STABILIZATION OF WAVE-WAVE TRANSMISSION PROBLEM WITH GENERALIZED ACOUSTIC BOUNDARY CONDITIONS

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ABSTRACT. We investigate the energy decay of hyperbolic system of wave-wave with generalized acoustic boundary conditions in d-dimensional space, with the equations being coupled through boundary connection. First, by spectrum approach combining with a general criteria of Arendt-Batty, we prove that our model is strongly stable. Then, after proving that this system lacks the exponential stability, we establish different type of polynomial energy decay rates provided that the coefficients of the acoustic boundary conditions satisfy some assumptions. Further, we present some appropriate examples and show that our assumptions have been set correctly. Finally, we prove that the obtained energy decay rate is optimal in particular case.

CONTENTS

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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^d$, with a boundary $\Gamma = \partial\Omega$ of class C^2. We denote by Ω_1, Ω_2 the open bounded sets of Ω, such that $\overline{\Omega}_2 = \Omega \backslash \overline{\Omega}_1$, with the interface $\mathcal{I} = \partial\Omega_1 \cap \partial\Omega_2$, and boundaries Γ_1, Γ_2 respectively, satisfying $\Gamma_j = \partial\Omega_j \backslash \mathcal{I}$ $(j = 1, 2)$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$, as shown in the figure below (see Figure 1).

For $m \in \mathbb{N}^*$, we further fix a vector valued function $C \in C^{0,1}(\Gamma_1; \mathbb{C}^m)$, a matrix valued function $B \in C^{0,1}(\Gamma_1; M_m(\mathbb{C}))$, and for every $x \in \Gamma_1$, an inner product $(\cdot,\cdot)_x$ in $\mathbb{C}^m$ such that

$$\Re(B(x)V, V)_x \leq 0, \quad \forall V \in \mathbb{C}^m.$$

For every $x \in \Gamma_1$, let $M(x) \in M_m(\mathbb{C})$ be the Hermitian positive-definite matrix associated with this inner product $(\cdot,\cdot)_x$, *i.e.*

$$(V_1, V_2)_x = \overline{V_2}^T M(x) V_1, \quad \forall V_1, V_2 \in \mathbb{C}^m.$$

From now on we further assume that M is Lipschitz continuous on Γ_1. For the sake of brevity, if there is no confusion we use the notation $(\cdot,\cdot)$ to denote $(\cdot,\cdot)_x$. The associated norm is denoted by $\|\cdot\|$.

The system that describes the model is the following

$$\begin{cases} u_{tt}(x,t) - a\Delta u(x,t) = 0, & \text{in} \quad \Omega_1 \times (0,\infty), \\ y_{tt}(x,t) - b\Delta y(x,t) = 0, & \text{in} \quad \Omega_2 \times (0,\infty), \\ u(x,t) - y(x,t) = 0, & \text{on} \quad \mathcal{I} \times (0,\infty), \\ a\partial_{\nu_1} u(x,t) + b\partial_{\nu_2} y(x,t) = 0, & \text{on} \quad \mathcal{I} \times (0,\infty), \\ a\partial_{\nu_1} u(x,t) - (\eta(x,t), C) = 0, & \text{on} \quad \Gamma_1 \times (0,\infty), \\ \eta_t(x,t) + Cu_t(x,t) - B\eta(x,t) = 0, & \text{on} \quad \Gamma_1 \times (0,\infty), \\ y(x,t) = 0, & \text{on} \quad \Gamma_2 \times (0,\infty), \end{cases} \tag{1.1}$$

where u and y are complex valued functions (representing the transverse displacement in the case $\Omega \subset \mathbb{R}$ and the potential velocity in the case $\Omega \subset \mathbb{R}^d$, with $d \geq 2$), a and b are two positive constants, $\nu_i(x)$, $i = 1, 2$ denotes the outer unit normal vector to the point $x \in \partial\Omega_i$, and ∂_{ν_i} is the corresponding normal derivative and η denotes the acoustic control variable. System (1.1) is considered with the following initial conditions:

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega_1, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in \Omega_2, \\ \eta(x,0) = \eta_0(x), \quad x \in \Gamma_1. \end{cases} \tag{1.2}$$

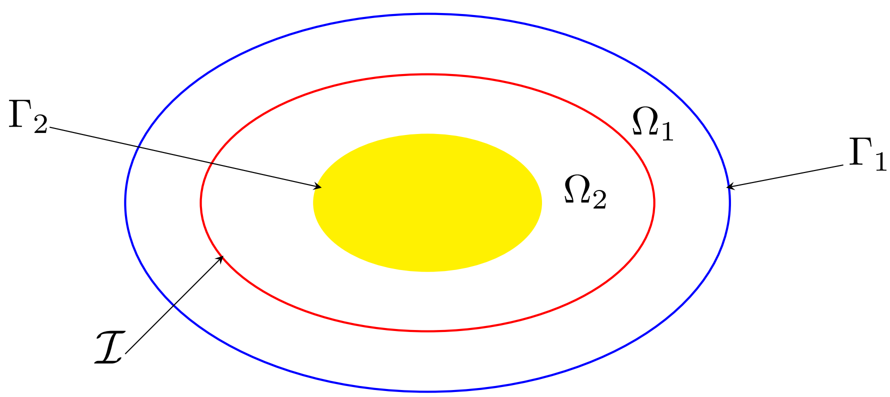

Figure 1. N-dimensional Model

Let u, y, η be smooth solutions of system (1.1). We define their associated energy by

$$E(t) = \frac{1}{2}\left(\int_{\Omega_1} |u_t|^2 dx + a\int_{\Omega_1} |\nabla u|^2 dx + \int_{\Omega_2} |y_t|^2 dx + b\int_{\Omega_2} |\nabla y|^2 dx + \int_{\Gamma_1} \|\eta\|^2 d\Gamma\right).$$

A direct calculation gives

$$\frac{d}{dt}E(t) = \int_{\Gamma_1} \Re(B(x)\eta, \eta) d\Gamma \leq 0,$$

and thus implies that system (1.1) is dissipative in the sense that the energy $E(t)$ is non-increasing with respect to time variable t.

1.1. **Motivation.** Acoustic conditions refer to approaches in many real life applications of mathematical physics and engineering. Since Morse [30] introduce such a damping on the boundary of wave equation, many authors were interested to study this problem ([8],[41],[13]). Recently, Abbas et el. [2], on a given open bounded domain Ω in $\mathbb{R}^d$, with a Lipschitz boundary Γ divided into two disjoint parts Γ_0 and Γ_1, consider the following system

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & \text{in} \quad \Omega, t > 0, \\ y(x,t) = 0, & \text{on} \quad \Gamma_1, t > 0, \\ \partial_{\nu_1} y(x,t) - (\delta(x,t), C) = 0, & \text{on} \quad \Gamma_0, t > 0, \\ \delta_t(x,t) + C y_t(x,t) - B\delta(x,t) = 0, & \text{on} \quad \Gamma_0, t > 0, \end{cases} \tag{1.3}$$

with the following initial conditions:

$$\begin{cases} y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in \Omega, \\ \delta(x,0) = \delta_0(x), \quad x \in \Gamma_0, \end{cases} \tag{1.4}$$

where y is a complex valued function and δ denotes the dynamical control variable, with $C \in C^{0,1}(\Gamma_0; \mathbb{C}^m)$, and $B \in C^{0,1}(\Gamma_0, M_m(\mathbb{C}))$, $m \in \mathbb{N}^*$. For every $x \in \Gamma_0$, $M(x) \in M_m(\mathbb{C})$ denotes a Hermitian positive-definite matrix associated with this inner product $(\cdot,\cdot)_x$, *i.e.*

$$(\delta_1, \delta_2)_x = \overline{\delta_2}^T M(x)\delta_1, \quad \forall\, \delta_1, \delta_2 \in \mathbb{C}^m.$$

Moreover, under an assumption on the behaviour of $\Re((isI - B)^{-1}C, C)$ for all large enough real numbers s, they established the polynomial energy decay rate of the system (1.3)-(1.4). Further, they presented some appropriate examples and showed that their assumptions have been set correctly. To our knowledge the obtained decay rate is not optimal, see for instance [36] and [43].

On the other hand, the wide range of attention taken nowadays in transmission problems, whether in modeling, control of engineering, physical interactive processes, or others, has motivated the authors to proceed with extensive studies ([23], [22],[19], [32]). Lately, Chai and Guo [16] consider two wave equations for a transmission problem with a dynamical boundary control. In a similar domain to that described in fig. 1, they deal with the following system

$$\begin{cases} u_i^{''}(x,t) - a_i \Delta u_i(x,t) = 0, & \text{in} \quad \Omega_i \times (0,\infty);\ i = 1,2, \\ m u_1^{''}(x,t) + a_1 \frac{\partial u_1(x,t)}{\partial \nu} = -\beta u_1^{'} - \gamma a_1 \frac{\partial u_1^{'}}{\partial \nu}, & \text{on} \quad \Gamma_1 \times (0,\infty), \\ u_2(x,t) = 0, & \text{on} \quad \Gamma_2 \times (0,\infty), \\ u_1(x,t) = u_2(x,t);\ a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu}, & \text{on} \quad \mathcal{I} \times (0,\infty), \end{cases} \tag{1.5}$$

with initial conditions

$$u_i(x,0) = u_i^0(x),\ u_i^{'}(x,0) = u_i^1(x), \quad \text{in} \quad \Omega_i \times (0,\infty);\ i = 1,2, \tag{1.6}$$

where ν denotes the unit normal on the boundary of Ω and $\mathcal{I}$, directed toward the exterior of Ω, $m \in L^\infty(\Gamma_1)$ with $m > 0$, and a_1, a_2, β and γ are positive constants, such that $\beta\gamma < m$. Furthermore, they discussed the well-posedness of the problem (1.5)-(1.6), and proved the exponential stability of the system under some geometric conditions, for the case where $m > 0$.

Here we raise two important questions we are interested in:

1. What are the consequences when two coupled waves are considered with a general acoustic boundary condition? Will the system maintain its strong stability, and polynomial stability?
2. Is it possible to improve the decay rate obtained by Abbas and Nicaise?

The main goal of this paper is to generalize the systems and improve the results of the above problems. In this regard, we propose a system of two waves in a transmission problem with a generalized acoustic boundary condition. More precise, we consider the system (1.1)-(1.2), and prove its well-posedness and strong stability. Moreover, we improve the polynomial decay rates, that were obtained in [2], using an assumption on the behaviour of (Bv, v). Additionally, we introduce some illustrative examples, to confirm our assumption. Furthermore, we prove the optimality in the one-dimensional case for some particular sample of the generalized system. The proposed idea is more general and superior to the prevailing ideas, and to our knowledge, this work is not done before.

1.2. **Literature.** In recent years, researchers have shown interest in studying the stability of systems, in particular coupled systems that describe the connection of materials that appears frequently in several fields such as engineering technology. The mathematical problem that deals with the propagation of the wave among different materials are called transmission problem (also known as interface problem, or problem with discontinuous coefficients), which in turn is of major importance for mathematical studies in many physical and living systems.

Among the applications of wave equations is the noise suppression in the structural acoustic systems, which is of great interest in physics and engineering. In fact, Acoustic controls deal with sound and vibration, for example in reduction of unwanted noise. That is to say, it is referred to as noise control. Morse, who has made several contributions to theoretical atomic physics, shows in his book "Vibration and Sound" that the new mathematical techniques which have been developed for the working out of quantum mechanics can be used to analyze such problems. A course given by him at the Massachusetts Institute of Technology led to writing his book. According to the preface to the first edition, which appeared in 1936, it is intended primarily as a textbook for students of physics, of mathematical physics, and of communication engineering. His book was first published in 1948, and since his book first appeared, the science of acoustic has expanded in many directions. It has been used increasingly in exploring the properties of matter, in plasma-physics, in meteorology and astrophysics, in quantum mechanics and many other fields. It takes new importance and significance both scientifically and technologically.

In this paper we consider the generalization of the so-called acoustic boundary conditions, that was introduced by Morse and Ingard [30] (for $m = 2$), where they use the model for a wave assumed to be at a definite frequency. Then in the 1970's, in a series of papers, Beale [8], [6],[7] proved the global existence and regularity of solutions in a Hilbert space of data with finite energy by means of semi group methods. In [41], Rivera and Qin establish the polynomial energy decay in $\mathbb{R}^3$. In contrast to other studies of acoustic/structure interaction, Graber [13] consider the non linear coupling, more precisely he considers a wave equation with non linear acoustic boundary conditions. Nicaise and Abbas [2] prove the polynomial stability of a wave system with generalized acoustic boundary control, where they show their results in $\mathbb{R}^d$, under an assumption on the behavior of $\Re((i\lambda I - B)^{-1}C, C)$ for all real numbers s with large enough modulus. Some particular cases of the generalized acoustic problem are the wave equation with dynamical control feedback, that has been treated in [44] (with $m = 1$ and $d = 1$), and in [43], [36] (with $m = 1$ and $d \geq 2$).

Transmission problems as well had received the attention, and there have been fruitful results concerning existence, regularity, controllability and decay estimate of the solutions of different types of such problems. For example, Lions [20] studied the existence, uniqueness and regularity of solutions for the transmission problem of wave equation with Dirichlet boundary condition, further he proved the exact controllability using Hilbert Uniqueness Method. Whereas, the exact controllability for plate equation were proved by Liu and Williams [24] and Aassila [1]. Besides Marzocchi's work [28] in which he proved the exponential decay of semi-linear problems in 1-dimensional space between elastic and thermo-elastic materials, and after that he extend his work to higher dimensions with the help of Naso [29]. For the transmission problem with frictional damping, Bastos and Raposo proved in [4] the well-posedness and exponential stability of the total energy.

The wide range of applications on these models, whether in modeling, control of engineering, physical interactive processes, or others, has motivated the authors to proceed with extensive studies. In 2000, Rivera and Oquendo [31] studied the wave propagation over materials consisting of elastic and viscoelastic components, where they confirmed that the corresponding solutions decay exponentially to zero no matter how small the interval of the viscoelastic part is. The following year, they checked out in [38] the asymptotic behavior of beams that are made of two different materials, with one of them having a localized thermoelastic effect. Their main objective was to show that the solutions decay exponentially to zero as time goes to infinity, no matter how small the interval where the thermal dissipation is effective. Later, in 2004 they proved the exponential stability under some geometric control conditions for the thermoelastic plates transmission problem [39]. At that time Zuazua and Zhang follow up on their searches that they started in 2003 and completed to get excess results and papers [45],[46], [47] on fluid-structure interactions with naive transmission condition at the interface. They showed the complexity of decay and control problems of such interaction, even in one space dimension, and proved a sharp polynomial decay rate of type $\frac{1}{t^2}$ for the energy of smooth solutions. Also, they worked on a similar model but with more natural transmission condition than the formal one. Then in 2005, they worked with Rauch in [37] on higher dimensions, analyzing the fluid-structure interaction model of the coupled equations at the interface with a suitable transfer condition, and obtained the result of the $\frac{1}{t}$ type polynomial decay, which was not sharp in general. Carrying on, in 2006 Duyckaerts [12] acted on fluid-structure interaction model, formed of heat and wave equations taking place in two distinct domains with an interface that is controlled by a transmission condition. He dealt with natural and naïve transmission problems, proved the polynomial decay result obtained by another author under interface geometrically controlling the wave domain, as well as he improved the speed of logarithmic decay for the solution of the system with a transmission condition. In [40], Rivera and Naso had exponential decay results for the thermoelasticity transmission problem. In the same year (2007), Rivera with Lapa [19] verified the existence of a global solution that decays exponentially for the nonlinear transmission problem for the wave equation with time-dependent coefficients and linear internal damping. Furthermore, Zhang and Zuazua [48] analyzed the long time behavior of coupled (wave-heat) equations evolved in two bounded domains with natural transmission condition at the interface, where they obtained a polynomial decay result for smooth solutions of the system under suitable geometric assumption guaranteeing that the heat domain envelopes the wave one. In absence of geometric conditions they got a logarithmic decay result for the system with more simplified transmission conditions at interface. In 2010, Georgi and Fernando investigated in [10] the large time behavior of the solutions of mixed boundary value problem, and found that the energy of the solutions of the transmission problem in bounded domains with dissipative boundary conditions decay exponentially. Recently, in 2016 Lulu and Wang [27] traced the transmission problem of Schrodinger equation with a viscous damped equation (which acts as a controller of the system), finding that this system achieves strong stability. In 2017, Zuazua and Han [17] discussed the asymptotic behavior of transmission problem solutions on star-shaped networks of interconnected elastic and thermoelastic rods, and demonstrated their exponential decay rate.

There have been a lot of works that can't be listed all, for that we mention only some of them, and still to end this paragraph we will mention some of the recent works done in the latest five years. Starting with the work of Nordstrom and Linder [33], done in 2018, in which they introduced the notion of transmission problems to describe a general class of problems with different dynamics coupled in time, besides to well-posedness and stability that were analyzed for continuous and discrete problems, using both strong and weak formulations, and a general transmission condition were obtained. In 2020, Coelho, Cavaleanti, and Valencia [11] in their proposal address the exponential stability of the solutions of a coupled system posed on an n-dimensional domain consisting of two parts: one made of viscoelastic material endowed with hereditary memory plus a localized elastic material and the other made of elastic material, where the common boundary is responsible for the transmission condition. Whereas, Guo and Chai [15] verified the exponential stabilization of transmission problem of wave equation with linear dynamical feedback control using classical energy methods and Multiplier technique in N-dimensional space. In 2021, Nonato, Raposo and Bastos [32] proved the exponential stability by energy method with the construction of a suitable Lyapunov functional for the transmission problem for one-dimensional waves with non-linear weights on the frictional damping and time varying delay. Finally, the work of Guo and Chai [16], in which they improved their work concerning the exponential stability of two wave equations with linear dynamical feedback control. They discussed the well-posedness of the problem (1.5)-(1.6), and proved the exponential decay of the energy of the system under some geometric conditions.

