_A + P \Longleftrightarrow \mathbf{y}^\gamma \in P.\end{aligned}$$

Hence, $P = \overline{J^{n-q}} : \mathbf{y}^\gamma$, and hence $P \in \operatorname{Ass} R/\overline{J^{n-q}}$. On the other hand, by Lemma 1.4 we have $R/\overline{J^{n-q}} \subseteq R/\overline{J^n}$. Thus, $P \in \operatorname{Ass} R/\overline{J^n}$, as claimed.

We now prove the reverse inclusion. Let $P \in \operatorname{Ass} R/\overline{J^n}$. Then $P = \overline{J^n} : \mathbf{y}^\beta$ for some monomial $\mathbf{y}^\beta \in R$. Let $Q := \mathfrak{m}_A + P$. In order to complete the proof it suffices to show that $Q = \overline{(\mathfrak{m}_A + J)^n} : \mathbf{x}^\alpha \mathbf{y}^\beta$.

Since $\mathbf{y}^\beta \notin \overline{J^n}$, by Lemma 1.8 we conclude that $\mathbf{y}^\beta \notin \overline{(\mathfrak{m}_A + J)^n}$. By Lemma 1.9 we have $\mathbf{y}^\beta \in \overline{J^{n-1}}$. Together with Lemma 1.8 again, we have $x_j \mathbf{y}^\beta \in \mathfrak{m}_A \overline{J^{n-1}} \subseteq$

$\overline{(\mathfrak{m}_A + J)^n}$. Now for any monomial $\mathbf{y}^\gamma \in R$ involving only variables in $y_1, \dots, y_t$ such that $\mathbf{y}^\gamma \in \overline{(\mathfrak{m}_A + J)^n} : \mathbf{y}^\beta$, then $\mathbf{y}^\beta \mathbf{y}^\gamma \in \overline{(\mathfrak{m}_A + J)^n}$. Together with Lemma 1.8, this fact forces $\mathbf{y}^\beta \mathbf{y}^\gamma \in \overline{J^n}$, so $\mathbf{y}^\gamma \in \overline{J^i} : \mathbf{y}^\beta = P$. It follows that $Q = \mathfrak{m}_A + P = \overline{(\mathfrak{m}_A + J)^n} : \mathbf{y}^\beta$, so $Q \in \operatorname{Ass} R/\overline{(\mathfrak{m}_A + J)^n}$, as required. □

2. Stability of Associated Prime Ideals

In this section for a graph G we always assume that $V(G) = [r]$; $R = k[x_1, \dots, x_r]$ is a polynomial ring over fields k and $\mathfrak{m} = (x_1, \dots, m_r)$ is the maximal homogeneous ideal of R. Recall that a vertex of degree one in G is a *leaf*; and an edge of G is incident with a leaf is called a leaf edge.

In this paper we will establish a bound of $\operatorname{dstab}(I(G))$ for any graph G. First we have some information about $\operatorname{dstab}(IG))$ when every component of G is nonbipartite.

Lemma 2.1. *Let G be a connected nonbipartite graph. Let $2m-1$ be the maximum length of odd cycles of G. Then for any $n \geqslant \upsilon(G) - \varepsilon_0(G) - m + 1$, there is a monomial f of degree $2n-1$ such that $\mathfrak{m} = I(G)^n : f$. Moreover, $f^2 \in I(G)^{2n-1}$.*

Proof. First we prove the existence of f. Let C be an old cycle of G with length $2m-1$. If C' is another cycle of G, then C' has an edge e that does not lie on the cycle C. Delete this edge from G, thereby obtaining a connected subgraph G' of G with $V(G') = V(G)$ and C is a cycle of G'. This process continues until we obtain a connected subgraph H of G such that $V(G) = V(H)$ and H has only one cycle C. Let $s := \upsilon(H) - \varepsilon_0(H) - m + 1$. By [29, Lemma 2.2] there is a monomial $g \in R$ such that $\deg g = 2s - 1$ and $x_i g \in I(H)^s$ for all $i = 1, \dots, r$. Because $I(H) \subseteq I(G)$, therefore

$$x_i g \in I(G)^s \text{ for all } i = 1, \dots, r.$$

As G is generated by quadric monomials and $\deg g = 2s - 1$, so $g \notin I(G)^s$. Thus, $\mathfrak{m} = I(G)^s : g$. Since $\upsilon(G) = \upsilon(H)$ and $\varepsilon_0(G) \leqslant \varepsilon_0(H)$, hence $s \leqslant \upsilon(G) - \varepsilon_0(G) - m + 1$.

