ON ALMOST P-STANDARD SYSTEM OF PARAMETERS OF IDEALIZATION AND APPLICATIONS

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In memory of Professor Shiro Goto

Abstract.[1] Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. In this paper, we construct almost p-standard systems of parameters (a very strict subclass of d-sequences) of the idealization $R \ltimes M$ of M over R. As applications, we build Cohen-Macaulay Rees algebras for idealizations, Cohen-Macaulay Rees modules for unmixed modules, then give precise formulas computing all the Hilbert coefficients of the idealization with respect to an almost p-standard system of parameters.

1 Introduction

Throughout this paper, $(R, \mathfrak{m})$ denotes a Noetherian local ring of dimension r. Let M be a finitely generated R-module with $\dim_R(M) = d$. The notion of d-sequence introduced by C. Huneke [15] makes a useful mean to study the powers of ideals [14, 15] and have important applications in the theory of Buchsbaum modules and generalized Cohen-Macaulay modules. In [6], N.T. Cuong introduced the notion of p-standard system of parameter (s.o.p for short). Note that if $x_1, \ldots, x_d$ is a p-standard s.o.p of M then it is a d-sequence on M and there exist non-negative integers $\lambda_0, \ldots, \lambda_d$ such that

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i$$

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for all $n_1, \ldots, n_d \geq 1$ (see [6, Theorem 2.6]). In generalized Cohen-Macaulay modules, every p-standard s.o.p is a standard s.o.p in the sense of [22], and in general, the notion of p-standard s.o.p plays a key role in the study of the singularity of Cohen-Macaulay type of Noetherian rings and modules (see [16, 17, 10]).

Let $x_1, \ldots, x_d$ be a s.o.p of M. If there exists non-negative integers $\lambda_0, \ldots, \lambda_d$ such that

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i$$

for all $n_1, \ldots, n_d \geq 1$, then $x_1^{n_1}, \ldots, x_d^{n_d}$ is a p-standard s.o.p for all $n_i \geq i$, for $i = 1, \ldots, d$ (see [7, Corollary 3.9]), however $x_1, \ldots, x_d$ is not necessary a p-standard s.o.p (see [8, Example 3.11]). This fact leads to the following notion (see [4, Definition 2.1]).

Definition 1.1. A s.o.p $x_1, \ldots, x_d$ of M is called *almost p-standard* if there exist non-negative integers $\lambda_0, \ldots, \lambda_d$ such that

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i$$

for all $n_1, \ldots, n_d \geq 1$.

Following [10, Theorem 1.2], R admits an almost p-standard s.o.p if and only if R is a quotient of a Cohen-Macaulay local ring, if and only if every finitely generated R-module admits an almost p-standard s.o.p. Note that every almost p-standard s.o.p is a d-sequence, this fact helps to compute several numerical invariants, the Hilbert coefficients, the partial Euler-Poincaré characteristics of the Koszul complex with respect to an almost p-standard s.o.p of M, see [4]. The notion of almost p-standard s.o.p makes an important role in the study of sequentially Cohen-Macaulay modules and sequentially generalized Cohen-Macaulay modules [7, 9].

The notion of the idealization was introduced by M. Nagata [20]. We provide a multiplication on the additive group $R \oplus M$

$$(a, x).(b, y) = (ab, ay + bx)$$

for all $(a, x), (b, y) \in R \oplus M$, then $R \oplus M$ forms a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$. This local ring is called the *idealization* of M over R and denoted by $R \ltimes M$. Note that $\dim(R \ltimes M) = \dim(R)$. The structure of the idealization and its applications have attracted the interest of mathematicians (see [2, 20, 13]).

The aim of this paper is to construct almost p-standard s.o.p of $R \ltimes M$. As applications, we build Cohen-Macaulay Rees algebras for $R \ltimes M$, Cohen-Macaulay Rees modules for unmixed module M, and find a tight relation between Macaulayfications of R and $R \ltimes M$ in several particular cases. Then we give precise formulas computing Hilbert coefficients of $R \ltimes M$ with respect to certain almost p-standard s.o.p.

The following theorem is the first main result of this paper.

Theorem 1.2. *Let $x_1, \ldots, x_r$ be elements in $\mathfrak{m}$. Set $u_i = (x_i, 0)$ for $i = 1, \ldots, r$ and $\underline{u} = u_1, \ldots, u_r$. The following statements are equivalent:*

(i) $\underline{u}$ *is an almost p-standard s.o.p of* $R \ltimes M$.

(ii) $x_1, \ldots, x_d$ *is an almost p-standard s.o.p of* M *and* $x_1, \ldots, x_r$ *is an almost p-standard s.o.p of* R *and* $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R(M)$.

As a consequence, we give a characterization for $R \ltimes M$ being a quotient of a Cohen-Macaulay local ring (Corollary 2.6).

Denote by $\widehat{R}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completion of R and M, respectively. Following M. Nagata [20], M is said to be *unmixed* if $\dim(\widehat{R}/\mathfrak{P}) = \dim_{\widehat{R}}(\widehat{M})$ for any $\mathfrak{P} \in \operatorname{Ass}_{\widehat{R}}(\widehat{M})$. Note that $R \ltimes M$ is unmixed if and only if $\dim(R) = \dim_R(M) = r$ and R, M are unmixed. The first application of Theorem 1.2 is to construct Cohen-Macaulay Rees algebras for the idealization in case where $\dim(R) = \dim_R(M) = r$.

Theorem 1.3. *Suppose that R is a quotient of a Cohen-Macaulay local ring, R and M are unmixed, and* $\dim_R(M) = \dim(R) = r > 1$. *Let $x_1, \ldots, x_r$ be an almost p-standard s.o.p of both R and M (such a s.o.p exists). For $i = 1, \ldots, r$, put $u_i = (x_i, 0)$, $P_i = (u_i, \ldots, u_r)$ and $P = P_1 P_2 \ldots P_{r-2}$. Then the Rees algebra $\mathfrak{R}(R \ltimes M, P)$ is Cohen-Macaulay.*

From an almost p-standard s.o.p of M, we can construct subquotient modules $U_M^{i,j}, \overline{U}_M^{i,j}$ which are independent of the choice of almost p-standard s.o.p (see [4, Proposition 2.2]). The second application of Theorem 1.2 is to clarify certain Hilbert coefficients of the idealization.

Theorem 1.4. *Let $x_1, \ldots, x_r$ be an almost p-standard s.o.p of R such that $x_1, \ldots, x_d$ is an almost p-standard s.o.p of M and* $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R(M)$. *Set $Q = (u_1, \ldots, u_r)$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$. Put $I = (x_1, \ldots, x_d)$ and $J = (x_1, \ldots, x_r)$. Then*

$$\ell((R \ltimes M)/Q^{n+1}) = e_0(Q, R \ltimes M)\binom{n+r}{r} + e_1(Q, R \ltimes M)\binom{n+r-1}{r-1} + \ldots + e_r(Q, R \ltimes M)$$

for all $n \geq 0$, where for $d = r$,

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}), & \text{if } 0 \leq i < r, \\ e_0(J, R) + e_0(J, M), & \text{if } i = r; \end{cases}$$

and for $d < r$,

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} e_0(J; R), & \text{if } i = r, \\ \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}), & \text{if } d < i < r, \\ \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}_R^{t,d+1}) + e_0(I, M), & \text{if } i = d, \\ \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}), & \text{if } 0 \leq i < d. \end{cases}$$

We also describe the Hilbert coefficients of $R \ltimes M$ in case where R and M are sequentially generalized Cohen-Macaulay (Corollary 4.4).

