ENTIRE HOLOMORPHIC CURVES INTO $\mathbb{P}^n(\mathbb{C})$ INTERSECTING $n+1$ GENERIC HYPERSURFACES

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ABSTRACT. Let $\{D_i\}_{i=1}^{n+1}$ be $n+1$ smooth hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, not all being hyperplanes, satisfying one precise geometric (generic) condition. Then, for every algebraically nondegenerate entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we show a weak Second Main Theorem:

$$\sum_{i=1}^{n+1} \delta_f(D_i) < n+1$$

in terms of defect inequality in Nevanlinna theory.

1. Introduction

Given a codimension one subvariety D in a complex manifold X such that the complement $X \setminus D$ satisfies certain complex hyperbolicity quality in spirit of the Kobayashi conjecture [Kob70] or the Green-Griffiths conjecture [GG80], one seeks to reach a quantivative interpretation in terms of Second Main Theorem in Nevanlinna theory, which relates, in certain proportional way, the "growth rate" of any algebraically nondegenerate holomorphic map $f : S \to X$ from certain source space S, usually being $\mathbb{C}$, to, the "intersection frequency" or "impact" of $f(S)$ with respect to D.

The classical result being Nevanlinna's celebrated work [Nev25] which quantifies Picard's little theorem about the hyperbolicity of $\mathbb{P}^1(\mathbb{C}) \setminus \{3 \text{ points}\}$. For higher dimensional target space X, for various source space S and (certain) holomorphic maps $f\colon S \to X$, we refer the readers to [Nev70, Sto77, Fuj93, NW14, Ru21] for developments.

In this paper, we study the case that $D = \cup_{i=1}^q D_i$ consists of $q = n+1$ smooth hypersurfaces $D_i \subset \mathbb{P}^n(\mathbb{C})$ of degrees $d_i \geqslant 1$ in general position. The algebraic degeneracy of entire holomorphic curves into the complement $\mathbb{P}^n(\mathbb{C}) \setminus D$ was established by Noguchi-Winkelmann-Yamanoi [NWY07].

Main Theorem. *Let $\{D_i\}_{i=1}^{n+1}$ be $n+1$ smooth hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, not all being hyperplanes, satisfying one precise geometric (generic) condition. Then, for every algebraically nondegenerate entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, the following defect inequality holds*

$$\sum_{i=1}^{n+1} \delta_f(D_i) < n+1. \tag{1.1}$$

Here we recall some standard notions in Nevanlinna theory. The order function

$$T_f(r) := \int_1^r \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\omega_{FS} \qquad (r>1),$$

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is a geometric equivalent version of Nevanlinna's characteristic function, historically discovered independently by Shimizu and Ahlfors [NW14, p. 11–12], measuring the area growth of the image of the disc $\mathbb{D}_r$ centered at 0 having radius r, with respect to the Fubini–Study metric ω_{FS}; and

$$N_f^{[k]}(r, D_i) \; := \int_1^r \sum_{|z|<t} \min\{k, \mathrm{ord}_z\, f^* D_i\} \frac{\mathrm{d}\, t}{t} \qquad (i = 1, ..., q;\ r > 1)$$

are the level–k truncated counting functions ($k \in \mathbb{N} \cup \{\infty\}$), which capture the intersection frequencies of $f(\mathbb{C}) \cap D_i$. The defect of f with respect to D_i is given by

$$\delta_f^{[k]}(r, D_i) := \liminf_{r\to\infty} \left(1 - \frac{N_f^{[k]}(r, D_i)}{\deg(D_i) T_f(r)}\right).$$

For brevity, when $k = \infty$, we write $N_f(r, D_i)$, $\delta_f(D_i)$ instead of $N_f^{[\infty]}(r, D_i)$, $\delta_f^{[\infty]}(D_i)$. By the First Main Theorem, one has

$$0 \leq \delta_f(r, D_i) \leq 1. \tag{1.2}$$

Moreover, $\delta_f(r, D_i) = 1$ if and only if

$$N_f(r, D_i) = o(T_f(r)), \tag{1.3}$$

namely the curve f does not meet D_i often. Theorefore (1.1) serves as a weak Second Main Theorem.