Write $n = s + t$ for some $t \geqslant 0$. Let us choose an quadratic monomial u of $I(G)$ and let $f := u^t g$. Since $\deg f = 2t + \deg g = 2t + 2s - 1 = 2n - 1$, so $f \notin I(G)^n$. It is clear that $f^2 \in I(G)^{2n-1}$ and $\mathfrak{m} = I(G)^n : f$, thus the lemma follows. □

Lemma 2.2. *Let G be a graph. Let $f_1, \dots, f_{2s}$ be monomials of R with $s \geqslant 1$. Assume that for all $i = 1, \dots, 2s$, we have $\deg f_i = 2n_i - 1$ and $f_i^2 \in I(G)^{2n_i - 1}$ where $n_i \geqslant 1$. Then,*

$$f_1 \cdots f_{2s} \in \overline{I(G)^n}$$

where $n = n_1 + \cdots + n_{2s} - s$.

Proof. For each $i = 1, \dots, 2s$, since $\deg f_i^2 = 2(n_i - 1)$ and $f^2 \in I(G_i)^{2n_i - 1}$, together with the fact that $I(G)$ is generated by quadric monomials we imply that there are $(2n_i - 1)$ quadric monomials, say $\mathbf{x}^{\alpha_{i,1}}, \dots, \mathbf{x}^{i, 2n_i - 1}$, of $I(G)$ such that

$$f_i^2 = \mathbf{x}^{\alpha_{i,1}} \cdots \mathbf{x}^{\alpha_{i,2n_i-1}}.$$

Let $f := f_1 \cdots f_{2s}$. Write $f := \mathbf{x}^\alpha$ with $\alpha \in \mathbb{N}^r$. Then,

$$2\alpha = \sum_{i=1}^{2s} \sum_{j=1}^{2i-1} \alpha_{i,j},$$

hence

$$\alpha = \sum_{i=1}^{2s} \sum_{j=1}^{2i-1} \frac{1}{2}\alpha_{i,j}.$$

Since

$$\sum_{i=1}^{2s} \sum_{j=1}^{2n_i-1} \frac{1}{2} = \sum_{i=1}^{2s} \frac{2n_i - 1}{2} = \sum_{i=1}^{2s} n_i - s = n,$$

by Formula (1.3) we conclude that $\mathbf{x}^\alpha \in \overline{I(G)^n}$, as required. □

Lemma 2.3. *Let G be a graph with connected components $G_1, \ldots, G_p$. Assume that all these components are nonbipartite. For each $i = 1, \ldots, p$, let $2m_i - 1$ be the maximum length of odd cycles of G_i. Let $2l - 1$ be the minimum length of odd cycles of G. Let*

$$n_0 := \begin{cases} \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^p m_i + s + 1 & \text{if } p = 2s+1 \text{ for some } s \geqslant 0; \\ \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^p m_i + s + l & \text{if } p = 2s \text{ for some } s \geqslant 1. \end{cases}$$

Then, $\mathfrak{m} \in \operatorname{Ass} R/\overline{I(G)^n}$ for any $n \geqslant n_0$. Moreover, there is a monomial f of degree $2n - 1$ such that $\mathfrak{m} \in \overline{I(G)^n} : f$

Proof. Case 1: $p = 2s + 1$. If $s = 0$, then lemma follows from Lemma 2.1. Thus we assume that $s \geqslant 1$. Let $n_i := \upsilon(G_i) - \varepsilon_0(G_i) - m_i + 1$ for $i = 1, \ldots, 2s$ and $n_{2s+1} := n-(n_1+\cdots+n_{2s}-s)$. Then we have $n_{2s+1} \geqslant \upsilon(G_{2s+1})-\varepsilon_0(G_{2s+1})-m_{2s+1}+1$. By Lemma 2.1, for each $i = 1, \ldots, 2s + 1$, there is a monomial f_i of degree $2n_i - 1$ such that

$$\mathfrak{m}_i = I(G_i)^{n_i} : f_i \text{ and } f_i^2 \in I(G_i)^{2n_i-1} \tag{2.1}$$

where $\mathfrak{m}_i = (x_j \mid j \in V(G_i))$.

