# TSUJI'S ITERATION ON PSEUDOCONVEX DOMAINS

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Abstract. We give an alternative proof of Tsuji's theorem on the construction of Kähler-Einsteins on strongly pseudoconvex domain, adapting the method of Berndtsson in the compact case.

## 1. Introduction

Let $\Omega$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary. In [CY82], Cheng-Yau proved the existence of a complete Kähler-Einstein metric $\omega_{KE}$ on $\Omega$ (see also [MY83] for its generalization on pseudoconvex domains). In [Tsu13], Tsuji showed that this metric is the limit of a sequence of Bergman metrics. This is the non-compact version of his earlier result on compact Kähler manifolds [Tsu10] (see also Song-Weinkove [SW10] and Berntdsson [Ber09b] for other proofs of this result with uniform convergence).

The aim of this note is to give an alternative proof of the result in [Tsu13] by adapting and refining the method of Berndtsson [Ber09b] in the case of compact Kähler manifolds. The main idea is to use a uniform asymptotic of Bergman kernel.

When $\Omega$ is a bounded domain in $\mathbb{C}^n$, then $K_\Omega$ is a trivial bundle, so we work on the space of holomorphic functions instead of holomorphic $(n,0)$-forms. We recall the definition of weighted Bergman kernels which will be used later. Let $\phi$ be a continuous plurisubharmonic function, then we can define a $L^2$ norm with respect to the weight $\phi$

$$\|u\|^2_{\phi,\mu} = \int_\Omega |u|^2 e^{-\phi}\mu$$

where $\mu$ is a positive measure. Define

$$L^2(\Omega,\phi,\mu) = \{u \,|\, \|u\|_\phi < +\infty\},$$

and

$$H_{\phi,\mu} = \{u \in \mathcal{O}(\Omega) |\, \|u\|_{\phi,\mu} < +\infty\} = L^2(\Omega,\phi,\mu) \cap \mathcal{O}(\Omega).$$

The projection from $L^2(\Omega,\phi,\mu)$ to $H_{\phi,\mu}$ is called the Bergman projection. Its kernel $\mathcal{K}(z,w)$ is given by

$$\mathcal{K}_{\phi,\mu}(z,w) = \sum u_j(z)\overline{u_j(w)},$$

where $\{u_j\}$ is any orthonormal basis for $H_{\phi,\mu}$. Let $\mathcal{K}_{\phi,\mu}$ be its associated Bergman kernel, defined by

$$\mathcal{K}_{\phi,\mu}(z) := \mathcal{K}_{\phi,\mu}(z,z) = \sum_j |u_j|^2(z).$$

*Date*: December 21, 2022.

We also have the extremal characterization

$$\mathcal{K}_{\phi,\mu}(z) = \sup_{u\in H_{\phi,\mu}} \frac{|u(z)|^2}{\|u\|^2_{\phi,\mu}}.$$

Consider the complex Monge-Ampère equation on $\Omega$:

$$(dd^c\phi)^n = e^{\phi}\nu, \tag{1.1}$$

where $\nu$ is a smooth volume form. With a special choice of $\Omega$, the solution $\phi$ of (1.1) gives us a complete Kähler-Einstein metric on $\Omega$ (cf. [CY82]). The corresponding Tsuji's iteration [Tsu10, Tsu13] can be defined as the following (see also [Ber09b]). Denote by $\mathcal{K}_{k\phi}$ the weighted Bergman kernel with respect to the weight $k\phi$ and $\mu = e^{\phi}\nu$, and define

$$\beta_k(\phi) = \frac{1}{k}(\log \mathcal{K}_{k\phi} - \log d_k),$$

where

$$d_k = (k/2\pi)^n.$$

Starting from any plurisubharmonic function $\phi_1$ on $\Omega$, we define the sequence of function $\phi_k$ by $\phi_{k+1} = \beta_k(\phi_k)$.