When D consists of $q \geq n + 2$ hyperplanes in general position, such a defect relation with truncation at level n is a corollary of the Second Main Theorem of H. Cartan [Car33]. When all components of D are hypersurfaces, such Second Main Theorems were obtained by Eremenko-Sodin [ES91] and Ru [Ru04], without effective truncation level.

When D has $q \leq n + 1$ components, few such Second Main Theorems were known. Following Siu's strategy [Siu04] for the (logarithmic) Kobayashi and Green-Griffiths conjectures [Dar16a], namely by using jet differential techniques [Blo26] and slanted vector fields [Siu02, Mer09, Dar16b], a Second Main Theorem was reached in the case $q = 1$ for general hypersurface of large degree $d \geq 15(5n + 1)n^n$ [HVX19]. By a breakthrough of Riedl-Yang [RY18], one can remove the Zariski dense assumption on $f(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$. Moreover, the exponential degree bound shall be lowered to certain polynomial growth by the recent advancement of Bérczi-Kirwan [BK19].

As a matter of fact, our initial motivation is to study the case of 3 conics in $\mathbb{P}^2(\mathbb{C})$ [GP85, DSW95]. See also [Bab84, Zai88, SY96, BD01, Rou09, Tib13] for nearby hyperbolicity results. When the number of targets is $\leq n+1$, Cartan's Wronskian method seems infertile. One possible approach is by using negatively twisted logarithmic k-jet differentials [HVX19, Theorem 3.1], the existence of which is guaranteed by certain Riemann-Roch calculation, albeit we might have no effective upper bound of $k \gg 1$ by current method (see e.g. [Mer15]). Consequently, the task of "controlling their base loci" is obscure, especially in low degree case, due to the lack of sufficient information of jet differentials for elimination process, even for the "baby" case of 3 conics in $\mathbb{P}^2(\mathbb{C})$ where k equals merely 2. Similar difficulties also appeared in an ampleness conjecture of Debarre [Deb05, Bro16, Xie18, BD18] for $k = 1$.

Backing to our Main Theorem, we will take an alternative geometric approach in which the number

$$n + 1 = \dim_{\mathbb{C}} \mathbb{P}^n(\mathbb{C}) + 1$$

of components of D is critical. Let us sketch the proof now. For simplicity, we assume that every hypersurface

$$D_i = (Q_i = 0) \subset \mathbb{P}^n(\mathbb{C}) \qquad {\scriptstyle (i=1,\ldots,n+1)}$$

is defined by some homogeneous polynomial $Q_i \in \mathbb{C}[z_0, \ldots, z_n]$ of equal degree d. Suppose on the contrary that (1.1) fails, i.e., by (1.2), all defect values reach maximum

$$\delta_f(r, D_i) = 1 \qquad {\scriptstyle (i=1,\ldots,n+1)}. \tag{1.4}$$

For the parabolic Riemann surface $\mathbb{C} \setminus f^{-1}(D)$, we will employ a non-smooth exhaustion function σ [PS14, Subsection 4.1] such that the weighted Euler characteristic $\mathfrak{X}_\sigma(r)$ is negligable compared with the parabolic order function

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f,\sigma}(r)} = 0. \tag{1.5}$$

The key trick is introducing the auxiliary hypersurface $\mathcal{V} \subset \mathbb{P}^n(\mathbb{C})$ defined by the Jacobian

$$\det \Big[\frac{\partial Q_i}{\partial z_j}\Big]_{0\leq i,j\leq n}$$

of degree

$$\sum_{i=0}^{n} d_i - (n+1).$$

Geometrically, $\mathcal{V}$ consists of the critical points of the endomorphism

$$F(z) = [Q_1(z) : Q_2(z) : \cdots : Q_{n+1}(z)] \quad : \quad \mathbb{P}^n(\mathbb{C}) \longrightarrow \mathbb{P}^n(\mathbb{C}).$$