In the next section, after giving some preliminaries on almost p-standard systems of parameters, we prove Theorem 1.2. In Section 3 and Section 4, we present the proofs of Theorem 1.3 and Theorem 1.4, respectively.

2 Almost p-standard system of parameters and idealization

We first recall some properties of almost p-standard s.o.p that will be used in the sequel, see [7, Corollaries 3.5, 3.6], [4, Lemma 2.9].

Lemma 2.1. *Let $x_1, \ldots, x_d$ be an almost p-standard s.o.p of M. For $i = 0, \ldots, d$, put $\lambda_i = e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_d)M})$. Then*

(i) $\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum\limits_{i=0}^{d} \lambda_i n_1 \ldots n_i$ *for all* $n_1, \ldots, n_d \geq 1$.

(ii) $N \cap (x_i, \ldots, x_d)M = 0$ *for any submodule N of M and any integer $i > \dim_R(N)$.*

Let $\underline{y} = x_1, \ldots, x_d$ be a s.o.p of M and $n_1, \ldots, n_d \geq 1$ be positive integers. We set $\underline{y}(\underline{n}) = x_1^{n_1}, \ldots, x_d^{n_d}$. The following function in $n_1, \ldots, n_d$ is very helpful in the study of almost p-standard s.o.p

$$\begin{aligned}\tilde{I}_{M,\,\underline{y}}(\underline{n}) := \ell(M/\underline{y}(\underline{n})M) - e(\underline{y}(\underline{n}); M)\\ - \sum_{i=0}^{d-1} n_1 \ldots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_d)M}).\end{aligned}$$

From Lemma 2.1 and [4, Proposition 2.6], we have the following properties of $\tilde{I}_{M,\,\underline{y}}(\underline{n})$.

Lemma 2.2. *Let $\underline{y} = x_1, \ldots, x_d$ be a s.o.p of M. Then*

(i) $\tilde{I}_{M,\,\underline{y}}(\underline{n})$ *is a non-decreasing function and* $\tilde{I}_{M,\,\underline{y}}(\underline{n}) \geq 0$ *for all* $n_1, \ldots, n_d \geq 1$.

(ii) $\underline{y}$ *is almost p-standard if and only if* $\tilde{I}_{M,\,\underline{y}}(\underline{n}) = 0$ *for all* $n_1, \ldots, n_d \geq 1$.

Lemma 2.3. *Let $x_1, \ldots, x_r$ be elements in $\mathfrak{m}$. For $i = 1, \ldots, r$, put $u_i = (x_i, 0)$. Then*

$$(0 : u_{i+1})_{(R\ltimes M)/(u_{i+2},\ldots,u_j)(R\ltimes M)} \simeq (0 : x_{i+1})_{R/(x_{i+2},\ldots,x_j)R} \times (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_j)M},$$

for all $0 \leq i < j \leq r$.

Proof. For all $0 \leq i < j \leq r$, we have

$$\begin{aligned}(0 : u_{i+1})_{(R\ltimes M)/(u_{i+2},\ldots,u_j)(R\ltimes M)} = [(u_{i+2}, \ldots, u_j)(R \ltimes M) :_{R\ltimes M} u_{i+1}]/(u_{i+2}, \ldots, u_j)(R \ltimes M);\\ (u_{i+2}, \ldots, u_j)(R \ltimes M) = (x_{i+2}, \ldots, x_j)R \times (x_{i+2}, \ldots, x_j)M.\end{aligned}$$

We claim that

$$[(u_{i+2}, \ldots, u_j)(R \ltimes M) :_{R\ltimes M} u_{i+1}] = [(x_{i+2}, \ldots, x_j)R :_R x_{i+1}] \times [(x_{i+2}, \ldots, x_j)M :_M x_{i+1}].$$

Indeed, take an element $(a, m) \in (u_{i+2}, \ldots, u_j)(R \ltimes M) :_{R\ltimes M} u_{i+1}$, then

$$(a, m)(x_{i+1}, 0) = (ax_{i+1}, x_{i+1}m) \in (u_{i+2}, \ldots, u_j)(R \ltimes M).$$

Hence $a \in (x_{i+2}, \ldots, x_j)R :_R x_{i+1}$ and $m \in (x_{i+2}, \ldots, x_j)M :_M x_{i+1}$. Conversely, let

$$(a, m) \in (x_{i+2}, \ldots, x_j)R :_R x_{i+1} \times (x_{i+2}, \ldots, x_j)M :_M x_{i+1}.$$

Then $ax_{i+1} \in (x_{i+2}, \ldots, x_j)R$ and $x_{i+1}m \in (x_{i+2}, \ldots, x_j)M$. Hence

$$\begin{aligned}(a, m)(x_{i+1}, 0) &= (ax_{i+1}, x_{i+1}m) \\ &\in (x_{i+2}, \ldots, x_j)R \times (x_{i+2}, \ldots, x_j)M = (u_{i+2}, \ldots, u_j)(R \ltimes M),\end{aligned}$$

therefore, $(a, m) \in (u_{i+2}, \ldots, u_j)(R \ltimes M) :_{R\ltimes M} u_{i+1}$, the claim is proved. Now, the result is clear by the claim. □

Lemma 2.4. *Let $\underline{x} = x_1, \ldots, x_r$ be a s.o.p of R. Set $\underline{u} = u_1, \ldots, u_r$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$. Then $\underline{u}$ is a s.o.p of $R \ltimes M$. Moreover, if $x_1, \ldots, x_d$ is a s.o.p of M and $(x_{d+1}, \ldots, x_r)M = 0$, then for any $n_1, \ldots, n_r \geq 1$ we have*

$$\tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, x_1,\ldots,x_d}(\underline{n}).$$

Proof. For a tuple of positive integers $\underline{n} = n_1, \ldots, n_r$, set $\underline{u}(\underline{n}) = u_1^{n_1}, \ldots, u_r^{n_r}$ and $\underline{x}(\underline{n}) = x_1^{n_1}, \ldots, x_r^{n_r}$. We have

$$(u_1^{n_1}, \ldots, u_r^{n_r})(R \ltimes M) \simeq (x_1^{n_1}, \ldots, x_r^{n_r})R \times (x_1^{n_1}, \ldots, x_r^{n_r})M.$$

Thus $\underline{u}$ is a s.o.p of $R \ltimes M$ and

$$\ell((R \ltimes M)/\underline{u}(\underline{n})(R \ltimes M)) = \ell(R/\underline{x}(\underline{n})R)) + \ell(M)/\underline{x}(\underline{n})M)).$$

It is clear that $e(\underline{u}; R \ltimes M) = e(\underline{x}; R) + e(\underline{x}; M)$, where $e(\underline{x}; M) = 0$ whenever $d < r$. So, by Lemma 2.3 we obtain

$$\begin{aligned}\tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) &= \ell((R \ltimes M)/\underline{u}(\underline{n})(R \ltimes M)) - n_1 \ldots n_r e(\underline{u}; R \ltimes M) \\ &\quad - \sum_{i=0}^{r-1} n_1 \ldots n_i e(u_1, \ldots, u_i; (0 : u_{i+1})_{(R\ltimes M)/(u_{i+2},\ldots,u_r)(R\ltimes M)}) \\ &= \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \ell(M/\underline{x}(\underline{n})M) - n_1 \ldots n_r e(\underline{x}; M) \\ &\quad - \sum_{i=0}^{r-1} n_1 \ldots n_i e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_r)M}).\end{aligned}$$

If $d = r$, then $\underline{x}$ is a s.o.p of M and the above equality gives

$$\tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, \underline{x}}(\underline{n}),$$

for all $n_1, \ldots, n_r \geq 1$. Let $d < r$. As $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R(M)$, we get $e(\underline{x}; M) = 0$ and

$$e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_r)M}) = 0$$

for $d < i < r$. Moreover,

$$\begin{aligned}e(x_1, \ldots, x_d; (0 : x_{d+1})_{M/(x_{d+2},\ldots,x_r)M}) &= e(x_1, \ldots, x_d; M); \\ e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_r)M}) &= e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2},\ldots,x_d)M})\end{aligned}$$

for $i < d$. From the above computations we have

$$\tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, x_1,\ldots,x_d}(\underline{n})$$

for all $n_1, \ldots, n_r \geq 1$. □

Now we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. *(i)* $\Rightarrow$ *(ii)*. Since $\underline{u}$ is a s.o.p of $R \ltimes M$, it follows that $\underline{x}$ is a s.o.p of R and $\underline{x}$ is a multiplicity system of M (i.e. $\ell(M/(x_1, \ldots, x_r)M) < \infty$).