1.3. **Organization of the Paper.** This paper is organized as follows: In Section 2, we prove the well-posedness property of system (1.1)-(1.2) using semi group theory, and the strong stability using a general criteria of Arendt-Batty, after formulating the system into evolution problem. After that we show the lack of exponential stability in section 3. Then in section 4, we prove the polynomial stability of the system while taking into consideration the different cases, so that two different rates are obtained. We add some illustrative examples in section 5, to verify our assumptions. Finally we draw a conclusion in the last section. For further concepts and definitions, you can refer to the appendix.

2. Well-Posedness, Strong Stability and Lack of Exponential Stability of the System

This section is devoted to study the well-posedness and the strong stability of system (1.1)-(1.2) by the semigroup approach.

2.1. **Well-Posedness.** We first introduce the following spaces:

$$\mathcal{H} = \{(u,v,y,z,\eta) \in H^1(\Omega_1) \times L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2) \times \left(L^2(\Gamma_1)\right)^m \quad | \quad u = y \quad \text{on } \mathcal{I}\},$$

where

$$H^1_{\Gamma_2}(\Omega_2) = \{y \in H^1(\Omega_2) \quad | \quad y = 0 \quad \text{on } \Gamma_2\}.$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$(U, U_1)_{\mathcal{H}} = \int_{\Omega_1} v\bar{v}_1 dx + a\int_{\Omega_1} \nabla u \cdot \nabla \bar{u}_1 dx + \int_{\Omega_2} z\bar{z}_1 dx + b\int_{\Omega_2} \nabla y \cdot \nabla \bar{y}_1 dx + \int_{\Gamma_1} (\eta, \eta_1) d\Gamma,$$

where $U = (u,v,y,z,\eta), U_1 = (u_1,v_1,y_1,z_1,\eta_1) \in \mathcal{H}$, the space $\mathcal{H}$ is a Hilbert space. We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longmapsto \mathcal{H}$ by:

$$\begin{aligned} D(\mathcal{A}) = \{(u,v,y,z,\eta) \in & \mathcal{H} :\ \Delta u \in L^2(\Omega_1),\ \Delta y \in L^2(\Omega_2),\ v \in H^1(\Omega_1),\ z \in H^1_{\Gamma_2}(\Omega_2), \\ & a\partial_{\nu_1} u + b\partial_{\nu_2} y = 0 \quad \text{on } \mathcal{I} \quad \text{and} \quad a\partial_{\nu_1} u - (\eta, C) = 0 \quad \text{on } \Gamma_1\}, \end{aligned} \tag{2.1}$$

$$\mathcal{A}(u,v,y,z,\eta)^T = (v, a\Delta u, z, b\Delta y, B\eta - \gamma(v)C)^T, \tag{2.2}$$

where $\gamma : H^1(\Omega_1) \longrightarrow L^2(\Gamma_1)$ is the trace operator. Now, setting $U = (u, u_t, y, y_t, \eta)$ as the state of system (1.1)-(1.2), we rewrite the problem into a first-order evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{2.3}$$

where $U_0 = (u_0, u_1, y_0, y_1, \eta_0)$.

Proposition 2.1. *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. For all $U = (u,v,y,z,\eta) \in D(\mathcal{A})$, we have

$$\begin{aligned} \Re(\mathcal{A}U, U) = \Re\Bigg\{ a\int_{\Omega_1} \nabla v \cdot \nabla \bar{u} dx + a\int_{\Omega_1} \Delta u \bar{v} dx + b\int_{\Omega_2} \nabla z \cdot \nabla \bar{y} dx \\ + b\int_{\Omega_2} \Delta y \bar{z} dx + \int_{\Gamma_1} (B\eta - \gamma(v)C, \eta)_{\mathbb{C}^m}\, d\Gamma \Bigg\}. \end{aligned} \tag{2.4}$$

It follows, from the boundary and the transmission conditions in (2.1), that

$$\Re(\mathcal{A}U, U) = \int_{\Gamma_1} \Re(B(x)\eta, \eta) d\Gamma \leq 0, \tag{2.5}$$

that implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. For this aim, if $\lambda > 0$ and $F = (f_1, g_1, f_2, g_2, h)^\top \in \mathcal{H}$, we look for $U = (u,v,y,z,\eta)^\top \in D(\mathcal{A})$ unique solution of

$$(\lambda I - \mathcal{A})U = F. \tag{2.6}$$

Equivalently, we have the following system

$$\lambda u - v = f_1, \tag{2.7}$$

$$\lambda v - a\Delta u = g_1, \tag{2.8}$$

$$\lambda y - z = f_2, \tag{2.9}$$

$$\lambda z - b\Delta y = g_2, \tag{2.10}$$

$$(\lambda I_m - B)\eta + \gamma(v)C = h, \tag{2.11}$$

where $I_m : \left(L^2(\Gamma_1)\right)^m \to \left(L^2(\Gamma_1)\right)^m$ is the identity mapping. As $\lambda \notin \sigma(B)$, equations (2.7) and (2.11) imply

$$\eta = (\lambda I_m - B)^{-1}\left(h + \gamma(f_1)C\right) - \lambda\gamma(u)(\lambda I_m - B)^{-1}C. \tag{2.12}$$

Now, eliminating v in (2.8) by (2.7) and z in (2.10) by (2.9), we obtain the following system

$$\lambda^2 u - a\Delta u \quad = \quad \lambda f_1 + g_1, \tag{2.13}$$

$$\lambda^2 y - b\Delta y \quad = \quad \lambda f_2 + g_2, \tag{2.14}$$

with the following boundary and transmission conditions

$$\begin{aligned} a\partial_{\nu_1} u + \left(\lambda u(\lambda I_m - B)^{-1}C, C\right) &= \left((\lambda I_m - B)^{-1}\left(h + \gamma(f_1)C\right), C\right), \quad \text{on } \Gamma_1, \\ y &= 0, \quad \text{on } \Gamma_2, \\ a\partial_{\nu_1} u + b\partial_{\nu_2} y = 0, \text{ and } u - y &= 0, \quad \text{on } \mathcal{I}. \end{aligned} \tag{2.15}$$

Set the Hilbert space H as

$$\mathsf{H} := \left\{(\mathsf{f},\mathsf{g}) \in H^1(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \mid \mathsf{f} = \mathsf{g} \quad \text{on} \quad \mathcal{I}\right\}, \tag{2.16}$$

equipped with the norm

$$\|(\mathsf{f},\mathsf{g})\|^2_{\mathsf{H}} = \|\nabla \mathsf{f}\|^2_{L^2(\Omega_1)} + \|\nabla \mathsf{g}\|^2_{L^2(\Omega_2)}. \tag{2.17}$$

Now, let $(\varphi, \psi) \in \mathsf{H}$. Multiplying (2.13) and (2.14) by φ and ψ respectively, then taking their integrals over their corresponding domain, and using the boundary and transmission conditions in (2.15), we obtain the following variational problem:

$$S_\lambda((u,y),(\varphi,\psi)) = L_\lambda(\varphi,\psi), \quad \forall\, (\varphi,\psi) \in \mathsf{H}, \tag{2.18}$$

where S and L are given by

$$\begin{aligned} S_\lambda((u,y),(\varphi,\psi)) = &\int_{\Omega_1} \lambda^2 u\overline{\varphi}dx + \int_{\Omega_2} \lambda^2 y\overline{\psi}dx + a\int_{\Omega_1} \nabla u \cdot \nabla\overline{\varphi}dx + b\int_{\Omega_2} \nabla y \cdot \nabla\overline{\psi}dx \\ &+ a\int_{\Gamma_1} \lambda((\lambda I_m - B)^{-1}C, C)\gamma(u)\gamma(\overline{\varphi})d\Gamma, \end{aligned} \tag{2.19}$$

$$L_\lambda(\varphi,\psi) = \int_{\Omega_1} (\lambda f_1 + g_1)\overline{\varphi}dx + \int_{\Omega_2} (\lambda f_2 + g_2)\overline{\psi}dx + a\int_{\Gamma_1} \left((\lambda I_m - B)^{-1}(\gamma(f_1)C + h), C\right)\gamma(\overline{\varphi})d\Gamma. \tag{2.20}$$

It is easy to see that S_λ is a sesquilinear and continuous form on the space $\mathsf{H} \times \mathsf{H}$. Besides, S_λ is coercive form on $\mathsf{H} \times \mathsf{H}$ as

$$\Re\left\{\left((\lambda I_m - B)^{-1}C, C\right)\right\} = \Re\left\{(Q, (\lambda I_m - B)Q)\right\} = \lambda||Q||^2 - \Re(Q, BQ) \geq 0,$$

where $Q = (\lambda I_m - B)^{-1}C$. Hence, we get that $\Re S_\lambda((u,y),(u,y)) \geq ||(u,y)||_{\mathsf{H}\times\mathsf{H}}$, thus the coercivity of S_λ. Moreover, L_λ is antilinear and continuous form on H. Then, it follows by Lax-Milgram's theorem that (2.18) admits a unique solution $(u,y) \in \mathsf{H}$. By choosing $\varphi \in C_c^\infty(\Omega_1)$, $\psi = 0$ in (2.18), and applying Green's formula, we have

$$\int_{\Omega_1} \left(\lambda^2 u - a\Delta u\right)\overline{\varphi}dx = \int_{\Omega_1} \left(\lambda f_1 + g_1\right)\overline{\varphi}dx, \qquad \forall \varphi \in C_c^\infty(\Omega_1),$$

which implies that the first equation of (2.13) holds in the sense of distributions in Ω_1, and hence it is satisfied in $L^2(\Omega_1)$. As $\lambda^2 u - \lambda f_1 - g_1$ belongs to $L^2(\Omega_1)$, the same holds for Δu, i.e., $\Delta u \in L^2(\Omega_1)$. In the same way, choosing $\varphi = 0$ and $\psi \in C_c^\infty(\Omega_2)$ in (2.18), we see that the second equation of (2.14) holds as equality in $L^2(\Omega_2)$, and therefore $\Delta y \in L^2(\Omega_2)$. Now, let us define the space

$$H^1_{*,I}(\Omega_1) = \left\{\mathsf{f} \in H^1(\Omega_1) \mid \ \mathsf{f} = 0 \ \text{ on } \ \mathcal{I}\right\}.$$

By taking $\varphi \in H^1_{*,I}(\Omega_1)$, $\psi = 0$, and applying Green's formula in (2.18), we obtain

$$\int_{\Gamma_1} \left(a\partial_{\nu_1} u + \left(\lambda u(\lambda I_m - B)^{-1}C, C\right)\right) \overline{\varphi} d\Gamma = \int_{\Gamma_1} \left((\lambda I_m - B)^{-1} \left(h + \gamma(f_1)C\right), C\right) \overline{\varphi} d\Gamma, \quad \forall \varphi \in H^{\frac{1}{2}}_{*,I}(\Gamma_1),$$

where $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is the corresponding trace space of $H^1_{*,I}(\Omega_1)$ through the trace operator γ. Since $H^{\frac{1}{2}}_{*,I}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, we deduce that u satisfies

$$a\partial_{\nu_1} u + \left(\lambda u(\lambda I_m - B)^{-1}C, C\right) = \left((\lambda I_m - B)^{-1} \left(h + \gamma(f_1)C\right), C\right) \quad \text{on} \quad \Gamma_1.$$

Coming back to (2.18), and again applying Green's formula, and using the fact that $u = y$ and $\varphi = \psi$ on $\mathcal{I}$, we get

$$\int_{\mathcal{I}} \left(a\partial_{\nu_1} u + b\partial_{\nu_2} y\right) \overline{\psi} d\Gamma = 0, \quad \forall \psi \in H^{\frac{1}{2}}_{*}(\mathcal{I}),$$

where $H^{\frac{1}{2}}_{*}(\mathcal{I})$ is the corresponding trace space of $H^1_*(\Omega_2)$ through the operator $\psi \longmapsto \psi|_{\mathcal{I}}$. Due to the density of $H^{\frac{1}{2}}_{*}(\mathcal{I})$ into $L^2(\mathcal{I})$, we can easily check that u and y satisfy the transmission conditions of $(2.15)_3$. Finally, by setting

$$v := \lambda u - f_1, \quad z := \lambda y - f_2, \quad \text{and} \quad \eta := (\lambda I_m - B)^{-1} \left(h + \gamma(f_1)C\right) - \lambda u(\lambda I_m - B)^{-1}C,$$

we conclude that there exists a unique $U = (u, v, y, z, \eta) \in D(\mathcal{A})$ solution of equation (2.6) and thus the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$. The proof is thus complete.

$\square$

According to Lumer-Philips theorem (see [34]), Proposition 2.1 implies that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ in $\mathcal{H}$, which gives the well-posedness of (2.3). Then, we have the following result:

Theorem 2.2. *For all $U_0 \in \mathcal{H}$, system (2.3) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the system (2.3) admits a unique strong solution

$$U(t) \in C^0(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

2.2. **Strong Stability.** This part will be specified for the prove of the strong stability of our system, without any geometric conditions. For this, we need to introduce some spaces and definitions.

$$\mathsf{H}_\star := \left\{(\mathsf{f}, \mathsf{g}) \in H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \mid \mathsf{f} = \mathsf{g} \quad \text{on} \quad \mathcal{I}\right\}, \tag{2.21}$$

equipped with the norm

$$\|(\mathsf{f}, \mathsf{g})\|^2_{\mathsf{H}} = \|\nabla \mathsf{f}\|^2_{L^2(\Omega_1)} + \|\nabla \mathsf{g}\|^2_{L^2(\Omega_2)}, \tag{2.22}$$

where

$$H^1_{\Gamma_1}(\Omega_1) = \{\mathsf{f} \in H^1(\Omega_1) \quad | \quad \mathsf{f} = 0 \quad \text{on}\, \Gamma_1\}.$$

Through the section, we will use the following set

$$\Sigma_m := \{\lambda \in \mathbb{C} : \exists x \in \Gamma_1 : (\lambda I_m - B(x)) \text{ is not invertible}\}.$$

From the continuity of B, Σ_m is a compact subset of $\mathbb{C}$. Define the linear unbounded operator $\mathcal{O}_{Dir} : D(\mathcal{O}_{Dir}) \longmapsto L^2(\Omega_1) \times L^2(\Omega_2)$ by

$$D(\mathcal{O}_{Dir}) = \left\{(\mathsf{f}, \mathsf{g}) \in \mathsf{H}_\star \ : \ \Delta \mathsf{f} \in L^2(\Omega_1), \ \Delta \mathsf{g} \in L^2(\Omega_2), \text{ and } a\partial_{\nu_1}\mathsf{f} + b\partial_{\nu_2}\mathsf{g} = 0 \quad \text{on} \ \mathcal{I}\right\}, \tag{2.23}$$

and

$$\mathcal{O}_{Dir}(\mathsf{f}, \mathsf{g}) = (-a\Delta \mathsf{f}, -b\Delta \mathsf{g}), \quad \forall\, (\mathsf{f}, \mathsf{g}) \in D(\mathcal{O}_{Dir}). \tag{2.24}$$

Lemma 2.3. *The linear unbounded operator $\mathcal{O}_{Dir}$ is a positive self-adjoint operator with a compact resolvent.*

Proof. In order to prove that $\mathcal{O}_{Dir}$ is a positive self-adjoint operator, we can instead prove that it is a symmetric and m-accretive operator, that in turn easy and clear. Moreover, due to sobolev embedding, the resolvent of the operator $\mathcal{O}_{Dir}$ is compact. Thus the proof is complete. □

Then, we can consider the sequence of eigenfunctions $(\varphi_{\mathcal{O},k}, \psi_{\mathcal{O},k})_{k\in\mathbb{N}^*}$ (that form an orthonormal basis of $L^2(\Omega_1) \times L^2(\Omega_2)$) of the operator $\mathcal{O}_{Dir}$, corresponding to the eigenvalues $(\lambda^2_{\mathcal{O},k})_{k\in\mathbb{N}^*}$ (repeated according to their multiplicity), such that $\lambda^2_{\mathcal{O},k}$ tends to infinity as k goes to infinity.