**Remark 1.1.** In [Ber09b], for a compact Kähler manifold $X$, the number $d_k$ was the dimension of $H^0(X, kK_X)$.

Then our main result is the following.

**Theorem 1.2.** *Let $\phi_\infty$ be a solution of (1.1) such that $(\Omega, dd^c\phi_\infty)$ has bounded geometry. Let $\phi_1$ is any continuous plurisubharmonic function on $\Omega$ such that $\sup_\Omega |\phi_1 - \phi_\infty| < \infty$. Let $\{\phi_k\}_{k\in\mathbb{N}^*}$ be the sequence defined by Tsuji's iteration above, then*

$$\sup_\Omega |\phi_k - \phi_\infty|$$

*uniformly converges to zero at the rate $1/k$.*

The main idea of the proof is to use a uniform asymptotic of Bergman kernel for Kähler manifolds of bounded geometry as in the compact case due to Berdtsson [Ber09b].

**Theorem 1.3.** [MM07, Problem 6.1] *Let $\phi$ is a smooth strictly psh function on $\Omega$ such that $(\Omega, \omega_\phi)$ has bounded geometry, then*

$$\mathcal{K}_{k\phi,\omega_\phi^n} e^{-k\phi} = \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{b_1}{2k} + O\left(\frac{1}{k^2}\right)\right), \tag{1.2}$$

*where $b_1(z,z) = S_{\omega_\phi}$, $S_{\omega_\phi}$ is the scalar curvature of $\omega_\phi$, and $O\left(\frac{1}{k^2}\right)$ denotes a quantity dominated by $C/k^{-2}$ on $\Omega$ with a uniform constant $C$ depending on the bounded geometry of $(\Omega, \omega_\phi)$ (see Definition 2.1).*

This result can be seen as a *uniform version* of the asymptotic of weighted Bergman kernels due to Engliš [Eng00]. In this note, we give a simple proof for this result. We follow some ideas from the asymptotic of Bergman kernels on compact Kähler manifolds in [Lu00, Cha15].

As in [Tsu10, Tsu13], an interesting corollary of Theorem 1.2 is the subharmonicity properties on the variation of Kähler-Einstein metrics on pseudoconvex domains (see Theorem 3.4).

## 2. Geometry of strongly pseudoconvex domains

We first recall the definition of bounded geometry for Kähler manifolds.

**Definition 2.1.** Suppose $(M,\omega)$ is a complete Kähler manifold. We say that $(M,\omega)$ has bounded geometry if there exists $r>0, c>0, A_k, k=1,2,\dots$ such that for any $p\in M$ there is a domain $U$ in $\mathbb{C}^n$ and a holomorphic maps $\phi: B_{\mathbb{C}^n}(0,r)\to M$ which is biholomorphic onto its image, satisfying

(1) $B_{\mathbb{C}^n}(0,r)\subset U$ and $\psi(0)=p$;
(2) On $U$ we have
$$c^{-1}\omega_{\mathbb{C}^n}\le \phi^*\omega\le c\omega_{\mathbb{C}^n};$$
(3) for any $k$ and any multi-indices $\alpha,\beta$ with $|\alpha|+|\beta|\le k$
$$\sup_{x\in B_{\mathbb{C}^n}(0,r)}\left|\frac{\partial^{|\alpha|+|\beta|}g_{i\bar{j}}(x)}{\partial z^\alpha\partial\bar{z}^\beta}\right|\le A_k,$$

where $g_{i\bar{j}}$ is the component of $\psi^*\omega$ on $U$ in terms of natural coordinates $(z^1,\dots z^n)$.

As consequence, we have the following normal coordinates for Kähler manifold of bounded geometry.