Whence if the entire curve f intersects $\mathcal{V}$ at a point $P \in \mathbb{C}$, the composite curve $g := F \circ f$ must tangent to $\mathcal{W} := F(\mathcal{V})$, i.e., having intersection multiplicity ≥ 2 at P. We will show that, for generic choices of polynomials Q_i, the hypersurfaces $\mathcal{V}$ and $\{H_i\}_{i=1}^{n+1}$ are in general position. Thus we can apply a Second Main Theorem of Min Ru [Ru04] to show that, under the presumed condition (1.4), the intersection frequency of the holomorphic curve $\widetilde{f} := f|_{\mathbb{C}\setminus f^{-1}(D)}$ with $\mathcal{V}$ must be high. This will contradict with another fact, to be obtained in Section 2 following a strategy of Noguchi-Winkelmann-Yamanoi [NWY08], that the parabolic holomorphic curve $\widetilde{g} := g|_{\mathbb{C}\setminus f^{-1}(D)}$ into the semi-abelian variety $(\mathbb{C}^*)^n$ cannot tangent to the effective divisor $\mathcal{W}$ very often. For details of the proof, see Section 5.

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2. Parabolic Nevanlinna theory in semi-abelian varieties

A non-compact Riemann surface $\mathcal{Y}$ is called *parabolic* if it admits a smooth exhaustion function

$$\sigma\colon \mathcal{Y} \to [1, \infty[$$

such that $\log \sigma$ is harmonic outside a compact subset of $\mathcal{Y}$. For every $r > 1$, we denote by

$$B_r := \{y \in \mathcal{Y} : \sigma(y) < r\}, \qquad S_r := \{y \in \mathcal{Y} : \sigma(y) = r\}$$

the open *parabolic ball* and the *parabolic sphere* of radius r respectively. By Sard's theorem, for almost every value $r \in \mathbb{R}_{>1}$, the sphere S_r is smooth. We donote its Euler characteristic by $\chi_\sigma(r)$, and we consider the induced length measure

$$\mathrm{d}\,\mu_r := \mathrm{d}^c \log \sigma|_{S_r}.$$

The *weighted Euler characteristic* $\mathfrak{X}_\sigma(r)$ is then defined by logarithmic average

$$\mathfrak{X}_\sigma(r) := \int_1^r \chi_\sigma(t)\,\frac{\mathrm{d}\,t}{t}.$$

Replacing the exhaustion $\mathbb{C} = \cup_{r>1} \mathbb{D}_r$ by $\mathcal{Y} = \cup_{r>1} B_r$, one can develop Nevanlinna theory for parabolic Riemann surfaces [Sto77, PS14]. Let X be a compact complex manifold. Let L be a holomorphic line bundle on X equipped with some hermitian metric $\|.\|$ with Chern $(1,1)$-form ω_L. Let E be an effective divisor defined by a global nonzero section s of L. In the parabolic context, the standard notions in Nevanlinna theory are defined as follows.

1. The k-truncated counting function
$$N^{[k]}_{f,\sigma}(r, E) \;:=\; \int_1^r \sum_{z \in B_r} \min\{k, \mathrm{ord}_z\, f^*E\}\,\frac{\mathrm{d}\,t}{t} \qquad (k \in \mathbb{N} \cup \{\infty\}; r > 1).$$
2. The proximity function
$$m_{f,\sigma}(r, E) \;:=\; \frac{1}{2\pi} \int_{S_r} \log \frac{1}{\|s \circ f\|}\,\mathrm{d}\,\mu_r.$$
3. The order function
$$T_{f,\sigma}(r, L) \;:=\; \int_1^r \frac{\mathrm{d}\,t}{t} \int_{B_t} f^*\omega_L \qquad (r > 1).$$

By the Jensen formula in the parabolic setting [PS14, Proposition 3.1], one has the following

Parabolic First Main Theorem. *Let $f\colon \mathcal{Y} \to X$ be a holomorphic map such that $f(\mathcal{Y}) \not\subset \mathrm{supp}\, E$. Then*

$$T_{f,\sigma}(r, L) \;=\; m_{f,\sigma}(r, E) + N_{f,\sigma}(r, E) + O(1).$$

For a parabolic Second Main Theorem, the weighted Euler characteristic natural appears.