Now will state the following theorem, but before we would like to add some assumptions that we will use through the section

$$\text{(SSC1)} \qquad \forall\, i\lambda \notin \Sigma_m,\ \lambda \in \mathbb{R}^*,\ \exists \alpha_\lambda > 0 \,:\, ((i\lambda I_m - B(x))^{-1}C(x), C(x)) \geq \alpha_\lambda,\ \forall x \in \Gamma_1.$$

$$\text{(SSC2)} \quad \forall\, i\lambda \in \Sigma_m,\ \lambda \in \mathbb{R}^*, : \forall\, M \subset \Gamma_1 \,:\, \text{mes}\,(M) > 0,\ \exists x \in M \,:\, (\eta, C(x)) \neq 0,\ \forall\, 0 \neq \eta \in \text{Ker}(i\lambda I_m - B(x)).$$

$$\text{(SSC3)} \qquad \forall\, i\lambda \in \Sigma_m,\ \lambda \in \mathbb{R}^*,\ C \notin \text{Ker}\,(i\lambda I_m + B^\star)^\perp \quad \text{on} \quad \Gamma_1.$$

$$\text{(SSC4)} \qquad \Sigma_m \cap \{\pm i\lambda_{\mathcal{O},k}, k \in \mathbb{N}^*\} = \emptyset.$$

Theorem 2.4. *Assume that (SSC1)-(SSC4) hold, $0 \notin \Sigma_m$ and $\Sigma_m \cap i\mathbb{R}^*$ is countable. Then, the C_0-semigroup of contraction $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on $\mathcal{H}$ in the following sense*

$$\lim_{t\to+\infty} ||e^{t\mathcal{A}}U_0||_{\mathcal{H}} = 0, \qquad \forall\, U_0 \in \mathcal{H}.$$

According to Theorem A.2, to prove Theorem 2.4, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of these results is not reduced to the analysis of the point spectrum of $\mathcal{A}$ on the imaginary axis since its resolvent is not compact. Hence the proof of Theorem 2.4 has been divided into the following two Lemmas.

Lemma 2.5. *Under the assumptions of Theorem 2.4, the operator $i\lambda I - \mathcal{A}$ is injective, for all $\lambda \in \mathbb{R}$.*

Proof. By contradiction argument, let $\lambda \in \mathbb{R}$, and assume that there exists $U = (u, v, y, z, \eta)^\top \in D(\mathcal{A}) \setminus \{0\}$, such that

$$\mathcal{A}U = i\lambda U. \tag{2.25}$$

Equivalently, we have the following system

$$v = i\lambda u, \tag{2.26}$$
$$a\Delta u = i\lambda v, \tag{2.27}$$
$$z = i\lambda y, \tag{2.28}$$
$$b\Delta y = i\lambda z, \tag{2.29}$$
$$B\eta - \gamma(v)C = i\lambda \eta. \tag{2.30}$$

Eliminating v in (2.27) and in (2.30) by (2.26), and eliminating z in (2.29) by (2.28), we obtain the following system:

$$\lambda^2 u + a\Delta u \quad = 0, \quad \text{in} \quad \Omega_1, \tag{2.31}$$
$$\lambda^2 y + b\Delta y \quad = 0, \quad \text{on} \quad \Omega_2, \tag{2.32}$$
$$(i\lambda I_m - B)\eta + i\lambda u C \quad = 0, \quad \text{on} \quad \Gamma_1, \tag{2.33}$$

with the following boundary and transmission conditions

$$\begin{aligned} a\partial_{\nu_1} u - (\eta, C) = 0, &\quad \text{on } \Gamma_1, \\ y = 0, &\quad \text{on } \Gamma_2, \\ a\partial_{\nu_1} u + b\partial_{\nu_2} y = 0, \text{ and } u - y = 0, &\quad \text{on } \mathcal{I}. \end{aligned} \tag{2.34}$$

In order to study the solution of (2.31)-(2.34), we will distinguish several cases.

Case 1. $i\lambda \notin \Sigma_m$**.** In this case we again distinguish between two cases.
Case 1.1. $\lambda = 0$**.** Then from (2.33) and $(2.34)_1$ we get

$$\partial_\nu u = 0 \quad \text{and} \quad \eta = 0, \quad \text{on} \quad \Gamma_1.$$

Multiplying equation (2.31) by $\overline{u}$ and equation (2.32) by $\overline{y}$, integrating by parts, and using (2.34), we get

$$a\int_{\Omega_1} \|\nabla u\|^2 dx + b\int_{\Omega_2} \|\nabla y\|^2 dx = 0.$$

It follows that $U = 0$, which contradicts the fact that $U \neq 0$.

Case 1.2. $\lambda \neq 0$**.** Then, by the proof of Proposition 2.1, we deduce that problem (2.31)-(2.34) admits a unique solution $(u, y) \in \mathsf{H}$, and it satisfies the following variational equation:

$$S_{i\lambda}((u,y),(\varphi,\psi)) = 0, \quad \forall\, (\varphi,\psi) \in \mathsf{H}, \tag{2.35}$$

where S is given by

$$\begin{aligned} S_{i\lambda}((u,y),(\varphi,\psi)) = &-\int_{\Omega_1} \lambda^2 u\overline{\varphi} dx - \int_{\Omega_2} \lambda^2 y\overline{\psi} dx + a\int_{\Omega_1} \nabla u \cdot \nabla\overline{\varphi} dx + b\int_{\Omega_2} \nabla y \cdot \nabla\overline{\psi} dx \\ &+ ia\int_{\Gamma_1} \lambda((i\lambda I_m - B)^{-1}C, C)\gamma(u)\gamma(\overline{\varphi}) d\Gamma. \end{aligned} \tag{2.36}$$

In particular, for $(\varphi,\psi) = (u,y)$, we have

$$\begin{aligned} &-\int_{\Omega_1} |\lambda u|^2 dx - \int_{\Omega_2} |\lambda y|^2 dx + a\int_{\Omega_1} |\nabla u|^2 dx + b\int_{\Omega_2} |\nabla y|^2 dx \\ &+ ia\int_{\Gamma_1} \lambda((i\lambda I_m - B)^{-1}C, C)|\gamma(u)|^2 d\Gamma = 0. \end{aligned} \tag{2.37}$$

Then, taking the imaginary part of (2.37), and using the fact that $\lambda \neq 0$, we get

$$\Re\left\{\int_{\Gamma_1} (i\lambda I_m - B)^{-1}C, C)|\gamma(u)|^2 d\Gamma\right\} = 0, \tag{2.38}$$

which together with (SSC1) condition yields

$$u = 0, \quad \text{on} \quad \Gamma_1.$$

On the other hand, using equation (2.33) and $(2.34)_1$, and the fact that $i\lambda \notin \Sigma_m$, we obtain

$$\eta = -i\lambda(i\lambda I_m - B)^{-1}Cu = 0, \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \quad \Gamma_1.$$

Consequently, u satisfies the following system:

$$\begin{cases} \lambda^2 u + a\Delta u = 0 \ \text{ in } \ \Omega_1, \\ u = \partial_{\nu_1} u = 0 \ \text{ on } \ \Gamma_1. \end{cases} \tag{2.39}$$

Hence, Holmgren uniqueness theorem (see [21]) yields

$$u = 0 \ \text{ in } \ \Omega_1. \tag{2.40}$$

It follows, from the transmission conditions, that

$$y = \partial_{\nu_2} y = 0 \ \text{ on } \ \mathcal{I}, \tag{2.41}$$

which together with (2.32) gives

$$\begin{cases} \lambda^2 y + b\Delta y = 0 \ \text{ in } \ \Omega_2, \\ y = \partial_{\nu_2} y = 0 \ \text{ on } \ \mathcal{I}. \end{cases} \tag{2.42}$$

Again, by the Holmgren uniqueness theorem we have

$$y = 0 \ \text{ in } \ \Omega_2. \tag{2.43}$$

Summing up, we have proved that $U=0$. This contradicts the fact that $U\neq 0$.

Case 2. $i\lambda\in\Sigma_m$**.** Assume that $\eta\neq 0$ (on the contrary we repeat the same proof in **Case 1.2.**). Then there exists $\Gamma_{\star,1}\subset\Gamma_1$ with $\mathrm{mes}(\Gamma_{\star,1})>0$, such that $\eta\neq 0$ on $\Gamma_{\star,1}$. We distinguish two cases.

Case 2.1. $u=0$ **in** Ω_1 **or** $y=0$ **in** Ω_2**.**
Assume that $u=0$ in Ω_1, then from the transmission condition $(2.34)_3$ we deduce that y satisfies system (2.42), and consequently $y=0$ in Ω_2. On the other hand, from (2.33) and $(2.34)_1$, we deduce that

$$(\eta,C(x))=0,\quad \text{and}\quad 0\neq\eta\in\mathrm{Ker}(i\lambda I_m-B(x)),\ \forall\, x\in\Gamma_{\star,1},$$

which is in contradiction with (SSC2). Similarly, if $y=0$. Then, $\eta=0$ and consequently $U=0$ which contradicts the fact that $U\neq 0$.

Case 2.2. $u\neq 0$ **in** Ω_1 **and** $y\neq 0$ **in** Ω_2**.** From equation (2.33), we deduce that

$$Cu\in \mathrm{R}\left(i\lambda I_m-B)\right)=\mathrm{Ker}\left(i\lambda I_m+B^\star\right)^\perp,\quad \text{on}\quad \Gamma_1,$$

which implies, from condition (SSC3), that

$$u=0\quad\text{on}\quad\Gamma_1.$$

Therefore, there exists $(u,y)\in D(\mathcal{O}_{Dir})$ such that

$$\mathcal{O}_{Dir}(u,y)=(-a\Delta,-b\Delta)(u,y)=\lambda^2(u,y),$$

hence $\exists\; k\in\mathbb{N}^\star$ such that $\lambda^2=\lambda^2_{Dir,k}$. Now going back to (2.33), we get

$$(\pm i\lambda_{Dir,k}-B)\eta=0\quad\text{on}\quad\Gamma_1.$$

Then using the assumption (SSC4), it implies that

$$\eta=0\quad\text{on}\quad\Gamma_1.$$

Consequently, we have from $(2.34)_1$ that

$$\partial_{\nu_1}u=0\quad\text{on}\quad\Gamma_1.$$

Then using Holmgren's theorem, we deduce that $u=0$, which is impossible. The proof is thus complete. □

Lemma 2.6. *Assume that* (SSC1) *is satisfied. Then we have*

$$\sigma(\mathcal{A})\cap i\mathbb{R}^*\subseteq\Sigma_m\cap i\mathbb{R}^*.$$

Proof. Let $\lambda\in\mathbb{R}^*$. Assume that $i\lambda\in\sigma(\mathcal{A})$ and $i\lambda\notin\Sigma_m$, we aim is to find a contradiction by proving that $i\lambda\in\rho(\mathcal{A})$. Indeed, under assumption (SSC1), using Lemma 2.5 **Case 1.2.**, we have $i\lambda-\mathcal{A}$ is injective, then it is left to prove the surjectivity of $i\lambda-\mathcal{A}$, i.e to prove

$$R(i\lambda I-\mathcal{A})=\mathcal{H}.$$

In fact, let $F=(f_1,g_1,f_2,g_2,h)^\top\in\mathcal{H}$, we look for $U=(u,v,y,z,\eta)^\top\in D(\mathcal{A})$ solution of

$$(i\lambda I-\mathcal{A})U=F. \tag{2.44}$$

Equivalently, we have the following system

$$i\lambda u-v=f_1, \tag{2.45}$$
$$i\lambda v-a\Delta u=g_1, \tag{2.46}$$
$$i\lambda y-z=f_2, \tag{2.47}$$
$$i\lambda z-b\Delta y=g_2, \tag{2.48}$$
$$i\lambda\eta-B\eta+\gamma(v)C=h, \tag{2.49}$$

with the following boundary and transmission conditions

$$u=y,\quad a\partial_{\nu_1}u=-b\partial_{\nu_2}y\ \ \text{on}\ \ \mathcal{I},\ \ a\partial_{\nu_1}u=(\eta,C)\ \ \text{on}\ \ \Gamma_1,\ \ \text{and}\ \ y=0\ \ \text{on}\ \ \Gamma_2. \tag{2.50}$$

Eliminating v in (2.49) by (2.45), and assuming $i\lambda \notin \sigma(B)$, we get

$$\eta = (i\lambda I - B)^{-1} \left(h - C(i\lambda\gamma(u) - \gamma(f_1))\right). \tag{2.51}$$

Eliminating v in (2.46) by (2.45), and z in (2.48) by (2.47), we get

$$\begin{cases} \lambda^2 u + a\Delta u = -g_1 - i\lambda f_1 \ \text{ in } \ \Omega_1, \\ \lambda^2 y + b\Delta y = -g_2 - i\lambda f_2 \ \text{ in } \ \Omega_2, \\ u = y, \quad \partial_{\nu_1} u = -\partial_{\nu_2} y \ \text{ on } \ \mathcal{I}, \\ a\partial_{\nu_1} u = (\eta, C) \ \text{ on } \ \Gamma_1, \\ y = 0 \ \text{ on } \ \Gamma_2. \end{cases} \tag{2.52}$$

Let $(\varphi, \psi) \in \mathsf{H}$, where H is defined by (2.16). Multiplying the first equation of (2.52) by φ and the second one by ψ, integrating and using by parts integration, yield

$$\begin{aligned} &-\lambda^2 \int_{\Omega_1} u\bar{\varphi}dx - \lambda^2 \int_{\Omega_2} y\bar{\psi}dx + a\int_{\Omega_1} \nabla u \cdot \nabla\bar{\varphi}dx + b\int_{\Omega_2} \nabla y \cdot \nabla\bar{\psi}dx \\ &+ i\lambda \int_{\Gamma_1} ((i\lambda I - B)^{-1}C, C)\gamma(u)\gamma(\bar{\varphi})d\Gamma = \int_{\Omega_1} (g_1 + i\lambda f_1)\bar{\varphi}dx \\ &+ \int_{\Omega_2} (g_2 + i\lambda f_2)\bar{\psi}dx + \int_{\Gamma_1} ((i\lambda I - B)^{-1}(h + C\gamma(f_1)), C)\gamma(\bar{\varphi})d\Gamma. \end{aligned} \tag{2.53}$$

Here we note that Lax-Milgram Lemma cannot be applied because the coercivity is not available. Therefore, we use a compact perturbation argument. For that purpose, let us introduce the sesquilinear form

$$a_\lambda((u, y), (\varphi, \psi)) = a\int_{\Omega_1} \nabla u \cdot \nabla\bar{\varphi}dx + b\int_{\Omega_2} \nabla y \cdot \nabla\bar{\psi}dx. \tag{2.54}$$

This sesquilinear form a_λ is continuous and coercive on H. Then, by Lax-Milgram Lemma, the operator

$$A_\lambda : \mathsf{H} \to \mathsf{H}' : (u, y) \to A_\lambda(u, y),$$

with $A_\lambda(u, y)((\varphi, \psi)) = a_\lambda((u, y), (\varphi, \psi))$ is an isomorphism. Now, let us set

$$R_\lambda : \mathsf{H} \to \mathsf{H}' : (u, y) \to R_\lambda(u, y),$$

with

$$R_\lambda(u, y)((\varphi, \psi)) = -\lambda^2 \int_{\Omega_1} u\bar{\varphi}dx - \lambda^2 \int_{\Omega_2} y\bar{\psi}dx + i\lambda \int_{\Gamma_1} ((i\lambda I - B)^{-1}C, C)\gamma(u)\gamma(\bar{\varphi})d\Gamma.$$

Due to the continuity of B and C and Cauchy-Schwarz's inequality, we see that

$$\begin{aligned} |R_\lambda(u, y)((\varphi, \psi))| \ &\leq \ \lambda^2 \, ||(u, y)||_{\mathsf{L}^2} ||(\varphi, \psi)||_{\mathsf{L}^2} + C_\lambda ||u||_{L^2(\Gamma_1)} ||\varphi||_{L^2(\Gamma_1)} \\ &\leq \ \lambda^2 \, ||(u, y)||_{\mathsf{L}^2} ||(\varphi, \psi)||_{\mathsf{L}^2} + C_\lambda ||(u, y)||_{\mathsf{L}^2_T} ||(\varphi, \psi)||_{\mathsf{L}^2_T}, \end{aligned} \tag{2.55}$$

where C_λ is a positive constant depending on λ and

$$\mathsf{L}^2 = L^2(\Omega_1) \times L^2(\Omega_2) \quad \text{and} \quad \mathsf{L}^2_T = L^2(\Gamma_1) \times L^2(\Gamma_2).$$

Now, for $\varepsilon \in (0, \frac{1}{2})$ we set the following space

$$H_\varepsilon = \left\{ (\varphi, \psi) \in H^{\frac{1}{2}+\varepsilon}(\Omega_1) \times H^{\frac{1}{2}+\varepsilon}(\Omega_2) \ : \ \varphi = \psi \ \text{ on } \ \mathcal{I} \right\}.$$

Then, by a trace theorem, equation (2.55) gives

$$|R_\lambda(u, y)((\varphi, \psi))| \ \leq \ \lambda^2 \, ||(u, y)||_{\mathsf{L}^2} ||(\varphi, \psi)||_{\mathsf{L}^2} + C_{\lambda,\varepsilon} ||(u, y)||_{\mathsf{H}} ||(\varphi, \psi)||_{\mathsf{H}_\varepsilon}, \tag{2.56}$$

with $C_{\lambda,\varepsilon}$ is a positive constant depending on λ and ε. If we introduce, for $\varepsilon \in (0, \frac{1}{2})$,

$$\mathsf{H}_{\varepsilon,\Gamma_2} = \left\{ (\varphi, \psi) \in \mathsf{H}_\varepsilon \ : \ \psi = 0 \ \text{ on } \ \Gamma_2 \right\},$$

that is clearly a Hilbert space equipped with the inner product of H_ε, we deduce, from (2.56), that

$$|R_\lambda(u,y)((\varphi,\psi))| \ \leq \ C_{\lambda,\varepsilon}||(u,y)||_{\mathsf{H}}||(\varphi,\psi)||_{H_\varepsilon}, \tag{2.57}$$

where $C_{\lambda,\varepsilon}$ is a new positive constant depending on λ and ε. Then, (2.57) means equivalently that

$$\sup_{(\varphi,\psi)\in\mathsf{H}_\varepsilon,(\varphi,\psi)\neq 0}\frac{|R_\lambda(u,y)((\varphi,\psi))|}{||(\varphi,\psi)||_{\mathsf{H}_\varepsilon}}\leq C_{\lambda,\varepsilon}||(u,y)||_{\mathsf{H}}.$$