If $d = r$, then $\underline{x}$ is a s.o.p of M. Using the assumption (i) together with Lemma 2.2(ii) and Lemma 2.4, we have

$$0 = \tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, \underline{x}}(\underline{n})$$

for all $n_1, \ldots, n_r \geq 1$. By Lemma 2.2(i), each term on the right hand side is non-negative. Therefore, $\tilde{I}_{R,\, \underline{x}}(\underline{n}) = \tilde{I}_{M,\, \underline{x}}(\underline{n}) = 0$ for all $n_1, \ldots, n_r \geq 1$. By Lemma 2.2(ii), $\underline{x}$ is an almost p-standard s.o.p of both M and R.

Suppose $d < r$. Via the canonical inclusion $\varepsilon : M \to R \ltimes M$ defined by $\varepsilon(x) = (0, x)$, each R-submodule of M can be identified with an $R \ltimes M$-submodule of $R \ltimes M$. Consider the submodule $\varepsilon(M) = 0 \times M$ of $R \ltimes M$. We have $\dim_{R\ltimes M}(0 \times M) = d < r$. Since $\underline{u}$ is an almost p-standard s.o.p of $R \ltimes M$, we get by Lemma 2.1(ii) that

$$0 \times (x_{d+1}, \ldots, x_r)M \subseteq (0 \times M) \cap (u_{d+1}, \ldots, u_r)(R \ltimes M) = 0.$$

Hence $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R(M)$. Set $\underline{y} = x_1, \ldots, x_d$. So from the assumption (i) together with Lemma 2.2(ii) and Lemma 2.4, we obtain

$$0 = \tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, \underline{y}}(\underline{n})$$

for all $n_1, \ldots, n_r \geq 1$. By Lemma 2.2(i), $\tilde{I}_{R,\, \underline{x}}(\underline{n}) = \tilde{I}_{M,\, \underline{y}}(\underline{n}) = 0$ for all $n_1, \ldots, n_r \geq 1$. By Lemma 2.2(ii), $\underline{x}$ is an almost p-standard s.o.p of R and $x_1, \ldots, x_d$ is an almost p-standard s.o.p of M.

(ii) $\Rightarrow$ *(i)*. Since $\underline{x}$ is an almost p-standard s.o.p of R and $\underline{y} = x_1, \ldots, x_d$ is an almost p-standard s.o.p of M, we get by Lemma 2.2(ii) that

$$\tilde{I}_{R,\, \underline{x}}(\underline{n}) = \tilde{I}_{M,\, \underline{y}}(\underline{n}) = 0$$

for all $n_1, \ldots, n_r \geq 1$. Therefore, we have by assumption (ii) and Lemma 2.4 that

$$\tilde{I}_{R\ltimes M,\, \underline{u}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, \underline{y}}(\underline{n}) = 0.$$

By Lemma 2.2(ii), $\underline{u}$ is an almost p-standard s.o.p of $R \ltimes M$. □

Theorem 1.2 leads to the following consequence for the existence of almost p-standard s.o.p of idealization.

Corollary 2.5. *The following statements are equivalent:*

(i) *R admits an almost p-standard s.o.p;*

(ii) *$R \ltimes M$ admits an almost p-standard s.o.p;*

(iii) *$R \ltimes M$ admits an almost p-standard s.o.p of the form $(x_1, 0), \ldots, (x_r, 0)$, where $x_1, \ldots, x_r$ is an almost p-standard s.o.p of R and $x_1, \ldots, x_d$ is an almost p-standard s.o.p of M.*

Proof. $(iii) \Rightarrow (ii)$ is clear.
$(ii) \Rightarrow (i)$. By assumption (ii), we get by [10, Theorem 1.2] that $R \ltimes M$ is a quotient of a Cohen-Macaulay local ring. Note that R is a quotient of $R \ltimes M$. Therefore, R is a quotient of a Cohen-Macaulay local ring. Now, the result follows by [10, Theorem 1.2].
$(i) \Rightarrow (iii)$. By assumption (i), we get by [10, Theorem 1.2] that R is a quotient of a Cohen-Macaulay local ring. Therefore, $\dim(R/\mathfrak{a}(N)) < \dim_R(N)$ for any finitely generated R-module N, where $\mathfrak{a}(N) = \mathfrak{a}_0(N)\mathfrak{a}_1(N)\ldots\mathfrak{a}_{\dim_R(N)-1}(N)$ and $\mathfrak{a}_i(N) = \mathrm{Ann}_R(H^i_{\mathfrak{m}}(N))$ for $i = 0, \ldots, \dim_R(N) - 1$. Therefore, by Prime Avoidance, there exists a p-standard s.o.p $x_1, \ldots, x_r$ of R such that $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R(M)$ and $x_1, \ldots, x_d$ is a p-standard s.o.p of M (see the definition of p-standard s.o.p in [6]). Hence $x_1, \ldots, x_r$ is an almost p-standard s.o.p of R and $x_1, \ldots, x_d$ is an almost p-standard s.o.p of M. By Theorem 1.2, $u_1, \ldots, u_r$ is an almost p-standard s.o.p of $R \ltimes M$, where $u_i = (x_i, 0)$ for all $i = 1, \ldots, r$. □

From Corollary 2.5 and [10, Theorem 1.2], we get immediately the following consequence.

Corollary 2.6. *A Noetherian local ring is a quotient of a Cohen-Macaulay local ring if and only if so is one of its idealization, if and only if so are all of its idealizations by finitely generated modules.*

3 Macaulayfication of idealization

In this section, we discuss an application of Theorem 1.2 to construct Cohen-Macaulay Rees algebras of idealization and then to prove the existence of Cohen-Macaulay Rees modules of unmixed modules.

Let I be an ideal of R and T be a variable over R. The *Rees algebra of* R with respect to I is the subring of $R[T]$ defined by

$$\mathfrak{R}(R, I) = R[IT] = \{\sum_{i=0}^{n} a_i T^i \mid n \in \mathbb{N}, a_i \in I^i\} = \bigoplus_{n \geq 0} I^n T^n,$$

where $I^0 = R$. Similarly, the *Rees module* of M with respect to I is defined by

$$\mathfrak{R}(M, I) = \{\sum_{i=0}^{n} a_i x_i T^i \mid n \in \mathbb{N}, a_i \in I^i, x_i \in M\} = \bigoplus_{n \geq 0} I^n M T^n,$$

where $I^0 M = M$. A Rees algebra $\mathfrak{R}(R, I)$ is called an *arithmetic Macaulayfication* of R if it is Cohen-Macaulay and I is of positive height. If $\mathfrak{R}(R, I)$ is an arithmetic Macaulayfication of R, then the canonical algebra homomorphism $R \to \mathfrak{R}(R, I)$ induces a morphism of Noetherian schemes $\mathrm{Proj}(\mathfrak{R}(R, I)) \to \mathrm{Spec}(R)$ which is called a *projective Macaulayfication*. More generally, a *Macaulayfication* of $\mathrm{Spec}(R)$ is a birational and proper morphism $X \to \mathrm{Spec}(R)$ where X is a Cohen-Macaulay locally Noetherian scheme.