Let $f := f_1 \ldots f_{2s+1}$, so that $\deg f = 2(n_1 + \cdots + n_{2s+1}) - (2s + 1) = 2n - 1$. It follows $f \notin \overline{I(G)^n}$. We now prove that $\mathfrak{m} = \overline{I(G)^n} : f$. It suffices to show $fx_i \in \overline{I(G)^n}$ for each $i = 1, \ldots, r$. In order to prove $fx_i \in \overline{I(G)^n}$, we may assume that $i \in V(G_1)$. By Formula (2.2) we have $f_1x_i \in \overline{I(G)^{n_1}}$. Let $J := I(G_2) + \cdots + I(G_{2s+1})$ and $m := n_2 + \cdots + n_{2s+1} - s$. By Lemma 2.2 one has $f_2 \cdots f_{2s+1} \in \overline{J^m}$. Notice that $n_1 + m = n$ and $J \subseteq I(G)$. Together with Lemma 1.7 we get

$$fx_i = (f_1x_i)f_2 \cdots f_{2s+1} \in \overline{I(G_1)^{n_1}} \cdot \overline{J^m} \subseteq \overline{I(G_1)^{n_1} J^m} \subseteq \overline{I(G)^{n_1+m}} = \overline{I(G)^n}.$$

It follows $\mathfrak{m} = \overline{I(G)^n} : f$ and the lemma holds for this case.

Case 2: $p = 2s$. The proof is almost the same as the previous case. We may assume that G_1 has a cycle of length $2l - 1$. Let $n_i := \upsilon(G_i) - \varepsilon_0(G_i) - m_i + 1$

for $i = 1, \dots, 2s-1$ and $n_{2s} := n - (l + n_1 + \cdots + n_{2s-1} - s)$. Then we have $n_{2s+1} \geqslant \upsilon(G_{2s+1}) - \varepsilon_0(G_{2s+1}) - m_{2s+1} + 1$. By Lemma 2.1, for each $i = 1, \dots, 2s$, there is a monomial f_i of degree $2n_i - 1$ such that

$$\mathfrak{m}_i = I(G_i)^{n_i} : f_i \text{ and } f_i^2 \in I(G_i)^{2n_i - 1} \tag{2.2}$$

where $\mathfrak{m}_i = (x_j \mid j \in V(G_i))$.

We can assume that G_1 has a cycle of length $2l-1$, say C. We also can assume that $V(G_1) = \{x_1, \dots, x_q\}$ for some $1 \leqslant q < r$ and the cycle C is $x_1, \dots, x_{2l-1}$. Let $g := x_1 x_2 \cdots x_{2l-1}$. Then $\deg g = 2l - 1$ and $g^2 \in I(C)^{2l-1}$. Let $f := g f_1 \dots f_{2s}$, so that $\deg f = (2l-1) + 2(n_1 + \cdots + n_{2s}) - 2s = 2n - 1$. It follows $f \notin \overline{I(G)^n}$. We now prove that $\mathfrak{m} = \overline{I(G)^n} : f$. It suffices to show $f x_i \in \overline{I(G)^n}$ for each $i = 1, \dots, r$. First if $i \in V(G_1)$, then by Formula (2.2) we have $f_1 x_i \in I(G)^{n_1}$. Let $J := I(C) + I(G_2) + \cdots + I(G_{2s+1})$ and $m := l + n_2 + \cdots + n_{2s} - s$. By Lemma 2.2 one has $g f_2 \cdots f_{2s} \in \overline{J^m}$. Notice that $n_1 + m = n$ and $J \subseteq I(G)$. Together with Lemma 1.7 we get

$$f x_i = (f_1 x_i) g f_2 \cdots f_{2s} \in \overline{I(G_1)^{n_1}} \cdot \overline{J^m} \subseteq \overline{I(G_1)^{n_1} \cdot J^m} \subseteq \overline{I(G)^{n_1+m}} = \overline{I(G)^n}.$$

For the case $i \in V(G_j)$ for some $2 \leqslant j \leqslant 2s$, we prove $f x_i \in \overline{I(G)^n}$ by the same way. Thus, the proof is complete. ☐

We are now in position to prove the main result of this section.

Theorem 2.4. *Let G be a graph. Let $G_1, \dots, G_s$ be all connected nonbipartite components of G. Let $2m-1$ be the minimum length of odd cycles of G. Let*

$$\overline{n}_0(G) := \begin{cases} 1 & \text{if } s = 0, \text{ i.e. } G \text{ is bipartite}, \\ \sum_{i=1}^{s} (\upsilon(G_i) - \varepsilon_0(G_i)) - m + 1. & \text{if } s \geqslant 1. \end{cases}$$

Then, $\overline{\operatorname{astab}}(I(G)) \leqslant \overline{n}_0(I(G))$. *In particular,* $\overline{\operatorname{astab}}(I(G)) < r$.

Proof. We start the proof with two remarks on $\overline{n}_0(G)$ that $\overline{n}_0(G) \geqslant 1$ and $\overline{n}_0(G) \geqslant \overline{n}_0(H)$ for any induced subgraph H of G.