Parabolic Logarithmic Derivative Lemma. *(*[PS14, Theorem 3.7]*) Let $f\colon \mathcal{Y} \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function. Then there exists some positive constant $C > 0$ such that the following estimate*

$$m_{\frac{f'}{f},\sigma}(r) \leq C\big(\log T_{f,\sigma}(r) + \log r\big) + \mathfrak{X}_\sigma(r)$$

holds true for all $r > 1$ outside an exceptional set of finite Lebesgue measure.

Using the above version of the classical Logarithmic Derivative Lemma, many results in the value distribution of entire holomorphic curves can be translated into the parabolic context. In the next parts, we will do this work in the case of semi-abelian varieties and the case of projective spaces.

3. Holomorphic curves from a parabolic Riemann surface into semi-abelian varieties

Throughout this section, we fix an exhaustion σ on the parabolic Riemann surface $\mathcal{Y}$, and for brevity, we will skip this notation. In [NWY08], Noguchi-Yamanoi-Winkelmann established a second type estimate for k-jet lifts of algebraically nondegenerate entire holomorphic curves f in semi-abelian varieties with the best truncation level one counting function, accepting an error term of the form $\epsilon T_f(r)$ (or equivalently $o(T_f(r))$, see [Yam13, Lemma 1.5]. The optimal truncation level in the counting function in their result yields several applications in studying the degeneracy of holomorphic curves [NWY08, NWY07].

This remarkable result can be translated into the parabolic context. But we need to take into account the weighted Euler characteristic $\mathfrak{X}(r)$ appearing each time when we apply the logarithmic derivative lemma. Hence throughout this section, it is necessary to put the following assumption:

$$\limsup_{r\to\infty} \frac{\mathfrak{X}(r)}{T_f(r)} = 0. \tag{3.1}$$

For application, we only need the result in the case of $(\mathbb{C}^*)^n$. Nevertheless, although we only need estimations for holomorphic curves, we must jump to higher order jets and establish a second main theorem type estimate, not only divisors, but also subvarieties of codimension ≥ 2 (see [NW14, 2.4.1] for definitions of standard notions in Nevanlinna theory for coherent ideal sheaves). For the notion of logarithmic k-jet bundle and its properties, the readers were referred to [DL01, Nog86].

With the assumption (3.1) about weighted Euler characteristic, [NW14, Thm. 6.5.1] for the special case $A := (\mathbb{C}^*)^n$ could be translated into the parabolic context as following

Theorem 3.1. *Let $\mathcal{Y}$ be a parabolic Riemann surface. Let $f\colon \mathcal{Y} \to A := (\mathbb{C}^*)^n$ be an algebraically nondegenerate holomorphic curve. For an integer $k \geq 0$, denote by $J_k f$ the k–jet lift of f and by $X_k(f)$ the Zariski closure of $J_k f$ in the k-jet space $J_k(A)$. Let Z be an algebraic reduced subvariety of $X_k(f)$.*

(1) *There exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*
$$T_{J_k f}(r, \omega_{\bar{Z}}) \leq N^{[1]}_{J_k f}(r, Z) + o(T_f(r)) \parallel,$$
where $\bar{Z}$ denotes the closure of Z in $\bar{X}_k(f)$.