The existence of arithmetic Macaulayfication and of Macaulayfication have been established by several authors. Kawasaki [17, Theorem 1.1] showed that a Noetherian local ring has an arithmetic Macaulayfication if and only if it is unmixed and all its formal fibers are

Cohen-Macaulay. Česnavičius [3] has introduced a notion of CM-quasi-excellent schemes as following.

Definition 3.1. A Noetherian scheme X is *CM-quasi-excellent* if

(a) Every formal fiber of local rings of X is Cohen-Macaulay, and

(b) Any integral subscheme of X has an open Cohen-Macaulay locus.

A Noetherian ring is *CM-quasi-excellent* if its prime spectrum is a CM-quasi-excellent affine scheme. In [3, Theorem 1.6], Česnavičius showed that if R is CM-quasi-excellent then $\operatorname{Spec}(R)$ admits a Macaulayfication.

Arithmetic Macaulayfication has been studied from other perspective by Kurano [19], Aberbach-Huneke-Smith [1], Cutkosky-Tai [12], Tai-Trung [21]. In [10], N.T. Cuong and D.T. Cuong extended Kawasaki's theorem for modules. They showed that there is an ideal I such that the Rees module $\mathfrak{R}(M, I)$ is Cohen-Macaulay if and only if M is unmixed and $R/\operatorname{Ann}_R(M)$ is a quotient of a Cohen-Macaulay ring.

Note that the idealization $R \ltimes M$ is a finite R-algebra (see, for example, [2, Proposition 2.2]). By Corollary 2.6, if R is a quotient of a Cohen-Macaulay ring, then so is $R \ltimes M$, therefore we get by [17, Theorem 1.1] that if R admits an arithmetic Macaulayfication and the idealization $R \ltimes M$ is unmixed then $R \ltimes M$ also admits an arithmetic Macaulayfication. Similarly, if R is CM-quasi-excellent then so is $R \ltimes M$ (see [3, Remark 1.5]). Česnavičius's theorem implies that in that case both $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R \ltimes M)$ admit Macaulayfications.

We now investigate further relations between arithmetic Macaulayfications and Macaulayfications respectively on R and $R \ltimes M$. We first prove Theorem 1.3.

Proof of Theorem 1.3. Since R, M are unmixed of the same dimension r, we get by [2, Theorem 4.11, 3.2] that the idealization $R \ltimes M$ is unmixed of dimension r. Since R is a quotient of a Cohen-Macaulay, R-module $R \oplus M$ admits an almost p-standard s.o.p $\underline{x} = x_1, \ldots, x_r$. By Lemma 2.2(ii),

$$0 = \tilde{I}_{R\oplus M,\, \underline{x}}(\underline{n}) = \tilde{I}_{R,\, \underline{x}}(\underline{n}) + \tilde{I}_{M,\, \underline{x}}(\underline{n}).$$

By Lemma 2.2(i), we get $\tilde{I}_{R,\, \underline{x}}(\underline{n}) = \tilde{I}_{M,\, \underline{x}}(\underline{n}) = 0$. Hence $x_1, \ldots, x_r$ is an almost p-standard s.o.p of both R and M by Lemma 2.2(ii). By Theorem 1.2, $(x_1, 0), \ldots, (x_r, 0)$ is an almost p-standard s.o.p of $R \ltimes M$. Therefore, Theorem 1.3 is then implied from [18, Proposition 8.2]. $\square$

Theorem 1.3 has an interesting application in constructing Cohen-Macaulay Rees module.

Let $x_1, \ldots, x_n,\ y_1, \ldots, y_m \in \mathfrak{m}$ and put $u_i = (x_i, 0), v_j = (y_j, 0) \in R \ltimes M$, for $i = 1, \ldots, n, j = 1, \ldots, m$. Denote $I = (x_1, \ldots, x_n)$, $J = (y_1, \ldots, y_m)$, and $P = (u_1, \ldots, u_n)$, $Q = (v_1, \ldots, v_m)$. The following properties are obvious

$$\begin{aligned} P + Q &= (I + J) \times (I + J)M, \\ PQ &= ((x_i y_j, 0))_{i,j} = IJ \times IJM, \\ P^t &= I^t \times I^t M, \end{aligned}$$

for all $t > 0$. They lead to the following lemma.

Lemma 3.2. *We have an algebra isomorphism*

$$\mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I).$$

Consequently, the Rees algebra $\mathfrak{R}(R \ltimes M, P)$ is Cohen-Macaulay if and only if $\mathfrak{R}(R, I)$ and $\mathfrak{R}(M, I)$ are Cohen-Macaulay of the same dimension.

Using Theorem 1.3 and Kawasaki's theorem on arithmetic Macaulayfication, we obtain another proof for the construction of Cohen-Macaulay Rees module in [10, Theorem 4.4].

Corollary 3.3. *Let R be a quotient of a Cohen-Macaulay local ring. Suppose that M is unmixed and of dimension $d > 1$. Then there is an ideal I such that the Rees module $\mathfrak{R}(M, I)$ is Cohen-Macaulay.*

Proof. Replace R by $R/\operatorname{Ann}_R(M)$, we may assume that R is unmixed of the same dimension with M. Since R is a quotient of a Cohen-Macaulay local ring, R admits an almost p-standard s.o.p. By Corollary 2.5 and Theorem 1.2, $R \ltimes M$ admits an almost p-standard s.o.p $u_1, \ldots, u_d$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, d$ such that $x_1, \ldots, x_d$ is an almost p-standard s.o.p of both R and M. Put $I_i = (x_i, \ldots, x_d)$ for $i = 1, \ldots, d$, and $I = I_1 \ldots I_{d-2}$. Also we denote $u_i = (x_i, 0)$, $P_i = (u_i, \ldots, u_d)$ for $i = 1, \ldots, d$, and $P = P_1 \ldots P_{d-2}$. Then $\mathfrak{R}(R, I)$ and $\mathfrak{R}(R \ltimes M, P)$ are Cohen-Macaulay. The Rees module $\mathfrak{R}(M, I)$ has the same dimension with $\mathfrak{R}(R, I)$ and $\mathfrak{R}(R \ltimes M, P)$. So the short exact sequence

$$0 \to \mathfrak{R}(M, I) \to \mathfrak{R}(R \ltimes M, P) \to \mathfrak{R}(R, I) \to 0,$$

implies that $\mathfrak{R}(M, I)$ is Cohen-Macaulay. ☐

Conversely, using [10, Theorem 4.4] we are able to give the second proof for Theorem 1.3 as following: Denote $I_i = (x_i, \ldots, x_r)$ and

$$I := I_1 \ldots I_{r-3} I_{r-2}.$$

Following [18, Proposition 8.2] and [10, Theorem 4.4], $\mathfrak{R}(R, I)$ and $\mathfrak{R}(M, I)$ are Cohen-Macaulay. By Lemma 3.2, $\mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I)$ which is thus Cohen-Macaulay, hence Theorem 1.3 is proved.

Another consequence of Theorem 1.3 is the following characterization for the existence of arithmetic Macaulayfication for idealizations.