Now we turn to prove the theorem. If G is bipartite, then for all $n \geqslant 1$, by Lemma 1.1 we have $I(G)^n = I(G)^{(n)}$. Since $I(G)$ is a square-free ideal, so $I(G)^n \subseteq \overline{I(G)^n} \subseteq I(G)^{(n)}$. This yields $\overline{I(G)^n} = I(G)^{(n)}$, so $\overline{\operatorname{astab}}(I(G)) = 1$. Thus, the theorem holds for this case.

We next prove the theorem by induction on $r = \upsilon(G)$. If $r \leqslant 2$, then $I(G)$ is obviously bipartite, and then the lemma holds by the argument above.

Assume that $r \geqslant 3$. By [28, Lemma 11] we have

$$\operatorname{Ass} R/\overline{I(G)^n} \setminus \{\mathfrak{m}\} = \bigcup_{i=1}^{r} \operatorname{Ass} R/\overline{I(G)^n_{\{i\}}}. \tag{2.3}$$

Now if G has a connected bipartite component, then [18, Proposition 3.3] forces $\mathfrak{m} \notin \operatorname{Ass} R/\overline{I(G)^m}$ for all $m \geqslant 1$. Thus,

$$\operatorname{Ass} R/\overline{I(G)^n} = \bigcup_{i=1}^{r} \operatorname{Ass} R/\overline{I(G)_{\{i\}}^n} \quad \text{for all } n \geqslant 1.$$

In particular, $\overline{\operatorname{astab}}(I(G)) \leqslant \max\{\overline{\operatorname{astab}}(I(G)_{\{1\}}), \ldots, \overline{\operatorname{astab}}(I(G)_{\{r\}})\}$.

If every connected component of G is nonbipartite, then by Lemma 2.3 we conclude that $\mathfrak{m} \in \operatorname{Ass} R/\overline{I(G)^n}$ for all $n \geqslant \overline{n}_0(G)$. Together this fact with Equation (2.3) we obtain

$$\operatorname{Ass} R/\overline{I(G)^n} = \{\mathfrak{m}\} \cup \bigcup_{i=1}^{r} \operatorname{Ass} R/\overline{I(G)_{\{i\}}^n} \quad \text{for all } n \geqslant \overline{n}_0(G).$$

In particular, $\overline{\operatorname{astab}}(I(G)) \leqslant \max\{\overline{n}_0(G), \overline{\operatorname{astab}}(I(G)_{\{1\}}), \ldots, \overline{\operatorname{astab}}(I(G)_{\{r\}})\}$.

Thus, $\overline{\operatorname{astab}}(I(G)) \leqslant \max\{\overline{n}_0(G), \overline{\operatorname{astab}}(I(G)_{\{1\}}), \ldots, \overline{\operatorname{astab}}(I(G)_{\{r\}})\}$ is always valid. So it remains to show that $\overline{\operatorname{astab}}(I(G)_{\{i\}}) \leqslant \overline{n}_0(G)$ for each $i = 1, \ldots, r$.

For any $1 \leqslant i \leqslant r$, we can assume $N_G(i) = \{1, \ldots, p\}$. Then $I(G)_{\{i\}} = (x_1, \ldots, x_p) + I(H)$ where $H = G \setminus (\{i\} \cup N_G(i))$ is an induced subgraph of G. Now by Lemma 1.10 we have $\overline{\operatorname{astab}}(I(G)_{\{i\}}) = \overline{\operatorname{astab}}(I(H))$. Since $v(H) < v(G)$, the induction hypothesis yields $\overline{\operatorname{astab}}(H) \leqslant \overline{n}_0(H)$. On the other hand, since H is an induced subgraph of G, so $\overline{n}_0(H) \leqslant \overline{n}_0(G)$. It follows that $\overline{\operatorname{astab}}(I(G)_{\{i\}}) \leqslant \overline{n}_0(G)$, as required. □

3. Stability of Depths

Let G be a graph. An independent set in G is a set of vertices no two of which are adjacent to each other. An independent set S in G is maximal (with respect to set inclusion) if the addition to S of any other vertex in the graph destroys the independence. Let $\Delta(G)$ be the set of independent sets of G. Then $\Delta(G)$ is a simplicial complex and this complex is the so-called independence complex of G; and facets of $\Delta(G)$ are just maximal independent sets of G. Note that $I(G) = I_{\Delta(G)}$.

If G is bipartite, then the vertex set of G can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; and such a partition (X, Y) is called a bipartition of G. Clearly then, X and Y are facets of $\Delta(G)$.