(2) *Assume furthermore that* $\operatorname{codim}_{X_k(f)} Z \geq 2$*, then*
$$T_{J_k f}(r, \omega_{\bar{Z}}) = o(T_f(r)).$$

(3) *In the case where $k = 0$ and Z is an effective divisor D on A, there exists a smooth compactification of A that is independent of f such that*
$$T_f(r, L(\overline{D})) \leq N^{[1]}_f(r, D) + o(T_f(r, L(\overline{D}))) \parallel .$$

This together with the First Main Theorem yields the following

Corollary 3.2. *Let $\mathcal{Y}$ be a parabolic Riemann surface. Let D be an effective divisor on $A := (\mathbb{C}^*)^n$. Let $f\colon \mathcal{Y} \to A$ be an algebraically nondegenerate holomorphic map. Then there exist a smooth compactification of A independent of f such that*
$$N_f(r, D) - N^{[1]}_f(r, D) = o(T_f(r, L(\overline{D}))) \qquad \parallel .$$

Now we provide a sketch proof of Theorem 3.1. We only recall the key steps in [NWY08], and emphasize some required modifications. First, the generalization of the Lemma on logarithmic forms [NW14, Lem 4.7.1] to the parabolic context is straightforward.

Lemma 3.3. *Let M be a complex projective algebraic manifold and let D be a reduced divisor on M. Let $f\colon \mathcal{Y} \to M$ be a holomorphic from a parabolic Riemann surface $\mathcal{Y}$ into M such that $f(\mathcal{Y}) \not\subset D$. Let ω be a logarithmic k-jet differential along D over M. Put $\xi := \omega(J_k f)$, then*

$$m_\xi(r) \leq S_f(r) + C\mathfrak{X}(r) = o(T_f(r)) \parallel .$$

For an integer $k \geq 0$, let $J_k(A)$ denote the k-jet bundle over A. There is a trivialization

$$J_k(A) = A \times J_{k,A} = A \times \mathbb{C}^{nk},$$

where $n = \dim A$, such that the natural induced A-action is given by $a\colon (x, v) \to (x + a, v)$ for all $x \in A, v \in \mathbb{C}^{nk}$. Denote by $J_k f$ the k–jet lift of f and by $X_k(f)$ the Zariski closure of $J_k f$ in the k-jet space $J_k(A)$. Let $\mathbf{B} := \mathrm{St}_A(X_k(f))$ be the stabilizer group with respect to the natural A-action and let $q\colon A \to A/\mathbf{B}$ be the natural projection. Then the jet projection method [NW14, Thm. 6.2.6] together with Lemma 3.3 yield $T_{q\circ f}(r) = o(T_f(r))$. Furthermore, we can assume $\dim \mathbf{B} > 0$, otherwise we would have $T_f(r) = o(T_f(r))$, which is impossible.

We will first establish a second main theorem for jet lifts. Let Z be an algebraic reduced subvariety of $X_k(f)$. Let $\mathbf{B}^0 = \mathrm{St}^0_A(X_k(f))$ denote the identity component of $\mathbf{B}$. Then

$$\dim \mathbf{B}^0 > 0 \quad \text{and} \quad T_{q_k^{\mathbf{B}^0}\circ J_k f}(r) = o(T_f(r)) \parallel, \tag{3.2}$$

where $q_k^{\mathbf{B}^0}\colon J_k(A) \to J_k(A)/\mathbf{B}^0 \cong (A/\mathbf{B}^0) \times J_{k,A}$ is the natural projection. This corresponds to [NW14, (6.5.9)] and hence, we can translate [NW14, Thm. 6.5.6] to the parabolic setting as follow

Lemma 3.4. *There exists a compactification $\bar{X}_k(f)$ of $X_k(f)$, and a positive integer ℓ_0 such that*

$$m_{J_k f}(r, \bar{Z}) = o(T_f(r)) \parallel,$$
$$T_{J_k f}(r, \omega_{\bar{Z}}) \leq N^{[\ell_0]}_{J_k f}(r, Z) + o(T_f(r)) \parallel,$$

where $\bar{Z}$ denotes the closure of Z in $\bar{X}_k(f)$.

Our next target is to show that the impact of $J_k f$ and a subvariety of $X_k(f)$ of codimension ≥ 2 is relatively small.