Corollary 3.4. *The idealization $R \ltimes M$ has an arithmetic Macaulayfication if and only if R has an arithmetic Macaulayfication and M is unmixed with $\dim(R) = \dim_R(M)$.*

Proof. Suppose $R \ltimes M$ has an arithmetic Macaulayfication. By [10, Corollary 5.4], $R \ltimes M$ is unmixed and is a quotient of a Cohen-Macaulay ring. Then R and M are unmixed of the same dimension and R is also a quotient of a Cohen-Macaulay ring. Using again [10, Corollary 5.4], R admits an arithmetic Macaulayfication.

Conversely, suppose that R has an arithmetic Macaulayfication and M is unmixed with $\dim_R(M) = \dim(R)$. Then R is a quotient of a Cohen-Macaulay local ring. Theorem 1.3 then implies that the idealization $R \ltimes M$ admits an arithmetic Macaulayfication. ☐

For Macaulayfication, we find a tight relation between certain Macaulayfications of R and $R \ltimes M$ in several particular cases.

First, suppose R and M are unmixed of the same dimension. If R is a quotient of a Cohen-Macaulay ring then by Theorem 1.3, there are arithmetic Macaulayfications of R, M and $R \ltimes M$ with relation

$$\mathfrak{R}(R \ltimes M, P) \simeq \mathfrak{R}(R, I) \ltimes \mathfrak{R}(M, I).$$

On the other hand, the canonical morphism $\mathfrak{R}(R, I) \to \mathfrak{R}(R \ltimes M, P)$ induces a morphism of R-schemes $\mathrm{Proj}(\mathfrak{R}(R \ltimes M, P)) \to \mathrm{Proj}(\mathfrak{R}(R, I))$ which is actually an isomorphism. Note that $\mathrm{Proj}(\mathfrak{R}(R \ltimes M, P))$ and $\mathrm{Proj}(\mathfrak{R}(R, I))$ are Cohen-Macaulay which are Macaulayfications of $\mathrm{Spec}(R \ltimes M)$ and $\mathrm{Spec}(R)$ respectively. Therefore in this case, the Macaulayfication of R and the idealization are isomorphic.

Now suppose that R is quasi-CM-excellent. The canonical map $R \ltimes M \to R$ induces a bijective morphism of affine schemes $\rho : \mathrm{Spec}(R) \to \mathrm{Spec}(R \ltimes M)$ (see [2, Theorem 3.2(b)]). Let $\mathfrak{p}$ be a minimal prime ideal of R, then $\rho(\mathfrak{p}) = \mathfrak{p} \ltimes M$ is the corresponding prime ideal of the idealization. By [2, Theorem 4.1], we have

$$(R \ltimes M)_{\mathfrak{p} \ltimes M} \simeq R_{\mathfrak{p}} \ltimes M_{\mathfrak{p}}.$$

In particular, if $\mathfrak{p}$ does not belong to in the support of M then

$$(R \ltimes M)_{\mathfrak{p} \ltimes M} \simeq R_{\mathfrak{p}}.$$

This proves the following proposition.

Proposition 3.5. *Assume that no associated prime ideals of M are minimal prime ideals of R. Then the morphism $\rho : \mathrm{Spec}(R) \to \mathrm{Spec}(R \ltimes M)$ is a birational morphism. Consequently, if $\varphi : X \to \mathrm{Spec}(R)$ is a Macaulayfication then $\varphi \circ \rho : X \to \mathrm{Spec}(R \ltimes M)$ is a Macaulayfication.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of R. Then $(R \ltimes M)_{\mathfrak{p} \ltimes M} \simeq R_{\mathfrak{p}}$. Since the morphism ρ is bijective, then it is clearly birational. Furthermore, ρ is obviously proper. So $\varphi \circ \rho$ is proper and birational, which is therefore a Macaulayfication of $\mathrm{Spec}(R \ltimes M)$. □

4 Hilbert function of idealization

Firstly, we recall the following property (see [4, Proposition 3.2, Corollary 3.5]).

Lemma 4.1. *Let $\underline{x} = x_1, \ldots, x_d$ be an almost p-standard s.o.p of M. Let i, j be integers such that $0 \leq i < j \leq d$. The following statements are true.*

(i) The subquotient module $U_M^{i,j} := (0 :_{M/(x_{i+2}^{n_{i+2}}, \ldots, x_j^{n_j})M} x_{i+1})$ is independent of the choice of the s.o.p $\underline{x}$ and of the exponents $n_{i+2}, \ldots, n_j \geq 2$.

(ii) If $j > i + 1$, then there is an injective homomorphism $\varphi_{i,j} : U_M^{i,j-1} \to U_M^{i,j}$ such that $\mathrm{Im}(\varphi_{i,j})$ is a direct summand of $U_M^{i,j}$. In particular, set $\overline{U}_M^{i,j} = \mathrm{Coker}(\varphi_{i,j})$, then

$$U_M^{i,j} \simeq \overline{U}_M^{i,j} \oplus \overline{U}_M^{i,j-1} \oplus \cdots \oplus \overline{U}_M^{i,i+2} \oplus\ U_M^{i,i+1}.$$

For an integer $0 \le i < d$, set $\overline{U}_M^{i,i+1} := U_M^{i,i+1}$. Note that $U_M^{d-1,d}$ is the largest submodule of M of dimension less than d, and $U_M^{0,1} = H_{\mathfrak{m}}^0(M)$. The subquotient modules $U_M^{i,j}, \overline{U}_M^{i,j}$ give a lot of information on structure of M. For example, M is Cohen-Macaulay if and only if $U_M^{i,j} = 0$ for all $i < j$, if and only if $\overline{U}_M^{i,j} = 0$ for all $i < j$. Moreover, M is generalized Cohen-Macaulay if and only if $\ell(U_M^{i,j}) < \infty$ for all $i < j$, if and only if $\ell(\overline{U}_M^{i,j}) < \infty$ for all $i < j$, see [4, Proposition 3.9].

From now on, we assume that R is a quotient of a Cohen-Macaulay local ring. Before proving Theorem 1.4, we compute the subquotient modules $U_{R\ltimes M}^{i,j}$ and $\overline{U}_{R\ltimes M}^{i,j}$ of the idealization.

Lemma 4.2. *The following statements are true.*

(i) If $d = r$, *then* $U_{R\ltimes M}^{i,j} \simeq U_R^{i,j} \times U_M^{i,j}$ *for all* $0 \le i < j \le r$.

(ii) If $d < r$, *then*

$$U_{R\ltimes M}^{i,j} \simeq \begin{cases} U_R^{i,j} \times U_M^{i,j} & \text{if } 0 \le i < j < d, \\ U_R^{i,j} \times U_M^{id} & \text{if } 0 \le i < d \le j \le r, \\ U_R^{i,j} \times M & \text{if } d \le i < j \le r. \end{cases}$$

Proof. Since R is a quotient of a Cohen-Macaulay local ring, R admits an almost p-standard s.o.p. By Corollary 2.5 and Theorem 1.2, $R\ltimes M$ admits an almost p-standard s.o.p $u_1, \ldots, u_r$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$ such that $x_1, \ldots, x_r$ is an almost p-standard s.o.p of R, $x_1, \ldots, x_d$ is an almost p-standard of M and $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R(M)$.

For integers $0 \le i < j \le r$, by Lemma 2.3 we have

$$\begin{aligned} U_{R\ltimes M}^{i,j} &:= (0 : u_{i+1})_{(R\ltimes M)/(u_{i+2}^2,\ldots,u_j^2)(R\ltimes M)} \\ &\simeq (0 : x_{i+1})_{R/(x_{i+2}^2,\ldots,x_j^2)} \times (0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M} \\ &\simeq U_R^{i,j} \times (0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M}. \end{aligned}$$

(i) If $d = r$, then $(0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M} \simeq U_M^{i,j}$ for $0 \le i < j \le r$, so $U_{R\ltimes M}^{i,j} \simeq U_R^{i,j} \times U_M^{i,j}$.