Lemma 3.1. *Let G be a graph with s connected bipartite components. Then,*

(1) $\min\{\operatorname{depth} R/\overline{I(G)^n} \mid n \geqslant 1\} = s$.

(2) $\lim_{n\to\infty} \operatorname{depth} R/\overline{I(G)^n} = s$.

Proof. By [1, Theorem 2.5] we imply that

(1) $\min\{\operatorname{depth} R/\overline{I(G)^n} \mid n \geqslant 1\} = \dim R - \ell(I(G))$.

(2) $\lim_{n\to\infty} \operatorname{depth} R/\overline{I(G)^n} = \dim R - \ell(I(G))$.

On the other hand, $s = \dim R - \ell(I(G))$ (see [30, Page 50]), and the lemma follows. □

We now ready to prove the main result of this section.

Theorem 3.2. *Let G be a graph. Let $G_1, \dots, G_s$ be all connected bipartite components of G and let $G_{s+1}, \dots, G_{s+t}$ be all connected nonbipartite components of G. Let $2k_i$ be the maximum length of cycles of G_i ($k_i = 1$ if G_i is a tree) for all $i = 1, \dots, s$; and let $2k_i - 1$ be the maximum length of odd cycles of G_i for every $i = s+1, \dots, s+t$; and let $2m-1$ be the minimum length of odd cycles of G. Let*

$$\overline{n}(G) := \begin{cases} \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1 & \text{if } t = 0, \text{ i.e. } G \text{ is bipartite,} \\ \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + m & \text{if } t = 2j \text{ for some } j \geqslant 1, \\ \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + 1 & \text{if } t = 2j+1 \text{ for some } j \geqslant 0. \end{cases}$$

Then, $\overline{\operatorname{dstab}}(I(G)) \leqslant \overline{n}(I(G))$. *In particular,* $\overline{\operatorname{dstab}}(I(G)) < r$.

Proof. If $t = 0$, i.e. G is a bipartite graphs, then by Lemma 1.1 we deduce that

$$\overline{I(G)^n} = I(G)^n \text{ for all } n \geqslant 1$$

hence $\overline{\operatorname{dstab}}(I(G)) = \operatorname{dstab}(I(G))$. The theorem now follows from [29, Theorem 4.6].

If $s = 0$, i.e. every connected component of G is nonbipartite. By Lemma 2.3 we have $\mathfrak{m} \in \operatorname{Ass} R/\overline{I(G)^n}$ for all $n \geqslant \overline{n}(I(G))$, i.e. $\operatorname{depth} R/\overline{I(G)^n} = 0$ with such n. Thus, $\overline{\operatorname{dstab}}(I(G)) \leqslant \overline{n}(I(G))$ and the theorem follows.

Assume that $s \geqslant 1$ and $t \geqslant 1$. In order to prove the theorem it suffices to show $\operatorname{depth} R/\overline{I(G)^n} = s$ for all $n \geqslant \overline{n}(G)$.

Let $H := G_1 \sqcup \cdots \sqcup G_s$ and $W := G_{s+1} \sqcup \cdots \sqcup G_{s+t}$. Then $G = H \sqcup W$ and $I(G) = I(H) + I(W)$. Note that $\upsilon(G) = \upsilon(H) + \upsilon(W)$ and $\varepsilon_0(G) = \varepsilon_0(H) + \varepsilon_0(W)$. Then we see that H is a bipartite graph with s components $H_1 = G_1, \dots, H_s = H_s$ and W is a nonbipartite graph with t components $G_{s+1}, \dots, G_{s+t}$.

We can assume that $V(W) = \{x_1, \dots, x_p\}$ and $V(H) = \{x_{p+1}, \dots, x_{p+q}\}$ where $q = r - q$. For simplicity, we set $y_1 := x_{p+1}, \dots, y_q := x_{p+q}$. Let $n_1 := \operatorname{dstab}(I(G))$ and $n_0 := n - n_0 + 1$. By [29, Theorem 4.6] we have $n_1 \leqslant \upsilon(H) - \varepsilon_0(H) - \sum_{i=1}^{s} k_i + 1$, so

(1) if $t = 2j$ for some $j \geqslant 1$, then $\overline{n}(G) = \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + m$, and then

$$n_0 = n - n_1 + 1 \geqslant \overline{n}(G) - n_1 + 1 \geqslant \upsilon(G) - \varepsilon_0(G) - \sum_{i=s+1}^{s+t} k_i + j + m.$$

(2) if $t = 2j+1$ for some $j \geqslant 0$, then $\overline{n}(G) = \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + j + 1$, and then

$$n_0 = n - n_1 + 1 \geqslant \overline{n}(G) - n_1 + 1 \geqslant \upsilon(G) - \varepsilon_0(G) - \sum_{i=s+1}^{s+t} k_i + j + 1.$$