**Lemma 2.2.** *Suppose $(M,\omega)$ is a Kähler manifold of bounded geometry. Then there exists $\epsilon>0, c>0, A_k>0, k=1,2,\dots$ such that for any point $p\in M$ we can choose a holomorphic chart $\psi: V=B_{\mathbb{C}^n}(0,\epsilon)\to M$ with $\psi(0)=p$ satisfies condition (2), (3) in Definition 2.1 and $\omega=dd^c\varphi$ with $\varphi_{i\bar{j}}(0)=\delta_{ij}$ and $\varphi_{i\bar{j}k}(0)=\varphi_{i\bar{j}\bar{k}}(0)=\varphi_{i\bar{j}\bar{k}\bar{l}}(0)=\varphi_{ijk\bar{l}}(0)=0$.*

*Proof.* For any $p\in M$ we choose the local chart $(U=B(0,r),\psi)$ at $p$ as in Definition 2.1. By complex linear transformation we can assume that $\psi^*\omega=dd^c\phi$ on $U$ with $\phi_{i\bar{j}}(0)=0$. Consider a biholomorphic map $\tilde{\psi}: B(0,\epsilon(p))\to U$ with $z^i=\tilde{\psi}(z)^i=w^iA^i_{k\ell}w^kw^\ell+B^i_{mnp}w^mw^nw^p$ where $A^i_{k\ell}, B^i_{mnp}$ will be chosen hereafter.

On $B(0,\epsilon)$, we have the pull-back metric $\tilde{\psi}^*(\psi^*\omega)=dd^c\varphi$ with
$$\varphi_{a\bar{b}}=\phi_{a\bar{b}}+2A^i_{ak}\phi_{i\bar{b}}w^k+2\overline{A^j_{b\ell}\phi_{a\bar{j}}}\overline{w}^\ell+3B^i_{amn}w^mw^n\phi_{i\bar{b}}+3\overline{B^j_{bpq}}\phi_{a\bar{j}}\overline{w}^p\overline{w}^q+4A^i_{ak}\overline{A^j_{b\ell}}\phi_{i\bar{j}}w_k\overline{w}_\ell+O(|w|^3)$$

Therefore we have
$$\varphi_{a\bar{b}c}(0)=\phi_{a\bar{b}c}(0)+2A^b_{ac};\phi_{a\bar{b}\bar{c}}=\phi_{a\bar{b}\bar{c}}+2\overline{A^b_{ac}};\varphi_{a\bar{b}cd}=\phi_{a\bar{b}cd}+6B^b_{acd}.$$

By choosing $A^b_{ac}=-\frac{1}{2}\phi_{a\bar{b}}(0)$ and $B^b_{acd}=-\phi_{a\bar{b}cd}(0)$ we get the local canonical coordinates for any $p\in M$. In particular, $A^a_{bc}$ and $B^a_{bcd}$ are uniformly bounded for any $p\in M$ by the definition of bounded geometry. Since $\epsilon(p)$ depends only on $A^a_{bc}$ and $B^a_{bcd}$ we can choose a uniform $\epsilon>0$ as required. ∎

We recall some properties of pseudoconvex domains (cf. [CY82]). Let $\Omega$ be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let $\rho$ be a smooth defining function for $\Omega$. We define $\varphi_\Omega = -(n+1)\log(-\rho)$, which is a strictly plurisubharmonic function on $\Omega$ and $\varphi_\Omega(z) \to +\infty$ as $z \to \partial\Omega$. It follows from [CY82] that $(\Omega, \omega_{\varphi_\Omega})$ is a complete Kähler manifold, where $\omega_{\varphi_\Omega} = dd^c\varphi_\Omega$. Since

$$\det((\varphi_\Omega)_{\bar{j}i}) = (n+1)^n \left(-\frac{1}{\rho}\right)^{n+1} \det(\rho_{\bar{j}i})(-\rho + |d\rho|^2),$$

we have the Ricci curvature

$$Ric(\omega_{\varphi_\Omega}) = -(\varphi_\Omega)_{\bar{j}i} + \partial_i\partial_{\bar{j}}F,$$

where $F = -\log[\det(\rho_{\bar{j}i})(-\rho + |d\rho|^2)]$ is a smooth function on $\bar{\Omega}$.