Lemma 3.5. *Let $Z \subset X_k(f)$ be a subvariety with $\mathrm{codim}_{X_k(f)} Z \geq 2$. Then*

$$T_{J_k f}(r, \omega_{\bar{Z}}) = o(T_f(r)) \parallel, \tag{3.3}$$

in particular we have

$$N_{J_k f}(r, Z) = o(T_f(r)) \parallel . \tag{3.4}$$

This result is an analog of [NW14, Thm. 6.5.17], where the proof follows the same lines, except a small needed modification in the first reduction. We reduce to the case where A admits a splitting $A = B \times C$, where B, C are semi-abelian varieties of positive dimensions with

$$B \subset \mathrm{St}(X_k(f))^0, \qquad \forall\, k \geq 0$$
$$\text{and } T_{q^B\circ f}(r) = o(T_f(r)) \parallel,$$

where $q^B\colon A \to A/B = C$ denotes the second projection. To do this, we consider the set $\mathfrak{B}$ of all semi-abelian subvarieties $B \subset A$ such that

$$T_{q^B\circ f}(r) = o(T_f(r)) \parallel .$$

We then use (3.2) and continue the arguments as in the proof of [NW14, Thm. 6.5.17]. Note that since we only work with $A = (\mathbb{C}^*)^n$, the result in [NW14, Lem.6.5.25] is automatically satisfied without using the simple connected condition on $\mathbb{C}$ to lift the curve to the universal coverings. In view of Lemma 3.4, it is enough to show that

$$N^{[1]}_{J_k f}(r, Z) = o(T_f(r)) \parallel .$$

Using the induction on the dimension of Z, it is enough to check the above estimate for the nonsingular part Z^{ns} of Z. Following the same lines as in [NW14, 6.5.3], we can find a sequence $n(\ell)$ such that $\lim_{\ell\to\infty} \frac{n(\ell)}{\ell} = 0$ and

$$(\ell+1) N^{[1]}_{J_k f}(r, Z^{ns}) \leq n(\ell)\, O(T_f(r)) + o(T_f(r)) \parallel,$$

which yields the required estimate. This finish the proof of the Lemma 3.5. □

End of the proof of Theorem 3.1. We follow the proof of [NW14, subsection 6.5.4]. It is enough to consider the case where Z is a reduced Weil divisor on $X_f(f)$ with the irreducible decomposition $Z = \sum_i Z_i$. Using 3.4, we have

$$\begin{aligned} T_{J_k f}(r, \omega_{\bar{Z}}) &\leq N^{[\ell_0]}_{J_k f}(r, Z) + o(T_f(r)) \parallel, \\ &\leq N^{[1]}_{J_k f}(r, Z) + \ell_0 \sum_{i<j} N^{[1]}_{J_k f}(r, Z_i \cap Z_j) + \ell_0 \sum_i N^{[1]}_{J_{k+1} f}(r, J_1(Z_i)) + o(T_f(r)) \parallel . \end{aligned} \tag{3.5}$$

Since $\mathrm{codim}_{X_k(f)}(Z_i \cap Z_j) \geq 2$ for $i \neq j$, the second term in the right hand side of (3.5) can be estimated using Lemma 3.5:

$$\ell_0 \sum_{i<j} N^{[1]}_{J_k f}(r, Z_i \cap Z_j) = o(T_f(r)).$$

We now treat the third term of (3.5). We consider two cases depending on the position of $\mathbf{B}^0_{k+1} := \mathrm{St}^0_A(X_{k+1}(f))$with respect to $\mathrm{St}^0(Z_i)$.