By Lemma 2.3 there is a vector $\beta = (\beta_1, \dots, \beta_p) \in \mathbb{N}^p$ such that $\deg \mathbf{x}^\beta = 2n_1 - 1$ and $(x_1, \dots, x_p) = \overline{I(W)^{n_0}} : \mathbf{x}^\beta$. Together with Lemma 1.9, this fact gives

$$\mathbf{x}^\beta \in \overline{I(W)^{n_0-1}} \text{ and } \mathbf{x}^\beta \in \overline{I(W)^{n_0}_F} \text{ whenener } \emptyset \neq F \subseteq [p]. \tag{3.1}$$

Since $n_1 = \operatorname{dstab}(I(H))$ and H is bipartite, by [29, Theorem 4.4 and Lemma 4.1] we conclude that there is $\alpha = (\alpha_1, \dots \alpha_q) \in \mathbb{N}^q$ such that

$$\widetilde{H}_{q-1}(\Delta_\alpha(I(H)^{n_1}); k) \neq \mathbf{0}, \quad \text{and} \sum_{i \notin V} \alpha_i = n_1 - 1 \quad \text{for all } V \in \mathcal{F}(\Delta_\alpha(I(G)^{n_1})). \tag{3.2}$$

Let $\gamma := (\beta_1, \dots, \beta_p, \alpha_1, \dots, \alpha_q) \in \mathbb{N}^r$ and $\mathbf{x}^\gamma := \mathbf{x}^\beta \mathbf{y}^\alpha \in R$. We now claim that

$$\Delta_\gamma(\overline{I(G)^n}) = \Delta_\alpha(I(H)^{n_1}). \tag{3.3}$$

Indeed, for all $F \in \Delta_\gamma(I(G)^n)$ we can partition F into $F = F_1 \cup F_2$ where $F_1 \in \Delta(W)$ and $F_2 \in \Delta(H)$. By Equation (1.1) we have

$$\mathbf{x}^\gamma = \mathbf{x}^\beta \mathbf{y}^\alpha \notin \overline{I(G)^n_F} = \overline{(I(W)_{F_1} + I(H)_{F_2})^n}. \tag{3.4}$$

Now, if $F_1 \neq \emptyset$, then by Formula (3.1) we would have $\mathbf{x}^\beta \in \overline{I(W)^{n_0}_{F_1}}$. On the other hand, by Lemma 1.7 we imply that

$$\overline{I(W)^{n_0}_{F_1}} \cdot \overline{I(H)^{n-n_0}_{F_2}} = \overline{I(W)^{n_0}_{F_1} \cdot I(H)^{n-n_0}_{F_2}} \subseteq \overline{(I(W)_{F_1} + I(H)_{F_2})^n} = \overline{(I(G)^n_F},$$

hence $\mathbf{y}^\alpha \notin \overline{I(H)^{n-n_0}_{F_2}} = \overline{I(H)^{n_1-1}_{F_2}}$. Since H is bipartite, $\overline{I(H)^{n_1-1}_{F_2}} = I(H)^{n_1-1}_{F_2}$, so $F_2 \in \Delta_\alpha(I(H)^{n_1-1})$. In particular, $\Delta_\alpha(I(H)^{n_1-1}) \neq \emptyset$. On the other hand, Formula (3.2) and Lemma 1.3 imply that every facet $V \in \Delta(H)$ satisfies:

$$\sum_{i \notin V} \alpha_i \geqslant n_1 - 1,$$

so $\Delta_\alpha(I(H)^{n_1-1}) = \emptyset$ by Lemma 1.3, a contradiction. Thus, $F_1 = \emptyset$, and thus $F = F_2$. Formula (3.4) now becomes

$$\mathbf{x}^\gamma = \mathbf{x}^\beta \mathbf{y}^\alpha \notin \overline{(I(W) + I(H)_F)^n}.$$

Together this fact with Formula (3.1) we imply that $\mathbf{y}^\alpha \notin \overline{I(H)^{n-n_0+1}_F} = \overline{I(H)^{n_1}_F}$, or equivalently, $F \in \Delta_\alpha(\overline{I(H)^{n_1}})$, and so $\Delta_\gamma(\overline{I(G)^n}) \subseteq \Delta_\alpha(\overline{I(H)^{n_1}}) = \Delta_\alpha(I(H)^{n_1})$.