As explained in [CY82] that $(\Omega, \omega_{\varphi_\Omega})$ has bounded geometry. The following theorem is due to Cheng-Yau [CY82].

**Theorem 2.3.** *There exists a unique complete Kähler-Einstein metric $\omega_{KE} = dd^c\phi_{KE}$ with*

$$Ric(\omega_{KE}) = -\omega_{KE},$$

*where $\phi_{KE}$ is smooth plurisubharmonic function satisfying*

$$(dd^c\phi_{KE})^n = e^{\phi_{KE} - \phi_\Omega + F}\omega^n_{\varphi_\Omega}, \tag{2.1}$$

*and $|\nabla^k_{\varphi_\Omega}(\phi_{KE} - \varphi_\Omega)| \leq C_k$ for any $k \in N$. Moreover $(\Omega, \omega_{KE})$ has bounded geometry.*

Then the following is straightforward from Theorem 1.2.

**Theorem 2.4.** *Let $\rho$ and $\varphi_\Omega = -(n+1)\log(-\rho)$ as above. Let $\phi_1$ is any continuous plurisubharmonic function on $\Omega$ such that $\sup_\Omega |\phi_1 - \varphi_\Omega| < \infty$. The the sequence $\{\phi_k\}_{k\in\mathbb{N}^*}$ defined by Tsuji's satisfies*

$$\sup_\Omega |\phi_k - \phi_{KE}|$$

*uniformly converges to zero at the rate $1/k$.*

## 3. Asymptotic of Bergman kernel and the main theorem

3.1. **A uniform asymptotic of Bergman kernel.** We first prove a uniform asymptotic of Bergman kernel on pseudoconvex domain which will be used in the proof of the main theorem. We refer to [Eng00] and references therein for the *local* version of this result.

**Theorem 3.1.** *Let $\phi$ is a smooth strictly psh function on $\Omega$ such that $(\Omega, \omega_\phi)$ has bounded geometry, then*

$$\mathcal{K}_{k\phi,\omega^n_\phi}e^{-k\phi} = \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{b_1}{2k} + O\left(\frac{1}{k^2}\right)\right), \tag{3.1}$$

*where $b_1(z) = S_{\omega_\phi}$, $S_{\omega_\phi}$ is the scalar curvature of $\omega_\phi$, and $O\left(\frac{1}{k^2}\right)$ denotes a quantity dominated by $C/k^2$ on $\Omega$ with a uniform constants $C$ depending on the bounded geometry of $(\Omega, \omega_\phi)$.*

We need the following Laplace approximation.

**Lemma 3.2.** [Hor90, Thm 7.7.5] *Let $K \subset \mathbb{R}^n$ be a compact set and $U$ an open neighborhood of $K$. If $u \in C^{2k}(K)$ and $f \in C^{3k+1}(U)$ then we have*

$$\left| \int u(x) e^{\lambda f(x)} dx - \frac{e^{\lambda f(x_0)}}{|-H_f(x_0)|^{-1/2}} \left(\frac{2\pi}{\lambda}\right)^{n/2} \sum_{j<k} \lambda^{-j} L_j u \right| \leq C\lambda^k \sum_{|\alpha|\leq 2k} \sup |D^\alpha u|$$

*for a constant $C$ depending on $\|f\|_{C^{3k+1}(U)}$, where*

$$L_j u = \sum_{s-r=j} \sum_{2s\geq 3r} 2^{-s} \left\langle -H_f(x_0)^{-1} D, D \right\rangle^s (g_{x_0}^r u)(x_0)/r!s!,$$

*with*

$$g_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2} \left\langle H_f(x_0)(x-x_0), x-x_0 \right\rangle.$$