(ii) Suppose that $d < r$. If $0 \le i < j < d$ then $(0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M} \simeq U_M^{i,j}$. Let $0 \le i < d \le j \le r$. Since $x_{d+1}, \ldots, x_r \in \operatorname{Ann}_R(M)$, we have

$$(0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M} = (0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_d^2)M} \simeq U_M^{i,d}.$$

It is clear that $(0 : x_{i+1})_{M/(x_{i+2}^2,\ldots,x_j^2)M} \simeq M$ for all $d \le i < j \le r$, the statement follows. ☐

For the subquotients $\overline{U}_{R\ltimes M}^{i,j}$ we have the following lemma.

Lemma 4.3. *The following statements are true.*

(i) If $d = r$, *then* $\overline{U}_{R\ltimes M}^{i,j} \simeq \overline{U}_R^{i,j} \times \overline{U}_M^{i,j}$ *for all* $0 \le i < j \le r$.

(ii) If $d < r$, then

$$\overline{U}^{i,j}_{R\ltimes M} \simeq \begin{cases} \overline{U}^{i,j}_R \times \overline{U}^{i,j}_M & \text{if } 0 \le i < j \le d, \\ \overline{U}^{i,j}_R & \text{if } 0 \le i < d < j \le r, \text{ or } d < i+1 < j \le r, \\ \overline{U}^{i,i+1}_R \times M & \text{if } d < i+1 = j \le r. \end{cases}$$

Proof. (i) Suppose that $d = r$ and $0 \le i < j \le r$. If $j = i+1$, then we get by Lemma 4.2(i)

$$\overline{U}^{i,i+1}_{R\ltimes M} = U^{i,i+1}_{R\ltimes M} \simeq U^{i,i+1}_R \times U^{i,i+1}_M = \overline{U}^{i,i+1}_R \times \overline{U}^{i,i+1}_M.$$

Let $j > i+1$. Then $U^{i,j-1}_{R\ltimes M} \simeq U^{i,j-1}_R \times U^{i,j-1}_M$ by Lemma 4.2(i), and hence

$$U^{i,j}_{R\ltimes M}/U^{i,j-1}_{R\ltimes M} \simeq U^{i,j}_R/U^{i,j-1}_R \times U^{i,j}_M/U^{i,j-1}_M.$$

We get by Proposition 4.1(ii) that

$$U^{i,j}_{R\ltimes M} \simeq \overline{U}^{i,j}_{R\ltimes M} \oplus U^{i,j-1}_{R\ltimes M},\ U^{i,j}_R \simeq \overline{U}^{i,j}_R \oplus U^{i,j-1}_R,\ U^{i,j}_M \simeq \overline{U}^{i,j}_M \oplus U^{i,j-1}_M.$$

Therefore

$$\overline{U}^{i,j}_{R\ltimes M} \simeq U^{i,j}_{R\ltimes M}/U^{i,j-1}_{R\ltimes M} \simeq U^{i,j}_R/U^{i,j-1}_R \times U^{i,j}_M/U^{i,j-1}_M \simeq \overline{U}^{i,j}_R \times \overline{U}^{i,j}_M.$$

(ii) Suppose that $d < r$ and $0 \le i < j \le r$. If $j \le d$, then by the same arguments as in the proof of (i), we have $\overline{U}^{i,j}_{R\ltimes M} \simeq \overline{U}^{i,j}_R \times \overline{U}^{i,j}_M$.

Let $j > d$. As in the proof of Lemma 4.2, there exists an almost p-standard s.o.p $x_1, \ldots, x_r$ of R such that $x_1, \ldots, x_d$ is an almost p-standard s.o.p of M, $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R(M)$ and

$$U^{i,j}_{R\ltimes M} \simeq U^{i,j}_R \times (0 : x_{i+1})_{M/(x^2_{i+2},\ldots,x^2_j)M}.$$

Note that $(0 : x_{i+1})_{M/(x^2_{i+2},\ldots,x^2_j)M} = U^{i,d}_M$ for all $i < d$ and $(0 : x_{i+1})_{M/(x^2_{i+2},\ldots,x^2_j)M} = M$ for all $i \ge d$. Therefore, if $i < d$ then

$$\overline{U}^{i,j}_{R\ltimes M} \simeq U^{i,j}_{R\ltimes M}/U^{i,j-1}_{R\ltimes M} \simeq U^{i,j}_R/U^{i,j-1}_R \times U^{i,d}_M/U^{i,d}_M \simeq \overline{U}^{i,j}_R.$$

If $j > i+1 > d$ then

$$\overline{U}^{i,j}_{R\ltimes M} \simeq U^{i,j}_{R\ltimes M}/U^{i,j-1}_{R\ltimes M} \simeq U^{i,j}_R/U^{i,j-1}_R \times M/M \simeq \overline{U}^{i,j}_R.$$

If $j = i+1 > d$ then

$$\overline{U}^{i,i+1}_{R\ltimes M} = U^{i,i+1}_{R\ltimes M} \simeq U^{i,i+1}_R \times M = \overline{U}^{i,i+1}_R \times M.$$

□

Proof of Theorem 1.4. Theorem 1.2 tells us that $\underline{u} = u_1, \ldots, u_r$ is an almost p-standard s.o.p of $R \ltimes M$. By [4, Theorem 4.7], we have

$$\ell((R\ltimes M)/Q^{n+1}) = e_0(Q, R\ltimes M)\binom{n+r}{r} + e_1(Q, R\ltimes M)\binom{n+r-1}{r-1} + \ldots + e_r(Q, R\ltimes M)$$

for all $n \geq 0$, where $e_{r-i}(Q, R \ltimes M) = \sum\limits_{t=0}^{i} e(u_1, \ldots, u_t; \overline{U}^{t,i+1}_{R\ltimes M})$ for all $0 \leq i \leq r-1$.

• Let $d = r$. Then J is a parameter ideal of M, therefore

$$e_0(Q, R \ltimes M) = e_0(J, R) + e_0(J, M).$$

Since $\overline{U}^{t,i+1}_{R\ltimes M} \simeq \overline{U}^{t,i+1}_{R} \times \overline{U}^{t,i+1}_{M}$ by Lemma 4.3, we get

$$e(u_1, \ldots, u_t; \overline{U}^{t,i+1}_{R\ltimes M}) = e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{R}) + e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{M})$$

for all $0 \leq t \leq i < r$. Therefore, for all $0 \leq i < r$ we have

$$e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{R}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{M}).$$

• Let $d < r$. Then $e_0(Q, R \ltimes M) = e_0(J, R)$. If $0 \leq i < d$, then $\overline{U}^{t,i+1}_{R\ltimes M} \simeq \overline{U}^{t,i+1}_{R} \times \overline{U}^{t,i+1}_{M}$ by Lemma 4.3 for all $t \leq i$, therefore,

$$e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{R}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{M}).$$

If $d \leq i < r$ then we get by Lemma 4.3 that

$$\overline{U}^{t,i+1}_{R\ltimes M} \simeq \begin{cases} \overline{U}^{t,i+1}_{R} & \text{if } 0 \leq t < i, \\ \overline{U}^{i,i+1}_{R} \times M & \text{if } t = i, \end{cases}$$

therefore,

$$e_{r-d}(Q, R \ltimes M) = \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}^{t,d+1}_{R}) + e_0(I, M)$$

and $e_{r-i}(Q, R \ltimes M) = \sum\limits_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}^{t,i+1}_{R})$ for all $i > d$. □