Accordingly, as H is subset in H_ε with densely compact embedding, we deduce that $R_\lambda(u,y)$ belongs to H'_ϵ with

$$\begin{aligned}||R_\lambda(u,y)||_{\mathsf{H}'_\varepsilon} &= \sup_{(\varphi,\psi)\in\mathsf{H}_\varepsilon,(\varphi,\psi)\neq 0}\frac{|R_\lambda(u,y)((\varphi,\psi))|}{||(\varphi,\psi)||_{\mathsf{H}_\varepsilon}}\\ &\leq \sup_{(\varphi,\psi)\in\mathsf{H},(\varphi,\psi)\neq 0}\frac{|R_\lambda(u,y)((\varphi,\psi))|}{||(\varphi,\psi)||_{\mathsf{H}_\varepsilon}}\\ &\leq C_{\lambda,\varepsilon}||(u,y)||_{\mathsf{H}}.\end{aligned}$$

As H is compactly and densely embedded into H_ε, by duality, H'_ε is compactly embedded into H', and therefore R_λ is a compact operator from H into H'. Thus, we deduce that $A_\lambda+R_\lambda$ is a Fredholm operator of index zero from H into H'. Now by setting

$$L_\lambda((\varphi,\psi))=\int_{\Omega_1}(g_1+i\lambda f_1)\bar{\varphi}dx+\int_{\Omega_2}(g_2+i\lambda f_2)\bar{\psi}dx+\int_{\Gamma_1}((i\lambda I-B)^{-1}(h+C\gamma(f_1)),C)\gamma(\bar{\varphi})d\Gamma.$$

We notice that (2.53) is equivalent to

$$(A_\lambda+R_\lambda)(u,y)=L_\lambda \quad \text{in} \quad \mathsf{H}'. \tag{2.58}$$

Hence problem (2.44) admits a unique solution (u,y) if and only if $A_\lambda+R_\lambda$ is invertible. But $A_\lambda+R_\lambda$ being a Fredholm operator it is enough to prove that $A_\lambda+R_\lambda$ is injective, i.e,

$$\ker(A_\lambda+R_\lambda)=\{0\}.$$

Let us now fix $(\boldsymbol{u},\boldsymbol{y})\in\ker(A_\lambda+R_\lambda)$, then it satisfies

$$\begin{aligned}-\lambda^2\int_{\Omega_1}\boldsymbol{u}\bar{\boldsymbol{\varphi}}dx-\lambda^2\int_{\Omega_2}\boldsymbol{y}\bar{\boldsymbol{\psi}}dx+a\int_{\Omega_1}\nabla\boldsymbol{u}\cdot\nabla\bar{\boldsymbol{\varphi}}dx+b\int_{\Omega_2}\nabla\boldsymbol{y}\cdot\nabla\bar{\boldsymbol{\psi}}dx\\ +i\lambda\int_{\Gamma_1}((i\lambda I-B)^{-1}C,C)\gamma(\boldsymbol{u})\gamma(\bar{\boldsymbol{\varphi}})d\Gamma=0.\end{aligned} \tag{2.59}$$

Thus, if we set $\boldsymbol{v}=i\lambda\boldsymbol{u}$, $\boldsymbol{z}=i\lambda\boldsymbol{y}$ and $\boldsymbol{\eta}=-i\lambda\gamma(\boldsymbol{u})(i\lambda I-B)^{-1}C$, we conclude that $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{v},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta})\in D(\mathcal{A})$ is a solution of

$$(i\lambda I-\mathcal{A})\boldsymbol{U}=\mathbf{0}.$$

Using Lemma 2.5, we deduce that $\boldsymbol{U}=0$. This shows that $A_\lambda+R_\lambda$ is invertible and therefore a unique solution $(u,y)\in\mathsf{H}$ of (2.58) exists. At this stage, by setting $v=i\lambda u-f_1$, $z=i\lambda y-f_2$ and $\eta=(i\lambda I-B)^{-1}\left(h-C(i\lambda\gamma(u)-\gamma(f_1))\right)$, we conclude that $U=(u,v,y,z,\eta)\in D(\mathcal{A})$ is a solution of (2.44) and consequently $(i\lambda I-\mathcal{A})$ is surjective. The proof is thus complete. □

Proof of Theorem 2.4. From Lemma 2.5, the operator $\mathcal{A}$ has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathcal{A})\cap i\mathbb{R}=\emptyset$). Moreover, from Lemma 2.5 and Lemma 2.6, $i\lambda I-\mathcal{A}$ is bijective for all $\lambda\in\mathbb{R}$ and since $\mathcal{A}$ is closed, we conclude with the help of the closed graph theorem that $i\lambda I-\mathcal{A}$ is an isomorphism for all $\lambda\in\mathbb{R}$, hence that $\sigma(\mathcal{A})\cap i\mathbb{R}=\emptyset$. Therefore, according to Theorem A.2, we get that the C_0-semigroup of contraction $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. The proof is thus complete. □

2.3. **Lack of Exponential Stability.** In this part, we will prove that system (1.1)-(1.2) is not exponentially stable. In other words we will prove the following theorem:

Theorem 2.7. *The C_0 semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is not uniformly stable in the energy space $\mathcal{H}$.*

According to Theorem A.3 due to Huang [18] and Prüss [35] it is sufficient to prove that the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis. For this aim, let us start with the following technical result.

Lemma 2.8. *Define the linear unbounded operator $\mathcal{O}_{\Delta,R} : D(\mathcal{O}_{\Delta,R}) \longmapsto L^2(\Omega)\times L^2(\Omega)$ by*

$$D(\mathcal{O}_{\Delta,R}) = \big\{(\mathsf{f},\mathsf{g}) \in \mathsf{H} : (\Delta \mathsf{f},\Delta \mathsf{g}) \in L^2(\Omega_1)\times L^2(\Omega_2),\ a\partial_{\nu_1}\mathsf{f} + b\partial_{\nu_2}\mathsf{g} = 0 \quad \text{on } \mathcal{I} \tag{2.60}$$
$$\text{and} \quad a\partial_{\nu_1}\mathsf{f} + (C,C)\mathsf{f} = 0 \quad \text{on } \Gamma_1\big\},$$

and

$$\mathcal{O}_{\Delta,R}(\mathsf{f},\mathsf{g}) = (-a\Delta \mathsf{f}, -b\Delta \mathsf{g}), \quad \forall\, (\mathsf{f},\mathsf{g}) \in D(\mathcal{O}_{\Delta,R}). \tag{2.61}$$

Then, $\mathcal{O}_{\Delta,R}$ is a positive self-adjoint operator with a compact resolvent.

Proof. To prove that $\mathcal{O}_{\Delta,R}$ is a positive self-adjoint operator, we will show that $\mathcal{O}_{\Delta,R}$ is a symmetric m-accretive operator. For this aim, we will divide the proof into steps.

Step 1. ($\mathcal{O}_{\Delta,R}$ is symmetric.) Indeed, for all $(f,g),(h,k) \in D(\mathcal{O}_{\Delta,R})$, we have

$$\begin{aligned}
\left(\mathcal{O}_{\Delta,R}(f,g),(h,k)\right)_{L^2(\Omega_1)\times L^2(\Omega_2)} &= -a\int_{\Omega_1}(\Delta f)\overline{h}dx - b\int_{\Omega_2}(\Delta g)\overline{k}dx\\
&= a\int_{\Omega_1}\nabla f\cdot\nabla\overline{h}dx + b\int_{\Omega_2}\nabla g\cdot\nabla\overline{k}dx + a\int_{\Gamma_1}(C,C)f\overline{h}d\Gamma\\
&= a\int_{\Omega_1}\nabla f\cdot\nabla\overline{h}dx + b\int_{\Omega_2}\nabla g\cdot\nabla\overline{k}dx - a\int_{\Gamma_1}f\partial_{\nu_1}\overline{h}d\Gamma\\
&= \left((f,g),\mathcal{O}_{\Delta,R}(h,k)\right)_{L^2(\Omega_1)\times L^2(\Omega_2)}.
\end{aligned} \tag{2.62}$$

Thus, $\mathcal{O}_{\Delta,R}$ is symmetric.

Step 2. $\mathcal{O}_{\Delta,R}$ is m-accretive. Indeed, for all $(f,g) \in D(\mathcal{O}_{\Delta,R})$, we have

$$\begin{aligned}
\Re\left(\mathcal{O}_{\Delta,R}(f,g),(f,g)\right)_{L^2(\Omega_1)\times L^2(\Omega_2)} &= -a\int_{\Omega_1}(\Delta f)\overline{f}dx - b\int_{\Omega_2}(\Delta g)\overline{g}dx\\
&= a\int_{\Omega_1}|\nabla f|^2dx + b\int_{\Omega_2}|\nabla g|^2dx + a\int_{\Gamma_1}(C,C)|f|^2d\Gamma \geq 0.
\end{aligned} \tag{2.63}$$

Thus, $\mathcal{O}_{\Delta,R}$ is accretive operator. Now, let $(F,G) \in L^2(\Omega_1)\times L^2(\Omega_2)$ and $\lambda > 0$, looking for $(f,g) \in D(\mathcal{O}_{\Delta,R})$ solution of

$$(\lambda I + \mathcal{O}_{\Delta,R})(f,g) = (F,G). \tag{2.64}$$

Equivalently, we have the following system

$$\lambda f - a\Delta f = F, \tag{2.65}$$
$$\lambda g - b\Delta g = G. \tag{2.66}$$

Taking $(\varphi,\psi) \in \mathsf{H}$, then integrating after multiplying (2.65) by φ and (2.66) by ψ, yields the two equations added in the following form

$$a\int_{\Omega_1}\nabla f\cdot\nabla\overline{\varphi}dx+b\int_{\Omega_2}\nabla g\cdot\nabla\overline{\psi}dx+\int_{\Gamma_1}(C,C)f\overline{\varphi}d\Gamma+\lambda\int_{\Omega_1}f\overline{\varphi}dx+\lambda\int_{\Omega_2}g\overline{\psi}dx=\int_{\Omega_1}F\overline{\varphi}dx+\int_{\Omega_2}G\overline{\psi}dx. \tag{2.67}$$

Letting

$$S((f,g),(\varphi,\psi))=a\int_{\Omega_1}\nabla f\cdot\nabla\overline{\varphi}dx+b\int_{\Omega_2}\nabla g\cdot\nabla\overline{\psi}dx+\int_{\Gamma_1}(C,C)f\overline{\varphi}d\Gamma+\lambda\int_{\Omega_1}f\overline{\varphi}dx+\lambda\int_{\Omega_2}g\overline{\psi}dx, \tag{2.68}$$

and

$$L(\varphi,\psi)=\int_{\Omega_1}F\overline{\varphi}dx+\int_{\Omega_2}G\overline{\psi}dx. \tag{2.69}$$

It is easy to see that S is a sesquilinear, continuous, and coercive form on the space $\mathsf{H}\times\mathsf{H}$, and L is antilinear and continuous form on H. Then, it follows by Lax-Milgram's theorem that $S((f,g),(\varphi,\psi))=L(\varphi,\psi)$ admits a unique solution $(f,g)\in\mathsf{H}$. By classical elliptic regularity, we deduce that $(f,g)\in D(\mathcal{O}_{\Delta,R})$ solution of system (2.65)-(2.66). Thus $\mathcal{O}_{\Delta,R}$ is m-accretive.

Step 3. $\mathcal{O}_{\Delta,R}$ has a compact resolvent.

$$R_\lambda(\mathcal{O}_{\Delta,R})=(\lambda I+\mathcal{O}_{\Delta,R})^{-1}.$$

Due to Sobolev embeddings, $R_0(\mathcal{O}_{\Delta,R})$ is compact. Then using the following resolvent identity

$$R_\lambda-R_\mu=(\mu-\lambda)R_\mu R_\lambda,$$

we deduce that the resolvent of the operator $(\lambda I+\mathcal{O}_{\Delta,R})^{-1}$ of $\mathcal{O}_{\Delta,R}$ is compact, and the proof is thus complete. □

Proof of Theorem 2.7 According to Theorem A.3 due to Huang [18] and Prüss [35], it is sufficient to show that the resolvent of $\mathcal{A}$ is not uniformly bounded on the imaginary axis. In other words, it is enough to show the existence of a positive real number M and some sequences $\lambda_n\in i\mathbb{R}$, $U_n=(u_n,v_n,y_n,z_n,\eta_n)^\top\in D(\mathcal{A})$, and $F_n=(f_{1,n},g_{1,n},f_{2,n},g_{2,n},h_n)^\top\in\mathcal{H}$, where $n\in\mathbb{N}$, such that

$$(\lambda_n I-\mathcal{A})U_n=F_n,\quad \forall n\in\mathbb{N}, \tag{2.70}$$

$$\|U_n\|_{\mathcal{H}}=M,\quad \forall n\in\mathbb{N}, \tag{2.71}$$

$$\lim_{n\to\infty}\|F_n\|_{\mathcal{H}}=0. \tag{2.72}$$

From Lemma 2.8, we can consider the sequence of eigenfunctions $(\varphi_n,\psi_n)_{n\in\mathbb{N}}$ (that form an orthonormal basis of $L^2(\Omega_1)\times L^2(\Omega_2)$) of the operator $\mathcal{O}_{\Delta,R}$, corresponding to the eigenvalues $(\mu_n^2)_{n\in\mathbb{N}}$, such that μ_n^2 tends to infinity as n goes to infinity. Consequently, for all $n\in\mathbb{N}$, they satisfy the following system

$$\begin{cases} -a\Delta\varphi_n=\mu_n^2\varphi_n, & \text{in } \Omega_1,\\ -b\Delta\psi_n=\mu_n^2\psi_n, & \text{in } \Omega_2,\\ \varphi_n-\psi_n=0, & \text{on } \mathcal{I},\\ a\partial_{\nu_1}\varphi_n+b\partial_{\nu_2}\psi_n=0, & \text{on } \mathcal{I},\\ \partial_{\nu_1}\varphi_n+(C,C)\varphi_n=0, & \text{on } \Gamma_1,\\ \psi_n=0 & \text{on } \Gamma_2, \end{cases} \tag{2.73}$$

with

$$\|(\varphi_n,\psi_n)\|^2_{L^2(\Omega_1)\times L^2(\Omega_2)}=\int_{\Omega_1}|\varphi_n|^2dx+\int_{\Omega_2}|\psi_n|^2dx=1. \tag{2.74}$$

Now, let us choose

$$u_n = \frac{\varphi_n}{i\mu_n}, \quad v_n = \varphi_n, \quad y_n = \frac{\psi_n}{i\mu_n}, \quad z_n = \psi_n, \quad \eta_n = -\frac{1}{i\mu_n} C\gamma(\varphi_n). \tag{2.75}$$

So, by setting $F_n = (0, 0, 0, 0, \frac{-i}{\mu_n} BC\gamma(\varphi_n))$, we deduce that

$$U_n = (u_n, v_n, y_n, z_n, h_n) \tag{2.76}$$

is the solution in $D(\mathcal{A})$ of the following equation

$$(i\mu_n I - \mathcal{A})U_n = F_n. \tag{2.77}$$

Now, multiplying equation $(2.73)_1$ and $(2.73)_2$ by φ_n and ψ_n respectively, integrating by parts, we get

$$\mu_n^{-2} a \int_{\Gamma_1} (C, C)|\varphi_n|^2 d\Gamma + \mu_n^{-2} a \int_{\Omega_1} |\nabla\varphi_n|^2 dx + \mu_n^{-2} b \int_{\Omega_2} |\nabla\psi_n|^2 dx = \int_{\Omega_1} |\varphi_n|^2 dx + \int_{\Omega_2} |\psi_n|^2 dx = 1. \tag{2.78}$$

This implies that

$$\|\varphi_n\|^2_{H^1(\Omega_1)} \lesssim \mu_n^2. \tag{2.79}$$

On the other hand, we have

$$\begin{aligned} \|U_n\|^2_{\mathcal{H}} &= \int_{\Omega_1} |\varphi_n|^2 dx + \int_{\Omega_2} |\psi_n|^2 dx \\ &\quad + \mu_n^{-2} \int_{\Omega_1} |\nabla\varphi_n|^2 dx + \mu_n^{-2} \int_{\Omega_2} |\nabla\psi_n|^2 dx + \mu_n^{-2} \int_{\Gamma_1} \|C\varphi_n\|^2_M d\Gamma \geq 1. \end{aligned} \tag{2.80}$$

By using the trace theorem of interpolation type (see Theorem 1.4.4 in [26] and Theorem 1.5.1.10 in [14]), equation (2.74) and equation (2.79), we obtain

$$\|F_n\|^2_{\mathcal{H}} = \mu_n^{-2} \int_{\Gamma_1} \|BC\varphi_n\|^2_M d\Gamma \lesssim \mu_n^{-2} \|\varphi_n\|^2_{L^2(\Gamma_1)} \lesssim \mu_n^{-2} \|\varphi_n\|_{H^1(\Omega)} \|\varphi_n\|_{L^2(\Omega)} \lesssim \mu_n^{-1} \to 0. \tag{2.81}$$

Then, the resolvent of the operator $\mathcal{A}$ is not uniformly bounded on the imaginary axis, and consequently our system is not uniformly (exponentially) stable. The proof is thus complete. □

3. Polynomial Stability

Since system (1.1)-(1.2) is not uniformly stable, we will look for a polynomial energy decay rate for smooth solutions. We assume that there exists a constant $\delta > 0$ and a point $x_0 \in \mathbb{R}^d$ such that, putting $r(x) = x - x_0$, we have

$$\text{(BMGC)} \quad (r \cdot \nu_1) \geq \delta^{-1}, \quad \forall x \in \Gamma_1, \quad (r \cdot \nu_2) \leq 0, \quad \forall x \in \Gamma_2, \quad \text{and} \quad (r \cdot \nu_1) \leq 0, \quad \forall x \in \mathcal{I},$$

where $(\cdot, \cdot)$ designates the scalar product in $\mathbb{R}^d$.