*In particular, $L_0 u = u(x_0)$ and*

$$L_1 u = \frac{1}{2}[u\{-f_{ik\ell} f_{jrs}(\frac{1}{4} f^{ij} f^{k\ell} f^{rs} + \frac{1}{6} f^{ij} f^{ks} f^{r\ell}) + \frac{1}{4} f^{ij} f^{kl} f_{ijkl}\} \quad (3.2)$$

$$+ f^{sq} f^{rp} f_{srq} u_p - \mathrm{Tr}(H_u H_f^{-1})]_{x=x_0}. \quad (3.3)$$

*Proof of Theorem 3.1.* The proof uses the method of peak sections due to [Tia90] (see also [Lu00, SW10]) and the approach using the Laplace approximation (cf. [Hor90]) by [Cha15] for compact Kähler manifolds.

For any $p \in \Omega$, we take $(U, z = (z^1, \ldots, z^n))$ the local canonical coordinates in Lemma 2.2 centered at $p$. Choose a holomorphic function $g$ such that $\tilde{\phi} = \phi - \log |g|^2$ has minimum at $p = (0, \ldots, 0)$ and

$$\tilde{\phi}(z) = \sum_{j=1}^n |z^j|^2 + \phi_{i\bar{j}k\bar{\ell}} z^i z^{\bar{j}} z^k z^{\bar{\ell}} + O(|z|^5) \quad (3.4)$$

Here for example, we can choose $a_i, b_{jk}, c_{lmn}, d_{pqrs}$ for $g(z) = \phi(0) + a_i z^i + b_{jk} z^j z^k + c_{lmn} z^m z^l z^n + d_{pqrs} z^p z^q z^r z^s$ to have (3.4).

Let $\eta : [0, \infty)$ be a cut-off function satisfying $\eta(t) = 1$ for $t \leq 1/2$, $\eta(t) = 0$ for $\geq 1$, $|\eta'(t)| \leq 4$ and $|\eta''(t)| \leq 8$. Define a $(1,0)$ form

$$\alpha = \bar{\partial} \left[ \eta \left( \frac{k|z|^2}{(\log k)^2} \right) \right] g^k,$$

which vanishes outside $A_k := \{z | (\log k)^2/(2k) \leq |z| \leq (\log k)^2/k\} \subset U$. It follows from the definition of bounded geometry that the weight $\psi := k\phi + \log(\omega_\phi^n/dV)$ satisfies

$$dd^c \psi = k\omega_\phi - Ric(\omega_\phi) \geq \frac{k}{C} \omega_\phi,$$

for a constant $C > 0$.

Applying Hömander's $L^2$ estimate (cf. [Hor73]) with the weight $\psi$: there exists a smooth function $u$ on $\Omega$ such that $\bar\partial u = \alpha, u(0) = 0$ and

$$\begin{aligned}\int_\Omega |u|^2 e^{-k\phi}\omega_\phi^n &\leq \frac{C}{k}\int_\Omega |\alpha|^2_{\omega_\phi} e^{-k\phi}\omega_\phi^n &(3.5)\\ &\leq C\frac{1}{(\log k)^2}\int_{A_k}|g|^{2k}e^{-k\phi}\omega_\phi^n &(3.6)\\ &= C\frac{1}{(\log k)^2}\int_{A_k} e^{-k\tilde\phi}\omega_\phi^n, &(3.7)\end{aligned}$$

here we used $|\alpha|^2_{\omega_\phi} = |\eta'\left(\frac{k|z|^2}{(\log k)^2}\right)|^2\frac{k^2}{(\log k)^4}\phi^{i\bar j}z^i\bar z^j|g|^{2k} \leq C\frac{k|g|^{2k}}{(\log k)^2}$ on $A_k$, otherwise $\alpha = 0$. Since on $A_k$, $e^{-k\tilde\phi} = (1-\sum_{j=1}^n |z^j|^2)^k + O(|z|^3)$, hence

$$e^{-k\tilde\phi}\omega_\phi^n \leq (1-\frac{1}{2}|z|^2)^k dV.$$

We infer that

$$\int_\Omega |u|^2 e^{-k\phi}\omega_\phi^n \leq C(\log k)^{2n-2}k^{-\log k/2-n}. \tag{3.8}$$

