Hermitian algebraic K–theory, Wagoner complex, and the root system D

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Abstract

This manuscript is a based on some recent results of ongoing project which is devoted to investigation of the role of root systems, Weyl and Coxeter groups in algebraic K-theory.

For the root system D we construct an analog of the Wagoner complex used in his proof of the equivalence of K_*^Q and K_*^{BN} (linear) algebraic K-theories. We prove that the corresponding K-theory KU_*^D for the even orthogonal group is naturally isomorphic to KU_*^{BN}-theory constructed by Yu.P. Solovyov and A.I. Nemytov. Also some open problems are raised.

Introduction

Let A be an associative ring. After pioneering works on the algebraic $K_n(A)$ groups for $n = 0, 1, 2$ several definitions of higher algebraic K-groups were proposed. The question of comparing the definitions of the higher K-groups was very natural. Most attention was paid to the sequence of functors and natural transformations described for example in [6]:

$$K_*^Q \to K_*^V \to K_*^S \to K_i^{K-V},$$

The first natural transformation lately was decomposed in [5, 7] into composition of two natural transformarions

$$K_*^Q(A) \xrightarrow{i} K_*^{BN}(A) \to K_*^V(A).$$

Some transformations were proved to be equivalences with less difficulties than other (for K_i^{K-V} one should assume that the argument ring A is left regular).

One of the most interesting case was the equivalence of the Quillen K-theory and the Volodin K-theory which was proved in [3]. Very remarkable proof of this equivalence was found later, see [4].

The groups K_*^{BN} were introduced by Wagoner in [7] and they are a version of the Volodin K–theory K_*^V. The proof in [5] uses the combinatoric of the root system A and the combinatoric of the corresponding partition of $\mathbb{R}^n$ into facettes.

In 1980 Yu.P.Solovyov and A.I.Nemytov for the rings with involution had established natural equivalence of KU_*^Q (Hermitian analog of the Quillen K-theory) and the Hermitian analog of K_*^{BN}-theory. Their construction and the proof were based on the combinatoric of the root system C. In this paper we consider an analog KU_*^D of the K_*^{BN}-theory which is constructed on the root system D and show that for the even orthogonal group this K–theory is equivalent to KU_*^Q.

1 Basic definitions

Let A be an associative ring with 1 equipped with an involution $a \mapsto a^*$ satisfying conditions: (1) $1^* = 1$; (2) $a^{**} = a$; (3) $(a+b)^* = a^* + b^*$; (4) $(ab)^* = b^* a^*$. Let us also fix a central element ε such that $\varepsilon^*\varepsilon = 1 = \varepsilon\varepsilon^*$. Fix an additive subgroup $\Lambda \subset A$ such that

(1) $a\Lambda a^* \subset \Lambda$ for all $a \in A$;

(2) $\Lambda_{min} = \{a - \varepsilon a^* \ : \ a \in A\} \subset \Lambda \subset \Lambda_{max} = \{a \in A \ : \ a = -\varepsilon a^*\}$.

Denote by Λ_{2n} the additive subgroup of $M_{2n}(A)$ consisting of matrices (x_{ij}) with elements satisfying the relations $x_{ij} = -\varepsilon x_{ji}^*$ и $x_{ii} \in \Lambda$.

The set of matrices

$$U_{2n}(A) = U_{2n}(A, \varepsilon, \Lambda) = \\ = \{X \in GL_{2n}(A) \ : \ X^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \operatorname{mod} \Lambda_{2n}\},$$

with matrix multiplication is call a *unitary group*. It depends on the choice of ε and Λ but for simplicity we shall denote it by $U_{2n}(A)$. Also it is often denoted by ${}_{\varepsilon}GQ_{2n}(A, \Lambda)$. This definition is due to Bak [**?**]. For particular choices of the parameters ε and Λ one can obtain classical groups like the general linear group, the symplectic group and the even orthogonal group etc.