Definition 3.1. *The matrix valued function $B \in C^{0,1}(\Gamma_1; M_m(\mathbb{C}))$ is said to be totally M-coercive if there exist $\alpha = (\alpha_j)_{1 \leq j \leq m}$, $\alpha_j > 0$, such that, for every $x \in \Gamma_1$,*

$$\Re(B(x)V, V)_x = \overline{V}^T M(x)B(x)V \geq \sum_{j=1}^m \alpha_j |v_j|^2 \quad \forall V = (v_1, \cdots, v_m) \in \mathbb{C}^m.$$

Definition 3.2. *For $j_0 \in \{1, 2, \cdots, m\}$, the matrix valued function $B \in C^{0,1}(\Gamma_1; M_m(\mathbb{C}))$ is said to be j_0-partially M-coercive if there exists an index $\alpha = (\alpha_j)_{1 \leq j \leq m, j \neq j_0}$, $\alpha_j > 0$, such that, for every $x \in \Gamma_1$,*

$$\Re(B(x)V, V)_x \geq \sum_{j=1, j\neq j_0}^m \alpha_j |v_j|^2 \quad \forall V = (v_1, \cdots, v_m) \in \mathbb{C}^m.$$

$$\text{(PSC1)} \quad \begin{cases} \text{The matrix valued function } -B \text{ is totally } M\text{-coercive, and} \\ \\ \exists\ j_1 \in \{1,2,\cdots,m\},\ c_{j_1,0} > 0\ :\ \Re\left\{c_{j_1}^2(x)\right\} \geq c_{j_1,0},\ \forall\ x \in \Gamma_1. \end{cases}$$

$$\text{(PSC2)} \quad \begin{cases} \exists\ j_0 \in \{1,2,\cdots,m\}\ :\ -B \text{ is } j_0\text{-partially } M\text{-coercive, and} \\ \\ \exists\, j_2, j_3 \in \{1,2,\cdots,m\},\ c_{j_2,0},\ c_{j_3,0} > 0\ :\ j_2 \neq j_3,\ \text{and } \Re\left\{c_{j_k}^2(x)\right\} \geq c_{j_k,0},\ \forall\, x \in \Gamma_1,\ k = 2,3. \end{cases}$$

$$\text{(PSC3)} \quad \begin{cases} \exists\ j_0 \in \{1,2,\cdots,m\}\ :\ -B \text{ is } j_0\text{-partially } M\text{-coercive, and} \\ \\ \exists\, j_4 \in \{1,2,\cdots,m\} \setminus \{j_0\},\ c_{j_4,0} > 0\ :\ \Re\left\{c_{j_4}^2(x)\right\} \geq c_{j_4,0},\ \forall\, x \in \Gamma_1,\ \text{and } c_k = 0,\ \text{for } k \neq j_4. \end{cases}$$

$$\text{(PSC4)} \quad \begin{cases} \exists\ j_0 \in \{1,2,\cdots,m\},\ c_{j_0,0} > 0\ :\ -B \text{ is } j_0\text{-partially } M\text{-coercive, and } \Re\left\{c_{j_0}^2(x)\right\} \geq c_{j_0,0},\ \forall\, x \in \Gamma_1. \\ \\ \exists\, j_5 \in \{1,2,\cdots,m\} \setminus \{j_0\},\ b_{j_0 j_5,0} > 0\ :\ \Re\left\{b_{j_0 j_5}^2(x)\right\} \geq b_{j_0 j_5,0},\ \forall\, x \in \Gamma_1,\ \text{and } c_k = 0,\ \text{for } k \neq j_0. \end{cases}$$

Theorem 3.3. *Let $b \geq a$. Assume that $i\mathbb{R} \subset \rho(\mathcal{A})$ and that the geometric conditions (BMGC) holds. Then, the C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable in the sense that, there exists a constant $\mathcal{C} > 0$ such that, for all $U_0 \in D(\mathcal{A})$, the energy of system (1.1)-(1.2) satisfies the following estimation*

$$E(t) \leq \frac{\mathcal{C}}{t^{2/\ell}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0 \tag{3.1}$$

with

$$\ell = \begin{cases} 2, & \textit{if} \quad \text{(PSC1)}, \quad \textit{or} \quad \text{(PSC2)}, \quad \textit{or} \quad \text{(PSC3)} \textit{ holds}, \\ 4, & \textit{if} \quad \text{(PSC4)} \textit{ holds}. \end{cases}$$

According to Theorem A.4, to prove Theorem 3.3, we need to prove the following condition

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{\lambda^\ell} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.2}$$

The condition (3.2) is proved by a contradiction argument. For this purpose, suppose that (3.2) is false, then there exists $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \eta_n)^\top)\}_{n\geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ with

$$|\lambda_n| \to \infty \quad \text{as } n \to \infty, \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \left\|(u_n, v_n, y_n, z_n, \eta_n)^\top\right\|_{\mathcal{H}} = 1, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

such that

$$(\lambda_n)^\ell (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, g_{1,n}, f_{2,n}, g_{2,n}, h_n)^\top \to 0 \quad \text{in} \quad \mathcal{H} \quad \text{as } n \to \infty. \tag{3.4}$$

We aim to prove that $\|U_n\|_{\mathcal{H}} = o(1)$ to get the desired contradiction. For this, we drop the index n for simplicity, and detail the equation (3.4), so that the following system is obtained

$$i\lambda u - v = \lambda^{-\ell} f_1 \quad \text{in} \quad H^1(\Omega_1), \tag{3.5}$$

$$i\lambda v - a\Delta u = \lambda^{-\ell} g_1 \quad \text{in} \quad L^2(\Omega_1), \tag{3.6}$$

$$i\lambda y - z = \lambda^{-\ell} f_2 \quad \text{in} \quad H^1(\Omega_2), \tag{3.7}$$

$$i\lambda z - b\Delta y = \lambda^{-\ell} g_2 \quad \text{in} \quad L^2(\Omega_2), \tag{3.8}$$

$$i\lambda\eta - B\eta + C\gamma(v) = \lambda^{-\ell} h \quad \text{in} \quad \left(L^2(\Gamma_1)\right)^m. \tag{3.9}$$

For clarity, we divide the proof of Theorem 3.3 into several Lemmas.

Lemma 3.4. *Under the same conditions of Theorem 3.3, the solution $U = (u, v, y, z, \eta)^\top \in D(\mathcal{A})$ of (3.5)-(3.9) satisfies the following estimations*

$$\int_{\Gamma_1} ||\eta||^2 d\Gamma = o(\lambda^{-2}), \quad \int_{\Gamma_1} |\partial_{\nu_1} u|^2 d\Gamma = o(\lambda^{-2}), \quad \textit{and} \quad \int_{\Gamma_1} |u|^2 d\Gamma = o(\lambda^{-2}). \tag{3.10}$$

Proof. Taking the inner product of (3.4) with U in $\mathcal{H}$, then using the fact that U is uniformly bounded in $\mathcal{H}$, we get

$$\Re\left((i\lambda I-\mathcal{A})U,U\right)_{\mathcal{H}}=-\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}}=-\int_{\Gamma_1}\Re\left(B(x)\eta,\eta\right)d\Gamma=o\left(\lambda^{-\ell}\right).\tag{3.11}$$

Case 1. If (PSC1) **holds.** Then, using (3.11) and the Definition 3.1, we get

$$\int_{\Gamma_1}|\eta_j|^2d\Gamma=o\left(\lambda^{-\ell}\right),\quad \forall\ j\in\{1,2,\cdots,m\}.\tag{3.12}$$

It follows, from the continuity of M that

$$\int_{\Gamma_1}||\eta||^2d\Gamma=\int_{\Gamma_1}(\eta,\eta)\,d\Gamma=\int_{\Gamma_1}\overline{\eta}^TM(x)\eta d\Gamma=o\left(\lambda^{-\ell}\right).\tag{3.13}$$

Besides, using (3.5) in (3.9) will implies

$$i\lambda\eta_{j_1}-\sum_{j=1}^{m}b_{j_1j}(x)\eta_j+i\lambda c_{j_1}(x)u=\lambda^{-\ell}h_{j_1}+\lambda^{-\ell}c_{j_1}(x)f_1.\tag{3.14}$$

Then, multiplying equation (3.14) by $c_{j_1}(x)\lambda\overline{u}$, integrating over Γ_1 and taking the imaginary part, we get

$$\begin{aligned}\int_{\Gamma_1}\Re\left\{c_{j_1}^2(x)\right\}|\lambda u|^2d\Gamma&=-\Re\left\{\int_{\Gamma_1}c_{j_1}(x)\frac{1}{\sqrt{2\varepsilon}}\lambda\eta_{j_1}\sqrt{2\varepsilon}\lambda\overline{u}d\Gamma\right\}\\&+\Im\left\{\sum_{j=1}^{m}\int_{\Gamma_1}c_{j_1}(x)b_{j_1j}(x)\lambda\eta_j\overline{u}d\Gamma+\int_{\Gamma_1}\lambda^{-\ell+1}c_{j_1}(x)h_{j_1}\overline{u}d\Gamma+\int_{\Gamma_1}\lambda^{-\ell+1}c_{j_1}^2(x)f_1\overline{u}d\Gamma\right\},\end{aligned}\tag{3.15}$$

where $\varepsilon>0$. Then, using (3.12) and the fact that $\|u\|_{L^2(\Gamma_1)}=O(1)$, $\|h_{j_1}\|_{L^2(\Gamma_1)}=o(1)$, $\|f_1\|_{L^2(\Gamma_1)}=o(1)$, and that $\Re\left\{c_{j_1}^2(x)\right\}\geq c_{j_1,0}>0$, we get

$$(c_{j_1,0}-\varepsilon)\int_{\Gamma_1}|\lambda u|^2d\Gamma=o(\lambda^{-\ell+2}).\tag{3.16}$$

By letting $\varepsilon=\frac{c_{j_1,0}}{2}$, we obtain the resulting estimate. The second estimation in (3.10) directly follows from the fact that $a\partial_{\nu_1}u(x,t)-(\eta(x),C)=0$ on Γ_1. Indeed,

$$||(\eta,C)||\lesssim||\eta||_{L^2(\Gamma_1)}||C||.\tag{3.17}$$

Thus we get the results in (3.10).

Case 2. If (PSC2) **holds.** Then, from (3.11) and the definition 3.2, we deduce that

$$\int_{\Gamma_1}|\eta_j|^2d\Gamma=o\left(\lambda^{-\ell}\right),\quad \forall\ j\in\{1,2,\cdots,m\}\setminus\{j_0\}.\tag{3.18}$$

Here we distinguish two cases:

Case 2.1. If $c_{j_0}=0$. Then, from the equation (3.9), we have

$$(i\lambda-b_{j_0j_0(x)})\eta_{j_0}-\sum_{j=1,j\neq j_0}^{m}b_{j_0j}(x)\eta_j=\ \lambda^{-\ell}h_{j_0},\quad\text{on}\quad\Gamma_1.\tag{3.19}$$

Now, multiplying equation (3.19) by $\lambda\overline{\eta}_{j_0}$, integrating over Γ_1, taking the imaginary part, using (3.18) and the boundedness of the entry b_{jj_0}, we get

$$\int_{\Gamma_1}|\lambda\eta_{j_0}|^2d\Gamma=\Im\left\{\int_{\Gamma_1}b_{j_0j_0}(x)\lambda|\eta_{j_0}|^2d\Gamma+\sum_{j=1,j\neq j_0}^{m}\int_{\Gamma_1}b_{j_0j}(x)\lambda\eta_j\overline{\eta}_{j_0}d\Gamma\right\}+o(\lambda^{-\ell+1}).\tag{3.20}$$

Using (3.18) and the fact that $\|b_{j_0j_0}\|_\infty\leq\lambda/2$ in (3.20), we get

$$\int_{\Gamma_1}|\eta_{j_0}|^2d\Gamma=o(\lambda^{-\ell})\quad\text{and consequently}\quad\int_{\Gamma_1}||\eta||^2d\Gamma=o(\lambda^{-\ell}).$$

On the other hand, using condition (PSC2) and equation (3.9), we get

$$i\lambda\eta_{j_2}-\sum_{j=1}^{m}b_{j_2j}(x)\eta_j+i\lambda c_{j_2}(x)u=\lambda^{-\ell}h_{j_2}+\lambda^{-\ell}c_{j_1}(x)f_1. \tag{3.21}$$

That, by repeating the same procedure used in **Case 1**, gives

$$\int_{\Gamma_1}|\lambda u|^2d\Gamma=o(\lambda^{-\ell+2})\quad\text{and}\quad\int_{\Gamma_1}|\partial_\nu u|^2d\Gamma=o(\lambda^{-\ell}).$$

Case 2.2. If $c_{j_0}\neq 0$. We need distinguish two cases.

Case 2.2.1 If $j_0\neq j_2$ and $j_0\neq j_3$. Then, we consider the following equations

$$i\lambda\eta_{j_0}-\sum_{j=1}^{m}b_{j_0j}(x)\eta_j+i\lambda c_{j_0}(x)u=\lambda^{-\ell}h_{j_0}+\lambda^{-\ell}c_{j_0}(x)f_1. \tag{3.22}$$

$$i\lambda\eta_{j_2}-\sum_{j=1}^{m}b_{j_2j}(x)\eta_j+i\lambda c_{j_2}(x)u=\lambda^{-\ell}h_{j_2}+\lambda^{-\ell}c_{j_2}(x)f_1. \tag{3.23}$$

Multiplying equation (3.22) by $c_{j_2}^2(x)$ and equation (3.23) by $c_{j_0}(x)c_{j_2}(x)$, we get

$$\begin{aligned}&i\lambda c_{j_2}^2(x)\eta_{j_0}-i\lambda c_{j_0}(x)c_{j_2}(x)\eta_{j_2}-\sum_{j=1}^{m}\left(b_{j_0j}(x)c_{j_2}^2(x)+b_{j_2j}(x)c_{j_0}(x)c_{j_2}(x)\right)\eta_j\\&=\lambda^{-\ell}\left(c_{j_2}^2(x)h_{j_0}-c_{j_0}(x)c_{j_2}(x)h_{j_2}\right).\end{aligned} \tag{3.24}$$

Now, multiplying equation (3.24) by $\lambda\overline{\eta}_{j_0}$, integrating over Γ_1, taking the imaginary part, using (3.18) and the continuity of the entries of the matrix B, we get

$$\begin{aligned}\int_{\Gamma_1}\Re\left\{c_{j_2}^2(x)\right\}|\lambda\eta_{j_0}|^2d\Gamma=&\int_{\Gamma_1}\Re\left\{c_{j_0}(x)c_{j_2}(x)\frac{1}{\sqrt{2\varepsilon}}\lambda\eta_{j_2}\sqrt{2\varepsilon}\lambda\overline{\eta_{j_0}}d\Gamma\right\}\\&+\Im\left\{\sum_{j=1}^{m}\int_{\Gamma_1}\left(b_{j_0j}(x)c_{j_2}^2(x)+b_{j_2j}(x)c_{j_0}(x)c_{j_2}(x)\right)\lambda\eta_j\overline{\eta_{j_0}}d\Gamma\right\}+o(\lambda^{-\ell+1}).\end{aligned} \tag{3.25}$$

Applying Cauchy Schwartz and Young inequalities, we get

$$\int_{\Gamma_1}|\eta_{j_0}|^2d\Gamma=o(\lambda^{-\ell})\quad\text{and consequently}\quad\int_{\Gamma_1}||\eta||^2d\Gamma=o(\lambda^{-\ell}).$$

Return to equation (3.23), and repeat the same steps as in **Case 1**, we obtain

$$\int_{\Gamma_1}|\lambda u|^2d\Gamma=o(\lambda^{-\ell+2})\quad\text{and}\quad\int_{\Gamma_1}|\partial_\nu u|^2d\Gamma=o(\lambda^{-\ell}).$$

Case 2.2.2 If $j_0=j_2$ or $j_0=j_3$. Assume $j_0\neq j_2$, so we consider the following equations

$$i\lambda\eta_{j_0}-\sum_{j=1}^{m}b_{j_0j}(x)\eta_j+i\lambda c_{j_0}(x)u=\lambda^{-\ell}h_{j_0}+\lambda^{-\ell}c_{j_0}(x)f_1. \tag{3.26}$$

$$i\lambda\eta_{j_2}-\sum_{j=1}^{m}b_{j_2j}(x)\eta_j+i\lambda c_{j_2}(x)u=\lambda^{-\ell}h_{j_2}+\lambda^{-\ell}c_{j_2}(x)f_1. \tag{3.27}$$

Then following the steps used in **Case 2.2.1**, we get the same results. Note here that in case we assume $j_0\neq j_3$, we will consider the following equations

$$i\lambda\eta_{j_0}-\sum_{j=1}^{m}b_{j_0j}(x)\eta_j+i\lambda c_{j_0}(x)u=\lambda^{-\ell}h_{j_0}+\lambda^{-\ell}c_{j_0}(x)f_1, \tag{3.28}$$

$$i\lambda\eta_{j_3}-\sum_{j=1}^{m}b_{j_3j}(x)\eta_j+i\lambda c_{j_3}(x)u=\lambda^{-\ell}h_{j_3}+\lambda^{-\ell}c_{j_3}(x)f_1. \tag{3.29}$$

Then following the same technique, we get the desired results.