It follows that the holomorphic function $f := \eta g^k - u$ satisfies $|f(0)|^2 = e^{-k\phi(0)}$ and

$$\|f\|^2_{k\phi} = \int_\Omega |f|^2 e^{-k\phi}\omega_\phi^n = \int_{U_k}|g|^{2k}e^{-k\phi}\omega_\phi^n + O(k^{-n-2}), \tag{3.9}$$

where $U_k = \{z : |z|^2 \leq (\log k)^2/k\}$. On $U_k$ we have $(\omega_\phi)^n = (1+\phi_{k\bar j j\bar l}z^k\bar z^l + o(|z|^3))dV$. Applying the Laplace approximation (Lemma 3.2), we have

$$\begin{aligned}\int_{U_k}|g|^{2k}e^{-k\phi}\omega_\phi^n &= \int_{U_k} e^{-k\tilde\phi}(1+\phi_{k\bar j j\bar l}z^k\bar z^l + o(|z|^3))dV\\ &= \left(\frac{2\pi}{k}\right)^n\left(1+\frac{1}{2k}\phi_{i\bar i j\bar j} + O(k^{-2})\right)\\ &= \left(\frac{2\pi}{k}\right)^n\left(1-\frac{1}{2k}S_{\omega_\phi}(0) + O(k^{-2})\right),\end{aligned}$$

where $O\left(\frac{1}{k^2}\right)$ dominated by $C/k^{-2}$ on $\Omega$ with a uniform constant $C$ depending only on the bound of $|D^\alpha\phi|$ on $U$ with $|\alpha| \leq 7$. Therefore

$$\mathcal{K}_{k\phi,\omega_\phi^n}(0) \geq \frac{|f(0)|^2}{\|f\|^2_{k\phi}} \geq \left(\frac{k}{2\pi}\right)^n e^{-k\phi(0)}\left(1+\frac{1}{2k}S_{\omega_\phi}(0)+O(k^{-2})\right), \tag{3.10}$$

hence we get a lower bound for $\mathcal{K}_{k\phi}$.

We now obtain an upper bound for $\mathcal{K}_{k\phi}$ following the strategy in [Cha15]. Let $f$ be any holomorphic function on $\Omega$. Since $|f/g^k|$ is subharmonic function on $U$, we have

$$|f(0)/g^k(0)|^2 \leq \frac{\int_{\Delta_R}|f/g^k|^2 e^{-k\tilde\phi_0}\rho dV}{\int_{\Delta_R} e^{-k\tilde\phi_0}\rho dV}, \tag{3.11}$$

where $\tilde{\phi}_0 = \sum |z^j|^2 + \phi_{i\bar{j}k\bar{\ell}} z^i z^{\bar{j}} z^k z^{\bar{\ell}}$, $\rho = 1 + \phi_{k\bar{j}j\bar{l}} z^k \bar{z}^l$ and $\Delta_R \subset U$ is a polydisc with radius $R$. Choosing $R = k^{-2/3}$, we have

$$-k\tilde{\phi}_0 \leq -k\tilde{\phi} + Ck^{-7/3}$$

and

$$\rho dV \leq (1 + Ck^{-2})\omega_\phi^n$$

hence

$$\begin{aligned}\int_{\Delta_R} |f/g^k|^2 e^{-k\tilde{\phi}_0} \rho dV \quad &\leq \quad (1 + Ck^{-2}) \int_{\Delta_R} |f/g^k|^2 e^{-k\tilde{\phi}} \omega_\phi^n \quad &(3.12)\\ &= \quad (1 + Ck^{-2}) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^n. \quad &(3.13)\end{aligned}$$