Passing to the limit with respect to the standard embedding $U_{2n}(A) \to U_{2(n+1)}(A)$ one obtains the group $U(A)$. Define the elementary subgroup to be the subgroup generated by elementary matricies that is the matrices of the form

$$
\begin{aligned}
s_{ij}(a) &= \begin{pmatrix} 1 + aE_{ij} & 0 \\ 0 & 1 - a^* E_{ji} \end{pmatrix}, \\
r_{ij}(a) &= \begin{pmatrix} 1 & aE_{ij} - \varepsilon^* a^* E_{ji} \\ 0 & 1 \end{pmatrix}, \\
t_{ij}(a) &= \begin{pmatrix} 1 & 0 \\ aE_{ij} - \varepsilon a^* E_{ji} & 1 \end{pmatrix}, \\
p_i(b) &= \begin{pmatrix} 1 & bE_{ii} \\ 0 & 1 \end{pmatrix}, \\
q_i(c) &= \begin{pmatrix} 1 & 0 \\ cE_{ii} & 1 \end{pmatrix},
\end{aligned}
$$

where $a \in A$, $b^*, c \in \Lambda$. It is known (see for example [2]) that $EU(A) = [U(A), U(A)] = [EU(A), EU(A)]$, that is $EU(A)$ is a perfect subgroup and it coincides with the commutant of $U(A)$.

Applying the plus-construction one obtains the definition of the Quillen hermitian K–theory: $KU_j^Q(A) = \pi_j(BU(A)^+)$.

Now let us remind the definition of $KU_*^{BN}(A)$ (see. [1]). Consider the hyperplanes in $\mathbb{R}^n$ given by the equations $e_i \pm e_j = 0$, $1 \leq i < j \leq n$, and $e_j = 0$, where e_i is the dual basis. Let us call by the facette of codimention j a component in the complement of the union of all $(j+1)$–fold intersection of the hyperplanes in the union of all j-fold intersections. Define the ordering of the facettes: $F < G$ iff $F \subseteq \overline{G}$.

Define $\mathcal{P}_C^n$ to be the simplicial complex with k–simplices of the form $F_0 < F_1 < \ldots < F_k$. The inclusion $\mathcal{P}_C^n \to \mathcal{P}_C^{n+1}$ is induced by the repetition of the last coordinate of a point in $\mathbb{R}^n$. Passing to the limit with respect to the inclusions one obtains the complex $\mathcal{P}_C$.

Let F be a facette in $\mathbb{R}^n$. Denote by $G_F \subset U_{2n}(A)$ the subgroup generated by the elements $s_{ij}(a)$ where $e_i - e_j > 0$ on F, $r_{ij}(a)$ where $e_i + e_j > 0$ on F, $t_{ij}(a)$ where $e_i + e_j < 0$ on F, $p_k(b)$ where $e_k > 0$ on F, $q_k(c)$ where $e_k < 0$ on F. This is so called unipotent subgroup corresponding to the facette F.

Define the ordering on the set of pairs (α, F), where $\alpha \in U_{2n}(A)$ and F is a facet: $(\alpha', F') < (\alpha'', F'')$ iff $\alpha' G_{F'} \subset \alpha'' G_{F''}$ and $F' \subset \overline{F''}$.

Denote by $U_{2n}^{BN}(A)$ the simplicial complex with k-simplexes of form $(\alpha_0, F_0) < (\alpha_1, F_1) < \ldots < (\alpha_k, F_k)$, where $F_0, F_1, \ldots, F_k$ are facettes and $\alpha_j \in U_{2n}(A)$ for all j. The sub complex defined by the condition $\alpha_j \in EU_{2n}(A)$ is denoted by $EU_{2n}^{BN}(A)$. Denote the limit groups by $U^{BN}(A)$ и $EU^{BN}(A)$ correspondingly. One can check that

$$U^{BN}(A) = KU_1^Q(A) \times EU^{BN}(A).$$

Let us define

$$KU_n^{BN}(A) = \pi_{n-1}(U^{BN}(A)), \text{ где } n \geq 1.$$

In [1] it was shown that the functors KU_n^{BN} and KU_n^Q ($n \geq 2$) are equivalent, and moreover, there is a natural homotopy equivalence $U^{BN}(A) \cong \Omega BU(A)^+$.