(1) In the case where $\mathbf{B}^0_{k+1} \not\subset \mathrm{St}^0(Z_i)$, we have ([NW14, Lem. 6.6.50]):

$$\mathrm{codim}_{X_{k+1}(f)}(X_{k+1}(f) \cap J_1(Z_i) \geq 2,$$

where we can apply Lemma 3.5 to obtain

$$N^{[1]}_{J_{k+1} f}(r, J_1(Z_i)) = o(T_f(r)).$$

(2) In the case where $\mathbf{B}^0_{k+1} \subset \mathrm{St}^0(Z_i)$, we consider the natural projection $q_k^{\mathbf{B}^0_{k+1}} : X_k(f) \to X_k(f)/\mathbf{B}^0_{k+1}$. The image of Z_i under this map is contained in a divisor on $X_k(f)/\mathbf{B}^0_{k+1}$, and hence, we can argue as in [NW14, Thm. 6.5.6, case (a)] to get

$$N^{[1]}_{J_{k+1} f}(r, J_1(Z_i)) \leq N_{J_{k+1} f}(r, J_1(Z_i)) = o(T_f(r)).$$

This finishes the proof of Theorem 3.1.

4. Holomorphic curves from a parabolic Riemann surface into projective spaces

A family $\{D_i\}_{1\leq i\leq q}$ of $q \geq n+2$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ is said to be *in general position* if any $n+1$ hypersurfaces in this family have empty intersection, namely

$$\bigcap\nolimits_{\imath\in I} D_i = \varnothing \quad \forall\, I \subset \{1,2,\dots,q\},\, |I| = n+1.$$

In [Ru04], the author confirms a conjecture of Shiffman by extending the classical Cartan's Second Main Theorem to the case of nonlinear targets. In the parabolic context, this result reads as follows.

Theorem 4.1. *Let $\mathcal{Y}$ be a parabolic Riemann surface. Let $\{D_i\}_{1\leq i\leq q}$ be a family of $q \geq n+2$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. Then for any algebraically nondegenerate holomorphic curve $f\colon \mathcal{Y} \to \mathbb{P}^n(\mathbb{C})$, there exists a positive constant C such that*

$$(q-n-1)T_f(r) \leq \sum_{i=1}^{q} \frac{N_f(r, D_i)}{\deg(D_i)} + C\mathfrak{X}(r) + o(T_f(r)) \qquad \| \,.$$

The proof follows the same lines as in [Ru04], where the filtration method of Corjava-Zannier [CZ04] was employed to reduce the problem to the linear case [Ru97, Voj97].

5. Proof of the Main Theorem

Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let $D = \sum_{i=1}^{n+1} D_i$ be a simple normal crossing divisor on $\mathbb{P}^n(\mathbb{C})$. Let Q_i be the homogeneous polynomials defining D_i. Let $F : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ be the endomorphism of degree $d = \operatorname{lcm}(d_1, \dots, d_{n+1})$ defined by

$$F(z) := [Q_1^{m_1}(z) : \cdots : Q_{n+1}^{m+1}(z)], \tag{5.1}$$

where $d_i m_i = d$. By construction, F sends the complement of D in $\mathbb{P}^n(\mathbb{C})$ to $(\mathbb{C}^*)^n$. Its critical points are consisted in the hypersurfaces D_i with multiplicities $m_i - 1$ and a hypersurface $\mathcal{V}$ of degree $\sum_{i=1}^{n+1} d_i - (n+1) > 0$ defined by

$$\det\left(\frac{\partial Q_i}{\partial z_j}\right)_{0\leq i,j\leq n} = 0.$$

Putting the generic condition that $\mathcal{V}$, together with the complements of D form a family of hypersurfaces in general position, namely $\mathcal{V} \cap (\cap_{i\in I} D_i) = \varnothing$ for any $I \subset \{1,\dots,n+1\}, |I| = n$. Set $g := F \circ f$ and $\mathcal{W} = F(\mathcal{V})$. It is not hard to check that

$$T_g(r) = O(T_f(r)).$$

We notice that if the curve f meets $\mathcal{V}$ at a point P, then g tangents with $\mathcal{W}$ at the point $F(P)$. More precisely, we have

Proposition 5.1. [NWY07, Thm 3.12]

$$\operatorname{ord}_z g^*\mathcal{W} \geq \operatorname{ord}_z f^*\mathcal{V} + 1 \qquad (\forall\, z\in\mathbb{C}).$$