Case 3. If (PSC3) **holds.** It follows from (3.11) and the Definition 3.2 that

$$\int_{\Gamma_1}|\eta_j|^2d\Gamma=o\left(\lambda^{-\ell}\right),\quad \forall\, j\in\{1,2,\cdots,m\}\setminus\{j_0\}. \tag{3.30}$$

Now repeating the same procedure as in **Case 2.1**, we get the resulting estimates.

Finally, by letting $\ell=2$ in **case 1**, **case 2**, and **case 3**, we obtain the results in (3.10).

Case 4. If (PSC4) **holds.** It follows from (3.11) and the Definition 3.2 that

$$\int_{\Gamma_1}|\eta_j|^2d\Gamma=o\left(\lambda^{-\ell}\right),\quad \forall\, j\in\{1,2,\cdots,m\}\setminus\{j_0\}. \tag{3.31}$$

Then using (PSC4), we have

$$i\lambda\eta_{j_5}-b_{j_5j_0}\eta_{j_0}-\sum_{j=1,j\neq j_0}^{m}b_{j_5j}\eta_j=\lambda^{-\ell}h_{j_5}. \tag{3.32}$$

Multiplying equation (3.32) by $b_{j_5j_0}(x)\overline{\eta_{j_0}}$, integrating over Γ_1 and taking the real part, we get

$$\int_{\Gamma_1}\Re\left\{b_{j_5j_0}^2(x)\right\}|\eta_{j_0}|^2d\Gamma=\Im\left\{\int_{\Gamma_1}b_{j_5j_0}(x)\lambda\eta_{j_5}\overline{\eta_{j_0}}d\Gamma\right\}-\Re\left\{\sum_{j=1,j\neq j_0}^{m}\int_{\Gamma_1}b_{j_5j_0}(x)b_{j_5j}(x)\eta_j\overline{\eta_{j_0}}d\Gamma\right\}+o(\lambda^{-\ell}).$$

We then obtain

$$\int_{\Gamma_1}|\eta_{j_0}|^2d\Gamma=o(\lambda^{-\ell+2}).$$

Now, going back to the following equation

$$i\lambda\eta_{j_0}-\sum_{j=1}^{m}b_{j_0j}(x)\eta_j+i\lambda c_{j_0}(x)u=\lambda^{-\ell}h_{j_0}+\lambda^{-\ell}c_{j_0}(x)f_1. \tag{3.33}$$

Multiplying it by $c_{j_0}(x)\lambda\overline{u}$, integrating over Γ_1 and taking the imaginary part, as done in (3.15), we get

$$\int_{\Gamma_1}|\lambda u|^2d\Gamma=o(\lambda^{-\ell+4}).$$

Thus,

$$\int_{\Gamma_1}|\partial_\nu u|^2d\Gamma=o(\lambda^{-\ell+2}).$$

Here, letting $\ell=4$ in **Case 4**, we get the results in (3.10). The proof of the Lemma is thus completed. $\square$

Now, substituting v from (3.5) into (3.6) and z from (3.7) into (3.8) gives the following system

$$\lambda^2u+a\Delta u = -\lambda^{-\ell}g_1-i\lambda^{-\ell+1}f_1, \tag{3.34}$$

$$\lambda^2y+b\Delta y = -\lambda^{-\ell}g_2-i\lambda^{-\ell+1}f_2. \tag{3.35}$$

Lemma 3.5. *Under the same assumptions of Theorem 3.3, the solutions $(u,v,y,z,\eta)\in D(A)$ of (3.5)-(3.9) satisfies the following estimation*

$$d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+d\int_{\Omega_2}|\lambda y|^2dx+b(2-d)\int_{\Omega_2}|\nabla y|^2dx=o(1). \tag{3.36}$$

Proof. Multiplying equation (3.34) by $2(r\cdot\nabla\bar{u})$, integrating over Ω_1, then taking the real part, we obtain

$$\Re\left\{2\lambda^2\int_{\Omega_1}u(r\cdot\nabla\bar{u})dx+2a\int_{\Omega_1}\Delta u(r\cdot\nabla\bar{u})dx\right\}=o(\lambda^{-\ell}). \tag{3.37}$$

Noting that, since $\|\lambda u\|_{L^2(\Omega_1)}$, $\|\nabla u\|_{L^2(\Omega_1)}$ are uniformly bounded, then using (3.10) and the fact that $f_1\to 0$ in $H^1(\Omega_1)$ and $g_1\to 0$ in $L^2(\Omega_1)$, we deduce

$$\begin{aligned}&-2\Re\left\{\int_{\Omega_1}(\lambda^{-\ell}g_1+i\lambda^{-\ell+1}f_1)(r\cdot\nabla\bar{u})dx\right\}=\Re\left\{-2\lambda^{-\ell}\int_{\Omega_1}g_1(r\cdot\nabla\bar{u})dx\right\}\\&+\Re\left\{2id\lambda^{-\ell}\int_{\Omega_1}f_1(\lambda u)dx+2i\lambda^{-\ell}\int_{\Omega_1}(r\cdot\nabla f_1)(\lambda u)dx-2i\lambda^{-\ell}\int_{\partial\Omega_1}(r\cdot\nu_1)f_1(\lambda u)d\Gamma\right\}=o(\lambda^{-\ell}).\end{aligned} \tag{3.38}$$

Making use of Green's formula and using the fact $r(x)=x-x_0$, we get

$$\Re\left\{2\lambda^2\int_{\Omega_1}u(r\cdot\nabla\bar{u})dx\right\}=-d\int_{\Omega_1}|\lambda u|^2dx+\int_{\Gamma_1}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\int_{\mathcal{I}}(r\cdot\nu_1)|\lambda u|^2d\Gamma, \tag{3.39}$$

$$\begin{aligned}\Re\left\{2a\int_{\Omega_1}\Delta u(r\cdot\nabla\bar{u})dx\right\}&=a(d-2)\int_{\Omega_1}|\nabla\bar{u}|^2dx-a\int_{\Gamma_1}(r\cdot\nu_1)|\nabla u|^2d\Gamma-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma\\&+\Re\left\{2a\int_{\Gamma_1}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma+2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma\right\}.\end{aligned} \tag{3.40}$$

Inserting (3.39) and (3.40) in (3.37), we obtain

$$\begin{aligned}&d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+a\int_{\Gamma_1}(r\cdot\nu_1)|\nabla u|^2d\Gamma\\&=\int_{\Gamma_1}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\Re\left\{2a\int_{\Gamma_1}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma+2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma\right\}\\&+\int_{\mathcal{I}}(r\cdot\nu_1)|\lambda u|^2d\Gamma-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma+o(\lambda^{-\ell}).\end{aligned} \tag{3.41}$$

It follows, by using Young's inequality, the first geometric condition in (BMGC) and equation (3.10), that

$$\begin{aligned}&d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+a(\delta^{-1}-\varepsilon R^2)\int_{\Gamma_1}|\nabla u|^2d\Gamma\\&\leq\int_{\Gamma_1}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\int_{\mathcal{I}}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\Re\left\{2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma\right\}-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma+o(\lambda^{-\ell}).\end{aligned} \tag{3.42}$$

where, $R=\|r\|_{L^\infty(\Omega)}$, and ε is an arbitrary positive constant to be fixed. Then, by taking $\varepsilon=\dfrac{\delta^{-1}}{2R^2}$, we get the following estimate

$$\begin{aligned}d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx&\leq\int_{\Gamma_1}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\int_{\mathcal{I}}(r\cdot\nu_1)|\lambda u|^2d\Gamma\\&+\Re\left\{2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma\right\}-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma+o(\lambda^{-\ell}).\end{aligned} \tag{3.43}$$

Multiplying (3.35) by $2r\cdot\nabla\bar{y}$, using Green's formula, and the boundary conditions of y on Γ_2 we obtain

$$\begin{aligned}d\int_{\Omega_2}|\lambda y|^2dx+b(2-d)\int_{\Omega_2}|\nabla y|^2dx&=\int_{\mathcal{I}}(r\cdot\nu_2)|\lambda y|^2d\Gamma-b\int_{\Gamma_2}(r\cdot\nu_2)|\nabla y|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\nabla y|^2d\Gamma\\&+\Re\left\{2b\int_{\Gamma_2}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma+2b\int_{\mathcal{I}}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma\right\}+o(\lambda^{-\ell}).\end{aligned} \tag{3.44}$$

Adding (3.43) and (3.44), we get

$$
\begin{aligned}
&d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+d\int_{\Omega_2}|\lambda y|^2dx+b(2-d)\int_{\Omega_2}|\nabla y|^2dx\\
&\leq\int_{\Gamma_1}(r\cdot\nu_1)|\lambda u|^2d\Gamma+\int_{\mathcal{I}}(r\cdot\nu_1)|\lambda u|^2d\Gamma-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma\\
&+\int_{\mathcal{I}}(r\cdot\nu_2)|\lambda y|^2d\Gamma-b\int_{\Gamma_2}(r\cdot\nu_2)|\nabla y|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\nabla y|^2d\Gamma\\
&+\Re\left\{2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma+2b\int_{\Gamma_2}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma+2b\int_{\mathcal{I}}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma\right\}+o(\lambda^{-\ell}).
\end{aligned}
\tag{3.45}
$$

Besides, using the conditions at the interface

$$
\begin{aligned}
&\Re\left\{2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma+2b\int_{\mathcal{I}}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma\right\}=2a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma\\
&+\Re\left\{2a\int_{\mathcal{I}}(r\cdot\tau)\partial_\tau u\partial_{\nu_1}ud\Gamma\right\}+2b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma+\Re\left\{2b\int_{\mathcal{I}}(r\cdot\tau)\partial_\tau y\partial_{\nu_2}yd\Gamma\right\}\\
&=2a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma+2b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma,
\end{aligned}
\tag{3.46}
$$

and

$$
\begin{aligned}
-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\nabla y|^2d\Gamma=&-a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_\tau u|^2d\Gamma-a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma\\
&-b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_\tau y|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma,
\end{aligned}
\tag{3.47}
$$

Adding (3.46) and (3.47) we get

$$
\begin{aligned}
&\Re\left\{2a\int_{\mathcal{I}}\partial_{\nu_1}u(r\cdot\nabla u)d\Gamma+2b\int_{\mathcal{I}}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma\right\}-a\int_{\mathcal{I}}(r\cdot\nu_1)|\nabla u|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\nabla y|^2d\Gamma\\
&=-a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_\tau u|^2d\Gamma+a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_\tau y|^2d\Gamma+b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma.
\end{aligned}
\tag{3.48}
$$

Moreover, when applying the conditions of y on Γ_2

$$
\begin{aligned}
2b\int_{\Gamma_2}\partial_{\nu_2}y(r\cdot\nabla y)d\Gamma-b\int_{\Gamma_2}(r\cdot\nu_2)|\nabla y|^2d\Gamma=&\,2b\int_{\Gamma_2}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma+\Re\left\{2b\int_{\Gamma_2}(r\cdot\tau)\partial_\tau y\partial_{\nu_2}yd\Gamma\right\}\\
&-b\int_{\Gamma_2}(r\cdot\nu_2)|\partial_\tau y|^2d\Gamma-b\int_{\Gamma_2}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma=b\int_{\Gamma_2}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma.
\end{aligned}
\tag{3.49}
$$

Thus, using the third estimation of (3.10) and (BMGC) we have

$$
\begin{aligned}
&d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+d\int_{\Omega_2}|\lambda y|^2dx+b(2-d)\int_{\Omega_2}|\nabla y|^2dx\\
&\leq-a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_\tau u|^2d\Gamma+a\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma-b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_\tau y|^2d\Gamma+b\int_{\mathcal{I}}(r\cdot\nu_2)|\partial_{\nu_2}y|^2d\Gamma+o(1)\\
&\leq(b-a)\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_\tau u|^2d\Gamma+\frac{a}{b}(b-a)\int_{\mathcal{I}}(r\cdot\nu_1)|\partial_{\nu_1}u|^2d\Gamma+o(1).
\end{aligned}
\tag{3.50}
$$

Finally, using the multiplier geometric condition on $\mathcal{I}$ in (BMGC) and from the fact that $b\geq a$, we can deduce that

$$
d\int_{\Omega_1}|\lambda u|^2dx+a(2-d)\int_{\Omega_1}|\nabla u|^2dx+d\int_{\Omega_2}|\lambda y|^2dx+b(2-d)\int_{\Omega_2}|\nabla y|^2dx=o(1).
$$

The proof is thus completed. □

Lemma 3.6. *Under the same assumptions of Theorem 3.3. The solutions $(u,v,y,z,\eta)\in D(\mathcal{A})$ of (3.5)-(3.9) satisfies the following*

$$(3.51)\qquad \int_{\Omega_1}|\lambda u|^2dx - a\int_{\Omega_1}|\nabla u|^2dx + \int_{\Omega_2}|\lambda y|^2dx - b\int_{\Omega_2}|\nabla y|^2dx = o(\lambda^{-2}).$$

Proof. Multiplying (3.34) and (3.35) by $\bar{u}$ and $\bar{y}$ respectively, integrating, then using integration by parts we get

$$(3.52)\qquad \int_{\Omega_1}|\lambda u|^2dx + a\int_{\partial\Omega_1}\partial_{n_1}u\, ud\Gamma - a\int_{\Omega_1}|\nabla u|^2dx = -\lambda^{-\ell}\int_{\Omega_1}g_1\bar{u}dx - i\lambda^{-\ell+1}\int_{\Omega_1}f_1\bar{u}dx,$$

and

$$(3.53)\qquad \int_{\Omega_2}|\lambda y|^2dx + b\int_{\partial\Omega_2}\partial_{n_2}y\, yd\Gamma - b\int_{\Omega_2}|\nabla y|^2dx = -\lambda^{-\ell}\int_{\Omega_2}g_2\bar{y}dx - i\lambda^{-\ell+1}\int_{\Omega_2}f_2\bar{y}dx.$$

Adding the equations (3.52) and (3.53), and by (3.10) and the fact that $F_n \to 0$, we have

$$(3.54)\qquad \int_{\Omega_1}|\lambda u|^2dx - a\int_{\Omega_1}|\nabla u|^2dx + \int_{\Omega_2}|\lambda y|^2dx - b\int_{\Omega_2}|\nabla y|^2dx = o(\lambda^{-\ell}) - \int_{\Gamma_1}\partial_{\nu_1}uud\Gamma = o(\lambda^{-2}).$$

Thus the proof is complete.

□

Lemma 3.7. *Under the same assumptions of Theorem 3.3, the solution $(u, v, y, z, \eta) \in D(\mathcal{A})$ of (3.5)-(3.9) achieves the following estimation*

$$(3.55)\qquad \int_{\Omega_1}|\lambda u|^2dx + \int_{\Omega_2}|\lambda y|^2dx + a\int_{\Omega_1}|\nabla u|^2dx + b\int_{\Omega_1}|\nabla y|^2dx = o(1).$$

Proof. Multiplying (3.51) by $(1-d)$, then adding it to (3.36), we obtain

$$(3.56)\qquad \int_{\Omega_1}|\lambda u|^2dx + \int_{\Omega_2}|\lambda y|^2dx + a\int_{\Omega_1}|\nabla u|^2dx + b\int_{\Omega_1}|\nabla y|^2dx = o(1).$$

□

Lemma 3.8. *Under the same assumptions of Theorem 3.3, the solution $(u, v, y, z, \eta) \in D(\mathcal{A})$ satisfies the following estimation*

$$(3.57)\qquad \int_{\Omega_1}|v|^2dx = o(1) \quad \textit{and} \quad \int_{\Omega_2}|z|^2dx = o(1).$$

Proof. Referring to (3.5) and (3.7), then using the fact that $f_1, f_3 \to 0$, and the results obtained in (3.55) we have

$$(3.58)\qquad \int_{\Omega_1}|v|^2dx = o(1) \quad \text{and} \quad \int_{\Omega_2}|z|^2dx = o(1).$$

□

Now, going back to find out $\|U\|_H$ by using (3.10),(3.55), and (3.57) we get

$$\|U\|_H = \int_{\Omega_1}|v|^2dx + \int_{\Omega_1}|\nabla u|^2dx + \int_{\Omega_2}|z|^2dx + \int_{\Omega_2}|\nabla y|^2dx + \int_{\Gamma_1}\eta^2d\Gamma = o(1).$$

Hence, we obtain $\|U\|_{\mathcal{H}} = o(1)$, which contradicts (3.3). Therefore the polynomial estimation of our system is proved.

4. Examples

In this section, we illustrate our general framework, by checking the assumptions for some particular examples.