2 K^D-groups and the even orthogonal group

The ideas presented in the previous section lead us to groups $K_*^D(A)$ whose construction is based on the root system D.

Consider facettes in $\mathbb{R}^n$ defined by the hyperplanes $e_i \pm e_j = 0$, $1 \leq i < j \leq n$. Denote these facettes by Φ_j to distinguish them the from the facettes defined by the root system C.

Let $\mathcal{P}_D^n$ denote the simplicial complex whose k–simplices are $(k+1)$-tuples $\Phi_0 < \Phi_1 < \ldots < \Phi_k$. For a D-facette $\Phi \subset \mathbb{R}^n$ denote by $G_\Phi \subset U_{2n}(A)$ the subgroup generates by the elements $s_{ij}(a)$ where $e_i - e_j > 0$ on Φ, $r_{ij}(a)$ where $e_i + e_j > 0$ on Φ, $t_{ij}(a)$ where $e_i + e_j < 0$ on Φ.

The set of all pairs $(\alpha; \Phi)$, where $\alpha \in U_{2n}(A)$ and Φ is a D-facet, is partially ordered by the condition that $(\alpha', \Phi') < (\alpha'', \Phi'')$ iff $\alpha' G_{\Phi'} \subseteq \alpha'' G_{\Phi''}$ and $\Phi' \subseteq \overline{\Phi''}$.

Let $U_{2n}^D(A)$ denote the simplicial complex whose k-simplices are $(\alpha_0, \Phi_0) < (\alpha_1, \Phi_1) < \ldots < (\alpha_k, \Phi_k)$ where $\Phi_0, \Phi_1, \ldots, \Phi_k$ are D-facettes and $\alpha_j \in U_{2n}(A)$. Also let $U^D(A) = \lim\limits_{\rightarrow} U_{2n}^D(A)$

Define

$$KU_n^D(A) = \pi_{n-1}(U^D(A)), \text{ где } n \geq 1.$$

Now let A be a commutative ring with 1. Let $a^* = a$, $\varepsilon = 1$, $\Lambda = \Lambda_{min} = 0$. Then the corresponding unitary group $U_{2n}(A, \varepsilon, \Lambda)$ coincides with the even orthogonal group $O_{2n}(A)$.

Theorem 1. *There exists a natural isomorphism $K_n^D(A) = K_n^{BN}(A)$.*

Remind (see. [6, 1]) that one has cartesian squares of spaces

$$\begin{array}{ccc} W_C(\alpha G_F) & \rightarrow & E(U(A)) \\ \downarrow & & \downarrow \\ W_C(A) & \rightarrow & BU(A) \end{array} \tag{1}$$

and

$$\begin{array}{ccc} W_D(\alpha G_\Phi) & \rightarrow & E(U(A)) \\ \downarrow & & \downarrow \\ W_D(A) & \rightarrow & BU(A) \end{array} \tag{2}$$

Let us describe the spaces from these diagrams. $W_C(A)$ is the realization of the simplicial space which in dimension k is the disjoint union of the spaces $(F_0 < \ldots < F_k) \times BG_{F_0}$. $W_C(\alpha G_F)$ is the realization of the simplicial space which in dimension k is the disjoint union of the spaces $((\alpha_0, F_0) < \ldots < (\alpha_k, F_k)) \times E(\alpha_0 G_{F_0})$. The definitions of $W_D(A)$ and $W_D(\alpha G_\Phi)$ are analogous. The universal covering $E(G) \to BG$ on the simplicial level is defined by the correspondence $(g_0, g_1, \ldots, g_k) \mapsto (g_0^{-1} g_1, \ldots, g_{k-1}^{-1} g_k)$. And finally $E(\alpha G_F)$ is the geometric realization of the simplicial subcomplex of $E(U(A))$ whose k-simplices are $(g_0, \ldots, g_k)$ where $g_k \in \alpha G_F$. The definition of the space $E(\alpha G_\phi)$ is analogous.