We now prove the reverse inclusion. Let F be a facet of $\Delta_\alpha(I(H)^{n_1})$. We need prove that $F \in \Delta_\gamma(\overline{I(G)^n})$. Indeed, $\mathbf{y}^\alpha \notin I(H)^{n_1}_F$ by Equation (1.1). If $\mathbf{x}^\gamma \in \overline{I(G)^n_F}$, then

$$\mathbf{x}^\gamma = \mathbf{x}^\beta \mathbf{y}^\alpha \in \overline{I(G)^n_F} = \overline{(I(W) + I(H)_F)^n}.$$

Note that F is a facet of $\Delta(H)$, so $I(H)_F = (y_i \mid i \notin F)$. By Lemma 1.8 we get

$$\overline{(I(W) + I(H)_F)^n} = \sum_{\nu=1}^{n} (y_i \mid i \notin F)^\nu \cdot \overline{I(W)^{n-\nu}}.$$

By Formula (3.2) we obtain

$$\sum_{i \notin F} \alpha_i = n_1 - 1,$$

so $\mathbf{y}^\alpha \in (y_i \mid i \notin F)^{n_1-1}$. Note that $\mathbf{y}^\alpha \notin (y_i \mid i \notin F)^{n_1}$ and

$$\mathbf{x}^\beta\mathbf{y}^\alpha \in \overline{(I(W)+I(H)_F)^n} = \sum_{\nu=1}^{n}(y_i \mid i \notin F)^\nu \cdot \overline{I(W)^{n-\nu}},$$

this fact forces $\mathbf{x}^\beta \in \overline{I(W)^{n-(n_1-1)}} = \overline{I(W)^{n_0}}$. But $I(W)^{n_0}$ generated by monomials of degree $2n_0$ while $\deg \mathbf{x}^\beta = 2n_0 - 1$, so $\mathbf{x}^\beta \in \overline{I(W)^{n_0}}$, a contradiction.

Hence, we must have $\mathbf{x}^\gamma \notin \overline{I(G)^n_F}$, i.e., $F \in \Delta_\gamma(\overline{I(G)^n})$, and hence $\Delta_\alpha(I(H)^{n_1}) \subseteq \Delta_\gamma(\overline{I(G)^n})$, and the claim follows.

Together Formulas (3.2) and (3.3) with Lemma 1.2 we obtain

$$\dim_k H^s_{\mathfrak{m}}(R/\overline{I(G)^n})_\gamma = \dim_k \widetilde{H}_{s-1}(\Delta_\gamma(\overline{I(G)^n}); k) = \dim_k \widetilde{H}_{s-1}(\Delta_\alpha(I(H)^m; k) \neq 0,$$

therefore $H^s_{\mathfrak{m}}(R/\overline{I(G)^n}) \neq \mathbf{0}$, and therefore $\operatorname{depth} R/\overline{I(G)^n} \leqslant s$. On the other hand, Lemma 3.1 gives $\operatorname{depth} R/\overline{I(G)^n} \geqslant s$. Thus, $\operatorname{depth} R/\overline{I(G)^n} = s$, as required. $\square$

Example 3.3. Let G be a unicyclic nonbipartite graph. Assume the unique cycle of G is of length $2m-1$. Then,

$$\overline{\operatorname{astab}}(I(G)) = \overline{\operatorname{dstab}}(I(G)) = \upsilon(G) - \varepsilon_0(G) - m + 1.$$

Proof. Let $\mathfrak{m} = (x_1, \ldots, x_r)$. We first claim that, for $n \geqslant 1$, then

$$\mathfrak{m} \in \operatorname{Ass} R/\overline{I(G)^n} \Longleftrightarrow n \geqslant \upsilon(G) - \varepsilon_0(G) - m + 1. \tag{3.5}$$

Indeed, ($\Longrightarrow$) if $\mathfrak{m} \in \operatorname{Ass} R/\overline{I(G)^n}$. Since $\operatorname{Ass} R/\overline{I(G)^n} \subseteq \operatorname{Ass} R/I(G)^n$ by [20, Proposition 3.17], we have $\mathfrak{m} \in \operatorname{Ass} R/I(G)^n$, so $\operatorname{depth} R/I(G)^n = 0$. By [29, Lemmas 1.1 and 5.2], we conclude that $n \geqslant \upsilon(G) - \varepsilon_0(G) - m + 1$. ($\Longleftarrow$) follows from Lemma 2.3, as claimed.

Notice that Formula (3.5) is actually equivalent to $\overline{\operatorname{dstab}}(I(G)) = \upsilon(G) - \varepsilon_0(G) - m + 1$.