4.1. **Example 1.** Let Ω be a domain in $\mathbb{R}^d$, organized in the same way as the domain in the introduction, and satisfying the BMGC conditions. Consider the following system

$$
\left\{
\begin{array}{lll}
u_{tt}(x,t) - a\Delta u(x,t) = 0, & \text{in} & \Omega_1 \times (0,\infty), \\
y_{tt}(x,t) - b\Delta y(x,t) = 0, & \text{in} & \Omega_2 \times (0,\infty), \\
u(x,t) - y(x,t) = 0, & \text{on} & \mathcal{I} \times (0,\infty), \\
a\partial_{\nu_1} u(x,t) + b\partial_{\nu_2} y(x,t) = 0, & \text{on} & \mathcal{I} \times (0,\infty), \\
a\partial_{\nu_1} u(x,t) + \eta(x,t) = 0, & \text{on} & \Gamma_1 \times (0,\infty), \\
\eta_t(x,t) - u_t(x,t) + \eta(x,t) = 0, & \text{on} & \Gamma_1 \times (0,\infty), \\
y(x,t) = 0, & \text{on} & \Gamma_2 \times (0,\infty).
\end{array}
\right.
\tag{4.1}
$$

The system above is nothing but the system (1.1), with

$$B = -1, \qquad M = 1, \qquad C = -1.$$

It is easy to check that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \phi$. Moreover, the matrix B satisfy

$$\Re(-Bv, v) = v^2. \tag{4.2}$$

Thus, $-B$ is Totally M-coercive. As $\Re\{c_1^2\} \geq 1 > 0$, then the condition (PSC1) holds. That implies the following energy decay estimation is satisfied

$$E(t) \leq \frac{\mathcal{C}}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0. \tag{4.3}$$

4.2. **Example 2.** Let Ω in $\mathbb{R}^3$, be a domain as the one considered in the introduction, satisfying the BMGC conditions. Consider the following system

$$
\left\{
\begin{array}{lll}
u_{tt}(x,t) - a\Delta u(x,t) = 0, & \text{in} & \Omega_1 \times (0,\infty), \\
y_{tt}(x,t) - b\Delta y(x,t) = 0, & \text{in} & \Omega_2 \times (0,\infty), \\
u(x,t) - y(x,t) = 0, & \text{on} & \mathcal{I} \times (0,\infty), \\
a\partial_{\nu_1} u(x,t) + b\partial_{\nu_2} y(x,t) = 0, & \text{on} & \mathcal{I} \times (0,\infty), \\
a\partial_{\nu_1} u(x,t) - \delta_t(x,t) = 0, & \text{on} & \Gamma_1 \times (0,\infty), \\
m\delta_{tt}(x,t) + d\delta_t(x,t) + k\delta(x,t) + \rho u_t(x,t) = 0, & \text{on} & \Gamma_1 \times (0,\infty), \\
y(x,t) = 0, & \text{on} & \Gamma_2 \times (0,\infty),
\end{array}
\right.
\tag{4.4}
$$

where ρ is a positive constant and m, d, k are positive and sufficiently smooth functions on Γ_1. We readily check that this system can be rewritten in the form of system (1.1) with $\eta = (\delta, \delta_t)^\top$ and

$$B(x) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{pmatrix}, \quad M(x) = \begin{pmatrix} \frac{k}{\rho} & 0 \\ 0 & \frac{m}{\rho} \end{pmatrix}, \quad C(x) = \begin{pmatrix} 0 \\ \frac{\rho}{m} \end{pmatrix}, \quad \forall\, x \in \Gamma_1.$$

For all $x \in \Gamma_1$, the matrix $B(x)$ is Hurwitz and thus $\Sigma_m \cap i\mathbb{R} = \phi$. Hence the assumptions (SSC2) to (SSC4) hold. Moreover, we can easily check (SSC1). Then we deduce by proposition (2.4) that the C_0- semi group of contraction $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable. Following Theorem 2.7, $(e^{t\mathcal{A}})_{t\geq 0}$ is not uniformly stable. In addition, we have

$$(-Bv, v) = \frac{d}{\rho} v_2^2, \tag{4.5}$$

which implies that $-B$ is 1-partially M-coercive. In the vector C, we have $c_1 = 0$, and $c_2 = \frac{\rho}{m}$, thus $\Re\{c_2^2\} \geq \rho^2 / \|m\|^2_{L^\infty(\Gamma_1)} > 0$. It follows that the condition (PSC3) holds. Thus the energy of the system (4.4) satisfies the following estimation

$$E(t) \leq \frac{\mathcal{C}}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0. \tag{4.6}$$

4.3. **Example 3.** Consider the following system defined on a domain satisfying the BMGC conditions

$$
\begin{cases}
u_{tt}(x,t) - a\Delta u(x,t) = 0, & x \in \Omega_1, \quad t > 0,\\
y_{tt}(x,t) - b\Delta y(x,t) = 0, & x \in \Omega_2, \quad t > 0,\\
u(x,t) - y(x,t) = 0, & x \in \mathcal{I}, \quad t > 0,\\
a\partial_{\nu_1} u(x,t) + b\partial_{\nu_2} y(x,t) = 0, & x \in \mathcal{I}, \quad t > 0,\\
a\partial_{\nu_1} u(x,t) - b_1\delta(x,t) - \delta_t(x,t) + \kappa(t) = 0, & x \in \Gamma_1, \quad t > 0,\\
\kappa_t(t) + b_2\kappa(t) - u_t(x,t) = 0, & x \in \Gamma_1, \quad t > 0,\\
\delta_{tt}(x,t) + b_1\delta_t(x,t) + b_0\delta(x,t) + b_0 u_t(x,t) = 0, & x \in \Gamma_1, \quad t > 0,\\
y(x,t) = 0, & x \in \Gamma_2, \quad t > 0,
\end{cases} \tag{4.7}
$$

with b_0, b_1, and b_2 are positive constants. Letting

$$\eta = \left(\frac{\delta_t + b_1\delta}{b_0}, -\delta, -\kappa\right)^\top,$$

then our system is nothing but (1.1) with

$$M = \begin{pmatrix} b_0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0\\ -b_0 & -b_1 & 0\\ 0 & 0 & -b_2\end{pmatrix}, \quad C = \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

As in the preceding example, it is easy to check that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \phi$ ((SSC1) holds and $\Sigma_m \cap i\mathbb{R} = \phi$), as well as

$$(-Bv, v) = b_1 v_2^2 + b_2 v_3^2, \tag{4.8}$$

that implies that $-B$ is 1-partially M-coercive. Besides, in the vector C exists c_1, c_3, with $\Re\{c_1^2\} \geq 1 > 0$, and $\Re\{c_3^2\} \geq 1 > 0$. Thus (PSC2) holds. Hence, the energy of the system (4.7) decays polynomially satisfying the following estimation

$$E(t) \leq \frac{\mathcal{C}}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0. \tag{4.9}$$

4.4. **Example 4.** On a domain satisfying the BMGC conditions, we consider the following system

$$
\begin{cases}
u_{tt}(x,t) - a\Delta u(x,t) = 0, & x \in \Omega_1, \quad t > 0,\\
y_{tt}(x,t) - b\Delta y(x,t) = 0, & x \in \Omega_2, \quad t > 0,\\
u(x,t) - y(x,t) = 0, & x \in \mathcal{I}, \quad t > 0,\\
a\partial_{\nu_1} u(x,t) + b\partial_{\nu_2} y(x,t) = 0, & x \in \mathcal{I}, \quad t > 0,\\
a\partial_{\nu_1} u(x,t) - b_1\delta(x,t) - \delta_t(x,t) = 0, & x \in \Gamma_1, \quad t > 0,\\
\delta_{tt}(t) + b_1\delta_t(t) + b_0\delta(t) + b_0 u_t(x,t) = 0, & x \in \Gamma_1, \quad t > 0,\\
y(x,t) = 0, & x \in \Gamma_2, \quad t > 0.
\end{cases} \tag{4.10}
$$

Set $\eta = (\frac{b_1\delta + \delta_t}{b_0}, -\delta)^\top$, we get a system of the form (1.1) with

$$M = \begin{pmatrix} b_0 & 0\\ 0 & 1\end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1\\ -b_0 & -b_1\end{pmatrix}, \quad C = \begin{pmatrix}1\\0\end{pmatrix}.$$

In this example, we have $\sigma(\mathcal{A}) \cap i\mathbb{R} = \phi$, and

$$(-Bv, v) = b_1 v_2^2. \tag{4.11}$$

Thus, $-B$ is 1-partially M-coercive. On the other hand, in the vector C, we have $\Re\{c_1^2\} \geq 1 > 0$ and $c_2 = 0$. Then as $\Re\{b_{12}^2\} \geq 1 > 0$, the condition (PSC4) holds, and the energy decay estimate of the system (4.10) is

$$E(t) \leq \frac{\mathcal{C}}{t^{1/2}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall\, t > 0. \tag{4.12}$$

5. Optimal Polynomial Decay Rate

The aim of this section is to prove that the energy decay rate obtained in Theorem 3.3 is optimal in the first dimension, for the case when $M = 1$, $B = -1$, and $C = -1$. Precisely, we will prove the following result:

Theorem 5.1. *Assume that $d = 1$. The energy decay rate (3.1) is optimal in the sense that for any $\varepsilon > 0$, we cannot expect the decay rate $\frac{1}{t^{1+\varepsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$ and for all $t > 0$.*

For the optimality, we search the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ and $U = (u, v, y, z, \eta) \in D(\mathcal{A})$ be an associated eigenfunction, then $\mathcal{A}U = \lambda U$. Equivalently, we have the following system:

$$\begin{cases} \lambda^2 u - u_{xx} = 0, & x \in (-1, 0), \\ \lambda^2 y - y_{xx} = 0, & x \in (0, 1), \\ u_x(-1) - \eta = 0, \\ y(1) = 0, \\ \lambda\eta - \lambda u(-1) + \eta = 0, \\ u(0) = y(0), \\ u_x(0) = y_x(0). \end{cases} \tag{5.1}$$

It is easy to see that $\lambda = 0$ and $\lambda = -1$ are not eigenvalues of the operator $\mathcal{A}$. Then, from now on we will assume that $\lambda \neq 0$ and $\lambda \neq -1$. It follows, from $(5.1)_5$, that

$$\eta = \frac{\lambda}{1+\lambda} u(-1). \tag{5.2}$$

A general solution of equation $(5.1)_1$ with boundary conditions $(5.1)_6 - (5.1)_7$ is given by:

$$u(x) = Ae^{\lambda x} + Be^{-\lambda x}, \quad x \in (-1, 0), \tag{5.3}$$

where

$$A = \frac{\lambda y(0) + y_x(0)}{2\lambda} \quad \text{and} \quad B = \frac{\lambda y(0) - y_x(0)}{2\lambda}. \tag{5.4}$$

A general solution of equation $(5.1)_2$ with boundary condition $(5.1)_4$ is given by:

$$y(x) = Ce^{\lambda x} - Ce^{2\lambda - \lambda x}, \quad x \in (0, 1), \tag{5.5}$$

where $C \in \mathbb{C}$ is a constant. So, combining (5.3), (5.4) and (5.5), we get

$$u(x) = Ce^{\lambda x} - Ce^{2\lambda - \lambda x}, \quad x \in (-1, 0). \tag{5.6}$$

Hence, a non trivial solution u exists if and only if $C \neq 0$. Finally, putting together $(5.1)_3$, (5.2) and (5.6), we get

$$\lambda + (\lambda + 2)e^{4\lambda} = 0. \tag{5.7}$$

Conversely, suppose that λ satisfies (5.7), and let η, y, u be defined by (5.2), (5.5) and (5.6) with $C \neq 0$. A simple calculus shows that (u, y, η) satisfies (5.1). Consequently, we have proved

$$\lambda \text{ is an eigenvalue of the operator } \mathcal{A} \Longleftrightarrow f(\lambda) := \lambda + (\lambda + 2)e^{4\lambda} = 0. \tag{5.8}$$

By complex analysis arguments, we easily see that the equation $f(z) = 0$ has an infinite number of solutions λ_n with $|\lambda_n| \to \infty$. In fact, if f has finite numbers of roots, we conclude from Hadamard's factorization theorem that

$$f(z) = P(z)e^{az}, a \in \mathbb{C},$$

for some polynomial P. Then, from the equality,

$$P(z)e^{az} = z + (z+2)e^{4z},\ \forall z \in \mathbb{C},$$

we conclude that $a = 4$, hence that $P(z) = z + 2$ and finally that

$$z = 0, \forall z \in \mathbb{C},$$

which is impossible. We can now state the following result

Lemma 5.2. *The number of eigenvalues, λ_n for $n \in \mathbb{Z}$, of $\mathcal{A}$ is infinite. Moreover, each eigenvalue is simple, and $|\lambda_n|$ goes to infinity as n goes to infinity.*

Proof. We only need to show that λ_n is simple (i.e, the algebraic multiplicity is equal to one). Let λ be an eigenvalue of A. Then we have

$$\ker(A - \lambda I) = \left\{ \frac{1}{\lambda}\phi(x), \phi(x), \frac{1}{\lambda}\psi(x), \psi(x), \frac{1}{\lambda+1}\phi(-1) \right\},$$

where

$$\phi(x) = e^{\lambda x} - e^{2\lambda - \lambda x}, \qquad -1 \leq x \leq 0,$$

and

$$\psi(x) = e^{\lambda x} - e^{2\lambda - \lambda x}, \qquad 0 \leq x \leq 1.$$

Assume that there exist $U = (u, v, y, z, \eta) \in \ker(A - \lambda I)^2 \setminus \ker(A - \lambda I)$. In other word, we have

$$AU - \lambda U = V \in \ker(A - \lambda I).$$

That is equivalent to

$$v - \lambda u = \tilde{u}, \tag{5.9}$$
$$u_{xx} - \lambda v = \tilde{v}, \tag{5.10}$$
$$z - \lambda y = \tilde{y}, \tag{5.11}$$
$$y_{xx} - \lambda z = \tilde{z}, \tag{5.12}$$
$$v(-1) - \eta - \lambda \eta = \tilde{\eta}. \tag{5.13}$$

We deduce that

$$u_{xx} - \lambda^2 u = \lambda \tilde{u} + \tilde{v} = 2(e^{\lambda x} - e^{2\lambda - \lambda x}), \tag{5.14}$$
$$y_{xx} - \lambda^2 y = \lambda \tilde{y} + \tilde{z} = 2(e^{\lambda x} - e^{2\lambda - \lambda x}), \tag{5.15}$$

besides to the following boundary conditions

$$u(0) = y(0), \tag{5.16}$$
$$u_x(0) = y_x(0), \tag{5.17}$$
$$y(1) = 0, \tag{5.18}$$
$$\eta = \frac{-\tilde{\eta}}{\lambda+1} + \frac{\lambda}{\lambda+1}u(-1) + \frac{1}{\lambda+1}\tilde{u}(-1). \tag{5.19}$$

Then we find that the general solution of u and y is given by

$$u = \frac{1}{\lambda}(e^{\lambda x} - e^{2\lambda - \lambda x}) + \frac{x}{\lambda}(e^{\lambda x} + e^{2\lambda - \lambda x}), \tag{5.20}$$

and

$$y = \frac{1}{\lambda}(e^{\lambda x} - e^{2\lambda - \lambda x}) + \frac{x}{\lambda}(e^{\lambda x} + e^{2\lambda - \lambda x}). \tag{5.21}$$

Now using boundary condition at $x = -1$ we have

$$\begin{aligned}
u_x(-1) = \eta &= \frac{\lambda}{\lambda+1}u(-1) - \frac{1}{(\lambda+1)^2}(e^{-\lambda} - e^{3\lambda}) \\
&= \frac{-2}{\lambda+1}e^{3\lambda} + \frac{1}{(\lambda+1)^2}e^{3\lambda} - \frac{1}{(\lambda+1)^2}e^{-\lambda}.
\end{aligned} \tag{5.22}$$

On the other hand, from (5.20) we have

$$u_x(-1) = 2e^{3\lambda} + \frac{1}{\lambda}e^{3\lambda} + \frac{1}{\lambda}e^{-\lambda}.$$

Thus we get

$$e^{4\lambda} = \frac{\lambda^2+3\lambda+1}{-2\lambda^3-7\lambda^2-5\lambda-1}.$$

Compared to (5.7) implies

$$\lambda^4 + 3\lambda^3 - 3\lambda - 1 = 0,$$

that in turn had the following solutions

$$\lambda = 1, \quad \lambda = -1, \quad \lambda = -\frac{3}{2} - \frac{\sqrt{5}}{2}, \quad \text{and} \quad \lambda = \frac{\sqrt{5}}{2} - \frac{3}{2}.$$

Then

$$e^4 = -\frac{1}{3},$$

which is impossible. This completes the proof. $\square$

Lemma 5.3. *(Asymptotic expansion) There exists $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{k\geq k_0}$ of simple roots of f (that are also simple eigenvalues of $\mathcal{A}$) and satisfying the following asymptotic behavior:*

$$\lambda_k = i\left(\frac{k\pi}{2} + \frac{\pi}{4} + \frac{4}{k\pi} - \frac{2}{k^2\pi}\right) - \frac{8}{k^2\pi^2} + o\left(\frac{1}{k^2}\right), \tag{5.23}$$

for k large enough.