Let the notations and assumptions be as in Theorem 1.4. Consider the case where R and M are generalized Cohen-Macaulay. We use Theorem 1.4 and [5, Lemma 2.4] to compute Hilbert coefficients of $R \ltimes M$. If $d = 0$ or $d = r$ then $R \ltimes M$ is generalized Cohen-Macaulay. In this case, if $d = r$ then

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \sum\limits_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(R)) + \sum\limits_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(M)) & \text{if } 0 \leq i < r, \\ e_0(J, R) + e_0(J, M) & \text{if } i = r. \end{cases}$$

and if $d = 0$ then

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \ell_R(H^0_{\mathfrak{m}}(R)) + \ell_R(M) & \text{if } i = 0, \\ \sum\limits_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(R)) & \text{if } 0 < i < r, \\ e_0(J, R) & \text{if } i = r. \end{cases}$$

If $0 < d < r$, then $R \ltimes M$ is not generalized Cohen-Macaulay. In this case we have

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} e_0(J; R) & \text{if } i = r, \\ \sum_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(R)) & \text{if } d < i < r, \\ \sum_{t=1}^{d} \binom{d-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(R)) + e_0(I; M) & \text{if } i = d, \\ \sum_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(R)) + \sum_{t=1}^{i} \binom{i-1}{t-1} \ell_R(H^t_{\mathfrak{m}}(M)) & \text{if } 0 \leq i < d. \end{cases}$$

Let $M_0 = H^0_{\mathfrak{m}}(M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$ be the dimension filtration of M, i.e. M_i is the largest submodule of M_{i+1} satisfying $\dim_R(M_i) < \dim_R(M_{i+1})$ for $i < t$. Following [11], M is *sequentially generalized Cohen-Macaulay* if each quotient M_{i+1}/M_i is generalized Cohen-Macaulay. Let $R_0 = H^0_{\mathfrak{m}}(R) \subsetneq R_1 \subsetneq \ldots \subsetneq R_s = R$ be the dimension filtration of R. For $i = 0, \ldots, s$ and $j = 0, \ldots, t$, put $d_i = \dim_R(R_i)$ and $d'_j = \dim_R(M_j)$. Denote $\Delta_R = \{d_1, \ldots, d_s\}$ and $\Delta_M = \{d'_1, \ldots, d'_t\}$ and set $\Delta := \Delta_R \cap \Delta_M$.

Corollary 4.4. *Let the notations and assumptions be as in Theorem 1.4. For $0 < i \leq r$, set $\underline{x}_i = x_1, \ldots, x_i$. Suppose that R and M are sequentially generalized Cohen-Macaulay.*

(i) If $d = r$ then for all $0 \leq i < r$ we have

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \ell(\overline{U}_R^{0,d_j+1}) + e(\underline{x}_{d_j}; R_j) + e(\underline{x}_{d_j}; M_j) + \ell(\overline{U}_M^{0,d_j+1}) & \text{if } i = d_j \in \Delta, \\ \ell(\overline{U}_R^{0,i+1}) + \ell(\overline{U}_M^{0,i+1}) & \text{if } i \notin \Delta_R \cup \Delta_M, \\ \ell(\overline{U}_R^{0,d_j+1}) + e(\underline{x}_{d_j}; R_j) + \ell(\overline{U}_M^{0,d_j+1}) & \text{if } i = d_j \in \Delta_R \setminus \Delta_M, \\ \ell(\overline{U}_R^{0,d'_j+1}) + e(\underline{x}_{d'_j}; M_j) + \ell(\overline{U}_M^{0,d'_j+1}) & \text{if } i = d'_j \in \Delta_M \setminus \Delta_R. \end{cases}$$

(ii) If $d < r$ then for $d < i < r$, we have

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \ell(\overline{U}_R^{0,d_j+1}) + e(\underline{x}_{d_j}; R_j) & \text{if } i = d_j \in \Delta_R, \\ \ell(\overline{U}_R^{0,i+1}) & \text{if } i \notin \Delta_R; \end{cases}$$

and for all $0 \leq i < d < r$ we have

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \ell(\overline{U}_R^{0,d_j+1}) + e(\underline{x}_{d_j}; R_j) + e(\underline{x}_{d_j}; M_j) + \ell(\overline{U}_M^{0,d_j+1}) & \text{if } i = d_j \in \Delta, \\ \ell(\overline{U}_R^{0,i+1}) + \ell(\overline{U}_M^{0,i+1}) & \text{if } i \notin \Delta_R \cup \Delta_M, \\ \ell(\overline{U}_R^{0,d_j+1}) + e(\underline{x}_{d_j}; R_j) + \ell(\overline{U}_M^{0,d_j+1}) & \text{if } i = d_j \in \Delta_R \setminus \Delta_M, \\ \ell(\overline{U}_R^{0,d'_j+1}) + e(\underline{x}_{d'_j}; M_j) + \ell(\overline{U}_M^{0,d'_j+1}) & \text{if } i = d'_j \in \Delta_M \setminus \Delta_R; \end{cases}$$

and finally for $i = d$ we have

$$e_{r-d}(Q, R \ltimes M) = \begin{cases} \ell(\overline{U}_R^{0,d+1}) + e(\underline{x}_d; R_j) + e_0(I, M) & \text{if } d = d_j \in \Delta_R, \\ \ell(\overline{U}_R^{0,d+1}) + e_0(I, M) & \text{if } d \notin \Delta_R. \end{cases}$$

Proof. We get by Lemma 4.1(ii) that

$$U_R^{i,n} \simeq \overline{U}_R^{i,n} \oplus \overline{U}_R^{i,n-1} \oplus \cdots \oplus \overline{U}_R^{i,i+2} \oplus \ U_R^{i,i+1} \text{ for all } 0 \leq i < n \leq r;$$
$$U_M^{j,m} \simeq \overline{U}_M^{j,m} \oplus \overline{U}_M^{j,m-1} \oplus \cdots \oplus \overline{U}_M^{j,j+2} \oplus \ U_M^{j,j+1} \text{ for all } 0 \leq j < m \leq d.$$

It follows by [8, Lemma 3.5] that $M_j = U_M^{i,i+1}$ for any integers i, j such that $d'_j \leq i < d'_{j+1}$, and $R_j = U_R^{i,i+1}$ for any integers i, j such that $d_j \leq i < d_{j+1}$. So, by [4, Proposition 2.9 (2)], $\overline{U}_M^{i,j} \oplus \overline{U}_M^{i,j-1} \oplus \cdots \oplus \overline{U}_M^{i,i+2}$ and $\overline{U}_R^{i,j} \oplus \overline{U}_R^{i,j-1} \oplus \cdots \oplus \overline{U}_R^{i,i+2}$ are of finite length. Hence

$$e(x_1, \ldots, x_i; \overline{U}_R^{i,n}) = \begin{cases} e(x_1, \ldots, x_{d_j}; R_j) & \text{if } n = i+1, i = d_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$e(x_1, \ldots, x_j; \overline{U}_M^{j,m}) = \begin{cases} e(x_1, \ldots, x_{d'_k}; M_k) & \text{if } m = j+1, j = d'_k, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Let $d = r$. By Theorem 1.4, we have

$$e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1})$$

for all $0 \leq i < r$. We divide into four cases.