Now put $\mathcal{E} = f^{-1}(D)$, which is a closed, countable set of points in $\mathbb{C}$. Suppose that $\mathcal{E} = \{a_i\}_{i=1}^{\infty}$. Note that the number of a_i in the disc $\mathbb{D}_t$ is exactly $n_f^{[1]}(t, D)$, which is finite. Denote by $\widetilde{f}$, $\widetilde{g}$ the restriction on

$\mathcal{Y} := \mathbb{C} \setminus \mathcal{E}$ of f, g respectively. Following [PS14, 4.1, example 2], let $(r_i)_i$ be a sequence of positive real numbers, such that all the discs or radii r_i centered at a_i are disjoints. We define the exhaustion σ for $\mathcal{Y}$ as

$$\log \sigma = \log^+ |z| + \sum_{i=1}^{\infty} \log^+ \frac{r_i}{|z - a_i|}.$$

Note that here the function $\tau = \log \sigma$ is not necessary outside compact set. Nevertheless, the Jensen formula is still valid under the following form

$$\int_1^r \frac{dt}{t} \int_{B_t} dd^c v = \int_{S_r} v d\mu_r - \int_{B_r} v dd^c \tau,$$

where $v \colon \mathcal{Y} \to [-\infty, +\infty)$ is a function defined on $\mathcal{Y}$ such that locally near any point of $\mathcal{Y}$ it can be written as a difference of two subharmonic functions. Furthermore, basic computation yields $B_r \subset \Delta_r$ and

$$\mathfrak{X}_\sigma(r) = N^{[1]}_{f,\sigma}(r, D) \leq N^{[1]}_f(r, D) + \log r. \tag{5.2}$$

Suppose on the contrary that (1.1) does not hold. Then the weighed Euler characteristic $\mathfrak{X}(r)$ satisfies

$$\limsup_{r \to \infty} \frac{\mathfrak{X}(r)}{T_{\widetilde{f},\sigma}(r)}.$$

This allow us to use all of the previous results. First, apply Theorem 4.1, we have

$$T_{\widetilde{f},\sigma}(r) \leq \frac{N_{\widetilde{f},\sigma}(r, \mathcal{V})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \parallel . \tag{5.3}$$

Applying Corollary 3.2 for $\widetilde{g}$, we have

$$N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W}) = o(T_{\widetilde{g},\sigma}(r)) \qquad \parallel . \tag{5.4}$$

On the other hand, it follows from Prop. 5.1 that

$$N_{\widetilde{f},\sigma}(r, \mathcal{V}) \leq N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W}). \tag{5.5}$$

Combining (5.3), (5.4), (5.5), we obtain

$$\begin{aligned} T_{\widetilde{f},\sigma}(r) &\leq \frac{N_{\widetilde{f},\sigma}(r, \mathcal{V})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \parallel \\ &\leq \frac{N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \parallel \\ &= o(T_{\widetilde{g},\sigma}(r)) + o(T_{\widetilde{f},\sigma}(r)) \parallel, \end{aligned}$$

which is a contradiction. This finishes the proof of the Main Theorem.

Remark 5.2. In the case where $f \colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is an algebraically nondegenerate holomorphic curve, $\mathcal{C}$ is the collection of two lines and one conic in $\mathbb{P}^2(\mathbb{C})$, in a private note, Noguchi could obtain a weak Second Main Theorem of the form

$$T_f(r) \leq \alpha N_f(r, \mathcal{C}) + [N^{[2]}_f(r, \mathcal{V}) - N^{[1]}_f(r, \mathcal{V})] + o(T_f(r)) \parallel,$$

where $\mathcal{V}$ is the critical curve of the endomorphism defined as above. Although the right hand side of the above inequality involves a quantity depending on $\mathcal{V}$ (which actually counts the number of tangent points of f and $\mathcal{V}$), this term is negligible when f omits $\mathcal{C}$.

Remark 5.3. Our result can be extended to the case of entire holomorphic curves into algebraic varieties of log-general type X with $\overline{q}(X) = \dim X$.

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