Proof. The complex λ is an eigenvalue of $\mathcal{A}$ if and only if $f(\lambda) = 0$. Then, we have

$$e^{4\lambda} = (-1)\frac{\lambda}{2+\lambda}. \tag{5.24}$$

It follows that

$$\lambda_k = \frac{i\pi}{4} + \frac{ik\pi}{2} + \ln\left(1 - \frac{2}{2+\lambda_k}\right). \tag{5.25}$$

As $|\lambda_k| \to \infty$, we obtain the following

$$\lambda_k = \frac{i\pi}{4} + \frac{ik\pi}{2} + o\left(\frac{1}{k}\right) \qquad \text{as } |k| \to \infty. \tag{5.26}$$

On the other hand, we have the following expansion

$$\ln\left(1 - \frac{2}{2+\lambda_k}\right) = -\frac{2}{\lambda_k} + \frac{2}{\lambda_k^2} + o\left(\frac{1}{\lambda_k^3}\right) \text{ as } |k| \to \infty. \tag{5.27}$$

Consequently, we get

$$\lambda_k = \frac{i\pi}{4} + \frac{ik\pi}{2} - \frac{2}{\lambda_k} + \frac{2}{\lambda_k^2} + o\left(\frac{1}{\lambda_k^3}\right) \text{ as } |k| \to \infty. \tag{5.28}$$

From (5.26), we get

$$\frac{-2}{\lambda_k} = \frac{4i}{k\pi} - \frac{2i}{k^2\pi} + o\left(\frac{1}{k^3}\right) \text{ as } |k| \to \infty, \tag{5.29}$$

and

$$
(5.30) \qquad \frac{2}{\lambda_k^2} = \frac{-8}{k^2\pi^2} + o\left(\frac{1}{k^3}\right) \text{ as } |k| \to \infty.
$$

Thus, inserting (5.29) and (5.30) into (5.28), we obtain the desired asymptotic expansion (5.23) and the proof is thus complete. ☐

Proof of Theorem 5.1. Let $\varepsilon > 0$, and set $l = \dfrac{\varepsilon}{1+\varepsilon}$. For $k \in \mathbb{N}^*$, let λ_k be an eigenvalue of the operator $\mathcal{A}$, and U_k the associated normalized eigenfunction. Consider the following sequences

$$
\beta_k = \frac{k\pi}{2} + \frac{\pi}{4} + \frac{4}{k\pi} - \frac{2}{k^2\pi},
$$

$$
(U_k) \subset D(\mathcal{A}).
$$

Using (5.23) we get

$$
\lim_{k\to+\infty} \beta_k^{2-2l} \|(i\beta_k - \mathcal{A})U_k\| = 0.
$$

By applying of Borichev Theorem A.4, we deduce that the trajectory $e^{t\mathcal{A}}u_0$ decays slower than $\dfrac{1}{t^{\frac{1}{2-2l}}}$ on the time $t \to \infty$. Then we cannot expect the energy decay rate $\frac{1}{t^{1+\varepsilon}}$. This ends the proof of Theorem 5.1 .

6. Conclusion

We have studied the stabilization of transmission problem of two coupled waves, with general acoustic conditions at the boundary of the first wave, while Dirichlet conditions set on the boundary of the second one. We proved the strong stability of the system using general criteria Arendt-Batty. In addition, we proved the lack of exponential stability. After that, we established two different polynomial energy decay rates, provided with some illustrative examples. Then we proved that the decay rate is optimal for some particular case.

Appendix A. Some Notions and Stability Theorems

In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

Definition A.1. Assume that A is the generator of C_0-semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on a Hilbert space H. The C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ is said to be

(1) Strongly stable if

$$
\lim_{t\to+\infty} \|e^{tA}x_0\|_H = 0, \quad \forall\, x_0 \in H.
$$

(2) Exponentially (or uniformly) stable if there exists two positive constants M and ε such that

$$
\|e^{tA}x_0\|_H \leq Me^{-\varepsilon t}\|x_0\|_H, \quad \forall\, t > 0,\ \forall\, x_0 \in H.
$$

(3) Polynomially stable if there exists two positive constants C and α such that

$$
\|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|Ax_0\|_H, \quad \forall\, t > 0,\ \forall\, x_0 \in D(A).
$$

☐

To show the strong stability of a C_0-semigroup we rely on the following result due to Arendt-Batty [3].

Theorem A.2. Assume that A is the generator of a C_0-semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on a Hilbert space H. If A has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable, where $\sigma(A)$ denotes the spectrum of A, then the C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ is strongly stable. ☐

Concerning the characterisation of exponential stability of a C_0-semigroup of contractions we rely on the following result due to Huang [18] and Prüss [35].

Theorem A.3. Let $A: D(A) \subset H \longrightarrow H$ generates a C_0-semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on H. Then, the C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

Finally for the polynomial stability of a C_0-semigroup of contractions we use the following result due to Borichev and Tomilov [9] (see also [5], [25], and the recent paper [42]).

Theorem A.4. Assume that A is the generator of a strongly continuous semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on $\mathcal{H}$. If $i\mathbb{R} \subset \rho(\mathcal{A})$, then for a fixed $\ell > 0$ the following conditions are equivalent

$$\limsup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} \frac{1}{|\lambda|^{\ell}} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty, \tag{A.1}$$

$$\|e^{tA}U_0\|_H^2 \leq \frac{C}{t^{\frac{2}{\ell}}}\|U_0\|^2_{D(A)},\ \forall t > 0,\ U_0 \in D(A),\ \text{for some } C > 0. \tag{A.2}$$

$\square$

Let us end up this appendix with the definition of our multiplier geometric control condition.

Definition A.5. We say that the partition (Γ_0, Γ_1) of the boundary Γ satisfies the multiplier geometric control condition **MGC** (see Fig. 2 for an illustration) if there exists a point $x_0 \in \mathbb{R}^2$ and a positive constant δ such that

$$h\cdot\nu \geq \delta^{-1} \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad h\cdot\nu \leq 0 \quad \text{on} \quad \Gamma_0, \tag{A.3}$$

where $h(x) = x - x_0$.

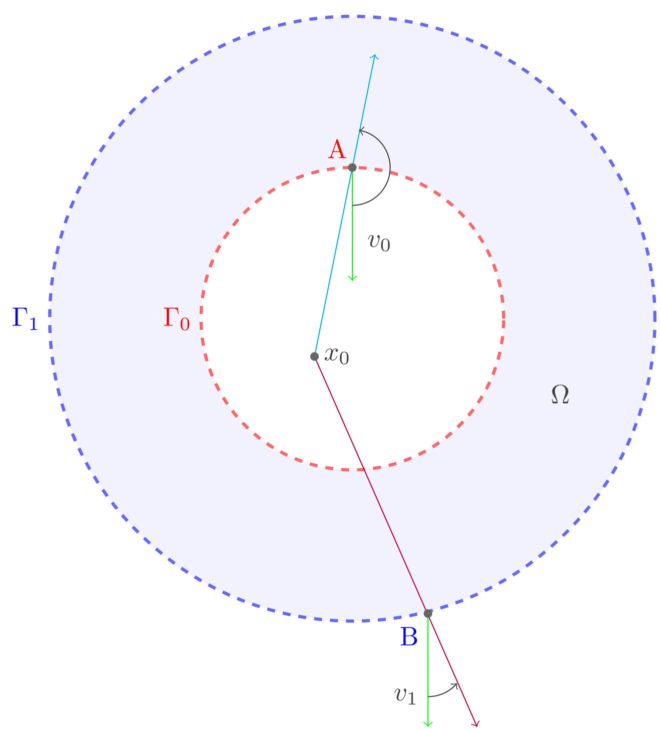

FIGURE 2. An example where the **MGC** boundary condition holds.

$\square$

Let

$$z^1(x,\rho,t) = u_t(x, t-\rho\tau_1) \quad \text{on} \quad \Gamma_1 \times (0,1) \times (0,\infty)$$

and

$$z^2(x,\rho,t) = \partial_\nu u_t(x, t-\rho\tau_2) \quad \text{on} \quad \Gamma_1 \times (0,1) \times (0,\infty) \tag{A.4}$$

Thus, we get

$$\tau_1 z^1_t + z^1_\rho = 0 \quad \text{on} \quad \Gamma_1.$$

and

$$\tau_2 z^2_t + z^2_\rho = 0 \quad \text{on} \quad \Gamma_1.$$

Energy:

$$E(t) = \frac{1}{2}\left\{ a(u,u) + \int_\Omega |u_t|^2 dx \, + \, \tau_1|\beta_2| \int_{\Gamma_1}\int_0^1 \left|z^1(\cdot,\rho,t)\right|^2 d\rho d\Gamma + \tau_2|\gamma_2| \int_{\Gamma_1}\int_0^1 \left|z^2(\cdot,\rho,t)\right|^2 d\rho d\Gamma \right\},$$

and

$$E'(t) \leq -(\beta_1 - |\beta_2|)\int_{\Gamma_1} |\partial_\nu u_t|^2 d\Gamma - (\gamma_1 - |\gamma_2|)\int_{\Gamma_1} |u_t|^2 d\Gamma$$

we choose

$$\mathcal{H} = H^2(\Omega) \times L^2(\Omega) \times (L^2(\Gamma_1 \times (0,1)))^2$$

.

Appendix B. Conclusion

We have studied the stabilization of transmission problem of two coupled waves, with dynamical feedback control at the boundary. On one part of the domain we consider a Dirichlet boundary condition, and on the other part we consider the dynamical one. We proved the strong stability of the system using general criteria Arendt-Batty. In addition, we proved the lack of exponential stability. After that, we established a polynomial energy decay rate. Then we proved that this decay rate is optimal in the one dimensional system.

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References

[1] M. Aassila. *E*xact boundary controllability of the plate equation. Differential and Integral Equations, 13(10-12):1413–1428, 2000.
[2] Z. Abbas and S. Nicaise. *T*he multidimensional wave equation with generalized acoustic boundary conditions ii: Polynomial stability. SIAM Journal on Control and Optimization, 53(4):2582–2607, 2015.
[3] W. Arendt and C. J. K. Batty. *T*auberian theorems and stability of one-parameter semigroups. Trans. Amer. Math. Soc., 306(2):837–852, 1988.
[4] W. D. Bastos and C. A. Raposo. *T*ransmission problem for waves with frictional damping. Electronic Journal of Differential Equations (EJDE)[electronic only], 2007:Paper–No, 2007.
[5] C. J. K. Batty and T. Duyckaerts. *N*on-uniform stability for bounded semi-groups on Banach spaces. J. Evol. Equ., 8(4):765–780, 2008.
[6] J. T. Beale. *S*pectral properties of an acoustic boundary condition. Indiana University Mathematics Journal, 25(9):895–917, 1976.
[7] J. T. BEALE. *A*coustic scattering from locally reacting surfaces. Indiana University Mathematics Journal, 26(2):199–222, 1977.
[8] J. T. Beale and S. I. Rosencrans. Acoustic boundary conditions. Bulletin of the american mathematical society, 80(6):1276–1278, 1974.
[9] A. Borichev and Y. Tomilov. *O*ptimal polynomial decay of functions and operator semigroups. Math. Ann., 347(2):455–478, 2010.
[10] F. Cardoso and G. Vodev. *B*oundary stabilization of transmission problems. Journal of mathematical physics, 51(2):023512, 2010.
[11] M. M. Cavalcanti, E. R. Coelho, and V. N. Domingos Cavalcanti. *E*xponential stability for a transmission problem of a viscoelastic wave equation. Applied Mathematics Optimization, 81(2):621–650, 2020.
[12] T. Duyckaerts. *O*ptimal decay rates of the energy of a hyperbolic–parabolic system coupled by an interface. Asymptotic Analysis, 51(1):17–45, 2007.
[13] P. J. Graber. *U*niform boundary stabilization of a wave equation with nonlinear acoustic boundary conditions and nonlinear boundary damping. Journal of Evolution Equations, 12(1):141–164, 2012.

[14] P. Grisvard. *E*lliptic problems in nonsmooth domains, volume 24 of Monographs and Studies in Mathematics. Pitman, Boston–London–Melbourne, 1985.
[15] Z. Guo and S. Chai. *T*he stabilization of the problem of transmission of the wave equation with dynamical control. 2020.
[16] Z. Guo and S. Chai. *E*xponential stabilization of the problem of transmission of wave equation with linear dynamical feedback control. Evolution Equations Control Theory, 2022.
[17] Z.-J. Han and E. Zuazua. *D*ecay rates for elastic-thermoelastic star-shaped networks. Networks Heterogeneous Media, 12(3):461, 2017.
[18] F. L. Huang. *C*haracteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. Ann. Differential Equations, 1(1):43–56, 1985.
[19] E. C. Lapa and J. E. Munoz Rivera. *A* nonlinear transmission problem with time dependent coefficients. Electronic Journal of Differential Equations (EJDE)[electronic only, 2007:Paper–No, 2007.
[20] J. Lions. Exact controllability, perturbations and stabilization of distributed systems. volume 1. 8, 1988.
[21] J. L. Lions. Exact controllability, stabilization and perturbations for distributed systems. SIAM Review, 30(1):1–68, 1988.
[22] W. Liu. *S*tabilization and controllability for the transmission wave equation. IEEE Transactions on Automatic Control, 46(12):1900–1907, 2001.
[23] W. Liu and G. Williams. *T*he exponential stability of the problem of transmission of the wave equation. Bulletin of the Australian Mathematical Society, 57(2):305–327, 1998.
[24] W. Liu and G. H. Williams. *E*xact controllability for problems of transmission of the plate equation with lower-order terms. Quarterly of Applied Mathematics, 58(1):37–68, 2000.
[25] Z. Liu and B. Rao. *C*haracterization of polynomial decay rate for the solution of linear evolution equation. Z. Angew. Math. Phys., 56(4):630–644, 2005.
[26] Z. Liu and S. Zheng. *S*emigroups associated with dissipative systems, volume 398 of Chapman Hall/CRC Research Notes in Mathematics. Chapman Hall/CRC, Boca Raton, FL, 1999.
[27] L. Lu and J.-M. Wang. *T*ransmission problem of schrödinger and wave equation with viscous damping. Applied Mathematics Letters, 54:7–14, 2016.
[28] A. Marzocchi, J. E. Mutoz Rivera, and M. Grazia Naso. *A*symptotic behaviour and exponential stability for a transmission problem in thermoelasticity. Mathematical methods in the applied sciences, 25(11):955–980, 2002.
[29] A. Marzocchi, J. E. M. Rivera, and M. G. Naso. *T*ransmission problem in thermoelasticity with symmetry. IMA Journal of Applied Mathematics, 68(1):23–46, 2003.
[30] P. M. Morse and K. U. Ingard. Theoretical acoustics. Princeton university press, 1986.
[31] J. E. Munoz Rivera and H. P. Oquendo. *T*he transmission problem of viscoelastic waves. Acta Applicandae Mathematica, 62(1):1–21, 2000.
[32] C. A. Nonato, C. A. Raposo, and W. D. Bastos. *A* transmission problem for waves under time-varying delay and nonlinear weight. arXiv preprint arXiv:2102.07829, 2021.
[33] J. Nordström and V. Linders. *W*ell-posed and stable transmission problems. Journal of Computational Physics, 364:95–110, 2018.
[34] A. Pazy. *S*emigroups of linear operators and applications to partial differential equations, volume 44 of Applied Mathematical Sciences. Springer-Verlag, New York, 1983.
[35] J. Prüss. *O*n the spectrum of C0-semigroups. Trans. Amer. Math. Soc., 284(2):847–857, 1984.
[36] B. Rao, L. Toufayli, and A. Wehbe. *S*tability and controllability of a wave equation with dynamical boundary control. Mathematical Control and Related Fields, 5(2):305, 2015.
[37] J. Rauch, X. Zhang, and E. Zuazua. *P*olynomial decay for a hyperbolic–parabolic coupled system. Journal de mathématiques pures et appliquées, 84(4):407–470, 2005.
[38] J. E. M. Rivera and H. P. Oquendo. *T*he transmission problem for thermoelastic beams. Journal of Thermal Stresses, 24(12):1137–1158, 2001.
[39] J. E. M. Rivera and H. P. Oquendo. *A* transmission problem for thermoelastic plates. Quarterly of Applied Mathematics, pages 273–293, 2004.
[40] J. M. Rivera and M. G. Naso. *A*bout asymptotic behavior for a transmission problem in hyperbolic thermoelasticity. Acta Applicandae Mathematicae, 99(1):1–27, 2007.
[41] J. M. Rivera and Y. Qin. *P*olynomial decay for the energy with an acoustic boundary condition. Applied Mathematics Letters, 16(2):249–256, 2003.
[42] J. Rozendaal, D. Seifert, and R. Stahn. *O*ptimal rates of decay for operator semigroups on Hilbert spaces. Advances in Mathematics, 346:359 – 388, 2019.
[43] L. Toufayli. *S*tabilisation polynomiale et contrôlabilité exacte des équations des ondes par des contrôles indirects et dynamiques. PhD thesis, Université de Strasbourg, 2013.
[44] A. Wehbe. *R*ational energy decay rate for a wave equation with dynamical control. Applied mathematics letters, 16(3):357–364, 2003.
[45] X. Zhang and E. Zuazua. *C*ontrol, observation and polynomial decay for a coupled heat-wave system. Comptes Rendus Mathematique, 336(10):823–828, 2003.
[46] X. Zhang and E. Zuazua. *P*olynomial decay and control of a 1- d model for fluid–structure interaction. Comptes Rendus Mathematique, 336(9):745–750, 2003.
[47] X. Zhang and E. Zuazua. *P*olynomial decay and control of a 1- d hyperbolic–parabolic coupled system. Journal of Differential Equations, 204(2):380–438, 2004.
[48] X. Zhang and E. Zuazua. *L*ong-time behavior of a coupled heat-wave system arising in fluid-structure interaction. Archive for rational mechanics and analysis, 184(1):49–120, 2007.