Using the Laplace approximation again we have

$$\int_{\Delta_R} e^{-k\tilde{\phi}_0} \rho dV = \left(\frac{2\pi}{k}\right)^n \left(1 - \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right). \quad (3.14)$$

Therefore we get

$$\begin{aligned}|f(0)/g^k(0)|^2 \quad &\leq \quad \frac{\int_{\Delta_R} |f/g^k|^2 e^{-k\tilde{\phi}_0} \rho dV}{\int_{\Delta_R} e^{-k\tilde{\phi}_0} \rho dV} \quad (3.15)\\ &= \quad \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^n.\end{aligned}$$

Since $|g(0)|^2 = e^{-\phi(0)}$, we imply

$$|f(0)|^2 \leq e^{k\phi(0)} \left(\frac{k}{2\pi}\right)^n \left(1 + \frac{1}{2k} S_{\omega_\phi}(0) + O(k^{-2})\right) \int_{\Delta_R} |f|^2 e^{-k\phi} \omega_\phi^n,$$

hence we get the upper bound for $\mathcal{K}_{k\phi}$. □

3.2. **Proof of the main Theorem.** We have the following lemma which is a local version of [Ber09b, Lemma 3].

**Lemma 3.3.** *Let $\phi_\infty$ be the solution of $(dd^c\phi_\infty)^n = e^{\phi_\infty}\nu$ such that $(\Omega, \omega_{\phi_\infty})$ has bounded geometry. Suppose that $C_1$, $C_2$ are two real numbers satisfying $C_1 \leq \phi - \phi_\infty \leq C_2$. Then we have*

$$\beta_k(\phi) \geq \phi_\infty + \frac{k-1}{k} C_1 - \varepsilon_k \quad (3.16)$$

*and*

$$\beta_k(\phi) \leq \phi_\infty + \frac{k-1}{k} C_2 + \varepsilon_k, \quad (3.17)$$

*where $\varepsilon_k$ depends on $\phi_\infty, k$ and tends to zero at rate $1/k^2$.*

In [Ber09b, Lemma 3], the author used the asymptotic of Bergman kernel for the Kähler-Einstein metric with the coefficient of $1/k$ is the scalar curvature of the metric. We give here a refined proof of this Lemma using only zero order asymptotic

$$\mathcal{K}_{k\phi,\omega_\phi^n} e^{-k\phi} = \left(\frac{k}{2\pi}\right)^n \left(1 + O\left(\frac{1}{k}\right)\right).$$

*Proof.* Since $(dd^c\phi_\infty)^n = e^{\phi_\infty}\nu$, using the first order asymptotic of Bergman kernels for the weight $\phi_\infty$ ( cf. Theorem 3.1), we have

$$\mathcal{K}_{k\phi_\infty, e^{\phi_\infty}\nu} e^{-k\phi_\infty} = \mathcal{K}_{k\phi_\infty, \omega^n_{\phi_\infty}} e^{-k\phi_\infty} \tag{3.18}$$

$$= (k/2\pi)^n \left(1 + O\left(\frac{1}{k}\right)\right). \tag{3.19}$$

By definition $d_k = (k/2\pi)^n$ then

$$\mathcal{K}_{k\phi_\infty} e^{-k\phi_\infty}/d_k = 1 + O\left(\frac{1}{k}\right). \tag{3.20}$$

This implies

$$\beta_k(\phi_\infty) = \frac{1}{k}\log(\mathcal{K}_{k\phi_\infty}/d_k) = \phi_\infty + \varepsilon_k \tag{3.21}$$

where $\varepsilon_k$ depends on $\phi_\infty, k$ and tends to zero at rate $1/k^2$.