On the level of bisimplicial sets the cartesian square (1) is defined by the correspondences

$$\begin{array}{ccc} ((\alpha_0, F_0) < \ldots < (\alpha_k, F_k)); (g_0, \ldots, g_l)) & \to & (g_0, \ldots, g_l) \\ \downarrow & & \downarrow \\ (F_0 < \ldots < F_k; (g_0^{-1} g_1, \ldots, g_{l-1}^{-1} g_l)) & \to & (g_0^{-1} g_1, \ldots, g_{l-1}^{-1} g_l) \end{array}$$

and the cartesian square (2) is defined by analogous correspondences with substitution of Φ_j instead of F_j.

The spaces $E(\alpha G_F)$ and $E(\alpha G_\Phi)$ are contractible therefore one has homotopy equivalences $W_C(\alpha G_F) \simeq U^{BN}(A)$ and $W_D(\alpha G_\Phi) \simeq U^D(A)$.

Hence to compare the groups $K_*^{BN}(A)$ and $K_*^D(A)$ one could try to compare the cartesian squares (1) and (2). So to prove theorem 1 it is sufficient to prove homotopy equivalence of the lower left corners of (1) and (2), that is to establish natural homotopy equivalence of the spaces $W_C(A)$ and $W_D(A)$.

Let us remind that a sheaf X of spaces over a simplicial complex K is a collection of spaces $\{X_\sigma : \sigma \in K\}$ and maps $i_{\sigma\tau} : X_\tau \to X_\sigma$ for all $\sigma < \tau$ such that $i_{\gamma\sigma} i_{\sigma\tau} = i_{\gamma\tau}$ whenever $\gamma < \sigma < \tau$. A simplicial subdivision K' of K induces a subdivision X' of X as follows: for $\sigma' \in K'$ define $X'_{\sigma'} = X_\sigma$ where $\sigma \in K$ is the smallest simplex containing σ. If $\sigma' < \tau'$ belong to K' and σ, τ are the smallest simplices of K containing σ', τ' respectively, then $\sigma < \tau$ and we let $i_{\sigma'\tau'} = i_{\sigma\tau}$.

The realization of a sheaf X is the space $|X|$ which is obtained from the disjoint union $\coprod_{\sigma \in K} \sigma \times X_\sigma$ by identification of points (s, x) and $(s, i_{\sigma\tau}(x))$ where $s \in \sigma < \tau$ and $x \in X_\tau$. The natural map $|X'| \to |X|$ is a homeomorphism.

Obviously the spaces $W_C(A)$ and $W_D(A)$ are the realization of some sheaves over $\mathcal{P}_C$ and $\mathcal{P}_D$ respectively. Denote these sheaves by W_C and W_D respectively.

Intersections of the unit sphere with D-facettes (C-facettes) define the complex $\mathcal{Q}_D^n$ ($\mathcal{Q}_C^n$ respectively). The complexes $\mathcal{P}_D^n$ and $\mathcal{P}_C^n$ are barycentric subdivisions of $\mathcal{Q}^n_D$ and $\mathcal{Q}^n_C$ respectively. The complex $\mathcal{Q}^n_C$ is a subdivision of $\mathcal{Q}^n_D$. More precisely, a simplex of $\mathcal{Q}^n_D$ is either a simplex of $\mathcal{Q}^n_C$ or is divided into two part by one of the hyperplanes $e_k = 0$. Namely, if two hyperplanes $e_k = 0$ and $e_l = 0$ intersect the simplex of $\mathcal{Q}^n_D$ transversally then in the corresponding facette there exist points such that $e_k + e_l > 0$ and points such that $-e_k - e_l < 0$ (or points such that $e_k - e_l > 0$ and points such that $-e_k + e_l < 0$).