- If $i = d_j \in \Delta$, then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}) = 0$ for all $t \notin \{0, d_j\}$. Hence

$$e_{r-i}(Q, A) = \ell(\overline{U}_R^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; R_j) + \ell(\overline{U}_M^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; M_j).$$

- If $i \notin \Delta_R \cup \Delta_M$, then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}) = 0$ for all $t \neq 0$. Hence

$$e_{r-i}(Q, R \ltimes M) = \ell(\overline{U}_R^{0,d_j+1}) + \ell(\overline{U}_M^{0,d_j+1}).$$

- If $i = d_j \in \Delta_R \setminus \Delta_M$, then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = 0$ for all $t \notin \{0, d_j\}$. Moreover, $e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}) = 0$ for all $t \neq 0$. Therefore,

$$e_{r-i}(Q, R \ltimes M) = \ell(\overline{U}_R^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; R_j) + \ell(\overline{U}_M^{0,d_j+1}).$$

- If $i = d'_j \in \Delta_M \setminus \Delta_R$, then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = 0$ for all $t \neq 0$; $e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}) = 0$ for all $t \neq \{0, d'_j\}$. Therefore

$$e_{r-i}(Q, A) = \ell(\overline{U}_R^{0,d'_j+1}) + e(x_1, \ldots, x_{d'_j}; M_j) + \ell(\overline{U}_M^{0,d'_j+1}).$$

(ii) Let $d < r$. We divide into three cases.

• Assume that $d < i < r$. By Theorem 1.4, $e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1})$. Note that if $i = d_j \in \Delta_R$ then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = 0$ for all $t \notin \{0, d_j\}$. Moreover, if $i \notin \Delta_R$ then $e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) = 0$ for all $t \neq 0$. Therefore,

$$e_{r-i}(Q, R \ltimes M) = \begin{cases} \ell(\overline{U}_R^{0,d_j+1}) + e(x_1, \ldots, x_{d_j}; R_j) & \text{if } i = d_j \in \Delta_R, \\ \ell(\overline{U}_R^{0,i+1}) & \text{if } i \notin \Delta_R. \end{cases}$$

• Assume that $0 \leq i < d$. Then by Theorem 1.4, we have

$$e_{r-i}(Q, R \ltimes M) = \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_R^{t,i+1}) + \sum_{t=0}^{i} e(x_1, \ldots, x_t; \overline{U}_M^{t,i+1}),$$

and the result follows by the same arguments as in the proof of (i).

• Assume that $i = d$. Then by Theorem 1.4, we have

$$e_{r-d}(Q, R \ltimes M) = \sum_{t=0}^{d} e(x_1, \ldots, x_t; \overline{U}_R^{t,d+1}) + e_0(I, M).$$

We note that if $d \notin \Delta_R$ then $e(x_1, \ldots, x_t; \overline{U}_R^{t,d+1}) = 0$ for all $t \neq 0$. Moreover, if $d \in \Delta_R$ then $e(x_1, \ldots, x_t; \overline{U}_R^{t,d+1}) = 0$ for all $t \notin \{0, d\}$. Therefore, the result follows. $\square$

Remark 4.5. Suppose that R, M are sequentially Cohen-Macaulay. Then $\overline{U}_M^{0,1} = H^0_{\mathfrak{m}}(M)$, $\overline{U}_R^{0,1} = H^0_{\mathfrak{m}}(R)$ and $\overline{U}_M^{0,i} = 0$, $\overline{U}_R^{0,i} = 0$ for all $i \geq 2$. Now, applying Corollary 4.4, we obtain a much better formula for Hilbert coefficients in this case.

We end this paper with an example of computing Hilbert coefficients of $R \ltimes M$ in case where R, M are sequentially generalized Cohen-Macaulay.

Example 4.6. Let $S = k[[x_1, x_2, x_3, x_4, x_5]]$ be the formal power series ring over a field k, let $\mathfrak{a} = (x_1, x_2) \cap (x_3, x_4, x_5)$ and $\mathfrak{b} = (x_1, x_2, x_3) \cap (x_3, x_4, x_5)$. Let $R = S/\mathfrak{a}$, $M = S/\mathfrak{b}$. Then $\dim(R) = 3$ and the filtration of R is $(0) = R_0 \subsetneq (x_1, x_2)R = R_1 \subsetneq R_2 = R$; $\dim_R(M) = 2$ and the filtration of M is $(0) = M_0 \subsetneq M_1 = M$. Denote by K_R^i is the i-th defficiency of R. Since $K_R^0 = 0$, K_R^1 is of length 1 and K_R^2 is Cohen-Macaulay of dimension 2, it follows by [11] that R is sequentially generalized Cohen-Macaulay, not sequentially Cohen-Macaulay. It is clear that M is generalized Cohen-Macaulay, not Cohen-Macaulay. Note that $U_R^{0,1} = 0$ and $U_M^{0,1} = 0$. We have $\Delta_R = \{2, 3\}$ and $\Delta_M = \{2\}$. We choose a_1, a_2, a_3 are respectively the image of $x_1 + x_4, (x_2 + x_5)^2, x_3$ in R. Then $a_3 \in \mathrm{Ann}_R(M)$ and

$$\ell(R/(a_1^{n_1}, a_2^{n_2}, a_3^{n_3})R) = 2n_1n_2n_3 + 2n_1n_2 + 1,$$
$$\ell(M/(a_1^{n_1}, a_2^{n_2})M) = 4n_1n_2 + 1,$$

for all $n_1, n_2, n_3 \geq 1$. Hence a_1, a_2, a_3 (resp. a_1, a_2) is an almost p-standard s.o.p of R (resp. M). Moreover $\ell(U_R^{0,3}) = \ell(\overline{U}_R^{0,3}) + \ell(\overline{U}_R^{0,2}) = 1$ and $\ell(U_M^{0,2}) = \ell(\overline{U}_M^{0,2}) = 1$, since $\overline{U}_M^{0,1} = 0$ and

$\overline{U}_R^{0,1} = 0$. Put $J = (a_1, a_2, a_3)$ and $I = (a_1, a_2)$. Then

$$\ell(R/J^{n+1}) = 2\binom{n+3}{3} + 2\binom{n+2}{2} + \binom{n+1}{1},$$
$$\ell(M/I^{n+1}M) = 4\binom{n+2}{2} + \binom{n+1}{1},$$

for all $n \geq 0$. Since a_1, a_2, a_3 is an almost p-standard s.o.p of R and $U_R^{2,3} = R_1$, we get

$$e_1(J, R) = \ell(\overline{U}_R^{0,3}) + e(a_1; \overline{U}_R^{1,3}) + e(a_1, a_2; R_1) = 2,$$
$$e_2(J, R) = \ell(\overline{U}_R^{0,2}) + e(a_1; \overline{U}_R^{1,2}) = 1.$$

Thus $\ell(\overline{U}_R^{0,3}) = e(a_1; \overline{U}_R^{1,3}) = 0$ and so $\ell(\overline{U}_R^{0,2}) = 1$. We set $Q = (u_1, u_2, u_3)$, where $u_i = (x_i, 0)$ for $i = 1, 2, 3$. By applying Corollary 4.4, we get $e_0(Q, R \ltimes M) = e_0(J, R) = 2$. Since $2 = \dim_R(M) \in \Delta_R \cap \Delta_M$,

$$e_1(Q, R \ltimes M) = \ell(\overline{U}_R^{0,2+1}) + e(a_1, a_2; R_1) + e_0(I, M) = 6.$$

Since $1 \notin \Delta_R \cup \Delta_M$, we have $e_2(Q, R \ltimes M) = \ell(\overline{U}_R^{0,1+1}) + \ell(\overline{U}_M^{0,1+1}) = 2$. Since $0 \notin \Delta_R \cup \Delta_M$, we get $e_3(Q, R \ltimes M) = \ell(\overline{U}_R^{0,0+1}) + \ell(\overline{U}_M^{0,0+1}) = 0$.

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