Next, it follows from the fact that $\phi \geq C_1 + \phi_\infty$ for some constant $C_1 > 0$, the extremal characterization of Bergman kernel implies that

$$\mathcal{K}_{k\phi} \geq e^{(k-1)C_1}\mathcal{K}_{k\phi_\infty}. \tag{3.22}$$

Therefore, combining with (3.21) we have

$$\beta_k(\phi) \geq \frac{k-1}{k}C_1 + \phi_\infty - \varepsilon_k$$

Similarly, we have

$$\beta_k(\phi) \leq \frac{k-1}{k}C_2 + \phi_\infty + \varepsilon_k$$

as required. □

*Proof of Theorem 1.2.* We define by recurrence $\phi_{k+1} = \beta_k(\phi_k)$. For any $k \geq 1$, denote $C_k$ the best constant in the inequality

$$\phi_k \geq C_k + \phi_\infty. \tag{3.23}$$

Lemma 3.3 thus implies that

$$C_{k+1} \geq \frac{k-1}{k}C_k - \varepsilon_k.$$

Since $\varepsilon_k$ is of order $1/k^2$, this implies that $kC_{k+1} = O(1)$, hence $C_k = O(1/k)$. By the same way, the last part of Lemma 3.3 gives a similar estimate that $\tilde{C}_k = O(1/k)$ where $\tilde{C}_k$ is the best constant such that

$$\phi_k \leq \phi_\infty + \tilde{C}_k$$

So we get the desired convergence. □

3.3. **Variation of Kähler–Einstein metrics.** Let $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ be the second projection and $\Omega$ be a smooth domain in $\mathbb{C}^{n+m}$ such that $\Omega_t := \Omega \cap \pi^{-1}(t)$ is a bounded pseudoconvex domain with smooth boundary. It follows from [CY82, MY83] that on each slice $\Omega_t$ there exists a unique complete Kähler-Einstein metric $\phi(z,t) := \phi_t(z)$, i.e

$$Ric(\omega_{\phi_t}) = -\omega_{\phi_t},$$

where $\omega_{\phi_t} = i\partial_{z^k}\partial_{\bar{z}^l}\phi_t(z)dz^k \wedge \bar{z}^l$.

**Theorem 3.4.** [Cho15a, Cho15b, Tsu13] *If $\Omega \subset \mathbb{C}^{n+m}$ is a bounded (strongly) pseudoconvex domain, the function $\phi(z,t)$ constructed above is a (strictly) plurisubharmonic function on $\Omega$.*

The result was proved in [Cho15a, Cho15b] using the boundary behavior of the geodesic curvature which satisfies a certain elliptic equation, and in [Tsu13] using Tsuji's construction of Kähler-Einstein metrics. For the reader's convenience we sketch the Tsuji's approach here.

*Proof.* Suppose first $\Omega$ is a bounded strongly pseudoconvex domain. Fix $\phi_1$ a continuous plurisubharmonic function on $\Omega$ such that $\sup_\Omega |\phi_1 - \varphi_\Omega| < \infty$. Denote $\phi_{1,t}$ the restriction of $\phi_1$ on $\Omega_t$, i.e $\phi_{1,t}(z) := \phi_1(z,t)$. Let $\phi_{k,t}$ be the weights constructed by Tsuji's iteration above starting from $\phi_{1,t}$ on $\Omega_t$. It follows from [Ber06] and the induction on $k$ that $(z,t) \mapsto \log K_{k\phi_{k,t}}(z)$ is psh on $\Omega$, hence $(z,t) \mapsto \phi_{k,t}(z)$ is psh on $\Omega$. Letting $k \to \infty$ and using Theorem 1.2, we imply that $\phi(z,t) = \lim_{k\to\infty} \phi_{k,t}(z)$ is strictly psh on $\Omega$.

In general when $\Omega$ is bounded pseudoconvex, the existence of complete Kähler-Einstein metric was constructed in [CY82, MY83] as the limit of Kähler-Einstein metrics on relatively compact subdomains. By this standard approximation process we also imply that $\phi(z,t)$ is psh on $\Omega$. □

**Acknowledgment.** Part of this work was done while the author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for financial support and hospitality.

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