One can check that for a C-facette F and the smallest D-facette Φ containing F one has $G_F = G_\Phi$. Note that in general case for rings with involution this isomorphism does not hold.

Therefore there exists a common subdivision $\hat{\mathcal{P}}^n$ of complexes $\mathcal{P}^n_D$ and $\mathcal{P}^n_C$ such that, induced sheaves W'_C and W'_D over it coincide. Taking the realizations we obtain our claim.

3 Further discussion

Assume $\Lambda = \Lambda_m in \neq 0$. This case is more difficult for the following reason. For a C-facette F and the smallest D-facette Φ containing F the inclusion $G_\Phi \subset G_F$ is presumably strict for most facettes. The reason is that for $\Lambda \neq 0$ there are so called long roots unipotent $p_i(b)$ and $q_i(c)$ which are not used as generators for elementary group for the root system D. This presumably shows that one cannot expect the group generated by the short root unipotents s_{ij}, t_{ij}, r_{ij} to be perfect and coinciding with the commutant of the corresponding unitary group. There is nontrivial example even in commutative case. For a commutative ring A, trivial involution, $\varepsilon = -1$ and $\Lambda = \Lambda_{max} = A$ one obtains symplectic K-theory. This leads for the following question: what part of the symplectic K-theory can be recovered from $KU^D(A)$?

Nevertheless the following statement shows that in the case $\Lambda = \Lambda_{min}$ the difference between the root systems C and D and corresponding generators of $EU(A)$ is more subtle.

Lemma 2. *Assume $\Lambda = \Lambda_{min}$. Then the group $EU(A)$ is generated by elementary matrices s_{ij}, r_{ij}, t_{ij}.*

Proof. In $EU_{2n}(A)$ for $n \geq 2$ one has the relations

$$[s_{ij}(a), r_{ji}(1)] = p_i(a - a^*\varepsilon^*)$$
$$[s_{ji}(a), t_{ij}(\varepsilon)] = q_i(a - a^*\varepsilon)$$

which shows that even in unstable range $EU_{2n}(A)$ is generated by the short root unipotents. □

Unfortunately there are no such statement for groups like G_F. To be more precise assume that for a facette F one has $e_i - e_j > 0$ and $e_i + e_j > 0$ on F. Hence $e_i > 0$ on F. Generators s_{ij}, r_{ij}, p_i belong to G_F and one can use the relation from the proof of the previous lemma to see that p_i can be excluded from the list of generators of G_F.

On the other hand this facette F has a face F_0 defined by the equation $e_i + e_j = 0$ and one no longer can apply the relation $[s_{ij}(a), r_{ji}(1)] = p_i(a - a^*\varepsilon^*)$ because there is no generator r_{ij} in G_{F_0}.

Moreover, consider for example the facette F defined by $e_1 = e_2 = \ldots = e_n > 0$. The corresponding group (in unstable range) G_F is abelian and is generated only by the long root unipotents p_i, $i = 1, \ldots, n$, while there are no short root generators.

For a C-facette F define the group G_F^D to be the subgroup of G_F generated only by short roots which are positive on F. Clearly for $F' \subset F$ on has the inclusions

$$\begin{array}{ccc} G_F & \supset & G_{F'} \\ \cup & & \cup \\ G_F^D & \supset & G_{F'}^D \end{array} \tag{3}$$

Now consider the space $\tilde{W}_C(A)$ which is defined in the same way as $W_C(A)$ using the groups G_F^D instead of G_F. Clearly one has a map $h : \tilde{W}_C(A) \to W_C(A)$ and the map $H : \tilde{\mathcal{P}}_C \to \mathcal{P}_C$ of the corresponding simplicial sheaves.

Hence to investigate whether the map h is a homotopy equivalence it is natural to investigate the map H. For that purpose it could be useful to consider a kind of cokernel of H because the quotient G_F/G_F^D is not too big and the generator of G_F which are not in G_F^D generate an abelian subgroup in G_F.

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