Now we turn to prove $\overline{\operatorname{astab}}(I(G)) = \upsilon(G) - \varepsilon_0(G) - m + 1$. Indeed, by (3.5) we imply $\overline{\operatorname{astab}}(I(G)) \geqslant \upsilon(G) - \varepsilon_0(G) - m + 1$. On the other hand, by Theorem 2.4 we have $\overline{\operatorname{astab}}(I(G)) \leqslant \upsilon(G) - \varepsilon_0(G) - m + 1$. Thus, $\overline{\operatorname{astab}}(I(G)) = \upsilon(G) - \varepsilon_0(G) - m + 1$, as required.

$\square$

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References

[1] Algebraic Group, *Stability of Depths and Cohen-Macaulayness of Integral Closures of Powers of Monomial Ideals*, Preprint.

[2] J. A. Bondy and U. S. R. Murty, Graph theory, Springer 2008.

[3] M. Brodmann, *The Asymptotic Nature of the Analytic Spread*, Math. Proc. Cambridge Philos Soc. **86** (1979), 35-39.

[4] M. Brodmann, *Asymptotic stability of* $\operatorname{Ass}(M/I^nM)$, Proc. AMS. **74** (1979), 16-18.

[5] W. Brun and J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge 1993.

[6] J. Chen, S. Morey and A. Sung, *The Stable Set of Associated Primes of the Ideal of a Graph*, Rocky Mountain J. Math. **32** (2002), 71-89.

[7] R.C. Cowsik and M.V. Nori, *Fibers of blowing up*, J. Indian Math. Soc. **40** (1976), 217-222.

[8] D. Eisenbud and C. Huneke, *Cohen-Macaulay Rees Algebras and their Specializations*, J. Algebra **81** (1983) 202-224.

[9] D. H. Giang and L. T. Hoa, *On local cohomology of a tetrahedral curve*, Acta Math. Vietnam., **35** (2010), no. 2, 229-241.

[10] M. Herrmann, S. Ikeda, U. Orbanz, Equimultiplicity and Blowing up, Springer-Verlag, 1988.

[11] J. Herzog and T. Hibi, *The Depth of Powers of an Ideal*, J. Algebra **291** (2005), 534-550.

[12] J. Herzog, A. Rauf and M. Vladoiu, *The stable set of associated prime ideals of a polymatroidal ideal*, J. Algebraic Combin. **37** (2013), no. 2, 289-312.

[13] J. Herzog and A. A. Qureshi, *Persistence and stability properties of powers of ideals*, arXiv:1208.4684 [math.AC]

[14] L.T. Hoa, *Stability of associated primes of monomial ideals*, Vietnam J. Math. **34** (2006), no. 4, 473-487.

[15] L. T. Hoa and T. N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Cambridge Philos Soc. **149** (2010), 1-18.

[16] M. Hochster, *Rings of Invariants of Tori, Cohen-Macaulay Rings Generated by Monomials, and Polytopes*, Ann. of Math. **96** (1972), 318-337.

[17] C. Huneke, *On the associated graded ring of an ideal*, Illinois J. Math. **26** (1982), 121-137.

[18] J. Martinez-Bernal, S. Morey, R. H. Villarreal, *Associated primes of powers of edge ideals*, Collect. Math. DOI 10.1007/s13348-011-0045-9.

[19] S. McAdam and P. Eakin, *The asymptotic Ass*, J. Algebra **61** (1979) 71-81.

[20] McAdam, S., Asymptotic Prime Divisors, Lecture Notes in Mathematics, vol. 103. Springer-Verlag, New York (1983)

[21] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of powers of two-dimensional squarefree monomial ideals*, J. Algebra **322** (2009), 4219-4227.

[22] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley-Reisner ideals*, Adv. Math. **226** (2011), no. 2, 1285-1306.

[23] S. Morey, *Depths of powers of the edge ideal of a tree*, Comm. Algebra **38** (2010), no. 11, 4042-4055.

[24] S. Morey, R. H. Villarreal, *Edge ideals: algebraic and combinatorial properties*, arXiv:1012.5329v3

[25] A. Simis, W.V. Vasconcelos and R.H. Villarreal, *On the Ideal Theory of Graphs*, J. Algebra **167** (1994), 389-416.

[26] R. P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhauser, Boston, MA, 1996.

[27] Y. Takayama, *Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327-344.

[28] T. N. Trung, *Stability of associated primes of integral closures of monomial ideals*, J. Combin. Ser. A., **116**(2009), 44-54.

[29] T. N. Trung, *stability of depths of powers of edge ideals*, preprint.

[30] W. Vasconcelos, Integral Closure, Rees Algebras, Multiplicities, Algorithms, Springer Monographs in Mathematics, Berlin, Springer-Verlag, 2005.

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